VERMA HOWE DUALITY AND LKB REPRESENTATIONS

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Abstract. We establish a version of quantum Howe duality with two general linear quantum enveloping algebras that involves a tensor product of Verma modules. We prove that the (colored higher) LKB representations arise from this duality and use this description to show that they are simple as modules for various subgroups of the braid group, including the pure braid group.

Contents

1. Introduction

Arguably the most classical form of ***Howe duality*** relates commuting actions of $\mathrm{GL}_m(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ on the symmetric algebra of $\mathbb{C}^m \otimes \mathbb{C}^n$, see [How89] or [How95].

Howe's approach turned out to be a game changer, even in fields beyond representation theory. For example, quantum versions of these dualities provide powerful and categorification-friendly descriptions of quantum invariants such as the colored Jones polynomial.

In this paper we prove a version of the Howe duality above where symmetric powers are replaced by Verma modules. We also extend the aforementioned application to quantum invariants: we show that the ***LKB (Lawrence–Krammer–Bigelow) representations*** and their colored and higher counterparts arise from this ***Verma Howe duality***, which in turn enables us to verify that the LKB representations are simple modules of various subgroups of Artin's braid group.

1A. **Schur–Weyl(–Brauer) and Howe dualities.** Three main themes in Weyl's seminal book ***"The classical groups"*** [Wey97] are the study of polynomial invariants for actions of the eponymous classical groups, and, more or less equivalent, decomposition of the tensor algebra for such an action, and, again more or less equivalent, the description of the invariants in the tensor algebra.

The two most prominent examples that fit into Weyl's setting are the celebrated ***Schur–Weyl duality*** [Sch01] for tensor invariants of $\mathrm{GL}_m(\mathbb{C})$ and ***Brauer duality*** [Bra37] for tensor invariants of $\mathrm{O}_m(\mathbb{C})$ and $\mathrm{SP}_m(\mathbb{C})$ (for the symplectic group m is even). Both of these were studied by using commuting actions of $\mathrm{GL}_m(\mathbb{C})$, and $\mathrm{O}_m(\mathbb{C})$, $\mathrm{SP}_m(\mathbb{C})$ on one side and the symmetric group S_n and the Brauer algebra, respectively, on the other side, both acting on a tensor product of the defining representation of the classical groups. In this commuting-action-approach, Schur–Weyl duality for example essentially reads:

(A) There are commuting actions of $\mathrm{GL}_m(\mathbb{C})$ and S_n on $(\mathbb{C}^m)^{\otimes n}$.

(B) The two actions generate each others centralizer.

(C) The $\mathrm{GL}_m(\mathbb{C})$-$S_n$ bimodule $(\mathbb{C}^m)^{\otimes n}$ can be explicitly decomposed into a direct sum of simple $\mathrm{GL}_m(\mathbb{C})$ modules tensored with simple S_n modules.

A statement of this form is what we call a ***double centralizer*** (a.k.a. double commutant) approach.

Howe [How89], [How95] studied polynomial invariants, e.g. via symmetric powers, of classical groups using a double centralizer approach, and the resulting dualities are called ***Howe dualities***

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in this paper. A prominent example is ***symmetric Howe duality*** where $\mathrm{GL}_m(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ act on $\bigoplus_{k\in\mathbb{Z}_{\geq 0}}\mathrm{Sym}^k(\mathbb{C}^m\otimes\mathbb{C}^n)$. Howe, albeit formulated differently, proves (A)-(C) as above for this and other dualities.

It is not surprising that Howe-type dualities have been at the heart of the representation theory of reductive groups every since, see also [CW12] for a summary of various such dualities, but are also pervasive in other fields. For example, in the early stages of quantum group theory Jimbo studied ***quantum Schur–Weyl duality*** [Jim86], which, in one way or the other, is central for the study of quantum invariants: That the Jones polynomial arises from the Temperley–Lieb calculus [Jon85] is an instance of quantum Schur–Weyl duality, although originally not formulated as such. And this is just the tip of the iceberg.

It did not take long for ***quantum Howe dualities*** to appear, see [NUW96] for an early reference. Also due to their relation to diagrammatics, quantum Howe dualities have been studied intensively since their first appearance in the 1990s, and also turned out to be very useful for the study of quantum invariants. For a few type A examples of such quantum Howe dualities, see [LZZ11], [CKM14] for quantum exterior and [RT16] for quantum symmetric Howe duality, and for some more "exotic type A settings", see [QS19], [TVW17] or [CW20], [BDK20].

Remark 1A.1. Quantum Howe dualities are of course not restricted to type A, but the reader should be warned at this stage: experience shows that quantum Howe dualities often run into quantization issues and nonstandard quantum objects tend to pop up. Examples are [NUW96], [ES18] or [ST19] where coideal subalgebras as in [NS95] appear. There are even such phenomena that are entirely in type A see e.g. [LTV22] and related quantization issues in [CK18], [QW18].

Quantum exterior and symmetric Howe dualities as well as their Verma counterparts are notable exceptions, and the quantization in these cases is not a big deal. In fact, our proofs will mostly stay in the non-quantum setting and the quantum case then follows using a flatness argument.

1B. **What this paper does.** The main theorem of this paper is Theorem 2B.3 where we formulate a ***(quantum) Verma Howe duality***. To explain the main points let us be less general than Theorem 2B.3 actually is. For example, as we wrote in Remark 1A.1, quantization is not an issue for us, and we stay in the classical case in this introduction. The classical, non-quantum, version of Theorem 2B.3 is then a bit more general then the following.

To work with Verma modules we go from the Lie group to the Lie algebras. For $\lambda_i\in\mathbb{C}\setminus\mathbb{Z}_{\geq 0}$, where $i\in\{1,\dots,n\}$, let $\mathtt{M}^{\lambda_i}$ be the $U(\mathfrak{gl}_2)$ Verma module of highest $\mathfrak{sl}_2$ weight λ_i. We take the tensor product $\mathtt{M}^{\lambda_1}\otimes\dots\otimes\mathtt{M}^{\lambda_n}$. For the same reason as for symmetric Howe duality, we then take a certain direct sum of the $\mathtt{M}^{\lambda_1}\otimes\dots\otimes\mathtt{M}^{\lambda_n}$. Call this direct sum $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_n)$.

Now, essentially by definition, $U(\mathfrak{gl}_2)$ acts on $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ and we also construct a dual action of $U(\mathfrak{gl}_n)$ on $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$. Using the double centralizer approach, Theorem 2B.3 states and proves (A)-(C) for the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$.

Since all symmetric powers for $U(\mathfrak{gl}_2)$ are quotients of Verma modules, we think of this Verma Howe duality as a generalization of symmetric Howe duality (with a caveat, see Section 1C below). Verma Howe duality is however much more difficult to prove: Firstly, the whole setting is, by its very nature, infinite dimensional and most of the classical statements need to be appropriately reformulated and adjusted to the infinite dimensional setting. Second, and more importantly, the simple $U(\mathfrak{gl}_n)$ appearing in (C) are neither highest nor lowest weight modules; they are simple ***dense*** weight modules in the sense of [Mat00]. This is very different from almost all Howe-type dualities in the literature and makes all calculations (for example actions of Casimir elements) much more involved. In particular, we need to take quite a detour to identify them explicitly and we crucially use results from [Maz03] and [MTL05], and implicitly compute help, to identify these dense modules. This is also our main reason to stay with $U(\mathfrak{gl}_2)$ instead of $U(\mathfrak{gl}_m)$.

As a direct application of Theorem 2B.3 we prove that the (colored higher) ***LKB*** constructed in [JK11] and [Mar20b] are simple as modules of the associated (colored) braid groups. This not just gives a new proof of [JK11, Theorem 3] but also strengthen the result of Jackson–Kerler quite a bit.

1C. **Outlook.** Separate from the evident question how to replace $U(\mathfrak{gl}_2)$ by $U(\mathfrak{gl}_m)$, here are a few directions one could try to explore:

(a) While (A) and (B) as above often hold in more generality, (C) is using that the underlying representation is semisimple. The non-semisimple versions of some of the above are known, see for example [DPS98] for an integral version of quantum Schur–Weyl duality. But these are also much more involved and often need some form of tilting theory.

A non-semisimple version Theorem 2B.3 would be a true generalization of quantum symmetric Howe duality since the cases where symmetric powers appear are precisely ruled out by our condition that $\lambda_i \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ (or by its quantum counterpart).

(b) Quantum Verma Howe duality originated in our attempt to understand invariants of links in handlebodies coming from handlebody braid groups, see e.g. [HOL02], [Ver98], [RT21], [TV21]. The associated Howe duality should involve a tensor product of Verma modules for the cores of the handlebodies, and also a tensor product of finite dimensional modules for the strands of the braids. Theorem 2B.3 captures the tensor product of Verma modules, while several papers discuss dualities involving one Verma and tensor products of finite dimensional modules, see e.g. [ILZ21] or [LV21]. It would be interesting to combine and compare these, also with an eye on categorification of the story as in [LNV21].

(c) Another interesting direction is the identification of the LKB representations with specialized parameters as cell representations of algebras within the symmetric web category from [RT16]. We suspect that this is a consequence of Verma Howe duality for (the quantum version of) $\lambda_i \in \mathbb{Z}_{\geq 0}$. Note that special cases of this are known: Jones' work [Jon85] implicitly showed the respective statement for the Burau representation and the Temperley–Lieb calculus, and [Zin01] implicitly showed an analog for the LKB representation. Note that Temperley–Lieb and the Brauer-type calculus used in [Jon85] and [Zin01], respectively, are special cases of the symmetric web calculus. (For the Temperley–Lieb calculus this is clear, while the Brauer-type calculus makes its appearance due to the "small number coincidence" that matches $\mathrm{SO}_3(\mathbb{C})$ representations and odd dimensional $\mathrm{SL}_2(\mathbb{C})$ representations.)

(d) An interesting question is how to categorify Verma Howe duality. We suspect this should be related to categorification of tensor products of infinite dimensional representations as in [DN21]. One could also hope, in some sense, that a categorification of LKB representations would be an upshot of such a categorical Verma Howe duality.

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2. A duality involving Verma modules

In this section we state a duality that we call ***(quantum) Verma Howe duality***.

Remark 2.1. We use colors in this paper, but these are a visual aid and do not have other significance. In particular, the paper is readable in black-and-white without restrictions.

2A. **Verma and dense modules.** Fix an arbitrary field $\mathbb{K}$ of characteristic zero and a transcendental element $q \in \mathbb{K}$ which we call the ***quantum parameter***.

We consider the ***quantum enveloping algebra*** $U_q(\mathfrak{gl}_2)$ ***of*** $\mathfrak{gl}_2$ over $\mathbb{K}$ with respect to the quantum parameter q. We specify our conventions later on in Section 3B and for now it is enough to know that $U_q(\mathfrak{gl}_2)$ is, as a $\mathbb{K}$ algebra, generated by E, F, $L_1^{\pm 1}$ and $L_2^{\pm 1}$.

From now on fix $n \in \mathbb{Z}_{\geq 1}$.

Notation 2A.1.

(a) We use a bold fond for tuples, e.g. $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n) \in \mathbb{K}^n$.

(b) Whenever an index of tuple is not defined but appears in a formula, then the associated element is zero, by convention. For example, $\lambda_{<1} = \lambda_{>n} = 0$ if we specify $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$.

(c) We will also use sums of the form $a_1 + a_2 + \ldots + a_{k-1} + a_k$ for $k \in \mathbb{Z}_{\geq 1}$ very often in this paper, and we abbreviate them to $\Sigma a_k = \sum_{i=1}^{k} a_i$.

(d) Denote by $\boldsymbol{\epsilon}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the tuple with the ith entry being 1, and $\boldsymbol{\alpha}_i = \boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_{i+1}$. We also use $\boldsymbol{\epsilon}_{ij}$ below meaning a matrix-style notation with only one nonzero entry.

Definition 2A.2. Given $\lambda \in \mathbb{K}$ we consider the field $\mathbb{K}_q^{\lambda} = \mathbb{K}(q^{\lambda})$. We define the ***quantum numbers*** as $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \in \mathbb{K}_q^{\lambda}$, where $x \in \lambda + \mathbb{Z}$. Similarly, for $\boldsymbol{\lambda} \in \mathbb{K}^n$ we use the field $\mathbb{K}_q^{\boldsymbol{\lambda}} = \mathbb{K}(q^{\lambda_1}, \ldots, q^{\lambda_n})$ and quantum numbers will be elements of $\mathbb{K}_q^{\boldsymbol{\lambda}}$.

Remark 2A.3. The tuple $\boldsymbol{\lambda} \in \mathbb{K}^n$ consist of the underlying ***parameters*** that we use. Note that our formulation includes the case where the quantum parameter q and the λ_i are formal variables by e.g. choosing $\mathbb{K} = \mathbb{Q}(Z, Z_1, \ldots, Z_n)$, for indeterminates Z and Z_i, and $q = Z$, $\lambda_i = Z_i$. In contrast, the parameters could be in $\mathbb{Z} \subset \mathbb{K}$, but we usually need to avoid that, see e.g. Definition 2A.17 below. It is allowed that some (or even all) of the λ_i are the same.

We consider $U_q(\mathfrak{gl}_2)$ also over fields such as $\mathbb{K}_q^{\boldsymbol{\lambda}}$ by scalar extension. The parameters only play a role for $U_q(\mathfrak{gl}_2)$ modules and not for $U_q(\mathfrak{gl}_2)$ itself.

Definition 2A.4. For any $\lambda \in \mathbb{K}$ the ***(quantum) Verma module*** $\mathtt{M}_q^{\lambda}$ of highest weight λ is $\mathtt{M}_q^{\lambda} = \mathbb{K}_q^{\lambda}\{m_i | i \in \mathbb{Z}_{\geq 0}\}$ as a $\mathbb{K}_q^{\lambda}$ vector space and the left $U_q(\mathfrak{gl}_2)$ action is

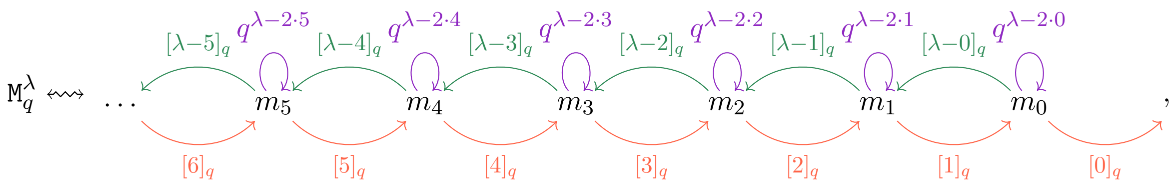
$$E \centerdot m_i = [i]_q \cdot m_{i-1}, \quad F \centerdot m_i = [\lambda - i]_q \cdot m_{i+1}, \quad L_1 \centerdot m_i = q^{\lambda - i} \cdot m_i, \quad L_2 \centerdot m_i = q^i \cdot m_i, \tag{2A.5}$$

where we use the quantum numbers from Definition 2A.2 and let $m_{-1} = 0$.

More generally, we define $\mathtt{M}_q^{\lambda,t}$ for $t \in \mathbb{K}$ by tensoring $\mathtt{M}_q^{\lambda-2t}$ with the one dimensional $U_q(\mathfrak{gl}_2)$ module of highest $\mathfrak{gl}_2$ weight (t, t).

Remark 2A.6. The highest weight of $\mathtt{M}_q^{\lambda}$ is strictly speaking (q^{λ}, q^0), but in Definition 2A.4, and throughout, we use the notion ***weight*** in the sense of classical $\mathfrak{sl}_2$ weight combinatorics. We however sometimes need to be more specific. For example, for $\mathtt{M}_q^{\lambda,t}$ we need the $\mathfrak{gl}_2$ weight notation and the classical highest weight of $\mathtt{M}_q^{\lambda,t}$ is $(\lambda - t, t)$ which is the same as $(\lambda - 2t, 0)$ when restricted to $\mathfrak{sl}_2$ weight notation. Whenever we use $\mathfrak{gl}_2$ notation we point that out.

Example 2A.7. The $U_q(\mathfrak{gl}_2)$ module $\mathtt{M}_q^{\lambda}$ is given by the usual picture but with slightly more generic quantum numbers in the action:



E moves to the right, F moves to the left, $K = L_1 L_2^{-1}$ is a loop.

The highest weight is λ. ◇

We will also use the ***quantum enveloping algebra*** $U_q(\mathfrak{gl}_n)$ ***of*** $\mathfrak{gl}_n$, again over $\mathbb{K}_q^{\boldsymbol{\lambda}}$, with conventions specified later on, see Section 3B. The generators are E_i, F_i for $i \in \{1, \ldots, n-1\}$, and $L_i^{\pm 1}$ for $i \in \{1, \ldots, n\}$.

Notation 2A.8. We consider $U_q(\mathfrak{gl}_2)$ and $U_q(\mathfrak{gl}_n)$ as different algebras, even if $n = 2$. All $U_q(\mathfrak{gl}_2)$ modules used in this paper are left $U_q(\mathfrak{gl}_2)$ modules, while all $U_q(\mathfrak{gl}_n)$ modules used in this paper are right $U_q(\mathfrak{gl}_n)$ modules. If we mean either $U_q(\mathfrak{gl}_2)$ or $U_q(\mathfrak{gl}_n)$, then we will write $U_q(\mathfrak{gl}_k)$. Similarly for their classical versions, and we will often drop the adjectives left and right.

We will need certain $U_q(\mathfrak{gl}_n)$ modules with bases indexed by:

Definition 2A.9. Fix $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{K}^n$ and $\boldsymbol{y} = (y_1, \ldots, y_{n-1}) \in \mathbb{K}^{n-1}$ such that $m_2 - m_3, \ldots, m_{n-1} - m_n \in \mathbb{Z}_{\geq 0}$ and $y_i - m_2 \notin \mathbb{Z}$ for all $i \in \{1, \ldots, n-1\}$. A ***GT (Gelfand–Tsetlin)***

pattern $GT_{\vec{x}}$ for $(\boldsymbol{m},\boldsymbol{y})$ is a triangular array of the form

$$GT_{\vec{x}} = \begin{array}{ccccccccc} x_{n1} & & x_{n2} & & \dots & & \dots & & x_{nn} \\ & \ddots & & \ddots & & \iddots & & \iddots & \\ & & x_{31} & & x_{32} & & x_{33} & & \\ & & & x_{21} & & x_{22} & & & \\ & & & & x_{11} & & & & \end{array} \tag{2A.10}$$

where $\vec{x} = (x_{n1}, ..., x_{nn}, x_{(n-1)1}, ...)$ (i.e. the pattern read row-wise) is such that:

(i) $x_{ni} = m_i$ for $i \in \{1, ..., n\}$ ($\boldsymbol{m}$ gives the top row),

(ii) $x_{i1} - y_i \in \mathbb{Z}$ for $i \in \{1, ..., n\}$ ($\boldsymbol{y}$ gives the first diagonal up to integers),

(iii) $x_{jk} - x_{(j-1)k} \in \mathbb{Z}_{\geq 0}$ and $x_{(j-1)k} - x_{j(k+1)} \in \mathbb{Z}_{\geq 0}$ for $j \in \{3, ..., n\}$ and $k \in \{2, ..., j\}$, that is,

$$\begin{array}{ccccc} x_{jk} & & \geq & & x_{j(k+1)} \\ & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & \\ & & x_{(j-1)k} & & \end{array}.$$

A ***two diagonal GT pattern (appearing in Verma Howe duality)*** is a GT pattern with $\boldsymbol{m} = (x_n = \Sigma\lambda_n + b, c, 0, ..., 0)$ with $b, c \in \mathbb{Z}$ to be chosen and $\boldsymbol{y}$ determined by $y_i = \Sigma\lambda_i$ for $i \in \{1, ..., n-1\}$. Letting $c_n = c$, we denote these by

$$GT_{\boldsymbol{x},\boldsymbol{c}} = \begin{array}{ccccccccc} x_n & & c_n & & 0 & & \dots & & 0 \\ & x_{n-1} & & c_{n-1} & & \ddots & & \iddots & \\ & & \ddots & & \ddots & & 0 & & \\ & & & x_2 & & c_2 & & & \\ & & & & x_1 & & & & \end{array}, \tag{2A.11}$$

with $\boldsymbol{x} = (x_1, ..., x_n) \in (\mathbb{K}_q^\lambda)^n$ as above and $\boldsymbol{c} = (c_2, ..., c_n) \in \mathbb{Z}_{\geq 0}^{n-1}$.

In (2A.10) and (2A.11) we have shaded the parts of the GT patterns which play significantly different roles. We call the shaded block to the right the ***integral part*** of the pattern since all the patterns we need will have integral entries in this part, and nonintegral entries otherwise.

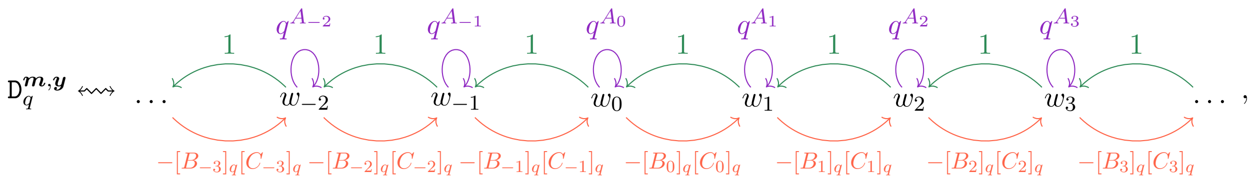
Definition 2A.12. Consider GT patterns as in Definition 2A.9. Define the ***dense module***, as a $\mathbb{K}_q^\lambda$ vector space, as $\mathtt{D}_q^{\boldsymbol{m},\boldsymbol{y}} = \mathbb{K}_q^\lambda\{GT_{\vec{x}} | GT_{\vec{x}} \text{ is a GT pattern for } (\boldsymbol{m},\boldsymbol{y})\}$ and the $U_q(\mathfrak{gl}_n)$ action is

$$\begin{aligned} GT_{\vec{x}} \centerdot E_i &= -\sum_{j=1}^{i} \frac{\prod_{k=1}^{i+1}[x_{ij} - j - x_{(i+1)k} + k]_q}{\prod_{k\neq j}[x_{ij} - j - x_{ik} + k]_q} \cdot GT_{\vec{x}+\boldsymbol{\epsilon}_{ij}}, \\ GT_{\vec{x}} \centerdot F_i &= \sum_{j=1}^{i} \frac{\prod_{k=1}^{i-1}[x_{ij} - j - x_{(i-1)k} + k]_q}{\prod_{k\neq j}[x_{ij} - j - x_{ik} + k]_q} \cdot GT_{\vec{x}-\boldsymbol{\epsilon}_{ij}}, \\ GT_{\vec{x}} \centerdot L_i &= q^{\sum_{j=1}^{i} x_{ij} - \sum_{j=1}^{i-1} x_{(i-1)j}} \cdot GT_{\vec{x}}, \end{aligned} \tag{2A.13}$$

where $\boldsymbol{\epsilon}_{ij}$ adds one to the x_{ij} entry of $\vec{x}$. Here we also use the convention that $GT_{\vec{x}\pm\boldsymbol{\epsilon}_{ij}}$ is zero if the GT conditions (i)-(iii) in Definition 2A.9 are violated.

In case of two diagonal patterns we write $\mathtt{D}_q^{b,c}$ for the respective dense module.

Example 2A.14. If $n = 2$, then the only entry in a GT patterns that is not completely determined by $(\boldsymbol{m},\boldsymbol{y})$ is x_{11}. The latter is some integer shift of y_1, so we can index a basis of $\mathtt{D}_q^{\boldsymbol{m},\boldsymbol{y}}$ as $\{w_i | i \in \mathbb{Z}\}$. Let $A_i = 2(y_1 + i) - (m_1 + m_2)$, $B_i = y_1 + i - m_1$ and $C_i = y_1 + i - m_2 + 1$. The picture is then



E moves to the right, F moves to the left, K is a loop.

The module $\mathtt{D}_q^{\boldsymbol{m},\boldsymbol{y}}$ has neither a highest nor a lowest weight. Note also that the conventions for the scalars in this example are different from Example 2A.7. (That is also why the basis vectors here are denoted by w_i and not by m_i.) But that can be fixed by appropriate base change. ◇

Remark 2A.15. For $U_q(\mathfrak{gl}_2)$ there are four ***interval-type pictures*** as in Example 2A.7 and Example 2A.14. First, a finite interval $[a,b]$, having a highest and a lowest weight, which corresponds to a finite dimensional $U_q(\mathfrak{gl}_2)$ module. One could also use $]-\infty,b]$ or $[a,\infty[$, and the associated $U_q(\mathfrak{gl}_2)$ modules are Verma and dual Verma modules, respectively. These have either a highest or a lowest weight. Finally, the interval $]-\infty,\infty[=\mathbb{R}$ corresponds to the dense modules and these have neither a highest nor a lowest weight. In this sense, dense modules are a natural family of $U_q(\mathfrak{gl}_2)$ modules.

More generally, dense modules appear in the study of weight modules for $U_q(\mathfrak{gl}_k)$. That is, every simple weight module of $U_q(\mathfrak{gl}_k)$ is dense or induced from a dense module, see [Fut87] and [Fer90], which reduces the classification of simple weight modules to dense modules. Hence, one could say that dense modules are prototypical weight modules.

Lemma 2A.16. *(2A.5) and (2A.13) endow $\mathtt{M}_q^\lambda$ and $\mathtt{D}_q^{\boldsymbol{m},\boldsymbol{y}}$, respectively, with structures of $U_q(\mathfrak{gl}_2)$ and $U_q(\mathfrak{gl}_n)$ modules.*

Proof. Well-known and easy. $\square$

Definition 2A.17. We call $\boldsymbol{\lambda}$ ***generic parameters*** if $\lambda_i, \Sigma\lambda_i \notin \mathbb{Z}_{\geq 0}$ for all $i \in \{1,...,n\}$.

We will need generic parameters because of Remark 2B.6 below and also because of:

Lemma 2A.18. *For generic parameters we have that the $U_q(\mathfrak{gl}_2)$ module $\mathtt{M}_q^\lambda$ and the $U_q(\mathfrak{gl}_n)$ module $\mathtt{D}_q^{b,c}$ are simple. Similarly, $\mathtt{M}_q^{\lambda,t}$ is a simple $U_q(\mathfrak{gl}_2)$ module if $\lambda-2t$ is generic.*

Proof. For the dense modules one can copy e.g. [Maz03, Lemma 3] while $\lambda \notin \mathbb{Z}_{\geq 0}$ implies simplicity of $\mathtt{M}_q^\lambda$, as usual in the theory. $\square$

2B. **Verma Howe duality.** Since we work with infinite dimensional $\mathbb{K}_q^{\boldsymbol{\lambda}}$ vector spaces and their homomorphisms, we need to be careful with respect to finite vs. infinite sums. To avoid convergence issues, we use the following definition, where rings, as throughout, are associative and unital.

Definition 2B.1. Let $\mathbb{S} \subset \mathbb{T}$ be two rings, and let $\mathtt{M}$ be a left (or right) $\mathbb{T}$ module. We call $\mathbb{S}$ a ***dense subring*** of $\mathbb{T}$ (with respect to $\mathtt{M}$) if for any $t \in \mathbb{T}$ and $m_1,...,m_k \in \mathtt{M}$ there exists $s \in \mathbb{S}$ such that $s \centerdot m_i = t \centerdot m_i$ (or $m_i \centerdot s = m_i \centerdot t$) for $i \in \{1,...,k\}$.

We say $\{s_i | i \in I\} \subset \mathbb{T}$ ***densely-generates*** $\mathbb{T}$ (with respect to a fixed $\mathtt{M}$) if $\{s_i | i \in I\}$ generates a dense subring of $\mathbb{T}$ and we write $\{s_i | i \in I\} \twoheadrightarrow_d \mathbb{T}$ in this case.

Notation 2B.2. We also write $\mathrm{End}_{\mathbb{S}}(\mathtt{M})$ instead of $\mathrm{End}_{\mathbb{S}^{op}}(\mathtt{M})$, i.e. we suppress the necessary but not enlightening appearance of the opposite ring.

Let $\mathbb{S}$ be a ring. For a left or right $\mathbb{S}$ module $\mathtt{M}$, the ring $\mathbb{S}' = \mathrm{End}_{\mathbb{S}}(\mathtt{M})$ is called the ***centralizer*** of $\mathbb{S}$ (on $\mathtt{M}$). We call the following theorem ***(quantum) Verma Howe duality***:

Theorem 2B.3.

(a) *There are commuting actions*

$$U_q(\mathfrak{gl}_2) \circlearrowright \mathtt{M}_q^{\oplus\boldsymbol{\lambda}} = \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n} \mathtt{M}_q^{\lambda_1+d_1} \otimes \ldots \otimes \mathtt{M}_q^{\lambda_n+d_n} \circlearrowleft U_q(\mathfrak{gl}_n).$$

(b) *Let ϕ_q^k be the algebra homomorphism induced by the $U_q(\mathfrak{gl}_k)$ actions from (a). Then, for generic $\boldsymbol{\lambda}$:*

$$\phi_q^2\colon U_q(\mathfrak{gl}_2) \twoheadrightarrow_d \mathrm{End}_{U_q(\mathfrak{gl}_n)}(\mathtt{M}_q^{\oplus\boldsymbol{\lambda}}), \quad \phi_q^n\colon U_q(\mathfrak{gl}_n) \twoheadrightarrow_d \mathrm{End}_{U_q(\mathfrak{gl}_2)}(\mathtt{M}_q^{\oplus\boldsymbol{\lambda}}).$$

That is, the two actions densely-generate the others centralizer.

(c) *For generic $\boldsymbol{\lambda}$ we have the decomposition of the $U_q(\mathfrak{gl}_2)$-$U_q(\mathfrak{gl}_n)$ bimodule $\mathtt{M}_q^{\boldsymbol{\lambda}}$ into*

$$\mathtt{M}_q^{\oplus\boldsymbol{\lambda}} \cong \bigoplus_{\substack{\boldsymbol{d}\in\mathbb{Z}^n \\ t\in\mathbb{Z}_{\geq 0}}} \mathtt{M}_q^{\Sigma\lambda_n+\Sigma d_n-t,t} \otimes \mathtt{D}_q^{\Sigma d_n-t,t}. \tag{2B.4}$$

All $\mathtt{M}_q^{\Sigma\lambda_n+\Sigma d_n-t,t}$ and $\mathtt{D}_q^{\Sigma d_n-t,t}$ are simple $U_q(\mathfrak{gl}_2)$ modules respectively $U_q(\mathfrak{gl}_n)$ modules.

There is also a similar statement in the non-quantum case which the reader can spell out easily by removing all q above.

The proof of Theorem 2B.3 is nontrivial and given in its own section, see Section 3 below.

Remark 2B.5. If $\mathtt{M}$ in Definition 2B.1 is finitely generated, then densely-generating the centralizer is the same as generating the centralizer. In this case Theorem 2B.3 is a classical Schur–Weyl(–Brauer) or Howe duality as in the introduction. The formulation above is copied from [AST17, Section 3], which also gives an an overview of Schur–Weyl(–Brauer) dualities.

Remark 2B.6. We suspect that Theorem 2B.3.(b) works without the genericity assumption, and we would expect tilting theory as in the proofs of Lemma 3A.8 and Lemma 3B.10 below to play a role. However, note that $\mathtt{M}_q^\lambda$ for $\lambda \in \mathbb{Z}_{\geq 0}$ is not tilting which makes the non-generic situation much more delicate. Note that Theorem 2B.3 for $\boldsymbol{\lambda} \in \mathbb{Z}^n_{\geq 0}$ could be used to generalize ***(quantum) symmetric Howe duality*** as in, for example, [How95, Theorem 2.1.2] and [RT16, Theorem 2.6].

Remark 2B.7. The GT patterns in Theorem 2B.3 always have many zeros, exactly as in (2A.11). This is because we consider $U_q(\mathfrak{gl}_2)$ and not $U_q(\mathfrak{gl}_m)$ for general $m \in \mathbb{Z}_{\geq 1}$.

Remark 2B.8. Verma Howe duality Theorem 2B.3 is formulated for $\big(U_q(\mathfrak{gl}_2), U_q(\mathfrak{gl}_n)\big)$. If the reader likes to work with the special linear group instead of the general linear group, then they can replace $\big(U_q(\mathfrak{gl}_2), U_q(\mathfrak{gl}_n)\big)$ with $\big(U_q(\mathfrak{sl}_2), U_q(\mathfrak{gl}_n)\big)$ or $\big(U_q(\mathfrak{gl}_2), U_q(\mathfrak{sl}_n)\big)$ in Theorem 2B.3.

3. The proof of Verma Howe duality

We first prove the classical version of Theorem 2B.3, and then use a flatness argument to get the quantum version.

3A. **The classical case.** We will need the Lie algebra $\mathfrak{gl}_k$ and it elements of the form E_{ij}. These are the $k \times k$ matrices with a one in the ith row and jth column and zeros otherwise.

Notation 3A.1.

(a) We also write $E_i = E_{i(i+1)}$, $F_i = E_{(i+1)i}$ and $L_i = E_{ii}$. For $\mathfrak{gl}_2$, we simplify this notation and use $E = E_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $F = F_1 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and we also have $L_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $L_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(b) We denote the operators used in actions by e.g. e_{ij} to distinguish then from the elements of the Lie algebras. The operators are always elements of some endomorphism space. The appearing operators will always be denoted using lowercase letters.

We will need the following realization of $\mathtt{M}^{\oplus \boldsymbol{\lambda}}$. Let $\mathbb{K}[\boldsymbol{X}^{\pm 1}, \boldsymbol{Y}]$ be the algebra generated by indeterminates $\boldsymbol{X}^{\pm 1} = (X_1^{\pm 1}, ..., X_n^{\pm 1})$ and $\boldsymbol{Y} = (Y_1, ..., Y_n)$. We shift the exponents of the X in $\mathbb{K}[\boldsymbol{X}^{\pm 1}, \boldsymbol{Y}]$ by $\boldsymbol{\lambda}$ so that powers of the variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ are now in $\boldsymbol{\lambda} + \mathbb{Z}^n$ and $\mathbb{Z}^n_{\geq 0}$, respectively. The resulting $\mathbb{K}$ vector space is denoted by $\mathtt{P}^{\boldsymbol{\lambda}} = \mathbb{K}[\boldsymbol{X}^{\boldsymbol{\lambda} + \mathbb{Z}^n}, \boldsymbol{Y}]$. We view $\mathtt{P}^{\boldsymbol{\lambda}}$ as a $\mathbb{K}[\boldsymbol{X}^{\pm 1}, \boldsymbol{Y}]$ bimodule, meaning that we allow multiplication by $X_i^{\pm 1}$ and by Y_i. We also use $\mathtt{P}^\lambda$ defined similarly.

Definition 3A.2. For $i \in \{1, ..., n\}$ we let operators ∂_{X_i} and ∂_{Y_i} act on $\mathtt{P}^\lambda$ as ***formal derivations***, i.e. for $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$ we define

$$\partial_{X_i} X_j^{\lambda + r} = \delta_{i,j}(\lambda + r) \cdot X_i^{\lambda + r - 1}, \partial_{X_i} Y_j^s = 0, \quad \partial_{Y_i} Y_j^s = \delta_{i,j} s \cdot Y_i^{s-1}, \partial_{Y_i} X_j^{\lambda + r} = 0,$$

and we then extend these rules to all of $\mathtt{P}^\lambda$ linearly and by the Leibniz rule.

We let the algebra $U(\mathfrak{gl}_2)$ act on $\mathtt{P}^\lambda$ by

$$E \mapsto e = X\partial_Y, \quad F \mapsto f = Y\partial_X, \quad L_1 \mapsto l_1 = X\partial_X, \quad L_2 \mapsto l_2 = Y\partial_Y. \tag{3A.3}$$

The action (3A.3) extends to an action of $U(\mathfrak{gl}_2)$ on

$$\mathtt{P}^{\boldsymbol{\lambda}} \cong \bigotimes_{i=1}^n \mathtt{P}^{\lambda_i}$$

by using the usual coproduct of $U(\mathfrak{gl}_2)$ determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{gl}_2$.

We have a dual action of $U(\mathfrak{gl}_n)$ on $\mathtt{P}^{\boldsymbol{\lambda}}$ determined by

$$E_{ij} \mapsto e_{ij} = X_i \partial_{X_j} + Y_i \partial_{Y_j}. \tag{3A.4}$$

In particular, E_i acts as $e_{i(i+1)}$, F_i acts as $e_{(i+1)i}$ and L_i acts as e_{ii}.

For $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$ let $X_i^{\lambda + r} Y_i^s$ be of degree $r + s$. This gives us a $\mathbb{Z}^n$ grading on $\mathtt{P}^{\boldsymbol{\lambda}}$. For $\boldsymbol{d} = (d_1, ..., d_n) \in \mathbb{Z}^n$ we denote the $\mathbb{Z}^n$ graded piece of $\mathtt{P}^{\boldsymbol{\lambda}}$ of degree $\boldsymbol{d}$ by $(\mathtt{P}^{\boldsymbol{\lambda}})_{\boldsymbol{d}}$.

Lemma 3A.5. *The graded $\mathbb{K}$ vector space*

$$\mathtt{P}^{\boldsymbol{\lambda}} \cong \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n} (\mathtt{P}^{\boldsymbol{\lambda}})_{\boldsymbol{d}}$$

is an $U(\mathfrak{gl}_2)$ module when endowed with (3A.3) that is isomorphic to $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ that decomposes as above. Moreover, it is also an $U(\mathfrak{gl}_n)$ module when endowed with (3A.4), and the two actions commute.

Proof. That (3A.3) defines a homogeneous action of $U(\mathfrak{gl}_2)$ is easy to see.

The resulting $U(\mathfrak{gl}_2)$ module is isomorphic to $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ as in the classical $\mathfrak{gl}_2$ theory: For $d = 0$ the basis elements of $(\mathtt{P}^{\lambda})_0$ are of the form $X^{\lambda-r}Y^r$ for $r \in \mathbb{Z}_{\geq 0}$ and e.g. $f(X^{\lambda-r}Y^r) = (\lambda - r)\cdot X^{\lambda-r-1}Y^{r+1}$. Comparing this with the classical version of Example 2A.7 shows that $(\mathtt{P}^{\lambda})_0 \cong \mathtt{M}^{\lambda}$. For general $d \in \mathbb{Z}$ the story is just shifted and we get $(\mathtt{P}^{\lambda})_d \cong \mathtt{M}^{\lambda+d}$. These isomorphisms extend to $\mathtt{P}^{\boldsymbol{\lambda}} \cong \mathtt{M}^{\oplus\boldsymbol{\lambda}}$ by using the coproduct.

That (3A.4) defines an $U(\mathfrak{gl}_n)$ action and that the two actions commute are direct calculations. □

We always use the two actions (3A.3) and (3A.4) for the reminder of this section. Note that Lemma 3A.5 gives us a $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule structure on $\mathtt{P}^{\boldsymbol{\lambda}}$.

Notation 3A.6. For $\boldsymbol{Z} = (Z_1, ..., Z_n)$ and $\boldsymbol{b} = (b_1, ..., b_n)$ we write $\boldsymbol{Z}^{\boldsymbol{b}} = Z_1^{b_1}\cdot ... \cdot Z_n^{b_n}$.

Lemma 3A.7. *The element $X^{\lambda+r}Y^s$ is annihilated by $E \in \mathfrak{gl}_2$ if and only if $s = 0$.*

Proof. This holds since $e = X\partial_Y$ so that $e(X^{\lambda+r}Y^s) = s\cdot X^{\lambda+r+1}Y^{s-1}$. □

An $U(\mathfrak{gl}_k)$ module is called ***countable semisimple*** if it is a countable direct sum of countable dimensional simple $U(\mathfrak{gl}_k)$ modules.

Lemma 3A.8. *For generic $\boldsymbol{\lambda}$ the $U(\mathfrak{gl}_2)$ module $\mathtt{P}^{\boldsymbol{\lambda}}$ is countable semisimple.*

Proof. Extending e.g. [Kåh10, Section 2] to $\mathbb{K}_q^{\boldsymbol{\lambda}}$, we let $\tilde{\mathcal{O}}$ denote enlarged category $\mathcal{O}$. We will not define $\tilde{\mathcal{O}}$ here as it can be defined, mutatis mutandis, as in [Kåh10, Section 2] with the same properties as therein. In particular, we have $\mathtt{P}^{\boldsymbol{\lambda}} \cong \mathtt{M}^{\oplus\boldsymbol{\lambda}} \in \tilde{\mathcal{O}}$.

By the usual Yoga, see [AST18, Section 2] and the extra notes for that paper in the arXiv version of it, tensor products of tilting $U(\mathfrak{gl}_2)$ modules decompose into direct sums of indecomposable tilting $U(\mathfrak{gl}_2)$ modules, and the latter are indexed by the same highest weights as the simples $U(\mathfrak{gl}_2)$ modules in $\tilde{\mathcal{O}}$. The usual Yoga also shows that the $\mathtt{M}^{\lambda_i}$ are tilting objects in $\tilde{\mathcal{O}}$ since they are standard and simple by Lemma 2A.18.

It then follows that $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ decomposes into a direct sum of indecomposable tilting $U(\mathfrak{gl}_2)$ modules in $\tilde{\mathcal{O}}$, and tracking the highest weight as in Lemma 3A.7 and using that $\boldsymbol{\lambda}$ is generic shows that these indecomposable tilting $U(\mathfrak{gl}_2)$ modules are actually simple and of the form $\mathtt{M}^{\mu}$ for generic $\mu \in \mathbb{K}$.

Finally, everything involved is clearly countable, so we are done. □

Lemma 3A.9. *Let $\boldsymbol{\lambda}$ be generic. As $U(\mathfrak{gl}_2)$ modules we have*

$$\mathtt{P}^{\boldsymbol{\lambda}} \cong \mathtt{M}^{\oplus\boldsymbol{\lambda}} \cong \bigoplus_{\substack{\boldsymbol{d}\in\mathbb{Z}^n\\ t\in\mathbb{Z}_{\geq 0}}} \mathtt{M}^{\Sigma\lambda_n+\Sigma d_n-t,t} \otimes \mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}, \tag{3A.10}$$

where $\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ is a multiplicity $\mathbb{K}$ vector space.

Proof. For generic $\lambda, \lambda' \in \mathbb{K}$ one can decompose $\mathtt{M}^{\lambda}\otimes\mathtt{M}^{\lambda'}$ explicitly, i.e. as $U(\mathfrak{gl}_2)$ modules we have

$$\mathtt{M}^{\lambda}\otimes\mathtt{M}^{\lambda'} \cong \bigoplus_{t\in\mathbb{Z}_{\geq 0}} \mathtt{M}^{\lambda+\lambda'-t,t}\otimes\mathfrak{D}^{\lambda+\lambda'-t,t}, \tag{3A.11}$$

for some countable dimensional multiplicity $\mathbb{K}$ vector space $\mathfrak{D}^{\lambda+\lambda'-t,t}$. This decomposition (3A.11) follows from Lemma 3A.7 and Lemma 3A.8 and the universal property of Verma modules.

The decomposition (3A.11) can then be applied inductively, since in each step one gets a direct sum of $\mathtt{M}^{\mu}$ for generic $\mu \in \mathbb{K}$. (Note that this needs that $\Sigma\lambda_k \notin \mathbb{Z}_{\geq 0}$ for $k \in \{1, ..., n\}$.) □

We now aim to identify $\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ from (3A.10) explicitly. To this end, we define a $\mathbb{K}$ sub vector space $\mathcal{D}^{b,c}$ of $\mathtt{P}^{\boldsymbol{\lambda}}$ that we will use for this purpose:

Definition 3A.12. Write $a_{ij} = \det\left(\begin{smallmatrix} x_i & y_i \\ x_j & y_j \end{smallmatrix}\right)$, $a_i = a_{i(i+1)}$, $\boldsymbol{a} = (a_1, ..., a_{n-1})$ and $\boldsymbol{l} = (l_1 ..., l_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$. For $b \in \mathbb{Z}$ and $c \in \mathbb{Z}_{\geq 0}$ let

$$\mathcal{D}^{b,c} = \mathbb{K}\mathrm{B}_{Det} \subset \mathtt{P}^{\boldsymbol{\lambda}},$$

where $\mathrm{B}_{Det} = \mathrm{B}_{Det}(b, c) = \{\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}} | \Sigma r_n = b - c, \Sigma l_{n-1} = c\}$.

Lemma 3A.13. *For fixed $c \in \mathbb{Z}_{\geq 0}$ let $\prod_{r,s} a_{rs}$ be a product of c determinants. Then $\prod_{r,s} a_{rs} \in \mathbb{K}[\boldsymbol{X}^{\pm 1}]\{\boldsymbol{a}^{\boldsymbol{l}} | \Sigma l_{n-1} = c\}$.*

Proof. The determinant of the singular matrix

$$\det\begin{pmatrix} X_r & X_r & Y_r \\ X_i & X_i & Y_i \\ X_s & X_s & Y_s \end{pmatrix} = X_r a_{is} - X_i a_{rs} + X_s a_{ri} = 0$$

gives the relation $a_{rs} = X_i^{-1}(X_r a_{is} + X_s a_{ri})$ for all $i \in \{1, ..., n\}$. This relation can then be successively applied to prove the statement. □

Lemma 3A.14. *The $U(\mathfrak{gl}_n)$ action from Lemma 3A.5 stabilizes $\mathcal{D}^{b,c} \subset \mathtt{P}^{\boldsymbol{\lambda}}$.*

Proof. A straightforward calculation gives

$$e_{ij}(a_{rs}) = \begin{cases} a_{is} & \text{if } j = r, \\ a_{ri} & \text{if } j = s, \\ 0 & \text{else.} \end{cases}$$

Using this we get

$$\begin{aligned} e_{ij}(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}) &= e_{ij}(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}})\boldsymbol{a}^{\boldsymbol{l}} + \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}} e_{ij}(\boldsymbol{a}^{\boldsymbol{l}}) \\ &= (\lambda_i + r_i) \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}+\boldsymbol{\epsilon}_i-\boldsymbol{\epsilon}_j}\boldsymbol{a}^{\boldsymbol{l}} + l_{j-1} \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}-\boldsymbol{\epsilon}_{j-1}} \underbrace{a_{(j-1)i}}_{\text{rewrite}} + l_j \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}-\boldsymbol{\epsilon}_j} \underbrace{a_{i(j+1)}}_{\text{rewrite}}. \end{aligned}$$

Now we use the rewriting as in the proof of Lemma 3A.13 on the marked terms, and we are done. Explicitly, we get

$$\begin{aligned} e_{ii}(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}) &= (\lambda_i + r_i + l_{i-1} + l_i) \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}, \\ e_{i(i+1)}(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}) &= (\lambda_i + r_i + l_{i+1}) \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}+\boldsymbol{\alpha}_i}\boldsymbol{a}^{\boldsymbol{l}} + l_{i+1} \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}-\boldsymbol{\alpha}_{i+1}}\boldsymbol{a}^{\boldsymbol{l}+\boldsymbol{\alpha}_i}, \\ e_{(i+1)i}(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}) &= (\lambda_i + r_i + l_{i-1}) \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}-\boldsymbol{\alpha}_i}\boldsymbol{a}^{\boldsymbol{l}} + l_{i-1} \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}+\boldsymbol{\alpha}_{i-1}}\boldsymbol{a}^{\boldsymbol{l}-\boldsymbol{\alpha}_i}. \end{aligned} \tag{3A.15}$$

Here we used $a_{i(i+2)} = X_{i+1}^{-1}(X_i a_{i+1} + X_{i+2} a_i)$ and $a_{(i-1)(i+1)} = X_i^{-1}(X_{i-1} a_i + X_{i+1} a_{i-1})$. □

We want to show that $\mathcal{D}^{b,c}$ is a dense module as in Theorem 2B.3. To do this we need an analog of the GT basis, and to define it we need to prepare the definition with some preliminaries.

Notation 3A.16.

(a) For $\boldsymbol{d} = (d_1, ..., d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$, we denote by $\binom{\boldsymbol{d}}{\boldsymbol{s}} \in \mathbb{Z}_{\geq 0}$ the ***multinomial-type*** number defined by the expansion $\prod_{i=1}^{n-1}(\Sigma X_i)^{d_i} = \sum_{\boldsymbol{s}} \binom{\boldsymbol{d}}{\boldsymbol{s}} \cdot \boldsymbol{X}^{\boldsymbol{s}}$.

(b) We let $(k)_l = k(k+1) \cdot ... \cdot (k+l-1)$ be the (increasing) ***Pochhammer symbol***.

(c) We write $\Pi(\boldsymbol{\lambda}, \boldsymbol{r}, \boldsymbol{d}, \boldsymbol{j})$ for $(\Sigma\lambda_1 + \Sigma r_1 - j_1 + 1)_{d_1+j_1-j_0} \cdot ... \cdot (\Sigma\lambda_{n-1} + \Sigma r_{n-1} - j_{n-1} + 1)_{d_{n-1}+j_{n-1}-j_{n-2}}$, where $\boldsymbol{j} \in \mathbb{Z}^n$ with $j_{n-1} = j_n = 0$.

Definition 3A.17. For the GT pattern

$$GT_{\boldsymbol{x},\boldsymbol{c}} = \begin{array}{cccccccc} x_n & & c_n & & 0 & & \ldots & 0 \\ & x_{n-1} & & c_{n-1} & & \ddots & & \iddots \\ & & \ddots & & \ddots & & 0 & \\ & & & x_2 & & c_2 & & \\ & & & & x_1 & & & \end{array}, \quad \begin{aligned} &c_1 = 0, c_n = c, \\ &\boldsymbol{d} = (c_2 - c_1, c_3 - c_2, ..., c_n - c_{n-1}), \\ &\Sigma r_i = x_i - \Sigma\lambda_i - \Sigma c_{i-1} + c_{i+1}, \end{aligned}$$

we define $\boldsymbol{d} = (d_1, ..., d_n)$ and $\boldsymbol{r} = (r_1, ..., r_n)$ as above. The associated ***GT vector*** is

$$GT_{\boldsymbol{d},\boldsymbol{r}} = \sum_{\boldsymbol{j} \in \mathbb{Z}_{\geq 0}^{n-2}} \binom{\boldsymbol{d}}{\boldsymbol{d} + \sum_{i=1}^{n-2} j_i \boldsymbol{\alpha}_i} \Pi(\boldsymbol{\lambda}, \boldsymbol{r}, \boldsymbol{d}, \boldsymbol{j}) \cdot \boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}-\sum_{i=1}^{n}(j_i - j_{i-2})\boldsymbol{\epsilon}_i} \boldsymbol{a}^{\boldsymbol{d}+\sum_{i=1}^{n-2} j_i \boldsymbol{\alpha}_i} \in \mathcal{D}^{b,c}. \tag{3A.18}$$

The set of these GT vectors is denoted by B_{GT}.

The following manipulation of one of the scalars defining $GT_{\boldsymbol{d},\boldsymbol{r}}$ will come in handy.

Lemma 3A.19. *We have* $\binom{\boldsymbol{d}}{\boldsymbol{d}+\Sigma j_{n-2}\boldsymbol{\alpha}_{n-2}} = \prod_{i=1}^{n-2}\binom{d_{i+1}+j_{i+1}}{j_i}$.

Proof. By using $(\Sigma X_i)^{d_i} = (\Sigma X_{i-1}+X_i)^{d_i} = \sum_{j=0}^{d_i}\binom{d_i}{j}(\Sigma X_{i-1})^{d_i-j}X_i^j$ recursively. □

Let $z_k = \sum_{1\leq j<i\leq k} e_{ji}e_{ij}$ for $k\in\{1,...,n\}$. We need what we call the ***Casimir elements*** of $\mathfrak{gl}_n$, which are defined for $k\in\{1,...,n\}$ as

$$Cas_k = \sum_{1\leq i\neq j\leq k} E_{ij}E_{ji} + \sum_{i=1}^{k} E_{ii}^2 = 2z_k + \sum_{1\leq j<i\leq k}(E_{ii}-E_{jj}) + \sum_{i=1}^{k} E_{ii}^2. \tag{3A.20}$$

The notation is such that Cas_k is the usual Casimir element of $\mathfrak{gl}_k$. We write cas_k for the associated operator. For Lemma 3A.22 below, which is the main lemma regarding the Casimir elements, we need the following formula for the action of z_k.

Lemma 3A.21. *We have* $z_k(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}) = s\cdot\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}} + e$ *with* $s\in\mathbb{K}$ *and an error term* e *given by*

$$\begin{aligned} s = &\sum_{i=1}^{k}\Big((\lambda_i+r_i)(\Sigma\lambda_{i-1}+\Sigma r_{i-1}+\Sigma l_n - l_{i-1}-l_i+i-1)\\ &+(i-1)l_i+(i-2)l_{i-2}+(l_{i-1}+l_i)\Sigma l_{i-2}\Big),\\ e =& l_k\sum_{i=1}^{k-1}(\lambda_i+r_i)\cdot\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}+\boldsymbol{\epsilon}_k+\boldsymbol{\epsilon}_{k+1}-\boldsymbol{\epsilon}_i-\boldsymbol{\epsilon}_{i+1}}\boldsymbol{a}^{\boldsymbol{l}-\boldsymbol{\epsilon}_k+\boldsymbol{\epsilon}_i}. \end{aligned}$$

Proof. A tedious calculation using the previous formulas. □

Lemma 3A.22. *Let* $\boldsymbol{\lambda}$ *be generic. The Casimir elements separate* B_{GT}*, and* B_{GT} *is a basis of* $\mathcal{D}^{b,c}$.

Proof. The proof splits into three steps.

Separation. We first assume that the Casimir elements act by a scalar on B_{GT}. We let $\lambda_k = y_k\boldsymbol{\epsilon}_1 + c_k\boldsymbol{\epsilon}_2$. We assume the scalar is

$$Cas_k \text{ acts on } GT_{\boldsymbol{d},\boldsymbol{r}} \text{ by } \langle\lambda_k+2\rho^{(k)},\lambda_k\rangle = y_k(y_k+k-1)+c_k(c_k+k-3), \tag{3A.23}$$

where $\rho^{(k)} = \frac{1}{2}\sum_{i=1}^{k}(k-2i+1)\cdot\boldsymbol{\epsilon}_i$ is the usual half-sum of the positive roots of $\mathfrak{gl}_n$. Note that implies that the Casimir elements separate, so it remains to verify (3A.23).

Scalar verification. We thus need to compute $cas_k(GT_{\boldsymbol{d},\boldsymbol{r}})$. The calculation that $cas_k(GT_{\boldsymbol{d},\boldsymbol{r}})$ equals (3A.23) boils down to a longish manipulation of symbols where one reindexes the sum defining $GT_{\boldsymbol{d},\boldsymbol{r}}$ appropriately. We sketch the main step in this calculation now.

(a) First, we use the second expression of Cas_k in (3A.20). As before, we use $z_k = \sum_{1\leq j<i\leq k} e_{ji}e_{ij}$ and we also write h_k for the Cartan part so that $cas_k = 2z_k + h_k$. By (3A.15), the Cartan part gives $h_k(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}) = s'\cdot\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}$ with scalar

$$s' = \sum_{1\leq j<i\leq k}(\lambda_i+r_i+l_{i-1}+l_i-\lambda_j-r_j-l_{j-1}-l_j) + \sum_{i=1}^{k}(\lambda_i+r_i+l_{i-1}+l_i)^2\in\mathbb{K}.$$

(b) We also have the scalar s from Lemma 3A.21. Thus, we get, again using Lemma 3A.21, that

$$cas_k(\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}}) = (2s+s')\cdot\boldsymbol{X}^{\boldsymbol{\lambda}+\boldsymbol{r}}\boldsymbol{a}^{\boldsymbol{l}} + e, \tag{3A.24}$$

where e is the error term in Lemma 3A.21.

(c) Next, we need to take the sum of (3A.24) as in the definition of the GT vectors. The resulting expression can then by manipulated as in the next few bullet points.

(d) We change the summation and use a few tricks to get the same expressions defining the GT vectors from (3A.18):
   - For the part with the multinomial scalars we use Lemma 3A.19 and the well-known formula $\binom{a-1}{b-1} = \frac{b}{a}\binom{a}{b}$ to rewrite e.g.

$$\binom{d_{i-1}+j_{i-1}\boxed{-1}}{j_{i-2}\boxed{-1}} = \frac{j_{i-2}}{d_{i-1}+j_{i-1}}\binom{d_{i-1}+j_{i-1}\boxed{+0}}{j_{i-2}\boxed{+0}}.$$

We further use $\binom{a-1}{b} = \frac{a-b}{a}\binom{a}{b}$ to rewrite for example

$$\binom{d_{i-1}+j_{i-1}\boxed{-1}}{j_{i-2}} = \frac{d_{i-1}+j_{i-1}-j_{i-2}}{d_{i-1}+j_{i-1}}\binom{d_{i-1}+j_{i-1}\boxed{+0}}{j_{i-2}}.$$

Here we marked the parts that we change to match the GT vectors. (If $a=0$ in these formulas, then we would use $a\binom{a-1}{b-1} = b\binom{a}{b}$ and $a\binom{a-1}{b} = (a-b)\binom{a}{b}$ which give $0=0$ so we can ignore these cases.)

- We rewrite the Pochhammer symbols as well, for example:

$$(\Sigma\lambda_i + \Sigma r_i - j_i\boxed{+2})_{d_i+j_i-j_{i-1}\boxed{-1}} = \frac{1}{\Sigma\lambda_i + \Sigma r_i - j_i + 1}(\Sigma\lambda_i + \Sigma r_i - j_i\boxed{+1})_{d_i+j_i-j_{i-1}\boxed{+0}}.$$

We again highlight the parts we change to match the expressions in the GT vectors.

(e) All introduced fractions disappear in the end. To elaborate, we get

$$\sum_{i=1}^{k-1}\Big(\prod_{l=i}^{k-1}\frac{j_l}{d_l+j_l}\Big)(d_i+j_i-j_{i-1})\frac{\prod_{l=i+1}^{k}(\Sigma\lambda_l+\Sigma r_l+d_l-j_{l-1}+1)}{\prod_{l=i}^{k-1}(\Sigma\lambda_l+\Sigma r_l-j_l+1)}(\Sigma\lambda_i+\Sigma r_i-j_i-j_{i-1}+1).$$

The sum of the two first terms is:

$$\frac{\prod_{i=2}^{k-1}j_i}{\prod_{i=3}^{k-1}(d_i+j_i)}\frac{\prod_{i=3}^{k}(\Sigma\lambda_i+\Sigma r_i+d_i-j_{i-1}+1)}{\prod_{i=3}^{k-1}(\Sigma\lambda_i+\Sigma r_i-j_i+1)}.$$

We continue, analyzing three, four etc. terms, until we find

$$j_{k-1}(\Sigma\lambda_k+\Sigma r_k+d_k-j_{k-1}+1).$$

(f) This implies that the overall scalar for the $\boldsymbol{j}\in\mathbb{Z}_{\geq 0}^{n-2}$ summand of $GT_{\boldsymbol{d},\boldsymbol{r}}$ is

$$\begin{aligned}\sum_{i=3}^{k}(\lambda_i+r_i+d_i+d_{i-1})(\Sigma d_{i-2}\boxed{+j_{i-2}}) + \sum_{i=2}^{k}\big((\Sigma\lambda_{i-1}+\Sigma r_{i-1}\boxed{-j_{i-1}-j_{i-2}})(d_i\boxed{+j_i-j_{i-1}}) \\ +(\lambda_i+r_i+d_i\boxed{+j_{i-2}-j_{i-1}})(\Sigma\lambda_{i-1}+\Sigma r_{i-1}\boxed{-j_{i-1}-j_{i-2}}+i-1) \\ +(i-2)(d_{i-1}\boxed{+j_{i-1}-j_{i-2}})\big)\boxed{+j_{k-1}(\Sigma\lambda_k+\Sigma r_k+d_k-j_{k-1}+1)}.\end{aligned} \tag{3A.25}$$

We marked the dependencies on $\boldsymbol{j}$.

(g) The dependencies on $\boldsymbol{j}$ in (3A.25) cancel and we get $cas_k(GT_{\boldsymbol{d},\boldsymbol{r}}) = s''\cdot GT_{\boldsymbol{d},\boldsymbol{r}}$ for the scalar

$$\begin{aligned}s'' = \sum_{i=3}^{k}(\lambda_i+r_i+d_i+d_{i-1})\Sigma d_{i-2} + \sum_{i=2}^{k}\big((\Sigma\lambda_{i-1}+\Sigma r_{i-1})d_i \\ +(\lambda_i+r_i+d_i)(\Sigma\lambda_{i-1}+\Sigma r_{i-1}+i-1)+(i-2)d_{i-1}\big) \\ +\sum_{i=1}^{k}\big((k-2i+1)(\lambda_i+r_i+d_{i-1}+d_i)+(\lambda_i+r_i+d_{i-1}+d_i)^2\big)\in\mathbb{K}.\end{aligned}$$

(h) Finally, matching s'' with (3A.23) is done by comparing the linear and the quadratic terms separately. This is again tedious, but straightforward.

Basis. The linear independence of B_{GT} follows from (3A.23) and that B_{GT} spans follows because the definition of $GT_{\boldsymbol{d},\boldsymbol{r}}$ implies that B_{GT} is uppertriangular (with an appropriate order) to B_{Det}. □

Remark 3A.26. We were able to guess the formulas in (3A.23) because of significant help of Magma and Mathematica, which were used to find the GT bases expressions in Definition 3A.17, as well as the formula given in [MTL05, (6.2)]. The computational expensive proof of Lemma 3A.22 was then also obtained by computer help. We however stress that everything can be done by hand and computers were only used to guess the various steps.

We now identify $\mathcal{D}^{b,c}$ with the simple dense $U(\mathfrak{gl}_2)$ module $\mathtt{D}^{b,c}$ from Definition 2A.12.

Lemma 3A.27. *For generic $\boldsymbol{\lambda}$ we have an isomorphism of $U(\mathfrak{gl}_n)$ modules $\mathcal{D}^{b,c}\cong\mathtt{D}^{b,c}$. In particular, $\mathcal{D}^{b,c}$ is a simple $U(\mathfrak{gl}_n)$ module and has a GT pattern realization.*

Proof. We first show that $\mathcal{D}^{b,c}$ is a simple $U(\mathfrak{gl}_n)$ module. To this end, we use that the Casimir elements separate the GT patterns in B_{GT}, see Lemma 3A.22, and then we use similar arguments as in [Maz03, Lemma 3]. That is, we claim that the e_{ij} act injectively (and thus, bijectively) on $GT_{\boldsymbol{d},\boldsymbol{r}}$ and also that the action graph of the e_{ij} action on B_{GT} is strongly connected.

The first claim follows from Lemma 3A.22, which implies that it is enough to show injectivity on B_{GT}, and the formulas for the action of e_{ij} on $GT_{\boldsymbol{d},\boldsymbol{r}}$ that we get from (3A.15).

For the second claim we compute that

$$\begin{aligned} e_{i(i+1)}(GT_{\boldsymbol{d},\boldsymbol{r}}) =& \frac{(\Sigma\lambda_i + \Sigma r_i + 1)\,(\Sigma\lambda_{i+1} + \Sigma r_{i+1} + d_{i+1})}{\Sigma\lambda_i + \Sigma r_i + d_i + 1} \cdot GT_{\boldsymbol{d},\boldsymbol{r}+\boldsymbol{\alpha}_i} \\ &+ \frac{d_i\,(\Sigma\lambda_{i+1} + \Sigma r_{i+1} + d_i + d_{i+1} + 1)}{\Sigma\lambda_i + \Sigma r_i + d_i + 1} \cdot GT_{\boldsymbol{d}+\boldsymbol{\alpha}_{i-1},\boldsymbol{r}-\boldsymbol{\alpha}_{i-1}}, \end{aligned}$$

with the second term being zero if $i = 1$ or if $\boldsymbol{d} + \boldsymbol{\alpha}_{i-1} \notin \mathbb{Z}^{n-1}_{\geq 0}$. There is also a similar formula for $e_{(i+1)i}(GT_{\boldsymbol{d},\boldsymbol{r}})$ with swapped signs in front of the $\boldsymbol{\alpha}_j$ and similar coefficients. Note that all appearing coefficients are nonzero since we have generic parameters.

Thus, the action graph is strongly connected and hence, the e_{ij} act bijectively and have a strongly connected action graph, showing that $\mathcal{D}^{b,c}$ is a simple $\mathfrak{gl}_n$ module.

Next, it follows from the definitions that $\mathcal{D}^{b,c}$ is a dense weight $U(\mathfrak{gl}_n)$ module in the sense of e.g. the introduction of [Maz03]. We have also verified that $\mathcal{D}^{b,c}$ is simple. Thus, we can use the classification of these modules from [Mat00], see also [Maz03, Section 2.3]. Comparing [Maz03, Section 2.3] (which has slightly different conventions but that does not play a role here) with the calculations in the proof of Lemma 3A.22 implies the claim. □

Proof of the classical version of Theorem 2B.3. There are three statements to verify.

Commuting actions. By Lemma 3A.5, we can consider the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule $\mathtt{P}^{\boldsymbol{\lambda}}$ and it remains to verify the centralizer property and the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule decomposition of Theorem 2B.3.

All parameters in this proof are generic from now on.

Bimodule decomposition. Let $b = \Sigma d_n - t$ and $c = t$. The $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule decomposition follows from Lemma 3A.27 after identifying $\mathcal{D}^{b,c}$ with the multiplicity space $\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ from Lemma 3A.9 as a $\mathbb{K}$ vector space. Note that $\mathcal{D}^{b,c}$ is a $U(\mathfrak{gl}_n)$ module so it has a $\mathbb{Z}^n$ grading coming from the $U(\mathfrak{gl}_n)$ weight spaces. At the same time, because $\mathtt{P}^{\boldsymbol{\lambda}}$ is an $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule, Lemma 3A.9 implies that $\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ is also a $U(\mathfrak{gl}_n)$ module, so we also have the notion of $U(\mathfrak{gl}_n)$ weight spaces. Explicitly, m in either $\mathcal{D}^{b,c}$ or $\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ is of degree $\boldsymbol{d} \in \mathbb{Z}^n$ if $e_{ii}(m) = (\lambda_i + d_i)\cdot m$. We apply this definition and (3A.15) to B_{Det} and get

$$\boldsymbol{Z}\mathrm{dim}_{\mathbb{K}}\mathcal{D}^{b,c} = \sum_{\boldsymbol{d}\in\mathbb{Z}^n} \delta_{b+c,\Sigma d_n} \tbinom{c+n-2}{c} \boldsymbol{Z}^{\boldsymbol{d}},$$

where we use $\boldsymbol{Z} = (Z_1, ..., Z_n)$ to keep track of the graded pieces and $\boldsymbol{Z}\mathrm{dim}_{\mathbb{K}}$ means graded dimensions. Moreover, using $\sum_{c=0}^{t} \binom{c+n-2}{c} = \binom{t+n-1}{t}$, we get

$$\boldsymbol{Z}\mathrm{dim}_{\mathbb{K}}\mathtt{P}^{\boldsymbol{\lambda}} = \sum_{\substack{\boldsymbol{d}\in\mathbb{Z}^n\\ t\in\mathbb{Z}_{\geq 0}}} \tbinom{t+n-1}{t} \boldsymbol{Z}^{\boldsymbol{d}} = \sum_{\substack{\boldsymbol{d}\in\mathbb{Z}^n\\ t\in\mathbb{Z}_{\geq 0}}} \Big(\sum_{c=0}^{t} \tbinom{c+n-2}{c} \Big) \boldsymbol{Z}^{\boldsymbol{d}}.$$

Thus, (3A.10) implies that

$$\boldsymbol{Z}\mathrm{dim}_{\mathbb{K}}\mathcal{D}^{b,c} = \boldsymbol{Z}\mathrm{dim}_{\mathbb{K}}\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t} = \sum_{\boldsymbol{d}\in\mathbb{Z}^n} \delta_{b+c,\Sigma d_n} \tbinom{c+n-2}{c} \boldsymbol{Z}^{\boldsymbol{d}}. \tag{3A.28}$$

Finally, note that $\mathcal{D}^{b,c} \subset \mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ because $e(\mathrm{B}_{Det}) = \{0\}$, as a simple calculation shows. Hence, $\mathcal{D}^{b,c} = \mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$.

Dense. Thus, it remains to prove that have dense subrings induced from the $U(\mathfrak{gl}_2)$ and $U(\mathfrak{gl}_n)$ actions. We denote their images in $\mathrm{End}_{\mathbb{K}}(\mathtt{P}^{\boldsymbol{\lambda}})$ by $\mathbb{S}$ and $\mathbb{T}$, respectively. The $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule decomposition implies that $\mathbb{T}$ is dense in $\mathbb{S}'$ since

$$U(\mathfrak{gl}_2) \twoheadrightarrow_d \mathrm{End}_{U(\mathfrak{gl}_n)}\big(\bigoplus_{\substack{\boldsymbol{d}\in\mathbb{Z}^n\\ t\in\mathbb{Z}_{\geq 0}}} \mathtt{M}^{\Sigma\lambda_n+\Sigma d_n-t,t} \otimes \mathtt{D}^{\Sigma d_n-t,t} \big) \cong \mathrm{End}_{U(\mathfrak{gl}_n)}(\mathtt{P}^{\boldsymbol{\lambda}}).$$

Similarly, with swapped roles of $U(\mathfrak{gl}_2)$ and $U(\mathfrak{gl}_n)$, we get that $\mathbb{S}$ is dense in $\mathbb{T}'$. □

Remark 3A.29. Note that (3A.28) implies that the dense modules we use have weight spaces of constant dimension. This is actually true in more generality, see e.g. [Maz03, Lemma 2].

3B. **The quantum case.** We now specify our quantum conventions.

Notation 3B.1.

(a) Let $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ denote a tuple of variables. Let $\mathbb{A}_v = \mathbb{Z}[v, v^{-1}]$ denote the ***A-form***, $\mathbb{A}'_v = \mathbb{Q}(v)$ the field of fractions of $\mathbb{A}_v$, $\mathbb{A}_v^{\boldsymbol{\mu}} = \mathbb{A}_v[\boldsymbol{\mu}, v^{\mu_1}, ..., v^{\mu_n}]$ and $\mathbb{A}_v^{\prime,\boldsymbol{\mu}}$ the field of fractions of $\mathbb{A}_v^{\boldsymbol{\mu}}$.

(b) For a fixed ring $\mathbb{S}$ and choices $\bar{v}, \bar{v}^{-1} \in \mathbb{S}$ and $\bar{\boldsymbol{\mu}} = (\bar{\mu}_1, ..., \bar{\mu}_n) \in \mathbb{S}^n$ such that $\bar{v}^{\bar{\mu}_1}, ..., \bar{v}^{\bar{\mu}_n} \in \mathbb{S}$, we can specialize any construction defined with coefficients from $\mathbb{A}_v^{\boldsymbol{\mu}}$ using $\_ \otimes_{\mathbb{A}_v^{\boldsymbol{\mu}}} \mathbb{S}$ where we see $\mathbb{S}$ as an $\mathbb{A}_v^{\boldsymbol{\mu}}$ module by $v^{\pm 1} \mapsto \bar{v}^{\pm 1}$ and $\boldsymbol{\mu} \mapsto \bar{\boldsymbol{\mu}}$. We will use this always without the bar notation. We can similarly specialize from $\mathbb{A}_v$ instead of $\mathbb{A}_v^{\boldsymbol{\mu}}$ in the very same way.

(c) The ***classical*** and the ***quantum specializations*** are $v \mapsto 1$ and $v \mapsto q$, respectively, and $\boldsymbol{\mu} \mapsto \boldsymbol{\lambda}$ for $\mathbb{S} = \mathbb{K}$ as fixed in Section 2A. Similarly for $\mathbb{A}_v$ instead of $\mathbb{A}_v^{\boldsymbol{\mu}}$.

For $k \in \mathbb{Z}_{\geq 1}$, let $U_{\mathbb{A}}(\mathfrak{gl}_k)$ denote the ***quantum enveloping algebra*** over $\mathbb{A}_v$ of $\mathfrak{gl}_k$. We use the conventions, excluding the Hopf algebra structure, from [Lus90] or [APW91] with $K_i^{\pm 1} = L_i^{\pm 1} L_{i+1}^{\mp 1}$. The $\mathbb{A}_v$ algebra $U_{\mathbb{A}}(\mathfrak{gl}_k)$ specializes to either $U(\mathfrak{gl}_k)$ for $v \mapsto 1$ and to $U_q(\mathfrak{gl}_k)$ for $v \mapsto q$. The classical specialization is the one we studied in Section 3A.

We use the same notation as in Definition 2A.2 for quantum numbers, but we see them as elements of $\mathbb{A}_v$ or $\mathbb{A}_v^{\boldsymbol{\mu}}$ in general, and these specialize to the ones in Definition 2A.2.

The $\mathbb{A}'_v$ algebra $U_v(\mathfrak{gl}_k)$ is generated by E_i, F_i for $i \in \{1, ..., k-1\}$, and $L_i^{\pm 1}$ for $i \in \{1, ..., k\}$ such that the $L_i^{\pm 1}$ commute with one another, L_i^{-1} is the inverse of L_i, and

$$L_i E_j = v^{\delta_{i,j} - \delta_{i,j+1}} \cdot E_j L_i, \quad L_i F_j = v^{-\delta_{i,j} + \delta_{i,j+1}} \cdot F_j L_i, \quad E_i F_j - F_j E_i = \delta_{i,j} \tfrac{L_i L_{i+1}^{-1} - L_i^{-1} L_{i+1}}{v - v^{-1}},$$
$$[2]_v \cdot E_i E_j E_i = E_i^2 E_j + E_j E_i^2, \text{ if } |i-j| = 1, \quad E_i E_j - E_j E_i = 0, \text{ if } |i-j| > 1,$$
$$[2]_v \cdot F_i F_j F_i = F_i^2 F_j + F_j F_i^2, \text{ if } |i-j| = 1, \quad F_i F_j - F_j F_i = 0, \text{ if } |i-j| > 1,$$

for all suitable i, j. We also choose the Hopf algebra structure on $U_{\mathbb{A}}(\mathfrak{gl}_k)$ given by

$$\Delta(E_i) = E_i \otimes L_i L_{i+1}^{-1} + 1 \otimes E_i, \quad \epsilon(E_i) = 0, \quad S(E_i) = -E_i L_i^{-1} L_{i+1},$$
$$\Delta(F_i) = F_i \otimes 1 + L_i^{-1} L_{i+1} \otimes F_i, \quad \epsilon(F_i) = 0, \quad S(F_i) = -L_i L_{i+1}^{-1} F_i,$$

with $L_i^{\pm 1}$ being group like.

Following [Lus90], $U_{\mathbb{A}}(\mathfrak{gl}_k)$ is the $\mathbb{A}_v$ subalgebra of $U_v(\mathfrak{gl}_k)$ generated by the ***divided powers*** for E_i and F_i, i.e.

$$E_i^{(j)} = \frac{E_i^j}{[j]_v!}, F_i^{(j)} = \frac{F_i^j}{[j]_v!}, i \in \{1, ..., k-1\}, j \in \mathbb{Z}_{\geq 0},$$

and also by some adjustments of the L_i, see [APW91]. As the Hopf algebra structure of $U_{\mathbb{A}}(\mathfrak{gl}_k)$ we take the one induced by $U_v(\mathfrak{gl}_k)$.

Remark 3B.2. The $\mathbb{A}_v$ algebra $U_{\mathbb{A}}(\mathfrak{gl}_k)$ specializes to $U(\mathfrak{gl}_k)$ for $v \mapsto 1$ and to $U_q(\mathfrak{gl}_k)$ for $v \mapsto q$. In both cases the divided power generators are only needed for $j = 1$.

We scalar extend $U_{\mathbb{A}}(\mathfrak{gl}_k)$ to an $\mathbb{A}_v^{\boldsymbol{\mu}}$ algebra, keeping the same notation. The additional parameters only play a role for $U_{\mathbb{A}}(\mathfrak{gl}_k)$ modules and not for $U_{\mathbb{A}}(\mathfrak{gl}_k)$ itself.

Lemma 3B.3. *Definition 2A.4 works verbatim over $\mathbb{A}_v^{\boldsymbol{\mu}}$ giving $U_{\mathbb{A}}(\mathfrak{gl}_2)$ modules.*

Proof. All appearing scalars can be interpreted in $\mathbb{A}_v^{\boldsymbol{\mu}}$. □

The $U_{\mathbb{A}}(\mathfrak{gl}_2)$ modules from Lemma 3B.3 are the ***integral Verma modules***. We denote these as before but using $\mathbb{A}$ as a subscript, e.g. $\mathtt{M}_{\mathbb{A}}^{\mu}$ is the integral version of $\mathtt{M}_q^{\lambda}$. Using the Hopf algebra structure we can then define $\mathtt{M}_{\mathbb{A}}^{\oplus \boldsymbol{\mu}}$ similarly as we defined $\mathtt{M}_q^{\oplus \boldsymbol{\lambda}}$.

Remark 3B.4. We now copy the approach taken in Section 3A. That is, we define a quantum polynomial algebra $\mathtt{P}_{\mathbb{A}}^{\boldsymbol{\mu}}$ on which $U_{\mathbb{A}}(\mathfrak{gl}_2)$ and $U_{\mathbb{A}}(\mathfrak{gl}_n)$ act by quantum derivatives in the spirit of e.g. [Kas95, Section VII.3]. This is done such that $\mathtt{P}_{\mathbb{A}}^{\boldsymbol{\mu}} \cong \mathtt{M}_{\mathbb{A}}^{\oplus \boldsymbol{\mu}}$ as $U_{\mathbb{A}}(\mathfrak{gl}_2)$-$U_{\mathbb{A}}(\mathfrak{gl}_n)$ bimodules. However, we do not use the language of quantum derivatives because of the various quantum parameters appear everywhere which make this setup cumbersome instead of helpful. For example, one would have relations of the form $Y_jX_i = X_iY_j + (v-v^{-1})X_jY_i$ and commutativity turns into quantum commutativity. In order to avoid these technical difficulties we decided to define $\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}}$ instead as a free $\mathbb{A}^{\boldsymbol{\mu}}_v$ module with an explicit biaction defined on basis elements. The reader is still invited to think of the below as quantum derivatives acting on a quantum polynomial algebra.

Definition 3B.5. We define the free $\mathbb{A}^{\boldsymbol{\mu}}_v$ module

$$\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}} = \mathbb{A}^{\boldsymbol{\mu}}_v\{\overline{\boldsymbol{X}}^{\boldsymbol{\mu}+\boldsymbol{r}}\overline{\boldsymbol{Y}}^{\boldsymbol{s}} | \boldsymbol{r}\in\mathbb{Z}^n, \boldsymbol{s}\in\mathbb{Z}^n_{\geq 0}\}$$

where we, as before, use formal parameters.

Write $X_i^{\mu_i+r_i} = [\mu_i+r_i]_v!\cdot\overline{X}_i^{\mu_i+r_i}$ and $Y_i^{s_i}=[s_i]_v!\cdot\overline{Y}_i^{s_i}$. We let $U_v(\mathfrak{gl}_2)$ act on the scalar extension of $\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}}$ to $\mathbb{A}^{\prime;\boldsymbol{\mu}}_v$ by

$$\begin{aligned}
E \centerdot \boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}} &= \sum_{i=1}^{n} v^{\Sigma\mu_n+\Sigma r_n-\Sigma\mu_i-\Sigma r_i-\Sigma s_n+\Sigma s_i}[s_i]_v\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}+\epsilon_i}\boldsymbol{Y}^{\boldsymbol{s}-\epsilon_i},\\
F \centerdot \boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}} &= \sum_{i=1}^{n} v^{-\Sigma\mu_{i-1}-\Sigma r_{i-1}+\Sigma s_{i-1}}[\mu_i+r_i]_v\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}-\epsilon_i}\boldsymbol{Y}^{\boldsymbol{s}+\epsilon_i},\\
L_1 \centerdot \boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}} &= v^{\Sigma\mu_n+\Sigma r_n}\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}},\\
L_2 \centerdot \boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}} &= v^{\Sigma s_n}\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}}.
\end{aligned}\tag{3B.6}$$

We also define an $U_v(\mathfrak{gl}_n)$ action on the scalar extension of $\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}}$ to $\mathbb{A}^{\prime;\boldsymbol{\mu}}_v$ by

$$\begin{aligned}
\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}} \centerdot E_i &= [\mu_{i+1}+r_{i+1}]_v v^{\mu_i+r_i-\mu_{i+1}-r_{i+1}}\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}+\boldsymbol{\alpha}_i}\boldsymbol{Y}^{\boldsymbol{s}} + [s_{i+1}]_v\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}+\boldsymbol{\alpha}_i},\\
\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}} \centerdot F_i &= [\mu_i+r_i]_v\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}-\boldsymbol{\alpha}_i}\boldsymbol{Y}^{\boldsymbol{s}} + v^{-\mu_i-r_i+\mu_{i+1}+r_{i+1}}[s_i]_q\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}-\boldsymbol{\alpha}_i},\\
\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}} \centerdot L_i &= v^{\mu_i+r_i+s_i}\cdot\boldsymbol{X}^{\boldsymbol{\mu}+\boldsymbol{r}}\boldsymbol{Y}^{\boldsymbol{s}}.
\end{aligned}\tag{3B.7}$$

As before, we also get graded pieces $(\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}})_{\boldsymbol{d}}$ for $\boldsymbol{d}\in\mathbb{Z}^n$.

Lemma 3B.8. *The graded free $\mathbb{A}_v$ module*

$$\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}} \cong \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n_{\geq 0}} (\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}})_{\boldsymbol{d}}$$

is an $U_{\mathbb{A}}(\mathfrak{gl}_2)$ module when endowed with the $\mathbb{A}^{\boldsymbol{\mu}}_v$-version of (3B.6) that decomposes as above and is isomorphic to $\mathtt{M}^{\oplus\boldsymbol{\mu}}_{\mathbb{A}}$. Moreover, it is also an $U_{\mathbb{A}}(\mathfrak{gl}_n)$ module when endowed with the $\mathbb{A}^{\boldsymbol{\mu}}_v$-version of (3B.7), and the two actions commute.

Proof. A direct calculation verifies that (3B.6) and (3B.7) give the scalar extension of $\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}}$ the structure of an $U_v(\mathfrak{gl}_2)$-$U_v(\mathfrak{gl}_n)$ bimodule. One then checks that (3B.6) and (3B.7) when successively applied to $E_i^{(j)}$ and $F_i^{(j)}$ has coefficients in $\mathbb{A}^{\boldsymbol{\mu}}_v$ when acting on the basis of the elements of the form $\overline{\boldsymbol{X}}^{\boldsymbol{\mu}+\boldsymbol{r}}\overline{\boldsymbol{Y}}^{\boldsymbol{s}}$ (to see this it is helpful to keep the quantum derivative picture from Remark 3B.4 and $\partial_X\frac{X^r}{r!}=\frac{X^{r-1}}{(r-1)!}$ in mind), so the biaction can be restricted and gives an $U_{\mathbb{A}}(\mathfrak{gl}_2)$-$U_{\mathbb{A}}(\mathfrak{gl}_n)$ bimodule structure on $\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}}$.

The final claim that $\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}}\cong\mathtt{M}^{\oplus\boldsymbol{\mu}}_{\mathbb{A}}$ as $U_{\mathbb{A}}(\mathfrak{gl}_2)$-$U_{\mathbb{A}}(\mathfrak{gl}_n)$ bimodules can be verified as in Lemma 3A.5. □

Definition 3B.9.

- **(a)** We call an $\mathbb{A}^{\boldsymbol{\mu}}_v$ module $\mathtt{M}$ ***(generically) flat*** if its free and all specializations to characteristic zero fields where $\boldsymbol{\mu}$ are specialized to generic parameters are of the same dimension.
- **(b)** We call an $U_{\mathbb{A}}(\mathfrak{gl}_k)$ module $\mathtt{M}$ ***(generically) flat*** if it is flat as an $\mathbb{A}^{\boldsymbol{\mu}}_v$ module and if all specializations of $\mathrm{End}_{U_{\mathbb{A}}(\mathfrak{gl}_k)}(\mathtt{M})$ to characteristic zero fields where $\boldsymbol{\mu}$ are specialized to generic parameters are of the same dimension.
- **(c)** By being ***(generically) flat as an*** $U_{\mathbb{A}}(\mathfrak{gl}_2)$-$U_{\mathbb{A}}(\mathfrak{gl}_n)$ ***bimodule*** we mean being flat as an $U_{\mathbb{A}}(\mathfrak{gl}_2)$ module and as an $U_{\mathbb{A}}(\mathfrak{gl}_n)$ module.

Lemma 3B.10. *The $U_{\mathbb{A}}(\mathfrak{gl}_2)$-$U_{\mathbb{A}}(\mathfrak{gl}_n)$ bimodule $\mathtt{M}^{\oplus\boldsymbol{\mu}}_{\mathbb{A}}\cong\mathtt{P}^{\boldsymbol{\mu}}_{\mathbb{A}}$ is flat, its classical specialization is the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule $\mathtt{M}^{\oplus\boldsymbol{\lambda}}\cong\mathtt{P}^{\boldsymbol{\lambda}}$ and its quantum specialization is the $U_q(\mathfrak{gl}_2)$-$U_q(\mathfrak{gl}_n)$ bimodule $\mathtt{M}^{\oplus\boldsymbol{\lambda}}_q\cong\mathtt{P}^{\boldsymbol{\lambda}}_q$.*

Proof. The quantum version of Lemma 3A.8 holds as well, with the same proof. This implies that $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}}$ is tilting when specialized to characteristic zero fields with $\boldsymbol{\mu} \mapsto \boldsymbol{\lambda}$ for $\boldsymbol{\lambda}$ generic parameters. Therefore $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}}$ is flat by the usual arguments, see e.g. the arXiv appendix to [AST18]. Moreover, comparison of formulas implies that the specializations are the claimed ones. □

Remark 3B.11. For the below note that, by their very construction, all $U_{\mathbb{A}}(\mathfrak{gl}_k)$ modules used in this paper are of ***type*** $(1, \ldots, 1)$ in the sense of e.g. [APW91, Section 1.4] or [Jan96, Section 5.2].

Proof of Theorem 2B.3. As we will see now, using flatness, the quantum Verma Howe duality theorem follows from the classical case. Our exposition below follows [ST19, Section 7A], but flatness arguments along the same lines are very common in the literature.

Commuting actions. To use Lemma 3B.10, one first needs to establish the existence of the commuting actions as in Theorem 2B.3 in the quantum case independently of the classical case. This is done in Lemma 3B.8, so we can focus on the $U_q(\mathfrak{gl}_2)$-$U_q(\mathfrak{gl}_n)$ bimodule decomposition.

As before in the classical case, all parameters in this proof are generic from now on.

Bimodule decomposition. We will now repeatedly use Lemma 3B.10. We compare the $U_q(\mathfrak{gl}_2)$ module $\mathtt{M}_q^{\oplus\boldsymbol{\lambda}}$ and the $U(\mathfrak{gl}_2)$ module $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$, and we see that the weights of these modules are the same under the usual identification of quantum and classical weights. Moreover, the weight multiplicities are also the same and all finite. It follows then from Lemma 3B.10 that we have

$$\mathtt{M}_q^{\boldsymbol{\lambda}} \cong \bigoplus_{\substack{\boldsymbol{d}\in\mathbb{Z}^n \\ t\in\mathbb{Z}_{\geq 0}}} \mathtt{M}_q^{\Sigma\lambda_n+\Sigma d_n-t,t} \otimes \mathfrak{D}_q^{\Sigma\lambda_n+\Sigma d_n-t,t}$$

as the quantum analog of (3A.10), where $\mathfrak{D}_q^{\Sigma\lambda_n+\Sigma d_n-t,t}$ are multiplicity $\mathbb{K}_q^{\boldsymbol{\lambda}}$ vector spaces. We actually know that these multiplicity $\mathbb{K}_q^{\boldsymbol{\lambda}}$ vector spaces are $U_q(\mathfrak{gl}_n)$ modules by the quantum specialization of the previously established $U_v(\mathfrak{gl}_2)$-$U_v(\mathfrak{gl}_n)$ bimodule structure.

We want to show that all appearing $\mathfrak{D}_q^{\Sigma\lambda_n+\Sigma d_n-t,t}$ are simple as $U_v(\mathfrak{gl}_n)$ modules. This is equivalent to the action giving a surjection

$$f\colon U_q(\mathfrak{gl}_n) \twoheadrightarrow \mathrm{End}_{U_q(\mathfrak{gl}_2)}(\mathtt{M}_q^{\boldsymbol{\lambda}}). \tag{3B.12}$$

Now, setting $v \mapsto 1$ or $v \mapsto q$, respectively, and $\boldsymbol{\mu} \mapsto \boldsymbol{\lambda}$ in the $\mathbb{A}_v^{\boldsymbol{\mu}}$ version, we can identify $\mathtt{M}_{\mathbb{A}}^{\oplus\boldsymbol{\mu}}$ with $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$, and the biactions of $U_{\mathbb{A}}(\mathfrak{gl}_2)/(v-1,\boldsymbol{\mu}-\boldsymbol{\lambda})$-$U_{\mathbb{A}}(\mathfrak{gl}_n)/(v-1,\boldsymbol{\mu}-\boldsymbol{\lambda})$ and $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ coincide under this specialization, and verbatim for $U_q(\mathfrak{gl}_2)$-$U_q(\mathfrak{gl}_n)$ instead of $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$. In particular, the images of these two actions agree. It follows now from the classical version of Theorem 2B.3 that the action map f is surjective classically. Thus, Lemma 3B.10 implies that (3B.12) holds and $\mathfrak{D}_q^{\Sigma\lambda_n+\Sigma d_n-t,t}$ are simple as $U_v(\mathfrak{gl}_n)$ modules.

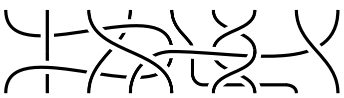
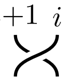
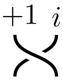
Next, comparison of definitions verifies that the classical version of $\mathfrak{D}_q^{\Sigma\lambda_n+\Sigma d_n-t,t}$ and $\mathtt{D}_q^{\Sigma d_n-t,t}$ are $\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ and $\mathtt{D}^{\Sigma d_n-t,t}$, respectively. By the classification recalled in [Maz03, Section 2.3] (originally proven in [Mat00]) we get also that $\mathfrak{D}^{\Sigma\lambda_n+\Sigma d_n-t,t} \cong \mathtt{D}^{\Sigma d_n-t,t}$, as before. Finally, $\mathfrak{D}_q^{\Sigma\lambda_n+\Sigma d_n-t,t}$ and $\mathtt{D}_q^{\Sigma d_n-t,t}$ are of type $(1, \ldots, 1)$, by construction, and true quantum deformation in the sense of [Maz03, Section 2.1] which implies $\mathfrak{D}_q^{\Sigma\lambda_n+\Sigma d_n-t,t} \cong \mathtt{D}_q^{\Sigma d_n-t,t}$.

Dense. By Lemma 3B.10 and the $U_q(\mathfrak{gl}_2)$-$U_q(\mathfrak{gl}_n)$ bimodule decomposition from (2B.4), the argument is now the same as in the classical case. □

4. The colored higher LKB representations are simple

Recall that we have fixed $n \in \mathbb{Z}_{\geq 1}$ and parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n$.

4A. **Pure and colored braids.** Let B_n denote the ***braid group*** with n strands which can be illustrated using the usual diagrammatics, e.g.

(4A.1)  , $\beta_{i,i+1} =$ $i{+}1 \;\; i$  $i \;\; i{+}1$, $\beta_{i,i+1}^{-1} =$ $i{+}1 \;\; i$  $i \;\; i{+}1$.

As displayed above, the transposition generators, crossing the ith and $(i+1)$th strand, of B_n are denoted by $\beta_{i,i+1}$ and $\beta_{i,i+1}^{-1}$.

Recall that the ***pure braid group*** is the subgroup $\mathrm{PB}_n \subset \mathrm{B}_n$ of all elements with the bottom and the top of each strand in the same position. More generally, we define:

Definition 4A.2. Let $\mathrm{P}(\{1,...,n\})$ be the set of partitions of $\{1,...,n\}$. For every $S \in \mathrm{P}(\{1,...,n\})$, the ***braid group*** that is ***pure on*** S is the subgroup $\mathrm{B}_n^S \subset \mathrm{B}_n$ such that the strands with bottom points in $A \in S$ have their top points in A as well.

Example 4A.3. We have $\mathrm{B}_n^{\{[1],...,[n]\}} = \mathrm{PB}_n$ and $\mathrm{B}_n^{\{[1,...,n]\}} = \mathrm{B}_n$, where we use square brackets for the parts of the partition. Moreover, the leftmost braid in (4A.1) is pure on the partition $S = \{[1],[2],[3,4,5,8,9],[6],[7]\}$, and S is the finest partition such that the braid is pure on it. $\diamond$

Definition 4A.4. We associate a partition $S(\boldsymbol{\lambda}) \in \mathrm{P}(\{1,...,n\})$ to $\boldsymbol{\lambda}$ by

$$i,j \text{ are in the same component of } S(\boldsymbol{\lambda}) \Leftrightarrow \lambda_i = \lambda_j.$$

We denote the corresponding subgroup of the braid group by $\mathrm{B}_n^{\boldsymbol{\lambda}} = \mathrm{B}_n^{S(\boldsymbol{\lambda})}$.

Example 4A.5. If all λ_i are different, then $\mathrm{B}_n^{\boldsymbol{\lambda}} = \mathrm{PB}_n$, and if $\boldsymbol{\lambda} = (\lambda,...,\lambda)$, then $\mathrm{B}_n^{\boldsymbol{\lambda}} = \mathrm{B}_n$. For the leftmost braid in (4A.1) the finest set of parameters is $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_3,\lambda_3,\lambda_6,\lambda_7,\lambda_3,\lambda_3)$ for pairwise distinct λ_i. $\diamond$

4B. **LKB representations.** We again work over $\mathbb{A}_v^{\boldsymbol{\mu}}$ and $\mathbb{K}_q^{\boldsymbol{\lambda}}$. We always consider $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}} = \mathtt{M}_{\mathbb{A}}^{\mu_1} \otimes ... \otimes \mathtt{M}_{\mathbb{A}}^{\mu_n}$ as a $U_{\mathbb{A}}(\mathfrak{gl}_2)$ module via (3B.6) (note that $\mathtt{P}_{\mathbb{A}}^{\boldsymbol{\mu}} \cong \mathtt{M}_{\mathbb{A}}^{\oplus\boldsymbol{\mu}} = \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n} \mathtt{M}_{\mathbb{A}}^{\mu_1+d_1} \otimes ... \otimes \mathtt{M}_{\mathbb{A}}^{\mu_n+d_n}$, but the direct sum is only needed for the dual action).

We now adjust the construction from [JK11, Section 3] (and the references to [Kas95] therein) to our setting. See also [Mar20b, Definition 2.19].

Definition 4B.1. Let $t^{\pm 1} \in \mathrm{End}_{\mathbb{A}_v^{\boldsymbol{\mu}}}(\mathtt{M}_{\mathbb{A}}^{\mu_i} \otimes \mathtt{M}_{\mathbb{A}}^{\mu_j})$ defined by

$$t^{\pm 1}(m_k \otimes m_l) = v^{\pm(-l\mu_i - k\mu_j + 2kl)} \cdot (m_k \otimes m_l).$$

Write $F^{[r]} = \frac{(v-v^{-1})^r}{[r]_v!} F^r$ for $r \in \mathbb{Z}_{\geq 0}$. Define the ***R matrix*** and its inverse on $\mathtt{M}_{\mathbb{A}}^{\mu_1} \otimes \mathtt{M}_{\mathbb{A}}^{\mu_2}$ as

$$\check{r}_{\mu_i,\mu_j} \colon \mathtt{M}_{\mathbb{A}}^{\mu_i} \otimes \mathtt{M}_{\mathbb{A}}^{\mu_j} \to \mathtt{M}_{\mathbb{A}}^{\mu_j} \otimes \mathtt{M}_{\mathbb{A}}^{\mu_i}, \quad \check{r}_{\mu_i,\mu_j} = s \circ t^{+1} \circ \left(\sum_{l=0}^{\infty} v^{l(l-1)/2} \cdot e^l \otimes f^{[l]} \right),$$

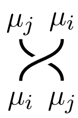
$$\check{r}_{\mu_i,\mu_j}^{-1} \colon \mathtt{M}_{\mathbb{A}}^{\mu_i} \otimes \mathtt{M}_{\mathbb{A}}^{\mu_j} \to \mathtt{M}_{\mathbb{A}}^{\mu_j} \otimes \mathtt{M}_{\mathbb{A}}^{\mu_i}, \quad \check{r}_{\mu_i,\mu_j}^{-1} = \left(\sum_{l=0}^{\infty} (-1)^l v^{-l(l-1)/2} \cdot e^l \otimes f^{[l]} \right) \circ t^{-1} \circ s,$$

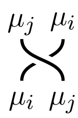
where s is the swap map $s(x \otimes y) = y \otimes x$.

Lemma 4B.2. *The operators $\check{r}_{\mu_i,\mu_j}$ and $\check{r}_{\mu_i,\mu_j}^{-1}$ are well-defined, i.e. the appearing summations are finite on every $m_k \otimes m_l$.*

Proof. This holds because the operator e is locally nilpotent. $\square$

Graphically we will denote these operators by

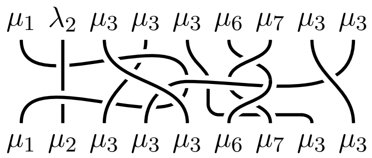
(4B.3) $\check{r}_{\mu_i,\mu_j} \leftrightsquigarrow$ 

, $\check{r}_{\lambda_i,\lambda_j}^{-1} \leftrightsquigarrow$ 

.

We now define a $\mathrm{B}_n^{\boldsymbol{\lambda}}$ action on $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\lambda}}$ by ***colored reading***. That is, one colors the strands of $\beta \in \mathrm{B}_n^{\boldsymbol{\lambda}}$ by $\boldsymbol{\lambda}$, and then we get an element of $\mathrm{End}_{\mathbb{A}_v^{\boldsymbol{\mu}}}(\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\lambda}})$ by composing the relevant version of (4B.3) from bottom to top. We call this element $\check{r}_\beta$.

Example 4B.4. For $\boldsymbol{\lambda} = (\mu_1,\mu_2,\mu_3,\mu_3,\mu_3,\mu_6,\mu_7,\mu_3,\mu_3)$ and the leftmost braid in (4A.1) we get



$\rightsquigarrow \check{r}_\beta = \check{r}_{\mu_2,\mu_1} \circ \check{r}_{\mu_3,\mu_1} \circ ... \circ \check{r}_{\mu_1,\mu_3}^{-1} \circ \check{r}_{\mu_1,\mu_2} \in \mathrm{End}_{\mathbb{K}_q^{\boldsymbol{\lambda}}}(\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\lambda}}).$

The endomorphism $\check{r}_\beta$ has eighteen R matrix factors in total. $\diamond$

Definition 4B.5. A ***refinement*** $\boldsymbol{\rho}$ of $\boldsymbol{\mu}$ is a set of parameters that gives a refined partition compared to $\boldsymbol{\mu}$ when applying Definition 4A.4. We write $\boldsymbol{\rho} \leq \boldsymbol{\mu}$ for refinements of $\boldsymbol{\mu}$.

Notation 4B.6. For $U_v(\mathfrak{gl}_2)$ we extend scalars to $\mathbb{A}'_v$ or $\mathbb{A}_v^{\prime;\boldsymbol{\mu}}$ but do not indicate this in the notation.

Similar to the braid group action in symmetric Howe duality, the braid group acts on one $\mathfrak{gl}_n$ weight space of $\mathtt{M}_{\mathbb{A}}^{\oplus\boldsymbol{\lambda}}$ and this action commutes with the $\mathfrak{gl}_2$ action:

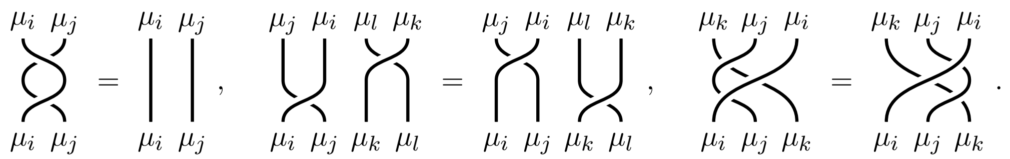
Lemma 4B.7.

(a) *(4B.3) and colored reading endows $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}}$ with the structure of a $\mathrm{B}_n^{\boldsymbol{\rho}}$ module for $\boldsymbol{\rho} \leq \boldsymbol{\mu}$.*

(b) *Colored reading commutes with the $U_v(\mathfrak{gl}_2)$ action coming from (3B.6).*

(c) *The image of $\mathrm{B}_n^{\boldsymbol{\rho}}$ under this module structure is in $\mathrm{End}_{U_v(\mathfrak{gl}_2)}(\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}})$.*

Proof. One first proves, by copying [JK11, Theorem 7], that (4B.3) satisfies the colored braid relations, e.g.



Hence, $\check{r}_\beta$ is independent of the choices in colored reading, and we obtain the claimed $\mathrm{B}_n^{\boldsymbol{\mu}}$ module structure. For $\mathrm{B}_n^{\boldsymbol{\rho}} \subset \mathrm{B}_n^{\boldsymbol{\mu}}$ this $\mathrm{B}_n^{\boldsymbol{\mu}}$ module structure restricts to $\mathrm{B}_n^{\boldsymbol{\rho}}$.

That the two actions commute follows because of the well-known fact (and easy calculation) that the R matrices are $U_v(\mathfrak{gl}_2)$ equivariant with respect to (3B.6).

The final claim follows since the action maps commute with the $U_v(\mathfrak{gl}_2)$ action on $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}}$. □

We thus have a $U_v(\mathfrak{gl}_2)$-$\mathrm{B}_n^{\boldsymbol{\rho}}$ bimodule structure on $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}}$.

Remark 4B.8. Lemma 4B.7 can be strengthened: the image of $\mathrm{B}_n^{\boldsymbol{\rho}}$ commutes with the action of the $\mathbb{A}_v^{\boldsymbol{\mu}}$ subalgebra of $U_{\mathbb{A}}(\mathfrak{gl}_2)$ generated by E, $F^{[r]}$ for $r \in \mathbb{Z}_{\geq 0}$, $L_1^{\pm 1}$ and $L_2^{\pm 1}$.

We now turn our attention to the (colored higher) LKB representations, which, following [JK11, Section 3], we define as follows:

Definition 4B.9. Let $\ker(e)$ and $\ker(k - \prod_{i=1}^n v^{\mu_i} v^{-2l}) = \ker(k - v^{\Sigma\mu_n - 2l})$ be the kernels of the indicated operators coming from the $U_v(\mathfrak{gl}_2)$ action on $\mathtt{M}_{\mathbb{A}}^{\boldsymbol{\mu}}$. For $l \in \mathbb{Z}_{\geq 0}$ the l***th LKB representation*** is defined as $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,l} = \ker(e) \cap \ker(k - v^{\Sigma\mu_n - 2l})$.

Lemma 4B.10. *For $\boldsymbol{\rho} \leq \boldsymbol{\mu}$ the free $\mathbb{A}_v^{\boldsymbol{\mu}}$ module $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,l}$ is stable under the $\mathrm{B}_n^{\boldsymbol{\rho}}$ action.*

Proof. Note first that the definition of $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,l}$ only involves the operators e, k and $\mathbb{A}_v^{\boldsymbol{\mu}}$ multiples of the identity. Hence, $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,l}$ is defined over $\mathbb{A}_v^{\boldsymbol{\mu}}$, by construction. The rest can be proven, mutatis mutandis, as in [JK11, Theorem 1]. □

Example 4B.11. With respect to the basis $\{m_i | i \in \mathbb{Z}_{\geq 0}\}$ in Example 2A.7, $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,1}$ has a basis given by an n fold tensor product of m_i with one entry being m_1 and all other entries being m_0. Hence, the $\mathbb{A}_v^{\boldsymbol{\mu}}$ rank is $\binom{n-1}{1}$. In general, $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,l}$ is of $\mathbb{A}_v^{\boldsymbol{\mu}}$ rank $\binom{n+l-2}{l}$. ◇

Remark 4B.12. The representation $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,l}$ has between two and $n+1$ parameters, depending on $\boldsymbol{\mu}$. For example, for $\boldsymbol{\mu} = (\mu, ..., \mu)$ one has v and μ as parameters.

Example 4B.13. The representation $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,0}$ is always trivial, while $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,1}$ is the (reduced) ***Burau representation*** of B_n, and $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,2}$ is its classical ***LKB representation*** as in [Law90], or closer to our formulation as, in [JK11].

Or, to be completely precise, $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,2}$ is a multiparameter version of the construction from [JK11], see also [Mar20b]. Moreover, the representation $\mathtt{LKB}_{\mathbb{A},\boldsymbol{\mu}}^{n,l}$ can then further be matched with it homological counterpart up to playing with parameters, see [Koh12, Theorem 6.1] and [Mar20a, Theorem 1.5] for a precise statement.

Using an appropriate ground field and quantum parameter, $\mathtt{LKB}^{n,2}_{q,\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda}=(\lambda,\dots,\lambda)$ is a faithful B_n module by [Big01] and [Kra02], and thus, $\mathtt{LKB}^{n,2}_{q,\boldsymbol{\lambda}}$ is also faithful for $\mathrm{B}^{\boldsymbol{\rho}}_n$ for all $\boldsymbol{\rho}$. $\diamond$

Specializing to the quantum case, the following is our main application of Theorem 2B.3:

Theorem 4B.14. *Assume that the parameters are generic. Then the representation* $\mathtt{LKB}^{n,l}_{q,\boldsymbol{\lambda}}$ *is a simple* $\mathrm{B}^{\boldsymbol{\rho}}_n$ *module for* $\boldsymbol{\rho}\leq\boldsymbol{\lambda}$ *and all* $l\in\mathbb{Z}_{\geq 0}$.

The proof of Theorem 4B.14 is given in Section 5.

Remark 4B.15. Theorem 4B.14 extends and generalizes [JK11, Theorem 3] at the same time. First, Theorem 4B.14 is a multiparameter version of [JK11, Theorem 3]. More importantly, even when we have only one parameter, i.e. $\boldsymbol{\lambda}=(\lambda,\dots,\lambda)$, Theorem 4B.14 is stronger than [JK11, Theorem 3] since we e.g. also prove that $\mathtt{LKB}^{n,l}_{q,\boldsymbol{\lambda}}$ is a simple PB_n module not just a simple B_n module. The proof given below is also very different from the one given in [JK11], and we do not know how to generalize the proof in [JK11] to include the various subgroups of B_n.

5. The proof of simplicity

Our proof of Theorem 4B.14 uses Verma versions of [LZ06, Theorem 5.5 and Remark 8.6].

Remark 5.1. We think of Verma modules as limits of symmetric powers and this was one of our main motivation to follow the approach taken in [LZ06].

5A. The classical case.

Our ground field in this section is $\mathbb{K}$.

Definition 5A.1. For $k\in\mathbb{Z}_{\geq 2}$ the ***infinitesimal pure braid group*** is the $\mathbb{K}$ algebra $\mathrm{PB}^{\varepsilon}_n$ generated by β^{ε}_{ij} for $1\leq i<j\leq k$ subject to

$$[\beta^{\varepsilon}_{ij},\beta^{\varepsilon}_{rs}]=[\beta^{\varepsilon}_{ir}+\beta^{\varepsilon}_{is},\beta^{\varepsilon}_{rs}]=[\beta^{\varepsilon}_{ij},\beta^{\varepsilon}_{ir}+\beta^{\varepsilon}_{jr}]=0,$$

for pairwise distinct i,j,r,s, and $[\_,\_]$ denotes the commutator. For $k=1$ we let $\mathrm{PB}^{\varepsilon}_n=\mathbb{K}$.

Remark 5A.2. The motivation to study $\mathrm{PB}^{\varepsilon}_n$ is that it gives rise to the so-called ***monodromy representation of the KZ equation*** of the pure braid group PB_n which works for very general tensor products of Lie algebra representations, see [Koh02, Proposition 2.3] for details.

Definition 5A.3. For all $1\leq i<j\leq n$ define operators on $\mathtt{M}^{\oplus\boldsymbol{\lambda}}\cong\mathtt{P}^{\boldsymbol{\lambda}}$ by

$$\gamma^{\varepsilon}_{ij}=X_iX_j\partial_{X_i}\partial_{X_j}+X_iY_j\partial_{Y_i}\partial_{X_j}+Y_iX_j\partial_{X_i}\partial_{Y_j}+Y_iY_j\partial_{Y_i}\partial_{Y_j}. \tag{5A.4}$$

Lemma 5A.5. *The assignment* $\beta^{\varepsilon}_{ij}\mapsto\gamma^{\varepsilon}_{ij}$ *endows* $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ *with the structure of a* $\mathrm{PB}^{\varepsilon}_n$ *module. This* $\mathrm{PB}^{\varepsilon}_n$ *action stabilizes* $(\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}$ *for all* $\boldsymbol{d}\in\mathbb{Z}^n$.

Proof. A direct calculation, see also [LZ06, Theorem 2.1]. $\square$

Remark 5A.6. The $\mathrm{PB}^{\varepsilon}_n$ action on $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ using (5A.4) factors through an $\mathrm{PB}^{\varepsilon}_n$ action on $U(\mathfrak{gl}_2)^{\otimes n}$, see [Koh02, Section 2]. This works as follows. Let $B=\{E,F,L_1,L_2\}$ be the usual basis of $\mathfrak{gl}_2$, and $E^{\star}=F$, $F^{\star}=E$, $L_i^{\star}=L_i$. Then define ***Casimir-type elements*** by $Cas_{ij}=\sum_{b\in B}1^{\otimes i-1}\otimes b\otimes 1^{\otimes j-i-1}\otimes b^{\star}\otimes 1^{\otimes n-j}\in U(\mathfrak{gl}_2)^{\otimes n}$ for $1\leq i<j\leq k$. Then $\beta^{\varepsilon}_{ij}\mapsto Cas_{ij}$ defines an $\mathrm{PB}^{\varepsilon}_n$ action that factors the $\mathrm{PB}^{\varepsilon}_n$ action given by $\beta^{\varepsilon}_{ij}\mapsto\gamma^{\varepsilon}_{ij}$.

We have $\mathtt{M}^{\oplus\boldsymbol{\lambda}}\cong\bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n}(\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}$ as $\mathrm{PB}^{\varepsilon}_n$ modules, by construction. But we in the end only need a fixed arbitrary direct summand $(\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\cong\mathtt{M}^{\lambda_1+d_1}\otimes\dots\otimes\mathtt{M}^{\lambda_n+d_n}$: Let $\mathrm{PE}^{\varepsilon}_n$ denote the image of the $\mathrm{PB}^{\varepsilon}_n$ action from Lemma 5A.5 restricted to $(\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}$. Note that $\mathrm{PE}^{\varepsilon}_n\subset\mathrm{End}_{\mathbb{K}}\big((\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$ but we will need the following stronger statement.

Lemma 5A.7. *Assume that the parameters are generic. We have* $\mathrm{PE}^{\varepsilon}_n\subset\mathrm{End}_{U(\mathfrak{gl}_2)}\big((\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$ *and* $\mathrm{PE}^{\varepsilon}_n\twoheadrightarrow_d\mathrm{End}_{U(\mathfrak{gl}_2)}\big((\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$.

Proof. The proof splits into several parts.

Containment. $\mathrm{PE}^{\varepsilon}_n\subset\mathrm{End}_{U(\mathfrak{gl}_2)}\big((\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$ follows from (5A.4) via a direct computation.

Applying Verma Howe duality. From Theorem 2B.3 we get commuting actions of $U(\mathfrak{gl}_2)$ and $U(\mathfrak{gl}_n)$ on $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ and the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule decomposition

$$(\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}} \cong \bigoplus_{t\in\mathbb{Z}_{\geq 0}} \mathtt{M}^{\Sigma\lambda_n+\Sigma d_n-t,t} \otimes (\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}},$$

with $\mathtt{M}^{\Sigma\lambda_n+\Sigma d_n-t,t}$ and $\mathtt{D}^{\Sigma d_n-t,t}$ being simple. It follows that

$$\mathrm{End}_{U(\mathfrak{gl}_2)}\big((\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big) \cong \mathrm{End}_{\mathbb{K}}\big(\bigoplus_{t\in\mathbb{Z}_{\geq 0}} (\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}\big).$$

It remains to show that all endomorphisms of $(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$ come from $\mathrm{PE}_n^{\varepsilon}$ in the dense sense.

The Casimir subalgebra. To this end, let $\mathrm{Cas}_n \subset \mathrm{End}_{\mathbb{K}}\big((\mathtt{M}_{\boldsymbol{d}}^{\oplus\boldsymbol{\lambda}})\big)$, by definition, be the $\mathbb{K}$ algebra generated by $\langle e_{ij}e_{ji}+e_{ji}e_{ij}, e_{kk} | 1\leq i<j\leq n, 1\leq k\leq n\rangle$ with the endomorphisms as in (3A.4). It is important to observe that the images of the Casimir operators Cas_k from (3A.20) are in Cas_n. Moreover, note that $\mathrm{Cas}_n \subset \mathrm{End}_{U(\mathfrak{gl}_2)}\big((\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$ by Theorem 2B.3 and the elements of Cas_n are homogeneous, and hence, Cas_n acts on $(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$.

Simplicity. We aim to show that $(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$ is simple as a Cas_n module. For this we use an analog of [MTL05, Theorem 6.1 and Remark 6.2], where the main observation is that the Casimir operators Cas_k from (3A.20) have a joint simple spectrum on $(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$ with diagonal basis given by the GT vectors. Indeed, by (3A.23) we know that Cas_k acts as $y_k(y_k+k-1)+c_k(c_k+k-3)$ on $GT_{\boldsymbol{d},\boldsymbol{r}}$ with y_k and c_k related to $\boldsymbol{d}$ and $\boldsymbol{r}$ as in Definition 3A.17. Observing that y_k^2 is different for all $k\in\{1,...,n\}$ and the only place where quadratic terms in the parameters show up, we get that the spectrum is simple.

Using the GT formulas from Definition 2A.12 it is not hard to see that the action graph of the Cas_k on $(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$ with vertices given by the relevant GT vectors is strongly connected. Hence, as soon as a nonzero Cas_n submodule $\mathtt{M}\subset(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$ contains at least one GT vector we have $\mathtt{M}=(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$. Finally, since the spectrum of the Cas_k is simple with diagonal basis given by the GT vectors by the above, every such $\mathtt{M}\subset(\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}$ contains indeed a GT vector.

Wrap-up. It follows that Cas_n generates the whole of $\mathrm{End}_{\mathbb{K}}\big((\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}\big)$. A calculation then verifies that $e_{ij}e_{ji}=\gamma_{ij}^{\varepsilon}+e_{ii}$ and e_{ii} acts as a scalar, which completes the proof since the elements $\gamma_{ij}^{\varepsilon}$ generate $\mathrm{PE}_n^{\varepsilon}$, by definition, and we then have

$$\mathrm{PE}_n^{\varepsilon} \twoheadrightarrow_d \mathrm{End}_{\mathbb{K}}\big(\bigoplus_{t\in\mathbb{Z}_{\geq 0}} (\mathtt{D}^{\Sigma d_n-t,t})_{\boldsymbol{d}}\big) \cong \mathrm{End}_{U(\mathfrak{gl}_2)}\big((\mathtt{M}^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$$

which gives denseness. □

5B. **The quantum case.** Instead of infinitesimal braids we come back to the pure braid groups. To this end, recall that $(\mathtt{M}_q^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}$ is a $\mathrm{B}_n^{\boldsymbol{\rho}}$ module for all $\boldsymbol{\rho}\leq\boldsymbol{\lambda}$ by Lemma 4B.7. In particular, $(\mathtt{M}_q^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}$ is a PB_n module. Let $\mathrm{PE}_{n,q}^{\varepsilon}$ be the image of this action, and similarly for generic parameter we let $\mathrm{PE}_{n,v}^{\varepsilon}$ be the respective image.

Lemma 5B.1. *The $\mathbb{K}_v^{\boldsymbol{\lambda}}$ algebra $\mathrm{PE}_{n,v}^{\varepsilon}$ contains an $\mathbb{A}_v^{\boldsymbol{\mu}}$ subalgebra $\mathrm{PE}_{n,v}^{\varepsilon}$ whose classical specialization contains $\mathrm{PE}_n^{\varepsilon}$ and whose quantum specialization is $\mathrm{PE}_{n,q}^{\varepsilon}$.*

Proof. First note that all appearing scalars are in $\mathbb{A}_v^{\boldsymbol{\mu}}\subset\mathbb{K}_v^{\boldsymbol{\lambda}}$, so the construction in Section 4B works verbatim over $\mathbb{A}_v^{\boldsymbol{\mu}}$.

The statement about the quantum specialization is then clear since this is how the quantum specialization is defined.

For the classical specialization the same argument as [LZ06, Proof of Theorem 7.5] works. □

Lemma 5B.2. *Assume that the parameters are generic. We have $\mathrm{PE}_{n,q}^{\varepsilon}\subset\mathrm{End}_{U_q(\mathfrak{gl}_2)}\big((\mathtt{M}_q^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$ and $\mathrm{PE}_{n,q}^{\varepsilon}\twoheadrightarrow_d\mathrm{End}_{U_q(\mathfrak{gl}_2)}\big((\mathtt{M}_q^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$.*

Proof. $\mathrm{PE}_{n,q}^{\varepsilon}\subset\mathrm{End}_{U_q(\mathfrak{gl}_2)}\big((\mathtt{M}_q^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}\big)$ follows from Lemma 4B.7 and the definition of the R matrices.

For the second statement recall that $\mathtt{M}_{\mathbb{A}}^{\oplus\boldsymbol{\mu}}$ is flat and specialize to $\mathtt{M}^{\oplus\boldsymbol{\lambda}}$ classically and to $\mathtt{M}_q^{\oplus\boldsymbol{\lambda}}$ in the quantum case, see Lemma 3B.10. Thus, on the side of the endomorphism algebra we can change between the classical and the quantum case. Moreover, Lemma 5B.1 shows that $\mathrm{PE}_{n,q}^{\varepsilon}$ is at least as big as $\mathrm{PE}_n^{\varepsilon}$. Taking both together and using Lemma 5A.7 implies then the claim. □

We are ready for the final proof of this paper:

Proof of Theorem 4B.14. It is enough to consider $\mathrm{B}_n^{\boldsymbol{\rho}} = \mathrm{PB}_n$, so we restrict to this case.

Note that $(\mathtt{M}_q^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}$ is a $U_q(\mathfrak{gl}_2)$-PB_n bimodule by Lemma 4B.7, and Lemma 5B.2 shows that $\mathrm{PE}_{n,q}^{\varepsilon}$ densely-generates the centralizer of $U_q(\mathfrak{gl}_2)$ on $(\mathtt{M}_q^{\oplus\boldsymbol{\lambda}})_{\boldsymbol{d}}$. Having this and the usual statements about simple modules of centralizers as e.g. in [GW09, Theorem 4.2.1], it remains to argue that the LKB representations are PB_n submodules of some $(\mathtt{D}^{\Sigma d_n - t,t})_{\boldsymbol{d}}$. (Note hereby that the LKB story is finite dimensional and densely generates turns into generates, and hence, [GW09, Theorem 4.2.1] applies.)

To see this we note that, as in the proof of the classical version of Theorem 2B.3, $\mathtt{D}^{\Sigma d_n - t,t}$ consists of highest weight vectors for the $U_q(\mathfrak{gl}_2)$ action, so the condition on $\mathtt{LKB}_{q,\boldsymbol{\lambda}}^{n,l}$ to be annihilated by e holds. Moreover, by (3B.6) we get that K acts on $\mathtt{D}^{\Sigma d_n - t,t}$ as the scalar $q^{\Sigma\lambda_n + \Sigma d_n}$. In particular, for e.g. $\boldsymbol{d} = (-2l, 0, ..., 0)$ we get $\ker(k - v^{\Sigma\lambda_n - 2l}) \subset \mathtt{D}^{\Sigma d_n - t,t}$. Hence, the LKB representations are PB_n submodules of some $(\mathtt{D}^{\Sigma d_n - t,t})_{\boldsymbol{d}}$ as desired. $\square$

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