# Systems with both constant and time-varying delays: a switched systems approach and application to observer-controller co-design

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***Abstract*— In this paper, we study the application of switched systems stability criteria to derive delay-dependent conditions for systems affected by both a constant and a time-varying delay. The main novelty of our approach lies on the use of path-complete Lyapunov techniques along with the proposition of a new modified functional to obtain convex analysis conditions while avoiding the need of computing a dwell time for each mode in a switched system representation, as usual in the *switched approach* for time-delay systems. Furthermore, we leverage the developed analysis to obtain LMIs for the closed-loop stabilization of systems with time-varying sensor delays by means of an observer-based compensator. A numerical example illustrates the proposed methods.**

## I. Introduction

Time delay appears in a wide variety of systems and is frequently caused by transport of mass/information. For a review of analysis and stabilization techniques for time-delay systems, we refer to [7].

Many works dealing with systems subject to time-varying delays have opted for a *switched* formulation, considering the value of the delay as a source of switching behavior; for an overview concerning switching systems analysis see [16]. Applications of this modeling strategy among the delay systems literature has been explored in multiple contexts. In [10], the equivalence between the existence of multiple Lyapunov functions for a switched delay-augmented representation and the existence of general quadratic Lyapunov-Krasovskii functionals (LKFs) for the original time-varying delay system is demonstrated. Similar equivalence results were proven, in the continuous-time setting, in [9]. Related connections between switched systems and delay systems Lyapunov-based stability conditions were studied more recently in [17], [5]. The relations between switched systems, systems with data-losses and delay systems is studied in [14], in which controllability and observability conditions are proposed. In addition to these general results, the switched-representation has been considered, while avoiding the so-called delay state-augmentation, in [13], for the continuous-time case, splitting the delay interval in multiple zones and then imposing a dwell time on each subsystem, following the approach of [16, Section 3]. In [21], stability of discrete time-delay systems is approached splitting the delay interval in two zones, leading to a 2-mode delay switched system representation, with one subsystem possibly unstable; the overall stability is then ensured imposing dwell time and persistence of switching assumptions.

In this paper we study stability of delay systems with *both* a bounded *time-varying* delay $d(k) \in [d_m, d_M]$ for $k \in \mathbb{N}$ and a *nominal* constant delay $d_n \in [d_m, d_M]$. The motivation to study this class of systems, and in particular to consider a term depending on a constant delay $d_n$, comes from the case of an observer-based control closed loop (for more details, see Section IV), where the observer is designed by considering an estimate of the unknown time-varying delay in an attempt to observe the non-delayed state $x(k)$. The introduction of the constant delay $d_n$ can improve the features of the closed-loop system in comparison with direct feedback of the output $y(k)$ which is affected by the time-varying delay $d(k)$. Nonetheless, the LKF-based stability conditions available in the literature do not model well the interaction between the constant $d_n$ and the time-varying delay $d(k)$, which motivates the switched systems representation used in this paper.

Differently from [10], we do not represent the considered delay system as a delay-free one, but rather as a switched system composed by two delay subsystems where the value of the time-varying delay $d(k)$ *with respect to* the constant delay $d_n$ is used to define the underlying switching signal. To study stability and stabilizability conditions, we make use of path-complete Lyapunov functions approach (introduced in [1] for delay-free switched systems). In this framework, the structure of the sufficient Lyapunov conditions is given by an underlying flexible combinatorial structure, a *path-complete graph*, which somehow encodes the switching signals the system will follow. These techniques have been proven to be less conservative than more classical multiple Lyapunov techniques for switched systems. For an overview see [1], [18] and references therein. We then leverage the developed conditions for the stabilization problem of systems with output time-varying delays. With the aid of Finsler's Lemma and algebraic manipulations, the proposed conditions for the design of stabilizing nominal-delay observers are rewritten in the form of linear matrix inequalities (LMIs).

Summarizing, although splitting the time-delay interval in sub-zones is a rather common idea in delay systems literature ([13], [21]), some important specificities of our manuscript can be enlisted: i) We propose a modified LKF structure (see Section III) that allows obtaining feasible convex conditions for the stability analysis in the arbitrary switching case (no dwell-time nor delay-free representations are needed,

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as commonly done in the literature). ii) The relation between a constant delay $d_n$ and a time-varying one $d(k)$ is taken into account by means of the switching signal, iii) the extension of path-complete Lyapunov criteria to LKFs, iv) the extension of the developed theory to derive design conditions in the form of LMIs for the observer-based *stabilization* of systems with unknown output time-varying delays.

**Notation.** Given $\mathrm{Y} \in \mathbb{R}^{n\times m}$, $\mathrm{Y}^\top$ denotes its transpose. Given $\mathrm{W} = \mathrm{W}^\top$, $\mathrm{Z} = \mathrm{Z}^\top$ in $\mathbb{R}^{n\times n}$, $\mathrm{W} \succ \mathrm{Z}$ ($\mathrm{W} \succeq \mathrm{Z}$) means that $\mathrm{W} - \mathrm{Z}$ is positive definite (positive semi-definite). With $\mathbb{S}_n^+$ we denote the set of symmetric positive definite matrices. I and 0 denote identity and null matrices of appropriate dimensions, with their dimensions explicitly stated whenever relevant. The operator $He\{\mathrm{Y}\}$ denotes $He\{\mathrm{Y}\} = \mathrm{Y} + \mathrm{Y}^\top$. The $\star$ symbol denotes symmetric blocks in the expression of a matrix. For matrices W and Z, $diag(\mathrm{W}, \mathrm{Z})$ corresponds to the block-diagonal matrix.

## II. Theoretical preliminaries

In this paper, we study stability of systems of the form

$$\begin{cases} x(k+1) = \mathrm{A}x(k) + \mathrm{A_n}x(k-d_n) \\ \qquad\qquad + \mathrm{A_d}x(k-d(k)), \ \forall k \geq 0 \\ x(k) = \phi(k), \ \forall k \in [-d_M, 0] \end{cases} \tag{1}$$

where $\phi(k)$ is the initial condition at the interval $[-d_M, 0]$, $x(k) \in \mathbb{R}^n$ is the plant state vector, $d(k)$ is a time-varying delay $1 \leq d_m \leq d(k) \leq d_M$ with known lower $d_m$ and upper $d_M$ limits, whereas the value of $d(k)$ at each sampling time $k$ is unknown. Furthermore, $d_n \in [d_m, d_M]$ is a constant delay within the same bounds. As a preliminary step in the analysis of (1), we study the stability for systems without constant delay (i.e. the case $\mathrm{A_n} = 0$ in (1)). This allows us to present, for a simpler system, the tools and techniques which will be generalized, in Section II, to the setting of system (1).

### A. Delay-dependent stability of time-delay systems

Consider the time-delay system

$$\begin{cases} x(k+1) = \mathrm{A}x(k) + \mathrm{A_d}x(k-d(k)), \ \forall k \geq 0 \\ x(k) = \phi(k), \ \forall k \in [-d_M, 0] \end{cases} \tag{2}$$

where $\phi(k)$ is the initial condition at the interval $[-d_M, 0]$, $x(k) \in \mathbb{R}^n$ is the plant state vector, and $d(k)$ is a time-varying delay $1 \leq d_m \leq d(k) \leq d_M$. When studying the stability of (2), a common approach is to search for a Lyapunov-Krasovskii functional [7], [8]. More precisely, we consider $V : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{d_M+1 \text{ times}} \to \mathbb{R}^+$, and, using the convention

$$\begin{aligned} &V(k) := V(\overline{x}(k)), \\ &\overline{x}(k) := \left[x^\top(k) \ x^\top(k-1) \ \ldots \ x^\top(k-d_M)\right]^\top, \end{aligned} \tag{3}$$

we require that $V(k) > 0$ for all $\overline{x}(k) \neq 0$ (and we say that $V$ is *positive definite*), and the forward difference of $V$ with respect to (2) is negative, i.e., $\Delta V(k) := V(k+1) - V(k) < 0$ along the trajectories of (2). In this case, we say that $V$ is a *Lyapunov Krasovskii functional (LKF)* for system (2). In order to obtain stability conditions in the form of LMIs, many LKF structures have been proposed in the literature, see [7], [8] and references therein. The manipulation of summation inequalities in the LKFs commonly lead to bounds of the form $\Delta V(k) \leq \xi^\top(k)\Phi(d(k), d_m, d_M)\xi(k)$, where $\Phi(d(k), d_m, d_M)$ is a matrix-valued function and $\xi(k)$ is a vector function depending explicitly on $x(k+1)$, $x(k)$, $x(k-d(k))$, and also on the states delayed by the minimum and maximum delays, i.e., $x(k-d_m)$ and $x(k-d_M)$ (see, for example, [20], [15], [19], [11]). LMI conditions are then obtained by either replacing $x(k+1)$ with the equation in (2) or by applying Finsler's Lemma (see Subsection II-B). Since standard manipulations already lead to bounds which depend on $x(k-d_m)$ and $x(k-d_M)$, the exact same LKF and manipulation procedure used to obtain stability conditions for (2) can be applied for systems of the more general form

$$\begin{cases} x(k+1) = \mathrm{A}x(k) + \mathrm{A_m}x(k-d_m) \\ \qquad + \mathrm{A_M}x(k-d_M) + \mathrm{A_d}x(k-d(k)), \ \forall k \geq 0 \\ x(k) = \phi(k), \ \forall k \in [-d_M, 0]. \end{cases} \tag{4}$$

### B. Standard analysis conditions with Finsler's Lemma

In this subsection we derive LMIs conditions, via a Lyapunov Krasovskii construction, for stability of (4). Since it represents a crucial tool in our algebraic manipulation, we recall here the celebrated Finsler's Lemma, [6].

**Lemma 1.** *[6] Consider $\Phi = \Phi^\top \in \mathbb{R}^{n_\xi \times n_\xi}$, and $\Gamma \in \mathbb{R}^{m_\xi \times n_\xi}$. The following statements are equivalent:*
*(i)* $\xi^\top \Phi \xi < 0, \ \forall \xi \neq 0$ *such that* $\Gamma\xi = 0$.
*(ii)* $\exists \mathfrak{J} \in \mathbb{R}^{n_\xi \times m_\xi} : \Phi + \mathfrak{J}\Gamma + \Gamma^\top \mathfrak{J}^\top \prec 0$.
*(iii)* ${\Gamma^\perp}^\top \Phi \Gamma^\perp \prec 0$*, where* $\Gamma\Gamma^\perp = 0$.

Next, we review analysis conditions for systems (4). We define $d_\Delta := d_M - d_m$, and the function $\gamma : \mathbb{N} \to \mathbb{R}$ given by

$$\begin{cases} \gamma(d) = 1, \text{ if } d = 1, \\ \gamma(d) = (d+1)/(d-1), \text{ if } d > 1. \end{cases} \tag{5}$$

Consider then the LKF-structure, inspired by [19], given by

$$V(k) = V_a(k) + V_b(k) + V_c(k), \tag{6}$$

$$\begin{aligned} V_a(k) &= w^\top(k)\mathrm{P}w(k), \\ V_b(k) &= \sum_{l=k-d_m}^{k-1} x^\top(l)\mathrm{Q}_1 x(l) + \sum_{l=k-d_M}^{k-d_m-1} x^\top(l)\mathrm{Q}_2 x(l), \\ V_c(k) &= d_m \sum_{l=-d_m+1}^{0} \sum_{i=k+l}^{k} \eta^\top(i)\mathrm{Z}_1\eta(i) \\ &\quad + d_\Delta \sum_{l=-d_M+1}^{-d_m} \sum_{i=k+l}^{k} \eta^\top(i)\mathrm{Z}_2\eta(i), \end{aligned} \tag{7}$$

with $w(k) := \left[x^\top(k) \ \sum_{l=k-d_m}^{k-1} x^\top(l) \ \sum_{l=k-d_M}^{k-d_m-1} x^\top(l)\right]^\top$ and $\eta(i) = x(i) - x(i-1)$. Supposing that $\mathrm{P} \in \mathbb{S}_{3n}^+$ and $\mathrm{Q}_1$, $\mathrm{Q}_2$, $\mathrm{Z}_1$, $\mathrm{Z}_2 \in \mathbb{S}_n^+$ implies that $V$ is positive definite. In what follows, we present a lemma allowing to perform stability analysis of (4) using the Lyapunov-Krasovskii structure introduced in (7). The proposed LMI conditions are equivalent in conservatism to the ones in [19, Theorem 5], but are slightly different due to the application of Lemma 1. In [19], instead, the stability conditions are obtained by direct substitution of the dynamics $x(k+1)$ in the manipulation of $\Delta V(k)$. The choice of using Lemma 1 will be justified in Section IV. Some steps of the proof of the lemma below, especially involving the bounding of the term $\Delta V_c(k)$, are not made completely explicit, in order to avoid repetition of the manipulations already presented in [19], to which we refer for the details.

**Lemma 2.** *Assume that there exist matrices* $\mathrm{P} \in \mathbb{S}^+_{3n}$, $\mathrm{Q}_1$, $\mathrm{Q}_2$, $\mathrm{Z}_1$, $\mathrm{Z}_2 \in \mathbb{S}^+_n$, $\mathrm{X} \in \mathbb{R}^{2n\times 2n}$ *such that*

$$\Psi_z \succ 0,\quad {\Gamma^{\perp}}^{\top}\Phi(d_m)\Gamma^{\perp} \prec 0,\quad {\Gamma^{\perp}}^{\top}\Phi(d_M)\Gamma^{\perp} \prec 0, \tag{8}$$

*hold with* $\Gamma^{\perp} = \begin{bmatrix} \mathrm{A} & \mathrm{A}_m & \mathrm{A}_{\mathrm{d}} & \mathrm{A}_M & 0_{n\times 3n} \\ & & \mathrm{I}_{7n} & & \end{bmatrix}$,

$$\begin{aligned}
&\Phi(d) = \Phi_1(d) + \mathcal{Q} + \Phi_3(d_m, d_M),\\
&\Phi_1(d) = \mathrm{W}_2^{\top}(d_m,d_M)\mathrm{P}\mathrm{W}_2(d_m,d_M)\\
&\qquad - \mathrm{W}_1^{\top}(d_m,d_M)\mathrm{P}\mathrm{W}_1(d_m,d_M)\\
&\qquad + \mathit{He}\left\{\mathrm{W}^{\top}(d)\mathrm{P}\left(\mathrm{W}_2(d_m,d_M) - \mathrm{W}_1(d_m,d_M)\right)\right\},\\
&\Phi_3(d_m,d_M) = \mathrm{W}_3^{\top}(d_m^2\mathrm{Z}_1 + d_{\Delta}^2\mathrm{Z}_2)\mathrm{W}_3\\
&\qquad - \mathrm{W}_s^{\top}\mathscr{Z}_1(d_m)\mathrm{W}_s - \mathrm{W}_{\Psi}^{\top}\Psi_z\mathrm{W}_{\Psi},\\
&\qquad\qquad \Psi_z = \begin{bmatrix} \mathscr{Z}_2 & \mathrm{X} \\ \star & \mathscr{Z}_2 \end{bmatrix},\\
&\mathscr{Z}_1(d_m) = \mathit{diag}\left(\mathrm{Z}_1, 3\gamma(d_m)\mathrm{Z}_1\right), \mathscr{Z}_2 = \mathit{diag}\left(\mathrm{Z}_2, 3\mathrm{Z}_2\right),
\end{aligned}$$

$\mathcal{Q} = \mathit{diag}(0, \mathrm{Q}_1, \mathrm{Q}_2 - \mathrm{Q}_1, 0, -\mathrm{Q}_2, 0, 0, 0)$, $\gamma$ *defined in* (5) *and* $\mathrm{W}_{\Psi}, \mathrm{M}, \mathrm{W}_s, \mathrm{W}_3, \mathrm{W}_1(d_m,d_M)$, $\mathrm{W}_2(d_m,d_M), \mathrm{W}(d)$ *given in Appendix I. Then, the LKF* $\mathrm{V}$ *defined in* (6)*-*(7) *is positive definite and satisfies* $\mathrm{V}(k+1) - \mathrm{V}(k) < 0$, *for all* $\overline{x}(k) \neq 0$. *In particular, system* (4) *is asymptotically stable for any time-varying delay* $d_m \leq d(k) \leq d_M$.

*Proof.* See [2]. □

## III. Switched representation

### A. Description of the considered delay-switched systems

We introduce an alternative representation of (1) in order to adapt the techniques presented in Section II in this setting. In particular, we rewrite system (1) in the form of a switched system composed by two delay subsystems, each one of the form (4). This is possible by noting that, when the time-varying delay $d(k)$ is between the minimum delay $d_m$ and the constant delay $d_n$, $d_n$ can be viewed as the maximum system delay for a first subsystem; when $d(k)$ is between $d_n$ and the maximum delay $d_M$, $d_n$ can be viewed as the lower bound on the delay for a second subsystem. Therefore, the following two-modes switched system can be defined:

$$\begin{cases} x(k+1) = f_{\sigma(k)}(\overline{x}(k), k),\ \forall k \geq 0 \\ x(k) = \phi(k),\ \forall k \in [-d_M, 0] \end{cases} \tag{9}$$

considering, for $j \in \{1,2\}$, the subsystems $f_j : \mathbb{R}^{n\times(d_M+1)} \times \mathbb{N} \to \mathbb{R}^n$ defined by

$$\begin{aligned} f_j(\overline{x}(k), k) :=& \mathrm{A}x(k) + \mathrm{A}_{m_j}x(k - d_{m_j}) \\ &+ \mathrm{A}_{M_j}x(k - d_{M_j}) + \mathrm{A}_{\mathrm{d}}x(k - d(k)), \end{aligned} \tag{10}$$

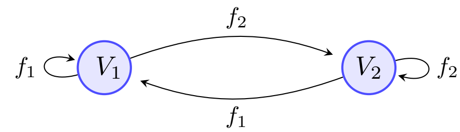

Fig. 1. Path-complete representation of inequalities in (13) in Lemma 3.

$$\begin{array}{l|l|l|l} \mathrm{A}_{\mathrm{m}_1} = 0 & \mathrm{A}_{\mathrm{M}_1} = \mathrm{A}_{\mathrm{n}} & d_{\mathrm{m}_1} = d_m & d_{\mathrm{M}_1} = d_n \\ \hline \mathrm{A}_{\mathrm{m}_2} = \mathrm{A}_{\mathrm{n}} & \mathrm{A}_{\mathrm{M}_2} = 0 & d_{\mathrm{m}_2} = d_n & d_{\mathrm{M}_2} = d_M \end{array} \tag{11}$$

The *switching rule* $\sigma : \mathbb{N} \to \{1,2\}$ is defined by

$$\sigma(k) = \begin{cases} 1, & \text{if } d_m \leq d(k) \leq d_n, \\ 2, & \text{if } d_n < d(k) \leq d_M. \end{cases} \tag{12}$$

With this convention, the time-varying delay $d(k)$ is such that $d_{\mathrm{m}_{\sigma(k)}} \leq d(k) \leq d_{\mathrm{M}_{\sigma(k)}}$, for any $k \in \mathbb{N}$. That means that the delay lower bound (noted $d_{\mathrm{m}_{\sigma(k)}}$) and the delay upper bound (noted $d_{\mathrm{M}_{\sigma(k)}}$) depend on the value of $\sigma(k)$ at each sampling time. The main advantage of this representation is that we get rid of the constant delay $d_n$, which is now seen as an upper/lower bound (depending on the active subsystems) for the time-varying delay $d(k)$. This allows us to restore in this setting the manipulation techniques described in Section II for the construction of Lyapunov-Krasovskii functionals even if the resulting system (9) is, at this stage, a switched system. However, this is not trivial since, as will be shown in the proof of Theorem 1, a new LKF modified from (6) along with new manipulations are proposed. Achieving convex conditions for the stability analysis of the switched system with arbitrary delay $d(k)$ defined in (9)-(12) is, thus, our first contribution.

### B. Stability Analysis

The search for a *common* Lyapunov-Krasovskii functional for the two delay subsystems defined by $f_1$ and $f_2$ can be computationally hard, leading to conservative or even structurally infeasible conditions. We propose a less conservative construction based on *multiple* Lyapunov-Krasovskii functionals, see [4] for an overview in the delay-free case. In particular we adapt, in this context, the ideas introduced in [18], [1], proposing a graph-based structure for the multiple Lyapunov-Krasovskii functionals, formally introduced in the following statement and graphically represented in Fig. 1.

**Lemma 3.** *Suppose there exist positive definite* $\mathrm{V}_j : \mathbb{R}^{n\times(d_M+1)} \to \mathbb{R}$, $j \in \{1,2\}$, *such that the following conditions are satisfied, for any* $\overline{x}(k) \neq 0$,

$$\mathrm{V}_1(k+1) < \mathrm{V}_1(k),\quad \textit{with}\quad x(k+1) = f_1(\overline{x}(k), k), \tag{13a}$$
$$\mathrm{V}_2(k+1) < \mathrm{V}_2(k),\quad \textit{with}\quad x(k+1) = f_2(\overline{x}(k), k), \tag{13b}$$
$$\mathrm{V}_1(k+1) < \mathrm{V}_2(k),\quad \textit{with}\quad x(k+1) = f_1(\overline{x}(k), k), \tag{13c}$$
$$\mathrm{V}_2(k+1) < \mathrm{V}_1(k),\quad \textit{with}\quad x(k+1) = f_2(\overline{x}(k), k), \tag{13d}$$

*using the notation introduced in* (3) *and* $f_j$ *defined by* (10) *and* (11). *Then, system* (9) *is asymptotically stable for all time-varying delays* $d : \mathbb{N} \to [d_m, d_M]$ *and* $\min_{j\in\{1,2\}}\{V_j(\overline{x})\}$ *is a LKF for* (9).

*Proof.* See [2]. □

In the following statement we propose LMI-based conditions, leading to *quadratic* Lyapunov-Krasovskii functionals $\mathrm{V}_1, \mathrm{V}_2$ satisfying the conditions (13) of Lemma 3, thus ensuring asymptotic stability of system (1). For better readability of the conditions, auxiliary matrices are defined in Appendix II.

**Theorem 1.** *Consider $j \in \{1, 2\}$ and assume that there exist matrices $\mathrm{P}_j \in \mathbb{S}^+_{3n+n(2-j)}$, $\mathrm{Q}_{1_j}$, $\mathrm{Q}_{2_j}$, $\mathrm{Q}_{3_1}$, $\mathrm{Z}_{1_j}$, $\mathrm{Z}_{2_j}$, $\mathrm{Z}_{3_1} \in \mathbb{S}^+_n$, $\mathrm{X}_j \in \mathbb{R}^{2n\times 2n}$ such that*

$$\Psi_{z_j} \succ 0, \quad \Gamma_j^{\perp\top}\Phi(d_{\mathrm{m}_j}, j)\Gamma_j^{\perp} \prec 0, \quad \Gamma_j^{\perp\top}\Phi(d_{\mathrm{M}_j}, j)\Gamma_j^{\perp} \prec 0, \tag{14}$$

*hold with*

$$\Gamma_j^{\perp} = \begin{bmatrix} \mathrm{A} & \mathrm{A_{m_j}} & \mathrm{A_d} & \mathrm{A_{M_j}} & 0_{n\times 3n} & 0_{n(2-j)\times 2n} \\ & & & \mathrm{I}_{7n+2n(2-j)} & & \end{bmatrix},$$

$$\Phi(d, j) = \Phi_1(d, j) + \mathcal{Q}_j + \Phi_3(j),$$

$$\Phi_1(d, j) = \mathrm{W}_{2_j}^\top \mathrm{P}_j \mathrm{W}_{2_j} - \mathrm{W}_{1_j}^\top \mathrm{P}_j \mathrm{W}_{1_j} + He\left\{\mathrm{W}_j^\top(d)\mathrm{P}_j\left(\mathrm{W}_{2_j} - \mathrm{W}_{1_j}\right)\right\},$$

$$\Phi_3(j) = \mathrm{W}_{3_j}^\top(d_{\mathrm{m}_j}^2 \mathrm{Z}_{1_j} + d_{\Delta_j}^2 \mathrm{Z}_{2_j} + \mathcal{Z}_{3_j})\mathrm{W}_{3_j} - \mathrm{W}_{s_j}^\top \mathscr{Z}_{1_j}(d_{\mathrm{m}_j})\mathrm{W}_{s_j} - \mathrm{W}_{\Psi_j}^\top \Psi_{z_j}\mathrm{W}_{\Psi_j} - \mathrm{W}_z^\top \mathscr{Z}_{3_j}\mathrm{W}_z,$$

*where $d_{\Delta_j} = d_{\mathrm{M}_j} - d_{\mathrm{m}_j}$, $\mathcal{Z}_{3_j} = (2-j)d_{\Delta_M}^2 \mathrm{Z}_{3_1}$, $d_{\Delta_M} = d_M - d_{M_j}$,*

$$\Psi_{z_j} = \begin{bmatrix} \mathscr{Z}_{2_j} & \mathrm{X}_j \\ \star & \mathscr{Z}_{2_j} \end{bmatrix}, \mathscr{Z}_{2_j} = diag\left(\mathrm{Z}_{2_j}, 3\mathrm{Z}_{2_j}\right),$$

$$\mathscr{Z}_{1_j}(d_{\mathrm{m}_j}) = diag\left(\mathrm{Z}_{1_j}, 3\gamma(d_{\mathrm{m}_j})\mathrm{Z}_{1_j}\right),$$

$$\mathscr{Z}_{3_j} = (2-j)diag\left(\mathrm{Z}_{3_1}, 3\mathrm{Z}_{3_1}\right),$$

$$\mathcal{Q}_j = diag\left(0, \mathrm{Q}_{1_j}, \mathrm{Q}_{2_j} - \mathrm{Q}_{1_j}, 0, \mathrm{R}_j\right),$$

$$\mathrm{R}_j = diag\left(\mathrm{Q}_{3_1}(2-j) - \mathrm{Q}_{2_j}, 0_{(3n+(2-j)2n)\times(3n+(2-j)2n)}\right),$$

*at the same time that*

$$\mathcal{L}^{\perp\top}(l_1)\left[diag(\mathrm{S}_1, 0_n) - diag(0_n, \mathrm{S}_2)\right]\mathcal{L}^{\perp}(l_1) \prec 0, \quad l_1 = d_{m_1}, ..., d_{M_1}, \tag{15}$$

$$\mathcal{L}^{\perp\top}(l_2)\left[diag(\mathrm{S}_2, 0_n) - diag(0_n, \mathrm{S}_1)\right]\mathcal{L}^{\perp}(l_2) \prec 0, \quad l_2 = d_{m_2}, ..., d_{M_2}, \tag{16}$$

*with $S_j = \mathrm{W}_{5_j}^\top \mathrm{P}_j \mathrm{W}_{5_j} + \mathcal{P}_j$ is also satisfied. Then, system (9) is asymptotically stable for any time-varying delay $d : \mathbb{N} \to [d_m, d_M]$.*

*Proof.* See [2]. □

## IV. Application to observer-controller design

In this section, with the aid of form (ii) of Lemma 1, we extend the conditions developed in the previous section to *co-design*, by means of LMIs, an observer and a controller for the stabilization of systems with output time-varying delays.

### A. Plant and controller description

Consider the plant described by the following equations

$$\begin{cases} x_p(k+1) = \mathrm{A}_p x_p(k) + \mathrm{B}_p u(k) \\ y(k) = x_p(k - d(k)) \end{cases} \tag{17}$$

where $x_p(k) \in \mathbb{R}^{n_p}$ is the plant state vector, $y(k) \in \mathbb{R}^{n_p}$ is the delayed measured output and $u(k) \in \mathbb{R}^m$ is the control input. Matrices $\mathrm{A}_p$, $\mathrm{B}_p$ are constant, known, and of appropriate dimensions, and the pair $(\mathrm{A}_p, \mathrm{B}_p)$ is controllable. The plant output delay is bounded and time-varying, satisfying $d_m \le d(k) \le d_M$. Furthermore, integers $d_m$ and $d_M$ are known, whereas the value of $d(k)$ at each sampling time is unknown. To control system (17) we consider the following Luenberger-type observer plus control pair

$$\begin{cases} \hat{x}_p(k+1) = \mathrm{A}_p \hat{x}_p(k) + \mathrm{B}_p u(k) + \mathrm{L}e_y(k) \\ u(k) = \mathrm{K}\hat{x}_p(k) + \mathrm{F}e_y(k) \end{cases} \tag{18}$$

where $e_y(k) = y(k) - \hat{x}_p(k - d_n)$, $d_n$ is the constant nominal delay for the observer, the term L is the classical observer corrector term, and K is related to state feedback control of the observed state. Since the time-varying delay $d(k)$ is unknown, the constant delay $d_n \in [d_m, d_M]$ represents, in this setting, a "guess" for $d(k)$ the designer provides to the observer-based controller.

### B. Closed-loop system and problem formulation

Consider the error signal defined by

$$e(k) = x_p(k) - \hat{x}_p(k). \tag{19}$$

By taking into account (17), (18), and by defining the extended vector $x(k) = \begin{bmatrix} x_p(k)^\top & e(k)^\top \end{bmatrix}^\top \in \mathbb{R}^n$, $n = 2n_p$, the closed-loop system can be written as (1) with

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_p + \mathrm{B}_p\mathrm{K} & -\mathrm{B}_p\mathrm{K} \\ 0 & \mathrm{A}_p \end{bmatrix}, \mathrm{A_d} = \begin{bmatrix} \mathrm{B}_p\mathrm{F} & 0 \\ -\mathrm{L} & 0 \end{bmatrix}, \mathrm{A_n} = \begin{bmatrix} -\mathrm{B}_p\mathrm{F} & \mathrm{B}_p\mathrm{F} \\ \mathrm{L} & -\mathrm{L} \end{bmatrix}. \tag{20}$$

### C. Observer-based controller design

The following corollary provides LMIs for the design of matrix gains $K$, $L$, and $F$ by leveraging the stability conditions developed in Section III.

**Corollary 1.** *Given scalar $\varepsilon \in [0, -1)$, assume that, for any $j \in \{1, 2\}$, there exist $\overline{\mathrm{P}}_j$ in $\mathbb{S}^+_{3n+(2-j)n}$, $\overline{\mathrm{Q}}_{1_j}$, $\overline{\mathrm{Q}}_{2_j}$, $\overline{\mathrm{Q}}_{3_1}$, $\overline{\mathrm{Z}}_{1_j}$, $\overline{\mathrm{Z}}_{2_j}$, $\overline{\mathrm{Z}}_{3_1}$ in $\mathbb{S}^+_n$, $\overline{\mathrm{X}}_j$ in $\mathbb{R}^{2n\times 2n}$, $\mathrm{J} = diag(\mathrm{U}, \mathrm{U})$ in $\mathbb{R}^{n\times n}$, U in $\mathbb{R}^{n_p\times n_p}$, $\overline{\mathrm{K}}$, $\overline{\mathrm{F}}$ in $\mathbb{R}^{m\times n_p}$, and $\overline{\mathrm{L}}$ in $\mathbb{R}^{n_p\times n_p}$ such that*

$$\overline{\Psi}_{z_j} \succ 0, \overline{\Phi}(d_{\mathrm{m}_j}, j) + \overline{\Upsilon}_j \prec 0, \overline{\Phi}(d_{\mathrm{M}_j}, j) + \overline{\Upsilon}_j \prec 0 \tag{21}$$

*where $\overline{\Phi}(d, j)$ has the same format of $\Phi(d, j)$ in Theorem 1 but with the "bar" matrices and where*

$$\overline{\Upsilon}_j = diag\left(\overline{\Upsilon}_{a_j}, 0_{(3n+(2-j)2n)\times(3n+(2-j)2n)}\right),$$

$$\overline{\Upsilon}_{a_j} = \begin{bmatrix} -\mathrm{J} - \mathrm{J}^\top & \overline{\mathrm{A}} - \varepsilon\mathrm{J} & \overline{\mathrm{A}_{\mathrm{m}_j}} & \overline{\mathrm{A}_{\mathrm{d}}} & \overline{\mathrm{A}_{\mathrm{M}_j}} \\ \star & \varepsilon\overline{\mathrm{A}} + \varepsilon\overline{\mathrm{A}}^\top & \varepsilon\overline{\mathrm{A}_{\mathrm{m}_j}} & \varepsilon\overline{\mathrm{A}_{\mathrm{d}}} & \varepsilon\overline{\mathrm{A}_{\mathrm{M}_j}} \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix},$$

*with* $\overline{\mathrm{A}} = \begin{bmatrix} \mathrm{A}_p\mathrm{U}^\top + \mathrm{B}_p\overline{\mathrm{K}} & -\mathrm{B}_p\overline{\mathrm{K}} \\ 0 & \mathrm{A}_p\mathrm{U}^\top \end{bmatrix}$, $\overline{\mathrm{A_d}} = \begin{bmatrix} \mathrm{B}_p\overline{\mathrm{F}} & 0 \\ -\overline{\mathrm{L}} & 0 \end{bmatrix}$,

$$\begin{cases} \overline{\mathrm{A}_{\mathrm{m}_1}} = 0,\ \overline{\mathrm{A}_{\mathrm{m}_2}} = \overline{\mathrm{A_n}} \\ \overline{\mathrm{A}_{\mathrm{M}_1}} = \overline{\mathrm{A_n}},\ \overline{\mathrm{A}_{\mathrm{M}_2}} = 0 \end{cases}, \quad \overline{\mathrm{A_n}} = \begin{bmatrix} -\mathrm{B}_p\overline{\mathrm{F}} & \mathrm{B}_p\overline{\mathrm{F}} \\ \overline{\mathrm{L}} & -\overline{\mathrm{L}} \end{bmatrix},$$

*hold at the same that the following inequalities*

$$\begin{aligned} &\left[diag(\overline{\mathrm{S}}_1, 0_n) - diag(0_n, \overline{\mathrm{S}}_2)\right] + He\{\mathcal{I}^\top \overline{\mathcal{L}}(l_1)\} \prec 0, \\ &\qquad\qquad\qquad\qquad l_1 = d_{m_1}, ..., d_{M_1}, \end{aligned} \quad (22)$$

$$\begin{aligned} &\left[diag(\overline{\mathrm{S}}_2, 0_n) - diag(0_n, \overline{\mathrm{S}}_1)\right] + He\{\mathcal{I}^\top \overline{\mathcal{L}}(l_2)\} \prec 0, \\ &\qquad\qquad\qquad\qquad l_2 = d_{m_2}, ..., d_{M_2}, \end{aligned} \quad (23)$$

*are also satisfied, where* $\mathcal{I} = \begin{bmatrix} \mathrm{I} & \varepsilon\mathrm{I} & 0 & \cdots & 0 \end{bmatrix}$ *and*

$$\begin{aligned} \overline{\mathcal{L}}(l) = &\begin{bmatrix} -\mathrm{J} & \overline{\mathrm{A}} & 0 & \cdots & 0 & \overline{\mathrm{A_n}} & 0 & \cdots & 0 \end{bmatrix} \\ &+ \overline{\mathrm{A_d}} \begin{bmatrix} 0 & 0 & \delta(1)\mathrm{I} & \cdots & \delta(d_M)\mathrm{I} \end{bmatrix},\ l \in [1, d_M], \\ &\text{with } \delta(i) = 1 \text{ if } i = l, \text{ and } \delta(i) = 0 \text{ if } i \neq l. \end{aligned}$$

*Then, matrices* $\mathrm{K} = \overline{\mathrm{K}}\mathrm{U}^{-T}$, $\mathrm{F} = \overline{\mathrm{F}}\mathrm{U}^{-T}$, $\mathrm{L} = \overline{\mathrm{L}}\mathrm{U}^{-T}$, *are such that the closed loop yielded by the connection between the plant* (17) *and the observer-controller* (18)*, i.e., system* (1) *with matrices given as in* (20)*, is asymptotically stable for any time-varying delay* $d : \mathbb{N} \to [d_m, d_M]$.

*Proof.* See [2]. □

### D. Numerical example

Consider the NCS studied in [12, Example 2]. By considering a sampling time of 0.5 seconds, an induced network time delay, we obtain the discrete-time model (17) with $\mathrm{A}_p = \left[\begin{smallmatrix} 0.6693 & -0.0042 \\ 0.4231 & 1.0501 \end{smallmatrix}\right]$, $\mathrm{B}_p = \left[\begin{smallmatrix} 0.1647 \\ 0.0960 \end{smallmatrix}\right]$. In [12], the control law is given by $u(k) = -\left[\,1.2625\ 1.2679\,\right] x_p(k - d(k))$, which guarantees closed-loop stability for a maximum induced delay of 1 second (or two samples $d_M = 2$), according to [12, Theorem 4]. More recently, the compensator strategy from [3] was able to stabilize this system for a time-varying delay in the range $1 \leq d(k) \leq 7$. By setting $\varepsilon = -0.995$ in Theorem 1, $d_n = 1$, we obtain matrices $\mathrm{K} = \left[\,-0.1925\ -0.1702\,\right]$, $\mathrm{F} = \left[\,-0.1755\ -0.1601\,\right]$, $\mathrm{L} = \left[\begin{smallmatrix} -0.0032 & -0.0007 \\ 0.0578 & 0.0525 \end{smallmatrix}\right]$ that guarantee stability for any time-varying delay $d(k)$ such that $1 \leq d(k) \leq 17$. That means that stability is guaranteed even for a time-varying delay maximum of 8.5 seconds. That is a substantial increase compared to the results obtained by [12] and [3], and further shows the advantage of introducing the delay $d_n$ and the observer strategy instead of simply feedbacking the delayed output $x(k - d(k))$. To illustrate the effects of the LMI tuning parameter $\varepsilon$ and of the constant delay $d_n$, Fig. 2 shows the maximum delay $d_M$ for which the conditions of Corollary 1 were feasible for different values of $\varepsilon$ and $d_n$ (the minimum delay was kept fixed at $d_m = 1$). It is observed that as $\varepsilon$ approaches $-1$, the maximum delay $d_M$ tends to increase. For $d_n$, it seems that the best results are achieved near to the bounds on the delay.

## V. Conclusion

In this work, a new technique for the stability analysis of systems with both a constant and a time-varying delay based on a specific switched representation has been studied

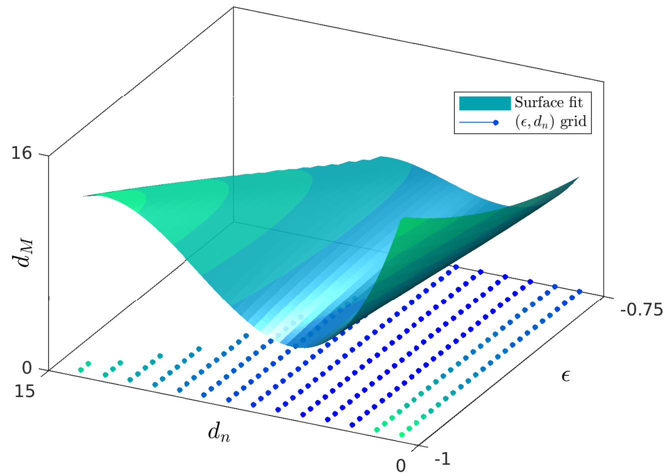


Fig. 2. Example 1: Relation between $\varepsilon$, $d_n$ and maximum delay $d_M$.

in conjunction with a new modified LKF (proposed in the proof of Theorem 1) that led to obtaining convex conditions in the form of LMIs. A stabilizing technique for discrete-time systems with sensor time-varying delays has been proposed as an extension. The obtained simulation results are promising, showing a substantial increase in the time-varying delay bounds for stabilization of an example from the literature. Further developments of the strategy can be envisaged to include, for example, the problem of control saturation and the study of stability conditions based on more elaborate (and possibly less conservative) path-complete criteria.

## References


[1] Ahmadi, A. A., Jungers, R. M., Parrilo, P. A., and Roozbehani, M., "Joint spectral radius and path-complete graph Lyapunov functions," *SIAM J Control Optim.*, vol. 52, no. 1, pp. 687–717, 2014.

[2] Alves Lima, T., Della Rossa, M., Gouaisbaut, F., Jungers, R., and Tarbouriech, S., "Systems with both constant and time-varying delays: a switched systems approach and application to observer-controller co-design," 2022, extended version. [Online]. Available: https://arxiv.org/abs/2209.04869

[3] Alves Lima, T., Tarbouriech, S., Gouaisbaut, F., Prudêncio de Almeida Filho, M., García, P., Claure Torrico, B., and Gonzalez Nogueira, F., "Analysis and experimental application of a dead-time compensator for input saturated processes with output time-varying delays," *IET Control. Theory Appl.*, vol. 15, no. 4, pp. 580–593, 2021.

[4] Branicky, M., "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, pp. 475–482, 1998.

[5] De Iuliis, V., D'Innocenzo, A., Germani, A., and Manes, C., "Stability conditions for linear discrete-time switched systems in block companion form," *IET Control. Theory Appl.*, vol. 14, pp. 3107–3115(8), 2020.

[6] de Oliveira, M. C. and Skelton, R. E., "Stability tests for constrained linear systems," in *Perspectives in robust control*, Moheimani, S. R., Ed. London: Springer London, 2001, pp. 241–257.

[7] Fridman, E., *Introduction to Time-Delay Systems*. Springer International Publishing, 2014.

[8] Gu, K., Kharitonov, V. L., and Chen, J., *Stability of Time-Delay Systems*. Birkhäuser Boston, 2003.

[9] Haidar, I., Mason, P., and Sigalotti, M., "Converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations," *Automatica*, vol. 62, pp. 263–273, 2015.

[10] Hetel, L., Daafouz, J., and Iung, C., "Equivalence between the Lyapunov–Krasovskii functionals approach for discrete delay systems and that of the stability conditions for switched systems," *Nonlinear Anal.: Hybrid Syst.*, vol. 2, no. 3, pp. 697 – 705, 2008.

[11] Hien, L. V. and Trinh, H., "New finite-sum inequalities with applications to stability of discrete time-delay systems," *Automatica*, vol. 71, pp. 197–201, 2016.

[12] Hu, L.-S., Bai, T., Shi, P., and Wu, Z., "Sampled-data control of networked linear control systems," *Automatica*, vol. 43, no. 5, pp. 903 – 911, 2007.

[13] Jiang, W., Kruszewski, A., Fridman, E., and Richard, J.-P., "Delay dependent stability analysis of interval time-delay systems," *IFAC Proceedings Volumes*, vol. 43, no. 2, pp. 313 – 318, 2010, 9th IFAC Workshop on Time Delay Systems.

[14] Jungers, R., Kundu, A., and Heemels, W. P. M. H., "Observability and controllability analysis of linear systems subject to data losses," *IEEE Trans. Automat. Contr.*, vol. 63, no. 10, pp. 3361–3376, 2018.

[15] Kwon, O., Park, M., Park, J. H., Lee, S., and Cha, E., "Stability and stabilization for discrete-time systems with time-varying delays via augmented Lyapunov–Krasovskii functional," *J. Frankl. Inst.*, vol. 350, no. 3, pp. 521 – 540, 2013.

[16] Liberzon, D., *Switching in systems and control*. Birkhaüser, 2003.

[17] Pepe, P., "Discrete-time systems with constrained time delays and delay-dependent Lyapunov functions," *IEEE Trans. Automat. Contr.*, vol. 65, no. 4, pp. 1724–1730, 2020.

[18] Philippe, M., Essick, R., Dullerud, G. E., and Jungers, R. M., "Stability of discrete-time switching systems with constrained switching sequences," *Automatica*, vol. 72, pp. 242–250, 2016.

[19] Seuret, A., Gouaisbaut, F., and Fridman, E., "Stability of discrete-time systems with time-varying delays via a novel summation inequality," *IEEE Trans. Automat. Contr.*, vol. 60, no. 10, pp. 2740–2745, 2015.

[20] Shao, H. and Han, Q. L., "New stability criteria for linear discrete-time systems with interval-like time-varying delays," *IEEE Trans. Automat. Contr.*, vol. 56, no. 3, pp. 619–625, 2011.

[21] Sun, X.-M., Liu, G.-P., Rees, D., and Wang, W., "Delay-dependent stability for discrete systems with large delay sequence based on switching techniques," *Automatica*, vol. 44, no. 11, pp. 2902 – 2908, 2008.

## Appendix I
## Auxiliary matrices for Lemma 2

$$\mathrm{W}_\Psi = \begin{bmatrix} 0_{2n\times 2n} & \mathrm{M} \\ 0_{2n\times n} \;\; \mathrm{M} & 0_{2n\times n} \end{bmatrix}, \mathrm{M} = \begin{bmatrix} 0 & \mathrm{I} & -\mathrm{I} & 0 & 0 & 0 \\ 0 & \mathrm{I} & \mathrm{I} & 0 & 0 & -2\mathrm{I} \end{bmatrix},$$

$$\mathrm{W}_s = \begin{bmatrix} \mathrm{M} & 0_{2n\times 2n} \end{bmatrix}, \mathrm{W}_3 = \begin{bmatrix} \mathrm{I} & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathrm{W}_1(d_m, d_M) = \begin{bmatrix} 0 & \mathrm{I} & 0 & 0 & 0 & \\ 0 & -\mathrm{I} & 0 & 0 & 0 & \mathrm{W}_4(d_m, d_M) \\ 0 & 0 & -\mathrm{I} & -\mathrm{I} & 0 & \end{bmatrix},$$

$$\mathrm{W}_2(d_m, d_M) = \begin{bmatrix} \mathrm{I} & 0 & 0 & 0 & 0 & \\ 0 & 0 & -\mathrm{I} & 0 & 0 & \mathrm{W}_4(d_m, d_M) \\ 0 & 0 & 0 & -\mathrm{I} & -\mathrm{I} & \end{bmatrix},$$

$$\mathrm{W}_4(d_m, d_M) = \begin{bmatrix} 0 & 0 & 0 \\ (d_m+1)\mathrm{I} & 0 & 0 \\ 0 & (1-d_m)\mathrm{I} & (d_M+1)\mathrm{I} \end{bmatrix},$$

$$\mathrm{W}(d) = \begin{bmatrix} 0_{2n\times 8n} \\ 0_{n\times 6n} \;\; d\mathrm{I}_n \;\; -d\mathrm{I}_n \end{bmatrix}.$$

## Appendix II
## Auxiliary matrices for Theorem 1

$$\mathrm{W}_{\Psi_j} = \begin{bmatrix} \mathrm{W}_\Psi & 0_{4n\times 2n(j-2)} \end{bmatrix}, \mathrm{M}_d = (d_{\Delta_M}+1)\mathrm{I}_{n(j-2)},$$

$$\mathrm{W}_z = \begin{bmatrix} 0_{n\times 4n} & \mathrm{I} & 0_{n\times 3n} & -\mathrm{I} & 0_{n\times n} \\ 0_{n\times 4n} & \mathrm{I} & 0_{n\times 3n} & \mathrm{I} & -2\mathrm{I} \end{bmatrix},$$

$$\mathrm{W}_{s_j} = \begin{bmatrix} \mathrm{W}_s & 0_{2n\times 2n(j-2)} \end{bmatrix}, \mathrm{W}_{3_j} = \begin{bmatrix} \mathrm{W}_3 & 0_{n\times 2n(j-2)} \end{bmatrix},$$

$$\mathrm{W}_{1_j} = \begin{bmatrix} \mathrm{W}_1(d_{m_j}, d_{M_j}) \;\; 0_{3n\times 2n(j-2)} \\ 0_{n\times 4n(j-2)} \;\; -\mathrm{I}_{n(j-2)} \;\; 0_{n\times 4n(j-2)} \;\; \mathrm{M}_d \end{bmatrix},$$

$$\mathrm{W}_{2_j} = \begin{bmatrix} \mathrm{W}_2(d_{m_j}, d_{M_j}) \;\; 0_{3n\times 2n(j-2)} \\ 0_{n\times 8n(j-2)} \;\; -\mathrm{I}_{n(j-2)} \;\; \mathrm{M}_d \end{bmatrix},$$

$$\mathrm{W}_j(d) = \begin{bmatrix} 0_{2n\times 8n} \;\; 0_{2n\times 2n(2-j)} \\ 0_{n\times 6n} \;\; d\mathrm{I}_n \;\; -d\mathrm{I}_n \;\; 0_{n\times 2n(2-j)} \\ 0_{n\times 10n(2-j)} \end{bmatrix},$$

$$\begin{aligned} \mathcal{L}(l) = &\begin{bmatrix} -\mathrm{I} & \mathrm{A} & 0 & \cdots & 0 & \mathrm{A}_n & 0 & \cdots & 0 \end{bmatrix} \\ &+ \mathrm{A}_d \begin{bmatrix} 0 & 0 & \delta(1)\mathrm{I} & \cdots & \delta(d_M)\mathrm{I} \end{bmatrix}, \; l \in [1, d_M], \\ &\text{with } \delta(i) = 1 \text{ if } i = l, \text{ and } \delta(i) = 0 \text{ if } i \neq l \end{aligned}$$

$$\begin{aligned} \mathcal{L}^\perp(l) &= \begin{bmatrix} \mathcal{L}_a^{\perp\top}(l) & \mathrm{I}_{d_M+1} \end{bmatrix}^\top, \\ \mathcal{L}_a^\perp(l) &= \begin{bmatrix} \mathrm{A} & 0 & \cdots & 0 & \mathrm{A}_n & 0 & \cdots & 0 \end{bmatrix} \\ &\quad + \mathrm{A}_d \begin{bmatrix} 0 & \delta(1)\mathrm{I} & \cdots & \delta(d_M)\mathrm{I} \end{bmatrix}, \; l \in [1, d_M], \\ &\quad \text{with } \delta(i) = 1 \text{ if } i = l, \text{ and } \delta(i) = 0 \text{ if } i \neq l. \end{aligned}$$

$$\mathrm{W}_{5_j} = \begin{bmatrix} \begin{matrix} \mathrm{I} \\ 0 \\ 0 \end{matrix} & \overbrace{\begin{matrix} 0 & \cdots & 0 \\ \mathrm{I} & \cdots & \mathrm{I} \\ 0 & \cdots & 0 \end{matrix}}^{d_{\mathrm{m}_j}\times} & \overbrace{\begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \mathrm{I} & \cdots & \mathrm{I} \end{matrix}}^{\left(d_{\mathrm{M}_j} - d_{\mathrm{m}_j}\right)\times} & \overbrace{\begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix}}^{\left(d_M - d_{\mathrm{M}_j}\right)\times} \\ \multicolumn{4}{c}{\mathrm{W}_{a_j}} \end{bmatrix},$$

$$\mathrm{W}_{a_j} = \begin{bmatrix} 0_{(2-j)n\times d_{M_j} n} & \mathrm{I}_{n(2-j)} & \cdots & \mathrm{I}_{n(2-j)} \end{bmatrix},$$

$$\mathcal{P}_j = \mathrm{diag}\left(\mathcal{H}_j, \overbrace{\mathrm{Q}_{3_1}, \cdots, \mathrm{Q}_{3_1}}^{\left(d_M - d_{\mathrm{M}_j}\right)\times}\right) + (2-j)\mathcal{H}_{aj},$$

$$\mathcal{H}_j = \begin{bmatrix} \mathcal{P}_0 & \mathcal{P}_{b_1} & 0 & \cdots & 0 & \cdots & 0 \\ \star & \mathcal{P}_{a_1} & \ddots & \ddots & \vdots & \ddots & \vdots \\ \star & \ddots & \ddots & \mathcal{P}_{b_{d_{\mathrm{m}_j}}} & 0 & \cdots & 0 \\ \vdots & \ddots & \star & \mathcal{P}_{a_{d_{\mathrm{m}_j}}} & \mathcal{P}_{d_1} & \ddots & \vdots \\ \star & \cdots & \star & \star & \mathcal{P}_{c_1} & \ddots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \mathcal{P}_{d_{d_{\Delta_j}}} \\ \star & \cdots & \star & \cdots & \star & \star & \mathcal{P}_{c_{d_{\Delta_j}}} \end{bmatrix}$$

$$\begin{aligned} \mathcal{P}_0 &= \mathrm{Z}_{1_j} d_{\mathrm{m}_j}^2 + \mathrm{Z}_{2_j} d_{\Delta_j}^2, \\ \mathcal{P}_{a_i} &= \mathrm{Q}_{1_j} + 2\mathrm{Z}_{2_j} d_{\Delta_j}^2 + \mathrm{Z}_{1_j} d_{\mathrm{m}_j}\left(2d_{\mathrm{m}_j} - 2i + 1\right), \\ \mathcal{P}_{b_i} &= -\mathrm{Z}_{2_j} d_{\Delta_j}^2 - \mathrm{Z}_{1_j} d_{\mathrm{m}_j}\left(d_{\mathrm{m}_j} - i + 1\right), \\ \mathcal{P}_{c_l} &= \mathrm{Q}_{2_j} + \mathrm{Z}_{2_j} d_{\Delta_j}\left(2d_{\Delta_j} - 2l + 1\right), \\ \mathcal{P}_{d_l} &= -\mathrm{Z}_{2_j} d_{\Delta_j}\left(d_{\Delta_j} - l + 1\right), \end{aligned}$$

for $i \in [1, d_{\mathrm{m}_j}]$ and $l \in [1, d_{\Delta_j}]$. Moreover, $\mathcal{H}_{aj}$ is a matrix of same format of $\mathcal{H}_j$ but with the following definitions:

$$\begin{aligned} &\mathcal{P}_0 = \mathrm{Z}_{3_1} d_{\Delta_M}^2, \mathcal{P}_{a_i} = 2\mathrm{Z}_{3_1} d_{\Delta_M}^2, \mathcal{P}_{b_i} = -\mathrm{Z}_{3_1} d_{\Delta_M}^2, \\ &\mathcal{P}_{c_l} = \mathrm{Z}_{3_1} d_{\Delta_M}\left(2d_{\Delta_M} - 2l + 1\right), \\ &\mathcal{P}_{d_l} = -\mathrm{Z}_{3_1} d_{\Delta_M}\left(d_{\Delta_M} - l + 1\right), \end{aligned}$$

for $i \in [1, d_{\mathrm{M}_j}]$ and $l \in [1, d_{\Delta_M}]$, where $d_{\Delta_M} = d_M - d_{\mathrm{M}_j}$.