OPTIMAL LIQUIDATION UNDER INDIRECT PRICE IMPACT WITH PROPAGATOR

Jean-Loup Dupret, Donatien Hainaut

REPRINT | 2025 / 09

UCLouvain

ISBA | LIDAM
Louvain Institute of Data Analysis and Modeling in economics and statistics

ISBA
Voie du Roman Pays 20 - L1.04.01
B-1348 Louvain-la-Neuve
Email : lidam-library@uclouvain.be
https://uclouvain.be/en/research-institutes/lidam/isba/publication

Quantitative Finance, 2025
Vol. 25, No. 3, 359–381, https://doi.org/10.1080/14697688.2025.2463368

Optimal liquidation under indirect price impact with propagator

JEAN-LOUP DUPRET *†‡ and DONATIEN HAINAUT†

†LIDAM-ISBA, Université Catholique de Louvain, Voie du Roman Pays 20, Louvain-La-Neuve, 1348, Belgium
‡Department of Mathematics and RiskLab, ETH Zurich, Rämistrasse 101, Zurich, 8092, Switzerland

(*Received 3 May 2024; accepted 1 February 2025*)

We propose in this paper a new framework of optimal liquidation strategies for a trader seeking to liquidate his large inventory based on a jump-dependent price impact model with propagator. This new jump-dependent price impact model best reproduces the empirical direct and indirect effects of market orders on the transaction price. More precisely, different choices of propagators are proposed and their implications in terms of temporary, permanent and transient impacts on the transaction price are discussed. For each choice of such kernels, we formulate the most relevant optimal liquidation problem faced by the trader, derive explicitly the related Hamilton-Jacobi-Bellman equation and solve it numerically. We then also show how our price impact model can be extended to incorporate the use of limit orders by the liquidating trader. Therefore, we aim with this paper to propose an alternative, more realistic and flexible description of the order book's dynamic, thereby contributing to bridging the gap between high-frequency price models and optimal liquidation problems.

Keywords: Optimal liquidation; HJB equation; Price impact model; Market impact; High-frequency trading

1. Introduction

A classical problem in finance is how an agent can sell (or buy) a large amount of shares and yet minimize adverse price movements generated by her own trades. *Large* means here that the amount the agent is interested in selling is too big to execute in a single trade. The agent must then take into account the balance between price impact (trade quicker) and price risk (take longer to complete all trades). Indeed, if an agent trades quickly, then her orders will walk through the buy side of the limit order book (LOB) and she will obtain lower prices for her orders, inducing a *temporary price impact*. Even if she splits each order into smaller trades to avoid walking the book and executes them rapidly in the market, then other traders will notice an excess of sell orders (or net order flow imbalance) and adjust their quotes. This will again result in a negative price impact, which is this time *transient* and/or *permanent*. On the other hand, if she trades slowly to avoid this price impact, then she will face the uncertainty of future prices, which could have dropped to a less desirable level by the time she completes the entire order. The aim of optimal liquidation is therefore to propose an optimal strategy that balances these two factors, based on a coherent price impact model reproducing the underlying asset price and the impact of the agent's transactions on this price. We refer to Gökay *et al.* (2011), Cartea *et al.* (2015), Guéant (2016), Bouchaud *et al.* (2018) and Donnelly (2022) for a review of the most important results concerning optimal execution problems.

Typically, three different time scales can be considered when modeling the price of an asset. At low frequencies, the dynamic of stock prices can often be reproduced by a diffusive process. At the other end, when dealing with very high frequencies, empirical features and statistical properties of the LOB dynamics have to be modeled (long-range dependence of trades, volatility clustering, high-frequency resilience of the price, etc.). At such very high frequencies, one then has to describe precisely the LOB dynamic. A variety of such models have been proposed in Abergel and Jedidi (2013), Cont and De Larrard (2013), Gareche *et al.* (2013) and Huang *et al.* (2015) and especially in Bacry and Muzy (2014) with a tick-by-tick price model ruled by Hawkes processes. Specifically, their model jointly describes the order flow and the price dynamic, reproducing numerous empirical properties of high-frequency markets. In between of low and high-frequency models, price impact models consider an intra-day intermediate time scale, anywhere from seconds to hours. They tend to discard most of the LOB events such as limit orders,

*Corresponding author. Email: jeanloup.dupret@math.ethz.ch, jean-loup.dupret@uclouvain.be

cancelations, market orders, etc. and instead concentrate on describing the price impact of the agents' transactions. Their goal is to be more tractable than high-frequency models and to offer analytical results on practical issues such as the optimal execution problems studied in this paper. The standard framework is well described in Gatheral (2010), where the execution price $S_t^{(\nu)}$ of a transaction is given by

$$S_t^{(\nu)} = S_0 + \sigma W_t + \int_0^t f(\nu_s)G(t-s)\,\mathrm{d}s, \quad t \in [0,T]. \quad (1)$$

with $T > 0$ the trading horizon, $S_0 > 0$ the initial price, ν_s the rate of trading of the liquidating agent at time s, $f(\nu)$ the instantaneous price impact of an agent trading at speed ν, G the *decay kernel* or *propagator* and $(W_t)_{t\geq 0}$ a noise process (typically a Brownian motion). As explained in Alfonsi and Blanc (2016), $f(\nu)G(0)$ captures the temporary/instantaneous impact of trades on the price, $f(\nu)G(+\infty)$ the permanent impact and $f(\nu)[G(0+) - G(+\infty)]$ the transient impact. That is why the two extreme cases where G is a Dirac's delta $\delta(t-s)$ on $[0,T]$ and when $G \equiv 1$ are referred in the literature as temporary price impact and permanent price impact, respectively. They are core features in the well-known model of Almgren and Chriss (1999, 2001), up to a multiplicative constant. Moreover, the price impact model (1) also allows to generalize most of the standard processes considered in the optimal liquidation literature, see Bertsimas and Lo (1998), Bouchaud *et al.* (2003), Almgren *et al.* (2005), Alfonsi *et al.* (2010), Forsyth *et al.* (2012), Obizhaeva and Wang (2013) and Donnelly (2022). Since we are interested in this paper in optimal execution problems, we will then work with such price impact models observed at middle frequency. However, our approach will expand on these models by incorporating more specific events and features of the LOB, as illustrated in figures 1 and 2. One of our primary goals is then to bridge this gap between high-frequency price models and the optimal execution framework by proposing a tractable model reproducing some empirical evidence of high-frequency microstructure markets. We refer to Alfonsi and Blanc (2016) or Cartea *et al.* (2014) who were among the first to propose such mixed-framework models incorporating both high-frequency and middle-frequency features of the LOB.

As already mentioned and explained in Gatheral (2010), Alfonsi and Blanc (2016) and Bouchaud *et al.* (2018) or Donnelly (2022), price impact can be temporary, transient and/or permanent. Market orders (MOs) that walk the LOB have a price impact because they obtain average execution prices worse than the best quote. This price impact is then temporary because it is assumed that the LOB will replenish very quickly and, after restocking, there is no effect on the midprice of the asset. On the other hand, investors' trading activity has a transient and permanent effect on midprices since the traders' order flow (i.e. the number and the size of MOs) conveys information that is impounded on the midprice of the asset, see Cont *et al.* (2014). In general, when there is positive net order flow (more buy than sell volume) the midprice tends to drift up, and when there is negative net order flow (more sell than buy volume) the midprice tends to drift down. Therefore, when an investor is executing a large market order, this adds one-sided pressure to the market's order flow and hence on midprices: up if the investor is buying or down if selling. In this paper, such downward permanent (and transient) pressure on the price will be modeled by an increase in the mean-reverting intensity of a point process with negative jump size, replacing the term $f(\nu_s)ds$ in the standard setting (1). This approach will thus help to capture the information about MOs' permanent and transient price impact. Moreover, various choices of decay kernel G will be investigated so as to combine different ways of modeling these temporary, transient and permanent price impacts.

More precisely, at any one time, we observe empirically that the number of shares displayed and available in the market at the quoted bid/ask is limited, see figures 1 and 2 below. The liquidating trader will then split its sell order in different trade sizes, some of which could be very small at a particular time compared to the depth of the LOB. If a market order at time t does not walk the LOB, i.e. the size of this transaction is smaller than the depth at the best bid, there will not be any direct change in the stock's midprice *(no temporary direct price impact from walking the LOB)*, even if this will lead to more pressure towards a price decrease *(indirect permanent and transient impact from the net order flow imbalance)*. As stated above, this downward pressure in the price will be modeled in this paper by an increase in the mean-reverting intensity of a jump process with negative jump size. Including a jump process for representing MO's price impact makes sense since the depth at the best bid is random/unpredictable at any time t. Most of the time a small sell MO will not lead to a direct shift in price (no jumps). However, if the quantity available at the best bid is really low at this time, this small order of the trader will walk the LOB and lead to an immediate shift (jump) in the midprice. Similarly, if the LOB is particularly deep at that time t, even a large market sell order could not lead to a direct shift in the midprice (*no temporary direct price impact*), but it will however put a lot of downwards pressure in the price (*indirect permanent/transient price impact*). This will hence translate in our model by a strong increase in the intensity of the jump process. These features of the LOB are confirmed in Cartea *et al.* (2015), Cartea and Jaimungal (2016), Bouchaud *et al.* (2018) and especially in Bacry and Muzy (2014) where a 4-dimensional Hawkes process is used to reproduce these direct and indirect price impacts. They are further empirically supported by the following figure 1, which represents the LOB for the AMZN share on June 21, 2012 at 10:36:47.016 and the related temporary price impact of a sell MO in function of volume. Similar figures and results are depicted in Cartea and Jaimungal (2016) for the SMH stock price.

We clearly see a plateau in the temporary price impact curve for small volumes, meaning that such small MOs do not walk the LOB (sell volume $<$ depth at best bid) and hence, do not lead to any direct temporary price impact. Indeed, for sell MOs with volume below 300, there is no direct temporary price impact besides the bid-ask spread. This is confirmed by the following figure 2 where multiple temporary price impact curves in function of the volume sold are depicted inside the same minute for the AMZN stock price.

In each price impact curve of figure 2, we again see plateaus for small volumes, translating the fact that small enough sell MOs (with volume lower than the depth at the best bid) do

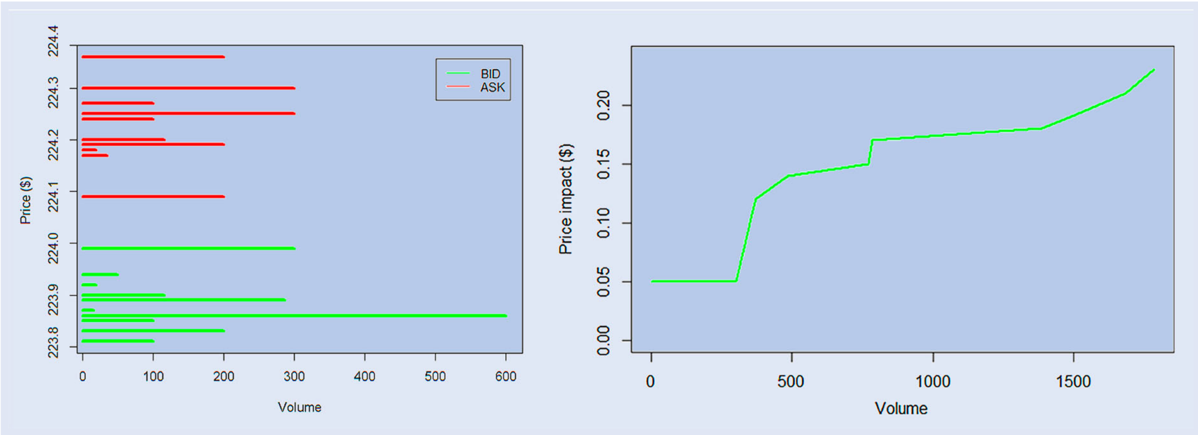

Figure 1. Limit order book of AMZN on the 21/06/2012 at 10:36:47.016 and related temporary price impact of a market sell order in function of the volume sold. In green, bid prices and in red, ask prices.

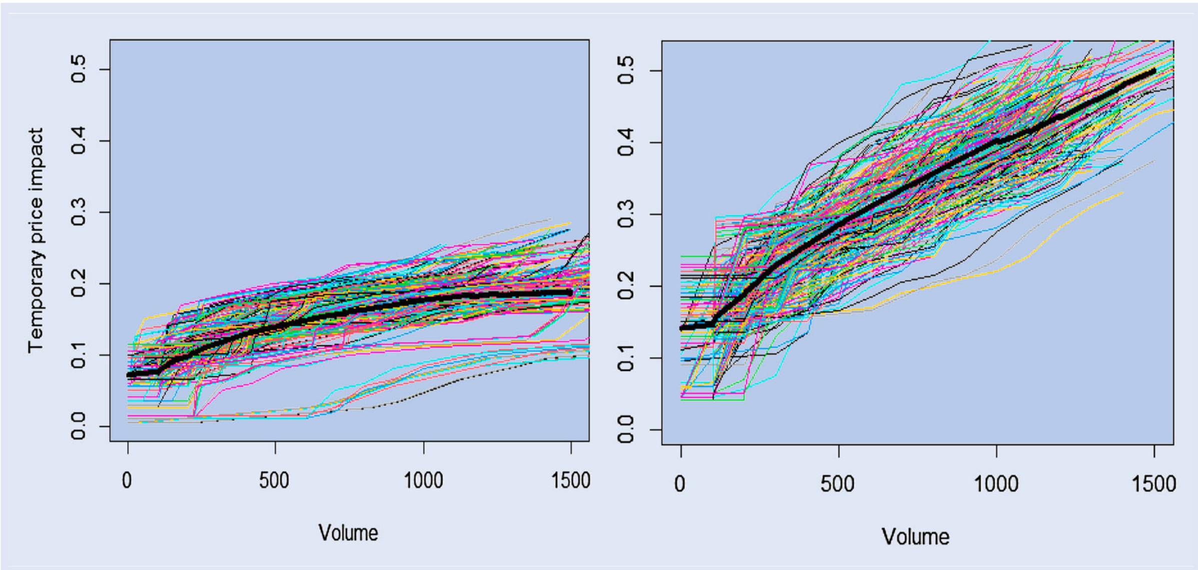

Figure 2. Temporary price impact in function of volume for AMZN on Jun 21, 2012 for every millisecond between 10:36 and 10:37 am (left plot) and between 09:43 and 09:44 am (right plot). In black wider line is depicted the mean temporary price impact over the corresponding minute.

not lead to any direct temporary price impact besides the bid-ask spread. More importantly, we see in both plots that even the average temporary impact curve over the minute (bold line) remains constant for sell volumes below 100. Besides, once the volume sold is large enough for triggering a price impact, this temporary impact seems to be best reproduced by a power-law function with exponent below 1, and more precisely between 0.4 and 0.8. This is consistent with the literature and the well-known square-root law, observed for a large set of assets and different trading conditions as investigated in Torre (1997), Bouchaud *et al.* (2003), Almgren *et al.* (2005) and Gatheral (2010). This behavior of the temporary impact curves justifies the use of an indirect price impact model based on a jump process, as we will introduce in the following section 2. This point process will indeed jump whenever the sell order is sufficiently large (here above 100) and the size of the negative jump will then scale as a power-law function of the volume sold. As mentioned above, the intensity of this jump process will also depend on the volume sold as each sell MO order puts a permanent downwards pressure on the price through the net order flow imbalance, even in the absence of a direct temporary impact (no jump). Despite being a universal feature of high-frequency markets, very few price impact models incorporate this absence of temporary negative impact for sufficiently low volumes (while still having a permanent impact from order flow imbalance), see e.g. the discussion in section 2.1.3 of Bacry and Muzy (2014). However, being able to trade with MOs without having any temporary price impact for small enough sell orders has a strong impact on the optimal strategy used by the liquidating trader. Moreover, we will see that this model extends the stochastic liquidity factor of Fruth *et al.* (2014, 2019) in the form of a trade-driven intensity process.

Finally, optimal liquidation problems based on jump processes have already been extensively studied in the literature, see e.g. Bacry and Muzy (2014), Bayraktar and Ludkovski (2014), Cartea *et al.* (2014), Guéant and Lehalle (2015), Da Fonseca and Malevergne (2021) and Sadoghi and Vecer (2022), or chapters 8.1–8.3 of Cartea *et al.* (2015). However, in these works, jump processes are systematically used to model the occurrence of market and limit orders sent by the liquidating trader. In contrast, our setting uses a jump process to explicitly model the occurrence of the temporary/permanent/transient price impacts arising from continuous trading with MOs, as not all trades lead to a direct negative impact on the execution price, see figure 2. In addition, we propose in section 3 an extension of our price impact model to include strategies that use both market and limit orders (based on a second jump process). This will hence provide a more complete and realistic price impact model, allowing for a better modeling of the LOB dynamic.

The contributions of this paper are therefore multiple. We first introduce a new jump-dependent price impact model with propagator allowing to best reproduce the empirical direct and indirect effects of market orders on the transaction price, as described in section 1. We then discuss in section 2 various ways of modeling in our setting the temporary, permanent and/or transient price impact of trades based on different choices of propagators G. For each choice of such kernels, we formulate the most relevant optimal liquidation problem faced by the agent, derive explicitly the related HJB equation and solve it numerically. This approach enables us to obtain and compare the optimal trading strategies in each scenario, while also discussing the impact of each model (and its parameters) on the agent's optimal trading rate. We finally extend in section 3 the initial price impact model based solely on market orders to include the possibility for the agent to use both market and limit orders. We hence provide with this paper a new jump-dependent indirect price impact model with stochastic liquidity factor, leading to a more realistic description of the underlying LOB dynamic and including specific high-frequency features of the order book.

The remainder of this study is organized as follows. Section 2 introduces the indirect price impact model with propagator. Sections 2.1, 2.2 and 2.3 consider various assumptions on the propagator G with, respectively, a temporary, a permanent and a transient kernel G. Section 3 generalizes the model by including limit orders. Section 4 provides an outline of the estimation methodology and section 5 concludes.

2. The indirect price impact model with propagator

We introduce the following liquidation model in continuous-time for a selling agent (which is also applicable to buying agents by symmetry). We assume at first that only MOs are posted by the trader (limit orders are then discussed in section 3). Furthermore, no bid-ask spread is taken into account in the following, but the model can easily be extended to include it. We first define the following quantities:

- T is the time horizon required for liquidating the shares, i.e. the trading horizon.
- $(\nu_t)_t := (\nu_t)_{0\le t\le T}$ is the trading rate process, i.e. the (non-negative) speed at which the agent is liquidating shares using MOs and which can be seen as the volume sold on each infinitesimal step of time. We denote by $(\nu_t)_t$ the whole strategy/process on $[0,T]$ and by ν a particular value in $\mathbb{R}^+$ of this strategy.
- $(Q_t)_t := (Q_t)_{0\le t\le T}$ is the agent's inventory of remaining shares to liquidate, affected by $(\nu_t)_t$,

$$\mathrm{d}Q_t = -\nu_t\,\mathrm{d}t, \quad Q_0 = Q > 0. \tag{2}$$

  We do not assume that the agent liquidates all the shares before T. Thus, it could be that $Q_T \ge 0$, i.e. the agent falls short of her target to liquidate all the shares before T, but in this case she incurs a terminal penalty (the form of which is detailed below). However, we do assume that the agent cannot liquidate more than the Q initial shares such that we cannot have $Q_T < 0$ (as implied by the stopping time τ in (4)).
- $(S_t^{(\nu)})_t := (S_t^{(\nu)})_{0\le t\le T}$ is the execution price affected by the chosen trading rate.
- $(X_t)_t := (X_t)_{0\le t\le T}$ is the agent's cash/wealth process from her execution strategy, given by

$$X_t = \int_0^t S_s^{(\nu)}\nu_s\,\mathrm{d}s. \tag{3}$$

We fix a filtered sprobability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ where T is the deterministic trading horizon. We then need to define the stopping time τ as

$$\tau := T \wedge \inf\{t \ge 0 : Q_t = 0\}, \tag{4}$$

which is the minimum of the maturity T and the first time the inventory hits zero (because then no more trading is necessary), see e.g. Chapters 7.2, 7.4, 8.4 of Cartea *et al.* (2015). This stopping time τ is therefore defined as the last trading time of the agent (after which the wealth $(X_t)_t$ does not vary anymore). The admissible set $\mathcal{A}$ of trading strategies consists in non-negative, $\mathbb{F}$-predictable and uniformly bounded from above on $[0,\tau)$ trading rates $(\nu_t)_t$,

$$\begin{aligned}\mathcal{A} = \{(\nu_t)_t \mid (\nu_t)_t \ge 0,\ \mathbb{F}-\text{predictable},\\ (\nu_t)_t \in L^\infty(\Omega\times[0,\tau))\}.\end{aligned} \tag{5}$$

Therefore, the strategy $(\nu_t)_t$ takes its values ν in some bounded set $\Gamma \subset \mathbb{R}^+$ on $[0,\tau)$. As an extension of the price impact model of Gatheral (2010), we further assume that the execution price $S_t^{(\nu)}$ at time $t \le \tau$ follows

$$S_t^{(\nu)} := S_t + \int_0^t G(t-s)\,\mathrm{d}L_s, \tag{6}$$

where $S_t = S_0 + \sigma W_t$ is a continuous martingale representing the unaffected stock price, and the decay kernel $G : [0,T] \to [0,+\infty]$ is such that G is a measurable function, square-integrable on $[0,T]$ and $G(0) := \lim_{t\to 0} G(t)$ exists. We will consider throughout the paper a $\mathbb{F}$-Brownian motion as the continuous martingale $(W_t)_t := (W_t)_{t\ge 0}$. We then further define the jump process for $t \in [0,\tau]$ by

$$L_t := -\int_0^t \kappa\,\nu_s^\gamma\,\mathrm{d}N_s, \tag{7}$$

where the jumps' size depends on the size of the trading rate through the exponent $\gamma \ge 0$ and where $(N_t)_t := (N_t)_{0\le t\le \tau}$ is a counting process independent from $(W_t)_t$ with mean-reverting intensity†

$$\begin{aligned} \mathrm{d}\lambda_t &= (\eta\nu_t - \alpha\lambda_t)\,\mathrm{d}t \quad \text{such that} \\ \lambda_t &= \lambda_0\,\mathrm{e}^{-\alpha t} + \eta\int_0^t \mathrm{e}^{-\alpha(t-s)}\nu_s\,\mathrm{d}s, \end{aligned} \tag{8}$$

where $\lambda_0, \eta, \alpha \geq 0$. Note that for any initial condition $(X_0, S_0^{(\nu)}, Q_0, \lambda_0) \in \mathbb{R}\times\mathbb{R}\times[0,Q]\times\mathbb{R}^+$ and any control $(\nu_t)_t \in \mathcal{A}$, all the state equations (2)–(3)–(6)–(8) admit a unique solution for $t \in [0,\tau)$. We will see in what follows that at $t = \tau$, despite the possible degenerate behavior of the state processes, the associated value function of the studied optimization problems will be well-defined. Moreover, since $(\nu_t)_t \in \mathcal{A}$ is non-negative, the mean-reverting intensity $(\lambda_t)_t := (\lambda_t)_{0\leq t\leq\tau}$ is also non-negative and bounded from below by $\lambda_{\min} := \lambda_0\,\mathrm{e}^{-\alpha T}$. In practice, this means that purchasing shares is not allowed in our model at any time during the trading horizon. Besides, this model induces that the execution price of a transaction depends both *indirectly* on the strategy $(\nu_t)_t$ chosen by the trader via the intensity $(\lambda_t)_t$ of $(N_t)_t$, as well as *directly* on $(\nu_t)_t$ through the jump size $(-\kappa\nu_t^{\gamma})$ of the point process $(L_t)_t$ and the propagator G in (6). Indeed, high trading rates increase the probability of jumps and hence of a negative impact on the transaction price, which then scales as a power-law function of the trade size ν with exponent $\gamma \leq 1$. Empirically, it makes sense that λ_t depends on the past trading rates $(\nu_s)_{0\leq s\leq t}$ since a sell MO puts a permanent and transient downward pressure on the price, which translates in our model by a higher probability of a negative jump, see again section 1. Therefore, we always have in this model a *permanent and transient* indirect impact of MOs on the price through the mean-reverting intensity $(\lambda_t)_t$, even if the decay kernel G ruling the direct price impact can either be temporary (cf. section 2.1) or take other forms (sections 2.2 and 2.3). This allows to obtain a more flexible setting than many liquidation models proposed in the literature. Note that the mean-reversion property of the intensity (8) induces an (indirect) transient price impact in our model while the fact that the mean-reversion level of $(\lambda_t)_t$ depends on ν_t leads to a permanent impact of the trades. As argued in Gatheral (2010) and Gatheral and Schied (2013), considering the martingale $S_t = S_0 + \sigma W_t$ (with $(W_t)_t$ Brownian motion) for the unaffected price then allows to implicitly incorporate the cumulative impact of others' trading (the trading crowd in the terminology of Huberman and Stanzl 2004) and is quite standard in the literature. Indeed, negative prices only occur with negligible probability at the considered time scales. Moreover, the drift term of $(S_t)_t$ is usually omitted for such small time scales. Finally, the agent's cash process $(X_t)_t$ satisfies $\mathrm{d}X_t = S_t^{(\nu)}\nu_t\,\mathrm{d}t = (S_t + \int_0^t G(t-s)\,\mathrm{d}L_s)\nu_t\,\mathrm{d}t$ and therefore, the expected revenues from the sale up to time $t \leq \tau$ is denoted

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}\left[\int_0^t S_s^{(\nu)}\nu_s\,\mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^t \left(S_s + \int_0^s G(s-u)\,\mathrm{d}L_u\right)\nu_s\,\mathrm{d}s\right]. \end{aligned} \tag{9}$$

It is important to note that the price impact model described in this section can be seen as a stochastic model for the liquidity factor, also known as Kyle's Lambda (1985). Specifically, it takes the form of a trade-driven intensity process $(\lambda_t)_t$ playing the role of a stochastic prefactor to the impact model (up to a constant κ and the propagator G). We will indeed prove in different settings that the optimal policy does not depend on the price jumps but only on its intensity. By focusing on this stochastic liquidity effect, we can hence compare our results with the deterministic liquidity case of Fruth *et al.* (2014) and Curato *et al.* (2017), or the stochastic case of Fruth *et al.* (2019) and Graewe and Horst (2017). The HJB PDE (15) we will obtain in the temporary case is indeed extremely similar as the one obtained in Ankirchner *et al.* (2014) or Graewe and Horst (2017).

Now that the price impact model is introduced for a general kernel G, we specify various suitable choices of such propagators in (6) to best fit the empirical observations of figures 1–2. Section 2.1 will then consider a temporary kernel, section 2.2 a permanent kernel and section 2.3 a transient kernel.

2.1. Temporary price impact kernel

In this section, we only consider a temporary direct impact of trading on the execution price, meaning that a sell MO at a time t only has a direct effect on the price of that transaction but not on the price of subsequent transactions. However, remember that the permanent and transient impacts of MOs are always included indirectly in our setting via the mean-reverting intensity $(\lambda_t)_t$ of the jump process (8). In this temporary case, it is best to start by defining the dynamic of the wealth process $(X_t)_t$ at the stopping time τ (4) in the following way

$$\begin{aligned} X_\tau &:= \int_0^\tau S_s\nu_s\,\mathrm{d}s - \int_0^\tau \kappa\nu_s^{\gamma+1}\,\mathrm{d}N_s \\ &= \int_0^\tau S_s\nu_s\,\mathrm{d}s - \sum_{j=1}^{N_\tau}\kappa\nu_{t_j}^{\gamma+1}. \end{aligned} \tag{10}$$

We then find using integration by parts with $Q_\tau = Q_t - \int_t^\tau \nu_s\,\mathrm{d}s$ and with $\tau \geq t$,

$$\int_t^\tau S_s\nu_s\,\mathrm{d}s = S_tQ_t - S_\tau Q_\tau + \int_t^\tau \sigma Q_s\,\mathrm{d}W_s.$$

Since $(Q_t)_t$ is bounded (and belongs more precisely to $[0,Q]$ in the interval $[0,\tau]$), the third term is a martingale and since

† One may consider instead as λ_0 a deterministic function of time so as to include time-of-day effects on the liquidity, as in Cont *et al.* (2014) and Fruth *et al.* (2014).

$\mathrm{d}X_t = S_t^{(\nu)}\nu_t\,\mathrm{d}t$, we find

$$\mathbb{E}[X_\tau \mid \mathcal{F}_t] = X_t + S_tQ_t + \mathbb{E}\left[-S_\tau Q_\tau - \kappa\int_t^\tau \nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]. \quad (11)$$

As $(\nu_t)_t \in \mathcal{A}$ is non-negative and $\mathbb{F}$-predictable, we indeed have from Lu (2013) that $\mathbb{E}[\int_t^\tau \kappa\nu_s^{\gamma+1}\,\mathrm{d}N_s \mid \mathcal{F}_t] = \mathbb{E}[\int_t^\tau \kappa\,\nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s \mid \mathcal{F}_t]$. The expression (10) for the agent's wealth appears to be the only reasonable way for modeling this temporary direct impact on the price within our framework. Indeed, since trading occurs continuously during the horizon and since the direct impact of a trade only affects the transaction price at this time, we have that the wealth is simply the proceeds of the sale at the unaffected price S_t minus the sum of these negative impulsive impacts $\kappa\nu_{t_j}^\gamma$ (each time they occur) times the quantity sold ν_{t_j} at that time. As explained in Gatheral (2010) and in Abi Jaber and Neuman (2022), considering only a temporary direct price impact amounts to use formally a Dirac's delta distribution $G(t-s) = \delta(t-s)$ on $[0,T]$ as propagator in the execution price $S_t^{(\nu)}$ in (6). We refer to Appendix 1 where we formally show in the temporary case the equivalence between the wealth process (10) and the execution price (6) with a Dirac's kernel G.

Now suppose that the agent is willing to optimize the following performance criterion

$$H^\nu(t,x,S,q,\lambda) = \mathbb{E}_{t,x,S,q,\lambda}\left[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau) - \phi\int_t^\tau Q_u^2\,\mathrm{d}u\right], \quad (12)$$

where $\mathbb{E}_{t,x,S,q,\lambda}[.]$ denotes the expectation conditional to $X_{t^-} = x$, $S_t = S$, $Q_t = q$, $\lambda_t = \lambda$ and where τ is the stopping time defined in (4). The corresponding value function at $t \le \tau$ is then

$$H(t,x,S,q,\lambda) = \sup_{(\nu_t)_t\in\mathcal{A}} \mathbb{E}_{t,x,S,q,\lambda}\left[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau) - \phi\int_t^\tau Q_u^2\,\mathrm{d}u\right], \quad (13)$$

where $\mathcal{A}$ is the admissible set (5) of trading strategies. ae first see that if the agent's strategy reaches the terminal date T with inventory left, then she must execute a final order to sell the required number of shares Q for a total revenue at T of $Q_T(S_T - \pi\lambda_T Q_T)$. π denotes the terminal liquidation penalty parameter† that includes the cost of walking the book at T and any other additional penalties that the agent must incur for the execution of the trade at the terminal date. The terminal penalty $Q_T(S_T - \pi Q_T)$ without the term λ_T is standard in the literature when the price impact is given by the model (1) and does not depend on a jump process, see chapter 6 of Cartea *et al.* (2015) and references therein. However, it is necessary in our context to adapt this terminal penalty. Indeed, depending on the shape of the LOB at maturity T, it could be that liquidating the Q_T remaining shares using MOs does not lead to a direct negative price impact. Therefore, the occurrence of this negative terminal penalty must again depend on the jump process $(N_t)_t$ with mean-reverting intensity $(\lambda_t)_t$. We again refer to Appendix 1 for the formal derivation of this terminal penalty $Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau)$ in the temporary case with Dirac propagator G. Secondly, the last term in (13) is a running inventory penalty, which is not a financial cost to the agent's strategy (and thus not dependent on λ_t), but which effectively captures the agent's urgency to execute the trade. It has been shown in Cartea *et al.* (2017) that this penalty is equivalent to a form of risk aversion from the agent. An alternative risk aversion penalty is also discussed in Appendix 2. Finally, as seen above in figure 2 and as shown in Torre (1997), Almgren *et al.* (2005), Gatheral (2010) and Cartea *et al.* (2015), we have empirically that the *temporary* (or instantaneous) price impact depends on the square root of the trading rate ($\gamma = 1/2$), and more generally on the power-law ν^γ with $\gamma \le 1$. Our setup in this section is in fact similar in terms of temporary impact as the one of Almgren (2003), Almgren *et al.* (2005), Gatheral (2010) where they also use $G(t-s) = \delta(t-s)$ but without jump process $(N_t)_t$ and based instead on equation (1) with $f(\nu) = -\kappa\nu^\gamma$, $\gamma \in [0.4, 0.8]$.

† We consider here the unaffected price S_T and not $S_T^{(\nu)}$ since the direct impact is temporary in this section, the terminal penalty being imputed here in place of this temporary impact.

2.1.1. Temporary impact with mean-reverting intensity.

We first consider that the jump process $(N_t)_t$ is ruled by the mean-reverting intensity (8). We will then show in section 2.1.2 that the optimization problem (13) can be simplified by choosing an alternative expression for $(\lambda_t)_t$. Recall first that the set $\mathcal{A}$ of admissible trading rates $(\nu_t)_t$ consists in uniformly bounded on $[0,\tau)$, non-negative and $\mathbb{F}$-predictable strategies. From equations (10) and (11), we can then rewrite the expected wealth (9) in the following way at the last trading time τ :

$$\mathbb{E}_{t,x,S,q,\lambda}[X_\tau] = x + qS + \mathbb{E}_{t,S,q,\lambda}\left[-S_\tau Q_\tau - \int_t^\tau \kappa\nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s\right],$$

with $X_{t^-} = x, S_t = S, Q_t = q, \lambda_t = \lambda$. When considering formally a Dirac delta $G(t-s) = \delta(t-s)$ in equation (6) for the execution price $S_t^{(\nu)}$ and using a Fubini's theorem for distribution (chapter 4, section 3, theorem IV. of Schwartz 1966), we find equivalently from equation (9) that

$$\begin{aligned}\mathbb{E}_{t,x,S,q,\lambda}[X_\tau] &= x + \mathbb{E}_{t,S,q,\lambda}\left[\int_t^\tau S_s\nu_s\,\mathrm{d}s - \int_t^\tau\left(\int_0^s G(s-u)\kappa\nu_u^\gamma\lambda_u du\right)\nu_s\,\mathrm{d}s\right]\\ &= x + qS + \mathbb{E}_{t,S,q,\lambda}\left[-S_\tau Q_\tau - \int_t^\tau \kappa\nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s\right].\end{aligned}$$

We can then rewrite the performance criterion (12) as

$$H^\nu(t,x,S,q,\lambda) = \mathbb{E}_{t,x,S,q,\lambda}\left[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau) - \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right]$$

$$= x + qS + \mathbb{E}_{t,q,\lambda}\left[-\int_t^\tau \kappa \nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s - \pi\lambda_\tau(Q_\tau)^2 - \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right].$$

And we finally have for the value function (13):

$$\begin{aligned} H(t,x,S,q,\lambda) &= x + qS + \sup_{(\nu_t)_t\in\mathcal{A}} \mathbb{E}_{t,q,\lambda}\left[-\int_t^\tau \kappa \nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s - \pi\lambda_\tau(Q_\tau)^2 - \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right] \\ &= x + qS - \inf_{(\nu_t)_t\in\mathcal{A}} \mathbb{E}_{t,q,\lambda}\left[\int_t^\tau \kappa \nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s + \pi\lambda_\tau(Q_\tau)^2 + \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right] \\ &= x + qS - \inf_{(\nu_t)_t\in\mathcal{A}} h^\nu(t,q,\lambda) \\ &= x + qS - h(t,q,\lambda). \end{aligned} \tag{14}$$

First, we observe that h is non-negative since h^ν is non-negative for every $(\nu_t)_t \in \mathcal{A}$. Secondly, since $(\nu_t)_t \in \mathcal{A}$ is bounded on $[0,\tau)$ (with values in Γ), we have from (8) that $(\lambda_t)_t$ is bounded on $[0,\tau)$ (and $(Q_t)_t$ bounded on $[0,\tau]$). Moreover, there always exists a $(\nu_t)_t \in \mathcal{A}$ such that $h^\nu(t,q,\lambda) < \infty$ and hence, $h(t,q,\lambda) := \inf_{(\nu_t)_t\in\mathcal{A}} h^\nu(t,q,\lambda) < \infty$ for all $(t,q,\lambda) \in \bar{\mathcal{S}} := [0,T] \times [0,Q] \times [\lambda_{\min},+\infty)$. Therefore, $H(t,x,S,q,\lambda)$ is also finite for any given values of $(t,q,\lambda,x,S) \in \bar{\mathcal{S}} \times \mathbb{R} \times \mathbb{R}$. However, no regularity results exist for value functions of the form (14). From the boundedness of h in $\bar{\mathcal{S}}$, we instead use the theory of viscosity solutions. More precisely, classical results from dynamic programming (see Bouchard 2007, Bouchard and Touzi 2011) imply that $h(t,q,\lambda)$ is a viscosity solution of the following HJB equation for all $(t,q,\lambda) \in \mathcal{S} := [0,T) \times (0,Q] \times [\lambda_{\min},+\infty)$,

$$\partial_t h(t,q,\lambda) + \phi q^2 + \min_{\nu\in\Gamma}\{(\eta\nu - \alpha\lambda)\partial_\lambda h(t,q,\lambda) - \nu\partial_q h(t,q,\lambda) + \kappa\nu^{\gamma+1}\lambda\} = 0, \tag{15}$$

subject to terminal and boundary conditions

$$h(T,q,\lambda) = \pi\lambda q^2, \quad h(t,0,\lambda) = 0 \quad \text{and} \quad \lim_{\lambda\to 0} h(t,q,\lambda) = 0,$$

since we must have $H(T,x,S,q,\lambda) = x + qS - \lambda\pi q^2$ and $H(t,x,S,0,\lambda) = x$. This last boundary condition is implied by the non-negativity of $(\nu_t^*)_t$ and by the fact that when the remaining inventory hits $q = 0$, the agent stops acquiring shares as of that stopping time and there is no penalty (she thus simply walks away with x in cash). Therefore, the first term of H in (14) is the accumulated cash of the strategy, the second is the marked-to-market book value of the remaining inventory, and h is an extra penalty due to price impact stemming from optimally liquidating the rest of the shares. Note that when λ_t tends to 0, the agent can liquidate all her remaining shares at time t without risking to have an adverse impact on price, which is an optimal strategy. We thus find the additional boundary condition $\lim_{\lambda\to 0} h(t,q,\lambda) = 0$ for the variable λ. However, specifying $\lambda_0 > 0$ in the mean-reverting intensity (8) prevents from attaining this boundary (even if this boundary condition still remains useful numerically).

From the HJB equation (15) with $\gamma > 0$, we obtain the following feedback form for the optimal trading rate ν^* for all $(t,q,\lambda) \in \mathcal{S}$,

$$\nu^*(t,q,\lambda) = \left(\frac{\partial_q h(t,q,\lambda) - \eta\partial_\lambda h(t,q,\lambda)}{(\gamma+1)\lambda\kappa}\right)^{1/\gamma}, \tag{16}$$

which is well defined (and admissible for $\gamma = 1/2$) since $\lambda \geq \lambda_{\min} > 0$. Hence, among all the admissible solutions for the trading strategy, the optimal one $(\nu_t^*)_t$ is deterministic as it does not depend on $(S_t)_t$, $(W_t)_t$, nor $(N_t)_t$. Note from the definition of the last trading time τ in (4), that we always have $\nu^*(t,q,\lambda) = 0$ for $t > \tau$. Upon substitution back in equation (15), we find that h is a viscosity solution for all $(t,q,\lambda) \in \mathcal{S}$ and $\gamma > 0$ of the PDE

$$\partial_t h(t,q,\lambda) + \phi q^2 - \frac{(\partial_q h(t,q,\lambda) - \eta\partial_\lambda h(t,q,\lambda))^{\frac{\gamma+1}{\gamma}}}{(\lambda\kappa(\gamma+1))^{1/\gamma}} \times \left(\frac{\gamma}{\gamma+1}\right) - \alpha\lambda\partial_\lambda h(t,q,\lambda) = 0, \tag{17}$$

with terminal condition $h(T,q,\lambda) = \pi\lambda q^2$ and boundary condition $h(t,0,\lambda) = 0$. Taking the square-root law $\gamma = 1/2$, we find for all $(t,q,\lambda) \in \mathcal{S}$ the following admissible optimal trading rate

$$\nu^*(t,q,\lambda) = \frac{4}{(3\lambda\kappa)^2}(\partial_q h(t,q,\lambda) - \eta\partial_\lambda h(t,q,\lambda))^2,$$

and the PDE

$$\partial_t h(t,q,\lambda) + \phi q^2 - \frac{4}{27(\lambda\kappa)^2}(\partial_q h(t,q,\lambda) - \eta\partial_\lambda h(t,q,\lambda))^3 - \alpha\lambda\partial_\lambda h(t,q,\lambda) = 0. \tag{18}$$

Therefore, in this temporary setting with mean-reverting intensity, the choice of a square-root law $\gamma = 1/2$ (as empirically motivated in the literature for temporary price impact) ensures that the optimal trading rate is non-negative and leads to consistent numerical results, as we will now show.

The PDE (17) cannot be solved analytically but is solved easily using a finite difference scheme. This scheme however requires that λ_0 is sufficiently large to ensure numerical convergence ($\lambda_0 > 0.1$). Since the optimal rate of trading is deterministic, the exact optimal strategy $(\nu_t^*)_t$ and $(Q_t^{(\nu^*)})_t$ can be computed at the inception $t = 0$ and are depicted in the following figures 3, 4 and 5 for various sets of parameters. If π and ϕ are sufficiently large (large terminal and running penalty), we first see on the figure 3 that the agent has an incentive to liquidate her shares as soon as possible and we thus find in this case that $Q_T^{(\nu^*)} = 0$ (e.g $\pi = 10$, $\phi = 5$). When π and ϕ are small, the agent has instead an incentive to liquidate her shares much slower and to keep a remaining quantity at maturity. Moreover, figure 4 shows that as the temporary penalty κ incurred with each trade increases, the

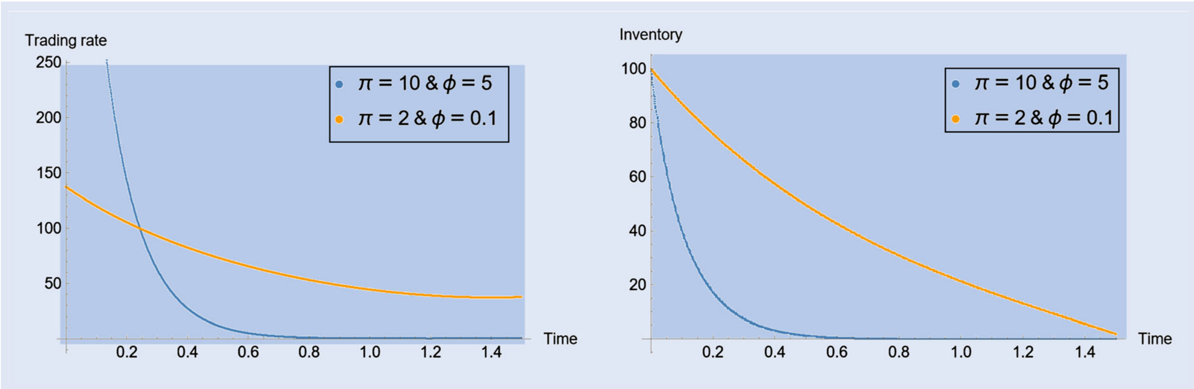

Figure 3. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a temporary (direct) price impact with $\pi = 10$ and $\phi = 5$ (blue line) compared to $\pi = 2$ and $\phi = 0.1$ (orange line) with $\kappa = 0.1$, $\eta = 0.002$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $T = 1.5$ and $Q = 100$.

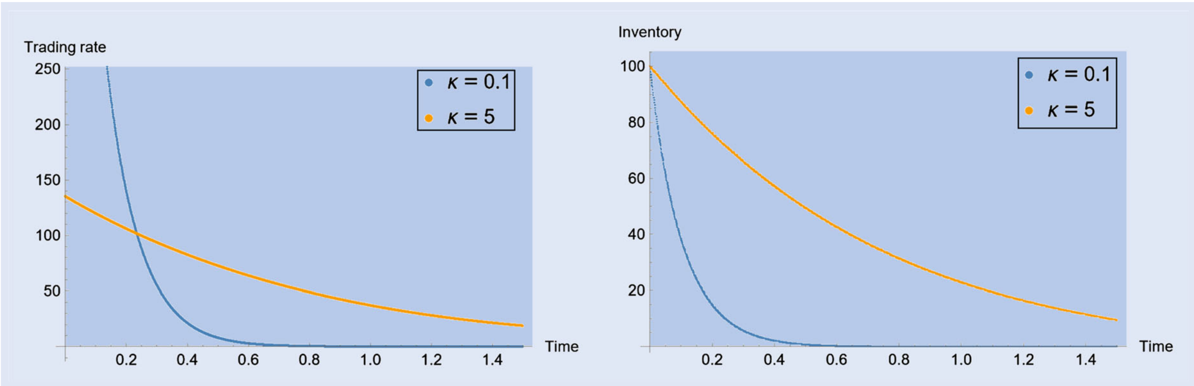

Figure 4. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a temporary (direct) price impact for different values of κ with $\pi = 10$, $\phi = 5$, $\eta = 0.001$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $T = 1.5$ and $Q = 100$.

optimal agent has an incentive to liquidate less shares. This way, he mitigates the negative impact on the transaction price and faces instead the terminal penalty, which becomes small relative to this large temporary impact κ.

Moreover, adjusting the initial level λ_0 of the mean-reverting intensity $(\lambda_t)_t$ in (8) leads to significant variations in the trend of the trading rate, even though it does not impact the terminal inventory $Q_T^{(\nu^*)}$. Notably, when η decreases and ϕ is large, the remaining inventory depletes much faster at the beginning of the trading horizon. Indeed, since the (permanent) impact on λ_t is much smaller, the agent can execute large trades early on while keeping the intensity relatively low (which is in accordance with the large urgency to trade ϕ) (figure 6). The impact of η, however, depends on the value of ϕ. When $\phi = 0$, we observe the opposite behavior (see the geometric intensity case 2.1.2 with no running penalty) since in this scenario, the agent has no urgency to trade and therefore no incentive to liquidate shares quickly at the beginning of the trading horizon (even when η is low). This confirms the empirical observations made in section 1 and figures 1 and 2. Indeed, when the intensity λ_t is low, the optimal trader may want to execute large trades rapidly, as the probability of incurring a negative temporary impact is reduced. However, this will then strongly increase the intensity process over the

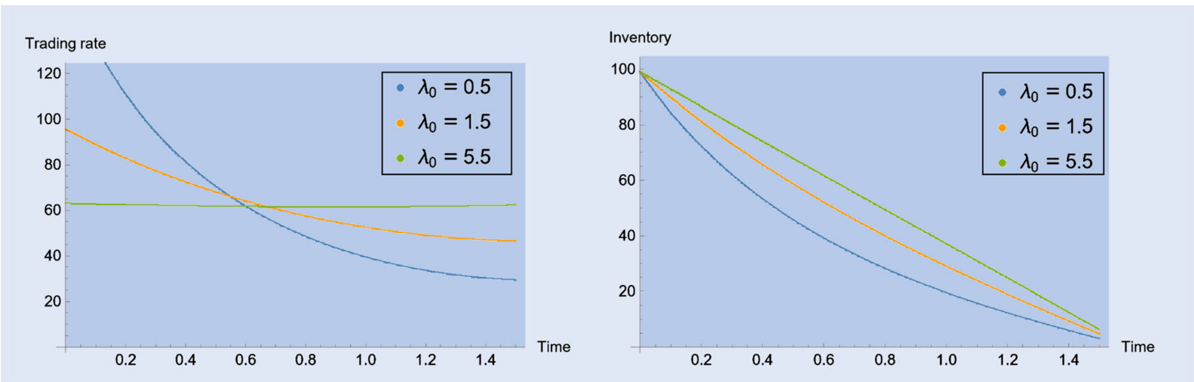

Figure 5. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a temporary (direct) price impact for various levels of λ_0 with $\pi = 10$, $\phi = 2$, $\eta = 0.001$, $\alpha = 0.5$, $\kappa = 0.1$, $T = 1.5$ and $Q = 100$.

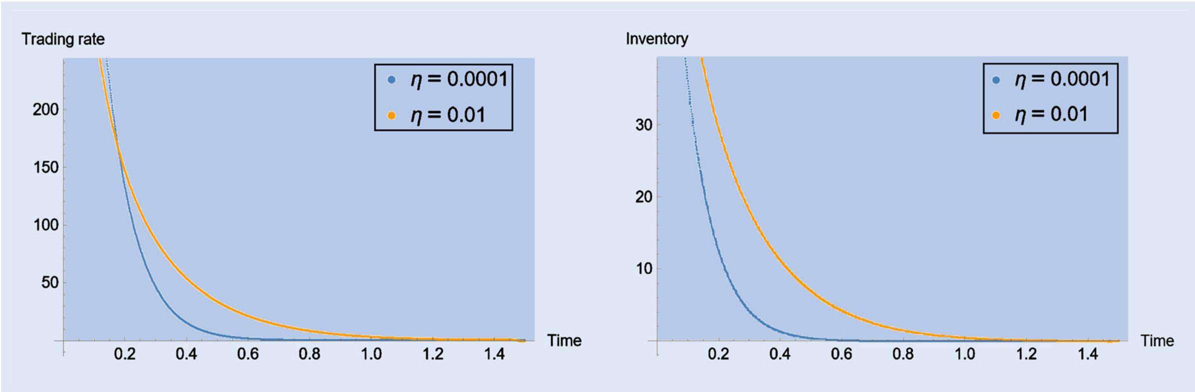

Figure 6. Optima trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a temporary price impact for different values η with $\pi = 10, \phi = 5, \kappa = 0.1, \alpha = 0.5, \lambda_0 = 1.5, T = 1.5, Q = 100$.

remaining trading horizon due to the permanent and transient effects of our indirect price impact model, which may in turn no longer be an optimal strategy depending on the parameters η, α, ϕ and π.

Finally, we compare the numerical results above with (a version of) the standard Almgren-Chriss model, which assumes power-law temporary impact function and no permanent impact, see Almgren (2003) and section 6.5 of Cartea *et al.* (2015). Figure A1 in Appendix 3 illustrates this comparison between the optimal trading strategies from our indirect price impact model (with Dirac's Delta kernel) and the power-law Almgren-Chriss model. We clearly observe significantly different optimal strategies, which are highly influenced by the intensity-related parameters. In particular, when the intensity level is low ($\lambda_0 = 0.5$), the optimal strategy from our model involves more aggressive trading at the start of the horizon compared to the Almgren-Chriss strategy, as the probability of inducing a jump (i.e. a temporary direct impact) is quite low. Conversely, the trader adopts a more conservative approach at the start with the indirect price impact model when the intensity level is high ($\lambda_0 = 2.5$).

2.1.2. Temporary impact with geometric intensity. Instead of the mean-reverting dynamic (8), we now consider that the intensity $(\lambda_t)_t$ of the jump process $(N_t)_t$ follows the following geometric dynamic for $t \leq \tau$:

$$\mathrm{d}\lambda_t = \eta \lambda_t \nu_t \,\mathrm{d}t \quad \text{such that} \quad \lambda_t = \lambda_0 \exp\left(\int_0^t \eta \nu_s \,\mathrm{d}s\right). \quad (19)$$

We need to impose $\eta > 0$ so that a higher volume sold ν leads to an increase of the occurrence probability of a negative price impact. We also assume no more running penalty for detaining stocks throughout the trading horizon ($\phi = 0$). The agent is then willing to optimize the following performance criterion, conditional to $X_{t^-} = x, S_t = S, Q_t = q, \lambda_t = \lambda$,

$$\begin{aligned} H(t,x,S,q,\lambda) &= \sup_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,x,S,q,\lambda}[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau)] \\ &= x + qS - \inf_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,q,\lambda}\left[\int_t^\tau \kappa \nu_s^{\gamma+1}\lambda_s \,\mathrm{d}s \right. \\ &\qquad \left. + \pi\lambda_\tau (Q_\tau)^2\right] \\ &= x + qS - \inf_{(\nu_t)_t \in \mathcal{A}} h^\nu(t,q,\lambda) \\ &= x + qS - h(t,q,\lambda), \end{aligned} \quad (20)$$

with again $h(T,q,\lambda) = \pi\lambda q^2$, $h(t,0,\lambda) = 0$ and $\lim_{\lambda \to 0} h(t,q,\lambda) = 0$ (since when the intensity is null, the agent can liquidate in an optimal way all her remaining shares at zero cost). Again, specifying $\lambda_0 > 0$ in (19) prevents from attaining this boundary, even if the limit $\lambda \to 0$ provides a boundary condition with respect to the variable λ necessary for solving numerically the associated HJB equation. The above reasoning of section 2.1.1 is still valid when replacing $(\eta\nu - \alpha\lambda)$ by $\eta\lambda\nu$. We therefore have that the value function h is a viscosity solution of the following HJB equation for all $(t,q,\lambda) \in \mathcal{S}$,

$$\begin{aligned} &\partial_t h(t,q,\lambda) + \min_{\nu \in \Gamma}\{\eta\lambda\nu\partial_\lambda h(t,q,\lambda) - \nu\partial_q h(t,q,\lambda) + \lambda\kappa\nu^{\gamma+1}\} \\ &\quad = 0, \end{aligned} \quad (21)$$

with $h(T,q,\lambda) = \lambda\pi q^2$ and $h(t,0,\lambda) = 0$. Taking $h(t,q,\lambda) = \lambda h(t,q)$, the PDE (21) becomes for all $(t,q) \in [0,T) \times (0,Q]$:

$$\partial_t h(t,q) + \min_{\nu \in \Gamma}\{\eta\nu h(t,q) - \nu\partial_q h(t,q) + \kappa\nu^{\gamma+1}\} = 0, \quad (22)$$

and hence, the optimization problem becomes independent of the intensity variable λ. This leads to

$$\nu^*(t,q) = \left(\frac{\partial_q h(t,q) - \eta h(t,q)}{(\gamma+1)\kappa}\right)^{1/\gamma}, \quad (23)$$

and finally to

$$\partial_t h(t,q) - \frac{(\partial_q h(t,q) - \eta h(t,q))^{\frac{\gamma+1}{\gamma}}}{((\gamma+1)\kappa)^{1/\gamma}}\left(\frac{\gamma}{\gamma+1}\right) = 0,$$

for all $(t,q) \in [0,T) \times (0,Q]$ and with $h(T,q) = \pi q^2$ and $h(t,0) = 0$. In summary, when considering the geometric intensity (19) without running penalty and with Dirac propagator G, we obtain a purely deterministic optimal rate $(\nu_t^*)_t$ that does not depend on the intensity variable λ.

We then find the following optimal strategies for various sets of parameters using the square-root law $\gamma = 1/2$.

Figure 7 shows that both the mean-reverting intensity model (8) from section 2.1.1 with $\lambda_0 = 0.5$ and the geometric intensity model (19) from this section lead to similar results. However, the convexity of the strategy is more pronounced when using a mean-reverting intensity $(\lambda_t)_t$ in the optimization problem, while the slope of the trading rate tends to remain constant under a geometric intensity. For various values of λ_0 in the mean-reverting dynamic (8), we see that the slope of $(\nu_t^*)_t$ with respect to time can change drastically, leading to different kinds of strategies. In contrast, the trading rate is always decreasing over time with a geometric intensity. As already mentioned, since we consider no running penalty ($\phi = 0$) in this section, a higher η leads to a faster decrease of the shares to liquidate (figure 8). Finally, it follows logically that a higher temporary impact κ results in a lower trading rate throughout the trading horizon, as the agent prefers in this case to incur the terminal penalty and to avoid these higher negative impacts κ (figure 9).

This temporary setting (10) with Dirac propagator G and mean-reverting intensity $(\lambda_t)_t$ appears as the most natural/consistent with the empirical observations of section 1 and figure 2. Indeed, this approach effectively disentangles the temporary direct price impact due to MOs walking the LOB from the indirect permanent and transient impacts resulting from the downward pressure in price due to changes in net order flow imbalance. However, we now propose in sections 2.2 and 2.3 alternative propagators that will allow to combine differently these temporary, transient and permanent (direct and indirect) effects.

2.2. *Permanent price impact kernel*

We consider in this section that the direct price impact ruled by the kernel G is permanent in the sense that a sell order permanently affects the price of the asset and thus the price at which all subsequent transactions are executed. This corresponds to setting $G \equiv 1$ in (6) which gives for $t \in [0, \tau]$,

$$\begin{aligned} \mathrm{d}X_t &= S_t^{(\nu)} \nu_t \,\mathrm{d}t = \left(S_t - \int_0^t \kappa \nu_s^\gamma \,\mathrm{d}N_s \right) \nu_t \,\mathrm{d}t \\ &=: (S_t - Z_t)\nu_t \,\mathrm{d}t, \end{aligned} \tag{24}$$

where $\mathrm{d}S_t^{(\nu)} = \mathrm{d}S_t - \mathrm{d}Z_t = \sigma \,\mathrm{d}W_t - \kappa \nu_t^\gamma \,\mathrm{d}N_t$. Z_t is thus the cumulative negative price impact from the trades up to time t. Notice that the execution price $S_t^{(\nu)} = S_t - Z_t$ here makes sense per se when $G \equiv 1$, compared with the price (A1) in the temporary case. Moreover, this setting accommodates both a permanent indirect impact (as before via $(\lambda_t)_t$) and a permanent direct impact on prices through this propagator $G \equiv 1$. We again suppose that the agent is willing to maximize the expected wealth at the last trading time τ,

$$\begin{aligned} \widetilde{H}(t, x, S^{(\nu)}, q, \lambda) = \sup_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,x,S^{(\nu)},q,\lambda} \Bigg[X_\tau + Q_\tau (S_\tau^{(\nu)} \\ - \pi \lambda_\tau Q_\tau) - \phi \int_t^\tau Q_u^2 \,\mathrm{d}u \Bigg], \end{aligned} \tag{25}$$

where $S^{(\nu)} := S_{t-}^{(\nu)}$ is a state variable (instead of S in the optimization (13) above) since the direct price impact in this section is permanent. Therefore, the price at which the agent can sell the Q_T remaining shares at the end of the trading horizon is $S_T^{(\nu)}$ (including the cumulative negative price impacts) minus the terminal penalty. Finally, as mentioned above and in Appendix 1, this terminal penalty is a temporary impact as it only affects the last trade at T, and hence does not depend in a direct way on the previous negative jumps of $(N_t)_{0 \le t \le T}$ (even if it still depends indirectly on the past trades $(\nu_t)_{0 \le t \le T}$ through the intensity λ_T). This again leads to the same final penalty $\mathbb{E}_{t,x,S^{(\nu)},q,\lambda}[Q_T(S_T^{(\nu)} - \pi \lambda_T Q_T)] = \mathbb{E}_{t,x,S^{(\nu)},q,\lambda}[Q_\tau(S_\tau^{(\nu)} - \pi \lambda_\tau Q_\tau)]$ as in section 2.1, but with state variable $S^{(\nu)}$ instead of S. From equation (24), we can rewrite the wealth process at τ by

$$X_\tau = X_t + \int_t^\tau (S_s - Z_s)\nu_s \,\mathrm{d}s.$$

Using integration by parts with $\mathrm{d}Q_t = -\nu_t \,\mathrm{d}t$, $\mathrm{d}S_t = \sigma \,\mathrm{d}W_t$ and $\mathrm{d}Z_t = \kappa \nu_t^\gamma \,\mathrm{d}N_t$, we have

$$\begin{aligned} X_\tau^\nu &= X_t - S_\tau Q_\tau + S_t Q_t + Z_\tau Q_\tau - Z_t Q_t + \int_t^\tau \sigma Q_s \,\mathrm{d}W_s \\ &\quad - \int_t^\tau \kappa Q_s \nu_s^\gamma \,\mathrm{d}N_s, \end{aligned}$$

which leads to the *proceeds* $R_\tau^{(\nu)}$ of the strategy at the stopping time τ (after which no more trading occurs),

$$\begin{aligned} R_\tau^{(\nu)} &:= X_\tau + Q_\tau (S_\tau^{(\nu)} - \pi \lambda_\tau Q_\tau) \\ &= X_t + S_t Q_t - Z_t Q_t + \int_t^\tau \sigma Q_s \,\mathrm{d}W_s \\ &\quad - \int_t^\tau \kappa Q_s \nu_s^\gamma \,\mathrm{d}N_s - \pi \lambda_\tau Q_\tau^2. \end{aligned}$$

Again from the boundedness of $(Q_t)_t$, the non-negativity of $(Q_t \nu_t^\gamma)_t$ and the $\mathbb{F}$-predictability of $(\nu_t)_t$,

$$\begin{aligned} &\mathbb{E}_{t,x,S^{(\nu)},q,\lambda}[X_\tau + Q_\tau (S_\tau^{(\nu)} - \pi \lambda_\tau Q_\tau)] \\ &\quad = x + qS^{(\nu)} + \mathbb{E}_{t,q,\lambda} \left[- \int_t^\tau \kappa Q_s \nu_s^\gamma \lambda_s \,\mathrm{d}s - \pi \lambda_\tau Q_\tau^2 \right]. \end{aligned}$$

We can then rewrite the value function (25) for $t \in [0, \tau]$ in the following way

$$\begin{aligned} &\widetilde{H}(t, x, S^{(\nu)}, q, \lambda) \\ &\quad = \sup_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,x,S^{(\nu)},q,\lambda} \Bigg[X_\tau + Q_\tau (S_\tau^{(\nu)} - \pi \lambda_\tau Q_\tau) \\ &\qquad - \phi \int_t^\tau Q_s^2 \,\mathrm{d}s \Bigg] \\ &\quad = x + qS^{(\nu)} - \inf_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,q,\lambda} \Bigg[\int_t^\tau \kappa Q_s \nu_s^\gamma \lambda_s ds + \pi \lambda_\tau Q_\tau^2 \\ &\qquad + \phi \int_t^\tau Q_s^2 \,\mathrm{d}s \Bigg] \\ &\quad = x + qS^{(\nu)} - \inf_{(\nu_t)_t \in \mathcal{A}} \widetilde{h}^\nu(t, q, \lambda) = x + qS^{(\nu)} - \widetilde{h}(t, q, \lambda), \end{aligned} \tag{26}$$

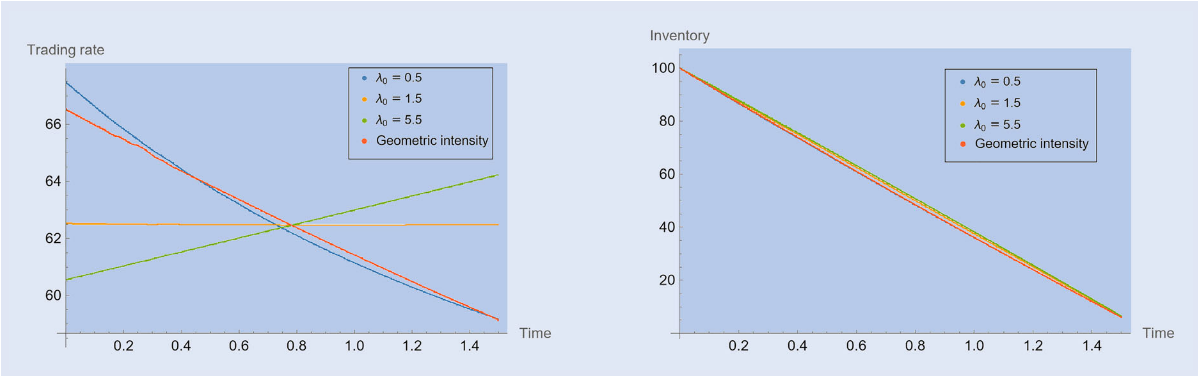

Figure 7. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a temporary (direct) price impact with the mean-reverting intensity (8) for various λ_0 (and $\phi = 0$), compared with the geometric intensity model (19) in red. Both models use $\pi = 8, \kappa = 0.8$, $\eta = 0.0025$, $\alpha = 0.05$, $Q = 100$ and $T = 1.5$.

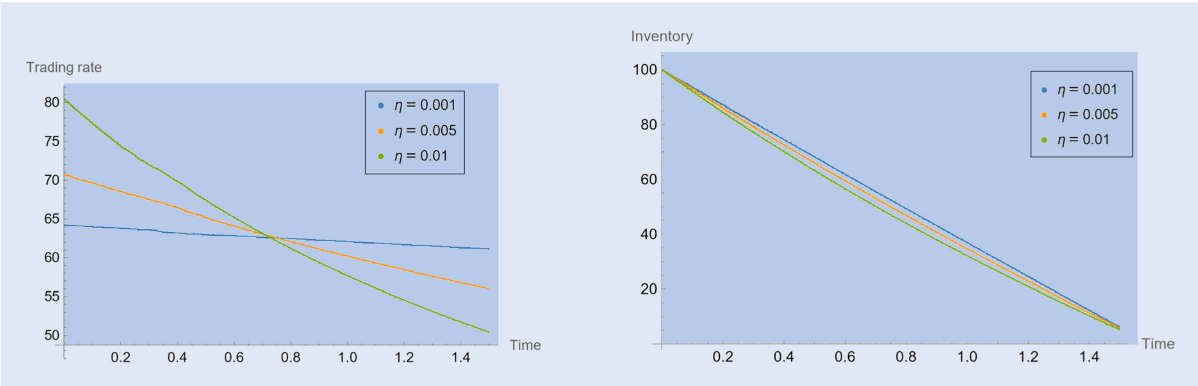

Figure 8. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a temporary (direct) price impact with geometric intensity (19) for various values of η with $\pi = 8$, $\kappa = 0.8$, $Q = 100$ and $T = 1.5$.

where $\widetilde{h}(t, q, \lambda)$ is again the cumulative penalty from optimally liquidating the rest of the shares, which is also bounded and non-negative in $[0, \tau]$ (and thus for all $(t, q, \lambda) \in \bar{\mathcal{S}}$). From the boundedness of $\widetilde{h}$ in $\bar{\mathcal{S}}$, we again use the theory of viscosity solutions. Besides, as we will now show, it is necessary to impose the condition $\gamma > 1$ in the permanent case. Consequently, the square-root law $\gamma = 1/2$ used in the temporary case can no longer hold with $G \equiv 1$.

2.2.1. Permanent impact with mean-reverting intensity. As in section 2.1.1, we first assume that the intensity $(\lambda_t)_t$ of $(N_t)_t$ is mean-reverting and satisfies (8). Since $\widetilde{h}$ is bounded on $\bar{\mathcal{S}}$, classical results from dynamic programming (see Bouchard and Touzi 2011) imply that $\widetilde{h}(t, q, \lambda)$ is the viscosity solution of the following HJB equation for all $(t, q, \lambda) \in \mathcal{S}$:

$$\partial_t \widetilde{h}(t, q, \lambda) + \phi q^2 + \min_{\nu \in \Gamma} \{(\eta\nu - \alpha\lambda)\partial_\lambda \widetilde{h}(t, q, \lambda) - \nu \partial_q \widetilde{h}(t, q, \lambda) + \kappa q \nu^\gamma \lambda\} = 0, \tag{27}$$

subject to terminal and boundary conditions

$$\widetilde{h}(T, q, \lambda) = \pi \lambda q^2, \quad \widetilde{h}(t, 0, \lambda) = 0 \quad \text{and}$$

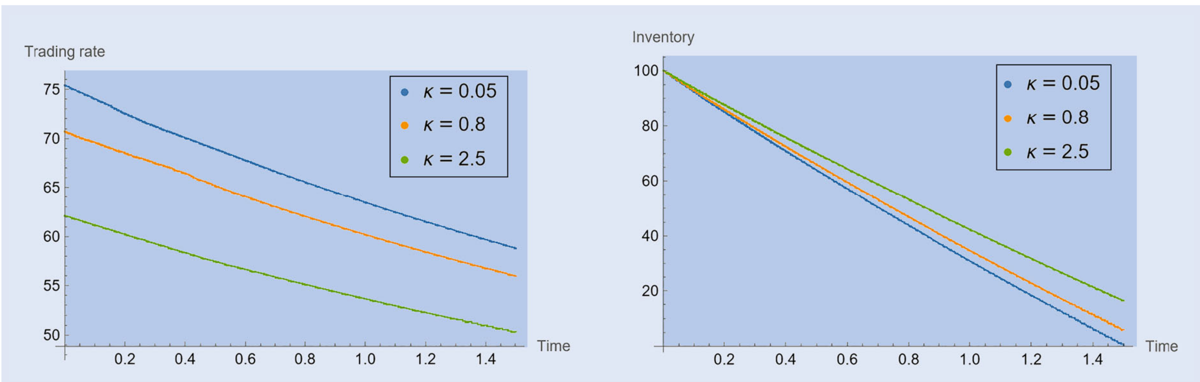

Figure 9. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a temporary (direct) price impact with geometric intensity (19) for various values of κ with $\pi = 8$, $\eta = 0.005$, $Q = 100$ and $T = 1.5$.

$$\lim_{\lambda \to 0} \widetilde{h}(t, q, \lambda) = 0.$$

Again, the first boundary condition comes from the non-negativity of $(\nu_t)_t$ and the stopping time (4), leading to $\widetilde{H}(t, x, S^{(\nu)}, 0, \lambda) = x$, while the second boundary condition is necessary numerically even if specifying $\lambda_0 > 0$ prevents from attaining this boundary. From (27) with $\gamma > 1$, the optimal trading rate is given explicitly in $\mathcal{S}$ by

$$\nu^*(t, q, \lambda) = \left(\frac{\partial_q \widetilde{h}(t, q, \lambda) - \eta \partial_\lambda \widetilde{h}(t, q, \lambda)}{\kappa q \lambda \gamma} \right)^{\frac{1}{\gamma - 1}}. \tag{28}$$

This optimal trading strategy $(\nu_t^*)_t$ is again deterministic (no dependence on $(N_t)_t$ nor $(W_t)_t$), and highly similar to (16) since we can denote $\gamma = \gamma' + 1 > 1$, where $\gamma' > 0$ is the exponent in (16). Note however that the remaining inventory q is now in the denominator and accounts for the permanent direct impact on the transaction price. This then leads to an exploding trading rate at the time of market drop-out since $\lim_{q \to 0} \nu^*(t, q, \lambda) = +\infty$, which prevents any smooth solution to the HJB equation (with again $\nu^*(t, q, \lambda) = 0$ for $t > \tau$). However, from (5), this trading rate is still admissible for $\gamma = 3/2$ and substituting (28) inside (27), we find that $\widetilde{h}$ is a viscosity solution for all $(t, q, \lambda) \in \mathcal{S}$ and $\gamma > 1$ of the PDE

$$\partial_t \widetilde{h}(t, q, \lambda) + \phi q^2 - \frac{(\partial_q \widetilde{h}(t, q, \lambda) - \eta \partial_\lambda \widetilde{h}(t, q, \lambda))^{\frac{\gamma}{\gamma - 1}}}{(\kappa \gamma q \lambda)^{1/(\gamma - 1)}} \times \left(\frac{\gamma - 1}{\gamma} \right) - \alpha \lambda \partial_\lambda \widetilde{h}(t, q, \lambda) = 0,$$

with terminal condition $\widetilde{h}(T, q, \lambda) = \pi \lambda q^2$ and boundary conditions $\widetilde{h}(t, 0, \lambda) = 0$, $\lim_{\lambda \to 0} \widetilde{h}(t, q, \lambda) = 0$. $\gamma = 3/2$ ensures the non-negativity of the optimal control since $\nu^*(t, q, \lambda) = \frac{4}{9}(\frac{\partial_q \widetilde{h}(t, q, \lambda) - \eta \partial_\lambda \widetilde{h}(t, q, \lambda)}{\kappa q \lambda})^2$ in $\mathcal{S}$, leading to

$$\partial_t \widetilde{h}(t, q, \lambda) + \phi q^2 - \frac{4}{27(\kappa q \lambda)^2} (\partial_q \widetilde{h}(t, q, \lambda) - \eta \partial_\lambda \widetilde{h}(t, q, \lambda))^3 - \alpha \lambda \partial_\lambda \widetilde{h}(t, q, \lambda) = 0.$$

Therefore, the obtained PDE for the permanent kernel is seemingly identical to the HJB equation in the temporary setting with square-root law (18), except with the quantity q now in the denominator $(\kappa q \lambda)^2$.

Choosing $\gamma = 3/2$ appears to be consistent with the square root law $\gamma' = 1/2$ from the previous temporary setting 2.1, as we now consider with $G \equiv 1$ the total/aggregated (direct) price impact of each MO. This choice leads to the following optimal strategies for various parameters. Another reasonable assumption would be to set γ close to 1, which would imply a linear permanent (direct) impact whenever $(L_t)_t$ jumps, in line with the linear permanent impact function $f(\nu)$ investigated in Almgren *et al.* (2005). Figures 10 and 11 again lead to financially consistent results when varying the penalty terms ϕ, π and κ. Additionally, for very low values of κ or high values of π and ϕ that force complete liquidation before T, the optimal strategy is to liquidate most shares right at the beginning of the trading horizon. However, changing the intensity-related parameters λ_0, α or η again strongly impacts the trend of the optimal strategy (see figure 12). Therefore, there is no clear pattern of optimal strategy with a permanent direct impact and a mean-reverting intensity. Moreover, we see that figure 11 (red line) confirms the exploding trading rate at the time of drop-out $q = 0$ when taking sufficiently low values κ to force complete liquidation. Finally, we again compare these numerical results with the Almgren-Chriss model based on a power-law temporary impact function and a linear permanent impact function, see Almgren (2003) and section 6.5 of Cartea *et al.* (2015). Figure A2 in Appendix 3 illustrates this comparison between the optimal trading strategies from our indirect price impact model (with permanent kernel $G \equiv 1$) and the power-law Almgren-Chriss model with permanent linear impact function. We again clearly observe significantly different optimal strategies, which are highly influenced by the intensity-related parameters.

2.2.2. Permanent impact with geometric intensity. As in section 2.1.2, we then assume for the permanent case $G \equiv 1$ no more running penalty ($\phi = 0$) and a geometric intensity linear in ν,

$$\mathrm{d}\lambda_t = \eta \lambda_t \nu_t \, \mathrm{d}t \quad \text{such that} \quad \lambda_t = \lambda_0 \exp\left(\int_0^t \eta \nu_s \, \mathrm{d}s \right).$$

We again find in this geometric case that $\widetilde{h}(t, q, \lambda)$ is a viscosity solution for all $(t, q, \lambda) \in \mathcal{S}$ of the HJB PDE

$$\partial_t \widetilde{h}(t, q, \lambda) + \min_{\nu \in \Gamma} \{ \eta \nu \lambda \partial_\lambda \widetilde{h}(t, q, \lambda) - \nu \partial_q \widetilde{h}(t, q, \lambda) + \kappa q \nu^\gamma \lambda \} = 0,$$

with $\widetilde{h}(T, q, \lambda) = \lambda \pi q^2$, $\widetilde{h}(t, 0, \lambda) = 0$. Taking $\widetilde{h}(t, q, \lambda) = \lambda \widetilde{h}(t, q)$ leads to

$$\partial_t \widetilde{h}(t, q) + \min_{\nu \in \Gamma} \{ \eta \nu \widetilde{h}(t, q) - \nu \partial_q \widetilde{h}(t, q) + \kappa q \nu^\gamma \} = 0, \tag{29}$$

with $\widetilde{h}(T, q) = \pi q^2$ and $\widetilde{h}(t, 0) = 0$. From (29) with $\gamma > 1$, the optimal trading rate satisfies for all $(t, q) \in [0, T) \times (0, Q]$:

$$\nu^*(t, q) = \left(\frac{\partial_q \widetilde{h}(t, q) - \eta \widetilde{h}(t, q)}{\kappa q \gamma} \right)^{\frac{1}{\gamma - 1}}, \tag{30}$$

so that the optimal strategy $(\nu_t^*)_t$ is independent of the intensity variable λ (and again deterministic). Again, the obtained solution is highly similar to (23), except with the remaining inventory q in the denominator. Therefore, this also leads to an exploding optimal trading rate $\nu^*(t, q, \lambda)$ at the time of market drop-out $q = 0$. Upon substitution back in the previous equation, we find that $\widetilde{h}(t, q)$ is a viscosity solution for all $(t, q) \in [0, T) \times (0, Q]$ and $\gamma > 1$ of the following PDE

$$\partial_t \widetilde{h}(t, q) - \frac{(\partial_q \widetilde{h}(t, q) - \eta \widetilde{h}(t, q))^{\frac{\gamma}{\gamma - 1}}}{(\kappa \gamma q)^{1/(\gamma - 1)}} \left(\frac{\gamma - 1}{\gamma} \right) = 0.$$

Choosing $\gamma = 3/2$ is the most sensible choice so that the total market impact is consistent with the square-root law used in

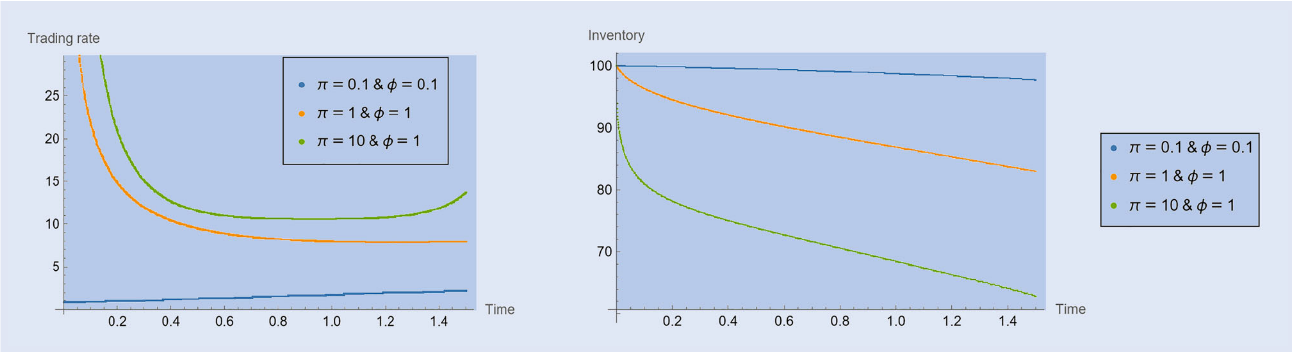

Figure 10. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a permanent (direct) price impact with mean-reverting intensity and $\gamma = 3/2$ for various values of π and ϕ with $\kappa = 1$, $\eta = 0.1$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

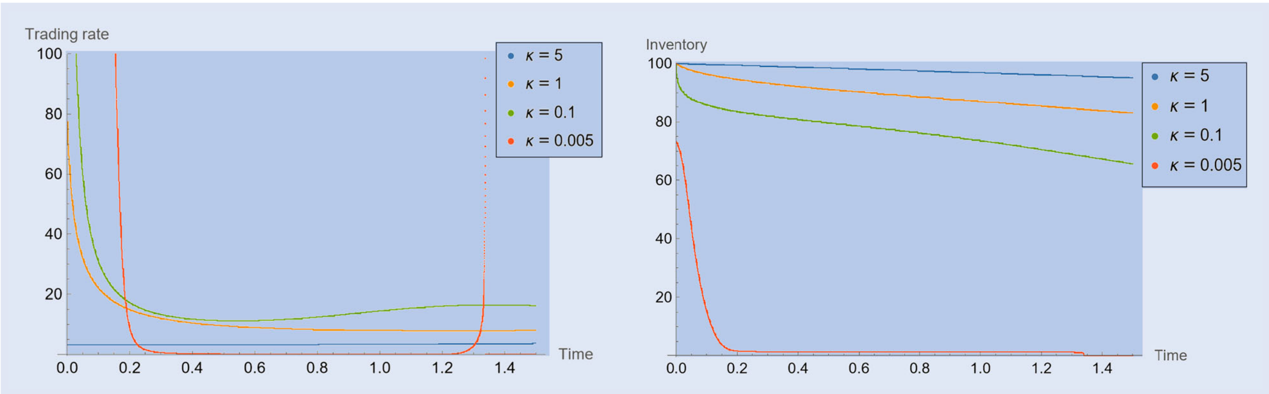

Figure 11. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a permanent (direct) price impact with mean-reverting intensity and $\gamma = 3/2$ for various values of κ with $\pi = 1$, $\phi = 1$, $\eta = 0.1$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

the temporary case, see Gatheral (2010). It also ensures the non-negativity and admissibility of $(\nu_t^*)_t$ in (30), leading to the following optimal strategies starting at time $t = 0$ for various sets of parameters. In figure 13, we first observe that when the terminal penalty π is sufficiently large and the permanent impact κ sufficiently small, the agent must liquidate all her shares before the trading horizon T to avoid the final penalty and to only incur the low permanent price impact κ during the trading horizon. However, to mitigate this cumulative permanent price impact on the whole trading horizon, the trading rate must be increasing with time, which is the opposite strategy than in the temporary setting 2.1.2. Moreover, when κ is sufficiently low (or π large) to force full liquidation before T, the optimal strategy consists here in liquidating most of these shares at the end of the trading horizon. This explosion of the trading rate when the number of remaining shares tends to zero can indeed be seen in the blue curve with $\pi = 30$. Additionally, when π becomes small relative to κ (i.e. the ratio π/κ is sufficiently small), the agent prefers to incur the terminal penalty and to liquidate fewer shares during the trading horizon in order to mitigate the permanent impact. We can indeed see in figure 13 (green line) a low constant trading rate resulting from the very small terminal penalty. A similar constant trading rate can be seen in figure 14, where the permanent impact κ is so large that the agent still prefers to incur the high terminal penalty π. Of course, as the penalty π increases, the trading rate tends to increase as well in order to end up with a lower terminal inventory $Q_T^{(\nu^*)}$. Finally, we again observe

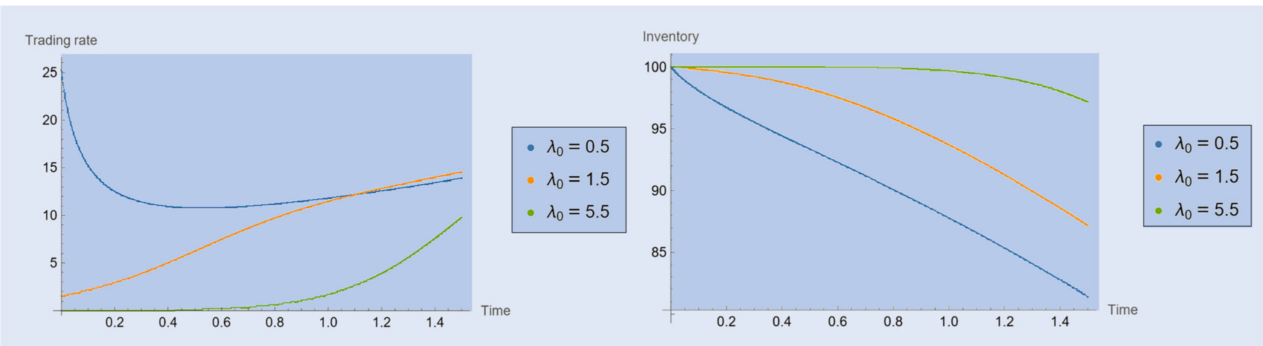

Figure 12. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a permanent (direct) price impact with mean-reverting intensity and $\gamma = 3/2$ for various values of λ_0 with $\pi = 1$, $\phi = 1$, $\kappa = 1$, $\eta = 0.1$, $\alpha = 1$, $Q = 100$ and $T = 1.5$.

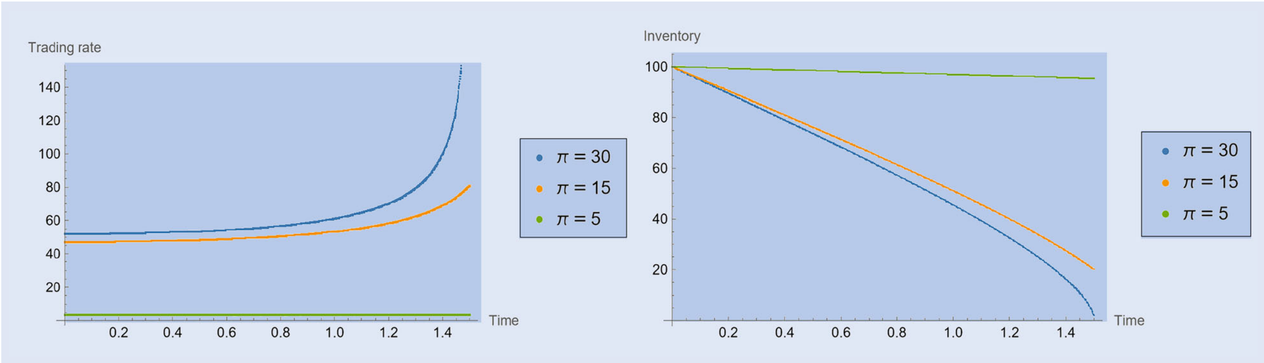

Figure 13. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a permanent (direct) price impact with $\gamma = 3/2$ for various values of π with $\kappa = 2$, $\eta = 0.01$, $Q = 100$ and $T = 1.5$.

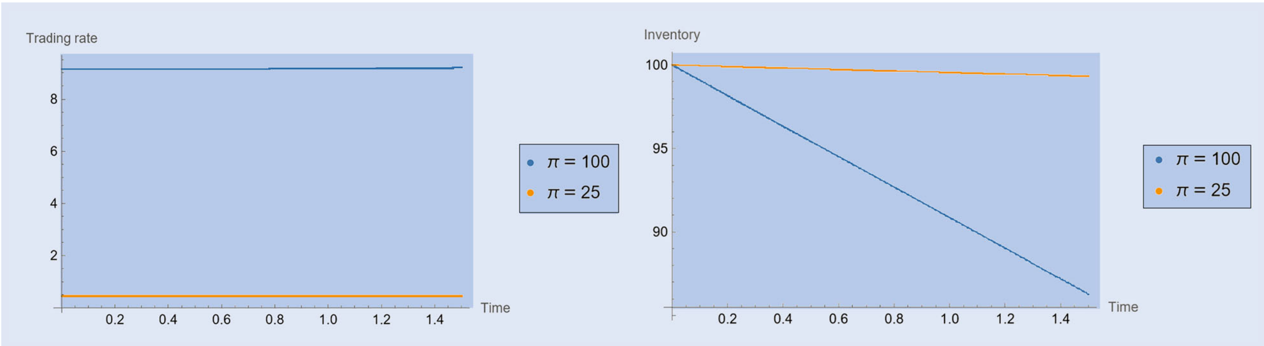

Figure 14. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a permanent (direct) price impact model with $\gamma = 3/2$ for various values of π with $\kappa = 25$, $\eta = 0.01$, $Q = 100$ and $T = 1.5$.

that the trend of the optimal strategy strongly depends on the intensity-related parameter (here η) (figure 15).

In between the temporary case $G \equiv \delta$ and the permanent one $G \equiv 1$, we now propose two different ways of describing a transient (direct) impact of the agent's transactions on the execution price.

2.3. *Transient impact function*

2.3.1. First alternative. We consider at first a direct impact kernel of the form $G(t-s) = \mathrm{e}^{-\beta(t-s)}$. Similarly as in section 2.2.1, we can rewrite equation (24) by taking S and Z as state variables for $t \leq \tau$,

$$\mathrm{d}X_t = \left(S_t - \kappa \int_0^t \mathrm{e}^{-\beta(t-s)} \nu_s^\gamma \,\mathrm{d}N_s\right) \nu_t \,\mathrm{d}t =: (S_t - Z_t)\nu_t \,\mathrm{d}t, \tag{31}$$

where $\mathrm{d}Z_t = -\beta Z_t \,\mathrm{d}t + \kappa \nu_t^\gamma \,\mathrm{d}N_t$ with $Z_0 = 0$ is again the cumulative negative impact from the trades on the transaction price. As in previous sections, we aim at maximizing the expected wealth of the agent given the set $\mathcal{A}$ of admissible trading strategies,

$$H^*(t,x,S,q,\lambda,Z) = \sup_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,x,S,q,\lambda,Z}\left[X_\tau + Q_\tau(S_\tau^{(\nu)} - \pi\lambda_\tau Q_\tau) - \phi \int_t^\tau Q_s^2 \,\mathrm{d}s\right].$$

Note that $S_\tau^{(\nu)} = S_\tau - Z_\tau$ is again used to define the terminal penalty since at the end of the trading horizon, the price at which we can sell the Q_τ remaining shares is $S_\tau^{(\nu)}$ (minus the penalty $\pi\lambda_T Q_\tau$) due to the transient direct price impact from G. We find the following proceeds $R_\tau^{(\nu)}$ using (31) and integration by parts,

$$\begin{aligned} R_\tau^{(\nu)} &= X_\tau + Q_\tau(S_\tau^{(\nu)} - \pi\lambda_\tau Q_\tau) \\ &= X_t + S_t Q_t - Z_t Q_t + \int_t^\tau Q_s(\sigma \,\mathrm{d}W_s + \beta Z_s \,\mathrm{d}s \\ &\quad - \kappa \nu_s^\gamma \,\mathrm{d}N_s) - \pi\lambda_\tau Q_\tau^2, \end{aligned} \tag{32}$$

since $\int_t^\tau Z_s \nu_s \,\mathrm{d}s = -Z_\tau Q_\tau + Z_t Q_t + \int_t^\tau Q_s(-\beta Z_s \,\mathrm{d}s + \kappa \nu_s^\gamma \,\mathrm{d}N_s)$ and $\int_t^\tau S_s \nu_s \,\mathrm{d}s = -S_\tau Q_\tau + S_t Q_t + \int_t^\tau Q_s \sigma \,\mathrm{d}W_s$. The term $\int_t^\tau Q_s \sigma \,\mathrm{d}W_s$ is again a martingale as $(Q_t)_t$ is bounded. We further have that $\mathbb{E}_{t,q,\lambda,Z}[\int_t^\tau \kappa Q_s \nu_s^\gamma \,\mathrm{d}N_s] = \mathbb{E}_{t,q,\lambda,Z}[\int_t^\tau \kappa Q_s \nu_s^\gamma \lambda_s \,\mathrm{d}s]$, see Lu (2013). We can hence rewrite H^* as

$$\begin{aligned} &H^*(t,x,S,q,\lambda,Z) \\ &= x + (S-Z)q + \sup_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau Q_s(\beta Z_s - \kappa \nu_s^\gamma \lambda_s)\,\mathrm{d}s \right. \\ &\quad \left. - \pi\lambda_\tau Q_\tau^2 - \phi \int_t^\tau Q_s^2 \,\mathrm{d}s\right] \\ &= x + S^{(\nu)} q - \inf_{(\nu_t)_t \in \mathcal{A}} \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau Q_s(\kappa \nu_s^\gamma \lambda_s - \beta Z_s)\,\mathrm{d}s \right. \\ &\quad \left. + \pi\lambda_\tau Q_\tau^2 + \phi \int_t^\tau Q_s^2 \,\mathrm{d}s\right] \end{aligned}$$

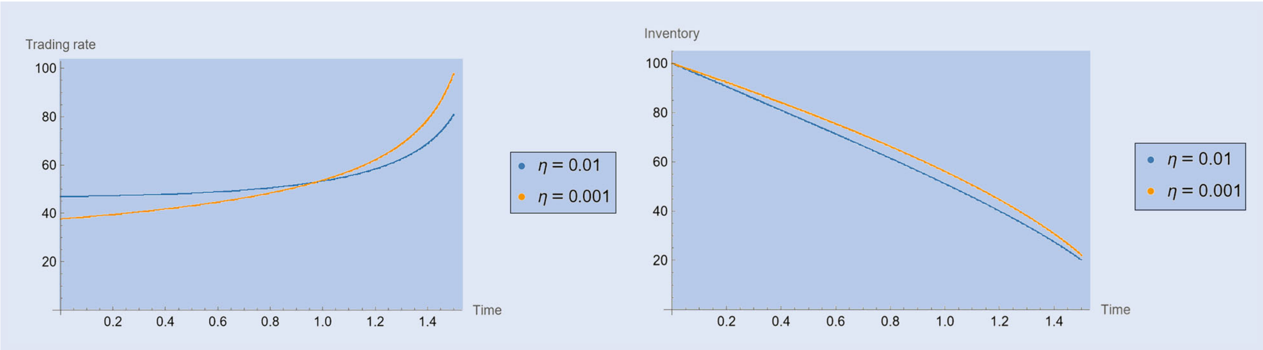

Figure 15. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a permanent (direct) price impact model with geometric intensity and $\gamma = 3/2$ for various values of η with $\kappa = 2$, $\pi = 15$, $Q = 100$ and $T = 1.5$.

$$
\begin{aligned}
&= x + S^{(\nu)}q - \inf_{(\nu_t)_t \in \mathcal{A}} h^{*,\nu}(t,q,\lambda,Z) \\
&= x + S^{(\nu)}q - h^*(t,q,\lambda,Z). \qquad (33)
\end{aligned}
$$

However, we cannot make use of the following HJB equation for $(t,q,\lambda,Z) \in \mathcal{S} \times \mathbb{R}^+$

$$
\begin{aligned}
&\partial_t h^*(t,q,\lambda,Z) + \phi q^2 - \beta Z \partial_Z h^*(t,q,\lambda,Z) - q\beta Z \\
&\quad + \min_{\nu \in \Gamma}\{(\eta\nu - \alpha\lambda)\partial_\lambda h^*(t,q,\lambda,Z) \\
&\quad - \nu \partial_q h^*(t,q,\lambda,Z) + q\kappa\nu^\gamma\lambda + \lambda[h^*(t,q,\lambda,Z+k\nu^\gamma) \\
&\quad - h^*(t,q,\lambda,Z)]\} = 0, \qquad (34)
\end{aligned}
$$

with $h^*(T,q,\lambda,Z) = \pi\lambda q^2$, $h^*(t,0,\lambda,Z) = 0$, $\lim_{\lambda\to 0} h^*(t,q,\lambda,Z) = 0$ and $\gamma > 1$. Indeed, sine the value function $h^{*,\nu}(t,q,\lambda,Z)$ is decreasing in Z, it is therefore not convex in ν. Consequently, ν^* is not a global minimum point of the argument in the minimization part of (34). Moreover, no explicit solution is available for the optimal trading rate ν^*, as the term $\kappa\nu^\gamma$ is now within $h^*(t,q,\lambda,\cdot)$ in (34). This makes the solving of (34) numerically very intensive and unstable when using classical numerical schemes such as finite differences or finite elements.

2.3.2. Second alternative. We now present some alternative (and more specific) assumptions on the transient direct price impact Z_t and on the terminal penalty, enabling us to obtain the optimal trading rate explicitly (and deterministically). In the wealth (31), we assume in a first step that the dynamic of the price impact Z_t is instead given by

$$
dZ_t = -\beta Z_{t^-}\,dt + \kappa Z_{t^-}\nu_t^\gamma\,dN_t, \qquad (35)
$$

with $\beta \geq 0$. Using Itô's lemma for jump processes, we obtain with $Z_0 > 0$,

$$
Z_t = Z_0\,e^{-\beta t}\prod_{0\leq s\leq t}^{\Delta N_s \neq 0}(1+\kappa\nu_s^\gamma),
$$

leading to an execution price of the form $S_t^{(\nu)} = S_t - Z_t$ (where Z_0 is the half bid-ask spread, assumed here to be non-zero). However, we can no longer write this execution price in the form (6) based on a propagator G. We further assume that the intensity $(\lambda_t)_t$ of the jump process $(N_t)_t$ follows the mean-reverting dynamic $d\lambda_t = (\eta\nu_t - \alpha\lambda_t)\,dt$ as in (8). We finally assume that the terminal penalty when $Q_T > 0$ is a given multiple π' of the total negative impact Z_T at time T. The penalized price $S_T^{(\pi')}$ at trading maturity T is thus defined formally as in (A2) by the random variable

$$
S_T^{\pi'} := S_T^{(\nu)} + dL_T^{(\pi')}/dt = S_T^{(\nu)} - \pi' Z_T Q_T \sum_{j=1}^{N_T}\delta(T-t_j), \qquad (36)
$$

where $L_T^{(\pi')} := -\pi'\int_0^T Q_s Z_{s^-}\,dN_s$ and with $(N_t)_t$ defined again by (8). We see in (36) that π is now replaced by $\pi' Z_T$. It makes then sense to impose $\pi' > 0$ since we expect additional penalties that the agent must incur for the execution of the trade at the terminal date, on top of the cumulative (transient) negative impact Z_T. Hence, we find as terminal penalty

$$
\begin{aligned}
\mathbb{E}_{t,x,S,q,\lambda,Z}[Q_T S_T^{(\pi')}] &= \mathbb{E}_{t,x,S,q,\lambda,Z}[Q_T(S_T^{(\nu)} - \pi' Z_T\lambda_T Q_T)] \\
&= \mathbb{E}_{t,x,S,q,\lambda,Z}[Q_\tau(S_\tau^{(\nu)} - \pi' Z_\tau\lambda_\tau Q_\tau)]. \qquad (37)
\end{aligned}
$$

By applying the exact same reasoning used for equation (32), but based on the dynamic (35) with $S_t^{(\nu)} = S_t - Z_t$, we derive the following optimization problem assuming $\phi = 0$ (no urgency to trade),

$$
\begin{aligned}
&H^*(t,x,S,q,\lambda,Z) \\
&= \sup_{(\nu_t)_t\in\mathcal{A}} \mathbb{E}_{t,x,S,q,\lambda,Z}[X_\tau + Q_\tau(S_\tau^{(\nu)} - \pi' Z_\tau\lambda_\tau Q_\tau)] \\
&= x + S^{(\nu)}q - \inf_{(\nu_t)_t\in\mathcal{A}} \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau Q_s(\kappa Z_s\nu_s^\gamma\lambda_s - \beta Z_s)\,ds \right. \\
&\quad \left. + \pi' Z_\tau\lambda_\tau Q_\tau^2\right] \\
&= x + S^{(\nu)}q - \inf_{(\nu_t)_t\in\mathcal{A}} h^{*,\nu}(t,q,\lambda,Z) \\
&= x + S^{(\nu)}q - h^*(t,q,\lambda,Z). \qquad (38)
\end{aligned}
$$

$h^*(t,q,\lambda,Z)$ is again finite for all $(t,q,\lambda,Z) \in \bar{\mathcal{S}} \times \mathbb{R}^+$ (as well as non-decreasing in Z and non-negative whenever $\beta \leq$

$\kappa\lambda_0\,\mathrm{e}^{-\alpha T}(\min_{s\in[t,\tau)}\nu_s^{\gamma})$). We then have that the value function $h^*(t,q,\lambda,Z)$ is a viscosity solution of the following HJB equation for $(t,q,\lambda,Z)\in\mathcal{S}\times\mathbb{R}^+$:

$$\begin{aligned}&\partial_t h^*(t,q,\lambda,Z)-\beta Z\partial_Z h^*(t,q,\lambda,Z)-q\beta Z-\alpha\lambda\partial_\lambda h^*\\&+\min_{\nu\in\Gamma}\{\eta\nu\partial_\lambda h^*-\nu\partial_q h^*+q\kappa Z\nu^{\gamma}\lambda\\&+\lambda[h^*(t,q,\lambda,Z+\kappa Z\nu^{\gamma})-h^*(t,q,\lambda,Z)]\}=0,\end{aligned}\tag{39}$$

with the new boundary and terminal conditions $h^*(T,q,\lambda,Z)=\pi'\lambda Zq^2$, $h^*(t,0,\lambda,Z)=0$, $h^*(t,q,\lambda,0)=0$ and $\lim_{\lambda\to 0}h^*(t,q,\lambda,Z)=0$. From the linearity of the value function (38) in Z, we then impose

$$h^*(t,q,\lambda,Z)=Zh_2^*(t,q,\lambda),$$

with $h_2^*(T,q,\lambda)=\pi'\lambda q^2$, $h_2^*(t,0,\lambda)=0$ and $\lim_{\lambda\to 0}h_2^*(t,q,\lambda)=0$. Indeed, in the previous sections 2.1 and 2.2 on temporary and permanent (direct) impacts, the value functions h and $\widetilde{h}$ could be seen as an extra penalty due to price impact stemming from optimally liquidating the rest of the shares. In this new transient setting where the terminal penalty is a multiple of the total negative impact, the value function h_2^* still represents the extra penalty from optimally selling the remaining shares, but expressed as a proportion of the total negative impact Z at that time. The value function $Zh_2^*(t,q,\lambda)$ is then again a viscosity solution for all $(t,q,\lambda,Z)\in\mathcal{S}\times\mathbb{R}^+$ of the PDE

$$\begin{aligned}&Z\partial_t h_2^*(t,q,\lambda)-\beta Zh_2^*(t,q,\lambda)-q\beta Z-\alpha\lambda Z\partial_\lambda h_2^*(t,q,\lambda)\\&+\min_{\nu\in\Gamma}\{\eta\nu Z\partial_\lambda h_2^*(t,q,\lambda)-\nu Z\partial_q h_2^*(t,q,\lambda)+q\kappa Z\nu^{\gamma}\lambda\\&+\lambda\kappa Z\nu^{\gamma}h_2^*(t,q,\lambda)\}=0,\end{aligned}$$

and therefore, h_2^* is the viscosity solution of

$$\begin{aligned}&\partial_t h_2^*(t,q,\lambda)-\beta h_2^*(t,q,\lambda)-q\beta-\alpha\lambda\partial_\lambda h_2^*(t,q,\lambda)\\&+\min_{\nu\in\Gamma}\{\eta\nu\partial_\lambda h_2^*(t,q,\lambda)-\nu\partial_q h_2^*(t,q,\lambda)\\&+\lambda\kappa\nu^{\gamma}(q+h_2^*(t,q,\lambda)))\}=0.\end{aligned}\tag{40}$$

It results that ν^* is a global minimum point of the minimization part of the PDE (40) in $\mathcal{S}$ provided that $\gamma>1$. We then have the following representation of the optimal trading rate for all $(t,q,\lambda)\in\mathcal{S}$ and $\gamma>1$,

$$\nu^*(t,q,\lambda)=\left(\frac{\partial_q h_2^*(t,\lambda,q)-\eta\partial_\lambda h_2^*(t,\lambda,q)}{\lambda\kappa\gamma(q+h_2^*(t,\lambda,q))}\right)^{\frac{1}{\gamma-1}}.\tag{41}$$

Again, we have that $\lambda\geq\lambda_{\min}>0$ due to the intensity (8) above, that $\lim_{q\to 0}\nu^*(t,q,\lambda)=+\infty$ and that $\nu^*(t,q,\lambda)=0$ for $t>\tau$. We finally obtain that h_2^* is a viscosity solution in $\mathcal{S}$ of the following explicit PDE when $\gamma>1$,

$$\begin{aligned}&\partial_t h_2^*(t,q,\lambda)-\beta(h_2^*(t,q,\lambda)+q)-\alpha\lambda\partial_\lambda h_2^*(t,q,\lambda)\\&-\frac{(\partial_q h_2^*(t,q,\lambda)-\eta\partial_\lambda h_2^*(t,q,\lambda))^{\frac{\gamma}{\gamma-1}}}{(\lambda\kappa\gamma(q+h_2^*(t,q,\lambda)))^{1/(\gamma-1)}}\left(\frac{\gamma-1}{\gamma}\right)=0.\end{aligned}$$

Choosing $\gamma=3/2$ as in section 2.2 on permanent impact is consistent with the square-root temporary impact and ensures the non-negativity and the admissibility of the optimal trading rate since $\nu^*(t,q,\lambda)=\frac{4}{9}(\frac{\partial_q h_2^*-\eta\partial_\lambda h_2^*}{\lambda\kappa(q+h_2^*)})^2$ and

$$\begin{aligned}&\partial_t h_2^*(t,q,\lambda)-\beta(h_2^*(t,q,\lambda)+q)-\alpha\lambda\partial_\lambda h_2^*(t,q,\lambda)\\&-\frac{4(\partial_q h_2^*(t,q,\lambda)-\eta\partial_\lambda h_2^*(t,q,\lambda))^3}{27(\lambda\kappa(q+h_2^*(t,q,\lambda)))^2}=0.\end{aligned}$$

To summarize, we obtain in this transient price impact model (35) a value function proportional to the total negative impact Z_t, which leads to the following optimal strategies. We again see in figure 16 that the intensity-related parameters strongly impact the trend of the optimal strategy. Moreover, when choosing in figure 17 a sufficiently small penalty κ to force full liquidation before T, the optimal strategy consists in selling most of the shares right at the beginning of the trading horizon. This figure confirms the exploding behavior of the optimal trading rate at the time of market drop-out $q=0$ when κ is really low (red line). Finally, the value of the drift β in (35) mostly influences the slope of the strategy at the beginning of the trading scheme, see figure 18. In section 2, we have so far considered that the liquidating agent could only post market (sell) orders. We now propose an extension of these price impact models that includes strategies using both market and limit orders.

3. Indirect optimal liquidation with limit and market orders

Therefore, we consider the possibility for the seller to also post limit sell orders (LOs) at a price $S_t+\delta_t$ that can be filled by MOs of buyers. The positive depth $(\delta_t)_t:=(\delta_t)_{t\in[0,T]}$ at which the LO is posted is then also a control process, with the aim of again maximizing the expected wealth of the selling agent. Based on Guéant and Lehalle (2015), Guéant (2016) and chapter 8.2 of Cartea *et al.* (2015), we expand the previous continuous setting $\mathrm{d}Q_t=-\nu_t\,\mathrm{d}t$ of section 2 by assuming instead

$$\mathrm{d}Q_t=-\nu_t\,\mathrm{d}t-\Delta\,\mathrm{d}M_t^{(\delta)}.\tag{42}$$

$M^{(\delta)}:=(M_t^{(\delta)})_{t\in[0,T]}$ is a new (controlled) counting process corresponding to the number of buy MOs which lift the agent's LO, i.e. MOs which walk the sell side of the book to a price greater than or equal to $S_t+\delta_t$. Each filled LO is of size $\Delta>0$. We consider that the number of arriving buy market orders (from other traders) follows a Poisson process $M^{(0)}$ with constant intensity Λ. The probability that the agent's LO at time t with depth δ_t will be lifted when a buy MO arrives (when $M_t^{(0)}$ jumps) is then assumed to be $\mathbb{P}(\delta_t)=\mathrm{e}^{-\rho\delta_t}$, $\rho>0$. As a result, the intensity of the counting process $M^{(\delta)}$ for filled limit orders is $\Lambda\,\mathrm{e}^{-\rho\delta_t}$. Besides, the model (42) now induces that the agent is continuously posting MOs at a rate $(\nu_t)_t$ and that her LOs at price $S_t+\delta_t$ are filled whenever $M_t^{(\delta)}$ jumps. For simplicity, we first assume as in section 2.1 that G is purely temporary such that the execution price is again given by $S_t^{(\nu)}=S_t-\kappa\nu_t^{\gamma}\,d\mathrm{d}N_t/\mathrm{d}t$, with $(N_t)_t$ the counting process defined by the mean-reverting

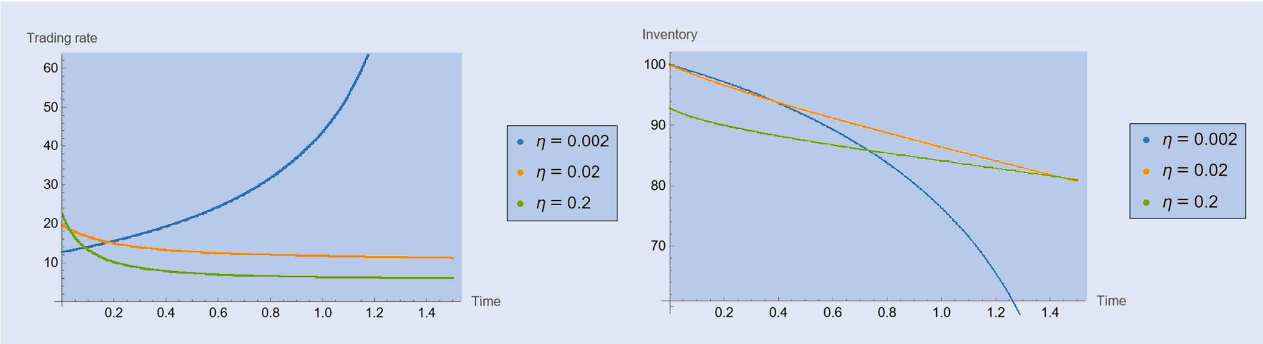

Figure 16. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a transient geometric (direct) price impact (35) with mean-reverting intensity and $\gamma = 3/2$ for various values of η with $\kappa = 0.006$, $\alpha = 0.5$, $\beta = 1$, $\pi' = 1000$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

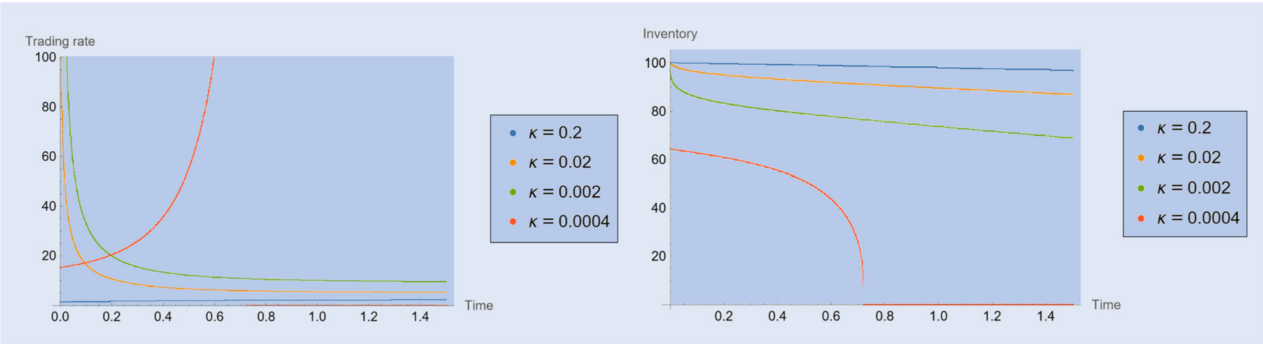

Figure 17. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a transient geometric (direct) price impact (35) with mean-reverting intensity and $\gamma = 3/2$ for various values of κ with $\alpha = 0.5$, $\eta = 0.1$, $\beta = 1$, $\pi' = 1000$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

intensity (8). The wealth process is then obtained by

$$\mathrm{d}X_t = (S_t + \delta_t)\Delta\, \mathrm{d}M_t^{(\delta)} + (S_t^{(\nu)} - \xi)\nu_t\, \mathrm{d}t,$$

where ξ is the half bid-ask spread (which is set to zero in the sequel, without loss of generality). We thus find for the wealth process at the last trading time τ on $[0, T]$,

$$X_\tau = X_t + \int_t^\tau (S_s + \delta_s)\Delta\, \mathrm{d}M_s^{(\delta)} + \int_t^\tau S_s \nu_s\, \mathrm{d}s - \int_t^\tau \kappa \nu_s^{\gamma+1}\, \mathrm{d}N_s. \tag{43}$$

The optimization problem becomes in this setting

$$\begin{aligned} H(t,x,S,q,\lambda) &= \sup_{((\nu_t)_t,(\delta_t)_t)\in\mathcal{A}'} H^{\nu,\delta}(t,x,S,q,\lambda) \\ &= \sup_{((\nu_t)_t,(\delta_t)_t)\in\mathcal{A}'} \mathbb{E}_{t,x,S,q,\lambda}\Big[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau) \\ &\qquad - \phi\int_t^\tau Q_s^2\, \mathrm{d}s\Big], \end{aligned}$$

where $\mathcal{A}'$ is defined as the set $\mathcal{A}$ in (5), enlarged with $\mathbb{F}$-predictable, bounded from below depths $(\delta_t)_t$. The notation $\mathbb{E}_{t,x,S,q,\lambda}$ represents expectation conditional on $X_{t^-} = x$, $S_t = S$, $Q_{t^-} = q$ and $\lambda_t = \lambda$. If the agent has inventory remaining at the end of the trading horizon, she liquidates it using a MO

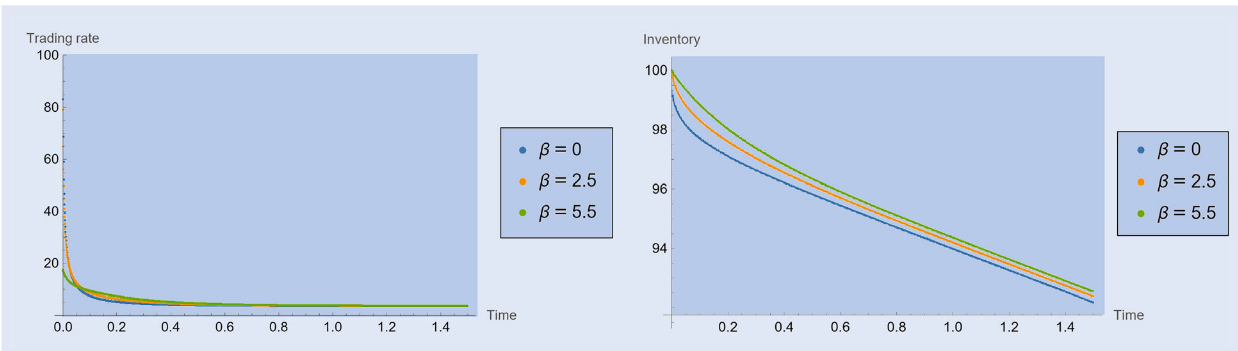

Figure 18. Optimal trading rate $(\nu_t^*)_t$ and inventory $(Q_t^{(\nu^*)})_t$ from a transient geometric (direct) price impact (35) with mean-reverting intensity and $\gamma = 3/2$ for various values of β with $\alpha = 1$, $\eta = 0.5$, $\kappa = 0.05$, $\pi' = 100$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

for which she obtains worse prices than the midprice (terminal penalty). As in section 2.1, we must again include the intensity λ_T in this terminal penalty, and the terminal price at which she can sell the Q_T shares is S_T (and not $S_T^{(\nu)}$) since the direct impact of MOs is here assumed to be temporary. Using integration by parts based on (42), we first find

$$\int_t^\tau S_s(\nu_s\,\mathrm{d}s + \Delta\,\mathrm{d}M_s^{(\delta)}) = -S_\tau Q_\tau + S_t Q_t + \int_t^\tau \sigma Q_s\,\mathrm{d}W_s.$$

Hence, we directly obtain that

$$\begin{aligned}
&\mathbb{E}_{t,x,S,q,\lambda}\left[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau) - \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right] \\
&= \mathbb{E}_{t,x,S,q,\lambda}\left[X_t + S_t Q_t + \int_t^\tau \sigma Q_s\,\mathrm{d}W_s - \int_t^\tau \kappa\nu_s^{\gamma+1}\,\mathrm{d}N_s \right.\\
&\quad \left. + \int_t^\tau \delta_s\Delta\,\mathrm{d}M_s^{(\delta)} - \pi\lambda_\tau Q_\tau^2 - \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right] \\
&= x + Sq - \mathbb{E}_{t,q,\lambda}\left[\int_t^\tau \kappa\nu_s^{\gamma+1}\lambda_s\,\mathrm{d}s \right.\\
&\quad \left. - \int_t^\tau \delta_s\Delta\Lambda\,\mathrm{e}^{-\rho\delta_s}\,\mathrm{d}s + \pi\lambda_\tau Q_\tau^2 + \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right] \\
&= x + Sq - h^{\nu,\delta}(t,q,\lambda),
\end{aligned}$$

which leads to

$$\begin{aligned}
H(t,x,S,q,\lambda) &= x + qS - \inf_{((\nu_t)_t,(\delta_t)_t)\in\mathcal{A}'} h^{\nu,\delta}(t,q,\lambda) \\
&= x + qS - h(t,q,\lambda). \qquad (44)
\end{aligned}$$

Given the admissible set $\mathcal{A}'$, $h(t,q,\lambda)$ is again bounded on $\bar{\mathcal{S}}$. Following section 2.1 with the PDE (15) and from Guéant and Lehalle (2015) (see also Chapter 8.4 of Cartea *et al.* 2015), we have that h is a viscosity solution of the following HJB equation for all $(t,q,\lambda)\in\mathcal{S}$:

$$\begin{aligned}
&\partial_t h(t,q,\lambda) + \phi q^2 - \alpha\lambda\partial_\lambda h(t,q,\lambda) \\
&\quad + \min_{\nu\in\Gamma}\{\eta\nu\partial_\lambda h(t,q,\lambda) - \nu\partial_q h(t,q,\lambda) + \kappa\nu^{\gamma+1}\lambda\} \\
&\quad + \min_{\delta\in\Gamma'}\{\Lambda\,\mathrm{e}^{-\rho\delta}[h(t,q-\Delta,\lambda) - h(t,q,\lambda)] - \delta\Delta\Lambda\,\mathrm{e}^{-\rho\delta}\} \\
&= 0, \qquad (45)
\end{aligned}$$

with terminal condition $h(T,q,\lambda) = \pi\lambda q^2$ and boundary conditions $h(t,0,\lambda) = 0$, $\lim_{\lambda\to 0} h(t,q,\lambda) = 0$. Finally, from the first order condition in (45) with $\gamma > 0$, we find the exact same expression for ν^* as equation (16),

$$\nu^*(t,q,\lambda) = \left(\frac{\partial_q h(t,q,\lambda) - \eta\partial_\lambda h(t,q,\lambda)}{(\gamma+1)\lambda\kappa}\right)^{1/\gamma}, \qquad (46)$$

with $\nu^*(t,q,\lambda) = 0$ for $t > \tau$, and for the optimal depth δ^* we obtain

$$\delta^*(t,q,\lambda) = \frac{1}{\rho} + \frac{h(t,q-\Delta,\lambda) - h(t,q,\lambda)}{\Delta}.$$

Substituting these optimal strategies back into the HJB equation (45), we finally find that h is a viscosity solution for all $(t,q,\lambda)\in\mathcal{S}$ and $\gamma > 0$ of the explicit PDE

$$\begin{aligned}
&\partial_t h(t,q,\lambda) + \phi q^2 - \alpha\lambda\partial_\lambda h(t,q,\lambda) \\
&\quad - \frac{(\partial_q h(t,q,\lambda) - \eta\partial_\lambda h(t,q,\lambda))^{\frac{\gamma+1}{\gamma}}}{(\lambda\kappa(\gamma+1))^{1/\gamma}}\left(\frac{\gamma}{\gamma+1}\right) \\
&\quad - \frac{\Lambda\,\mathrm{e}^{-1}\Delta}{\rho}\exp\left(-\rho\left[\frac{h(t,q-\Delta,\lambda) - h(t,q,\lambda)}{\Delta}\right]\right) = 0, \qquad (47)
\end{aligned}$$

which can be solved numerically in a very efficient way using the method of lines (with a delayed PDE in q). Again, choosing the square root law $\gamma = 1/2$ appears to be the most consistent choice in the temporary case. As the remaining inventory q now depends on the counting process $M^{(\delta)}$ through (42), the optimal trading rate $\nu^*(t,q,\lambda)$ and the optimal depth $\delta^*(t,q,\lambda)$ in the LOB are not deterministic anymore. In summary, we now have a more realistic price impact model where the liquidating agent can both use MOs and LOs. We then obtain in figure 19 the following optimal strategies $(\nu_t^*)_t$ and $(\delta_t^*)_t$ for various sets of parameters.

We see on the first plot that the optimal trading rate $(\nu_t^*)_t$ is logically increasing with time: the closer the maturity, the more the agent must trade to avoid the terminal penalty. It is also increasing with the remaining inventory: the larger the inventory q, the more the agent needs to trade to liquidate the remaining shares. In contrast, the optimal depth $(\delta_t^*)_t$ is instead decreasing in time due to the agent becoming more averse to holding inventories as the trading horizon approaches, and also decreasing in inventory. Indeed, if the agent's inventory is large, she is willing to accept a lower premium δ for providing liquidity so as to increase the probability that her order is filled. At the same time, this ensures that she may complete the liquidation of the Q shares before T and to avoid paying a terminal penalty. However, if inventories are low, the agent prefers setting a large depth δ because with low inventory, the terminal penalty she incurs when crossing the spread with MOs will be moderate. Moreover, this setting still reproduces the empirical observations of section 1, as already discussed at the end of sections 2.1.1 and 2.1.2. Finally, figure 20 above confirms that the optimal trading rate and optimal depth both depend on the jumps of the counting process $M^{(\delta)}$ and are therefore not deterministic anymore.

This framework can easily be extended to a more general price impact model $S_t^{(\nu)} = S_t - Z_t$, with Z_t the cumulative price impact at t, by using instead a wealth process that depends on $(S_t^{(\nu)})_t$, i.e.:

$$\mathrm{d}X_t = (S_t^{(\nu)} + \delta_t)\Delta\,\mathrm{d}M_t^{(\delta)} + (S_t^{(\nu)} - \xi)\nu_t\,\mathrm{d}t.$$

We can then maximize the expected wealth at τ (with $S_\tau^{(\nu)}$ in the terminal penalty),

$$\begin{aligned}
&\mathbb{E}_{t,x,S,q,\lambda,Z}\left[X_\tau + Q_\tau(S_\tau^{(\nu)} - \pi\lambda_\tau Q_\tau) - \phi\int_t^\tau Q_s^2\,\mathrm{d}s\right] \\
&\quad = x + S^{(\nu)}q - \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau Q_{s^-}\,\mathrm{d}Z_s - \int_t^\tau \delta_s\Delta\Lambda\,\mathrm{e}^{-\rho\delta_s}\,\mathrm{d}s\right.
\end{aligned}$$

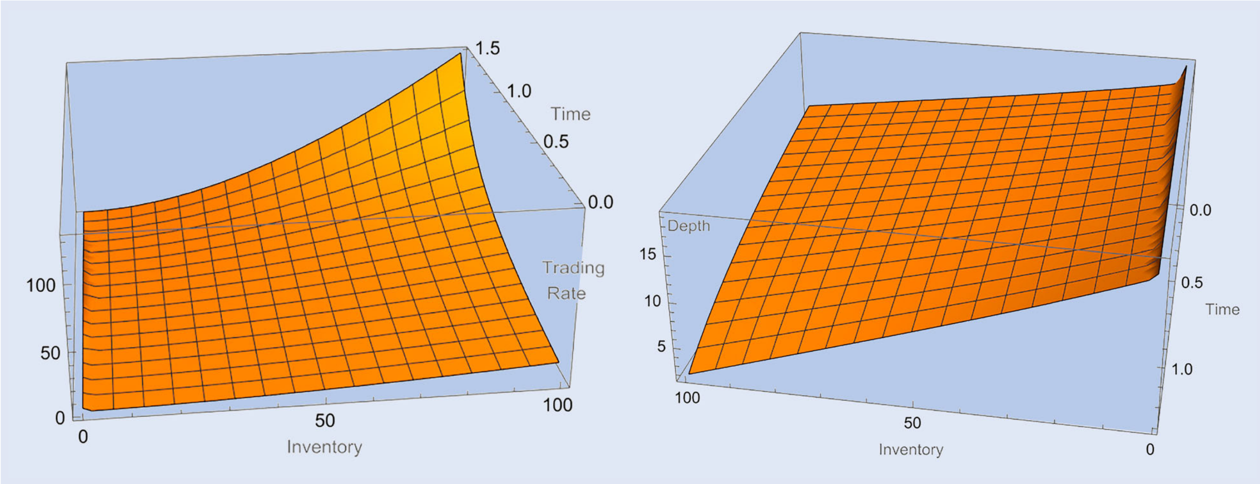

Figure 19. Optimal trading rate $(\nu_t^*)_t$ (left plot) and depth $(\delta_t^*)_t$ (right plot) in function of the remaining inventory q and time t at $\lambda = 2$ from a temporary (direct) price impact model allowing MOs and LOs with $\gamma = 1/2$, $\alpha = 0.5$, $\eta = 0.01$, $\kappa = 0.5$, $\pi = 0.05$, $\phi = 0$, $\Lambda = 1$, $\rho = 0.08$, $\Delta = 1$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

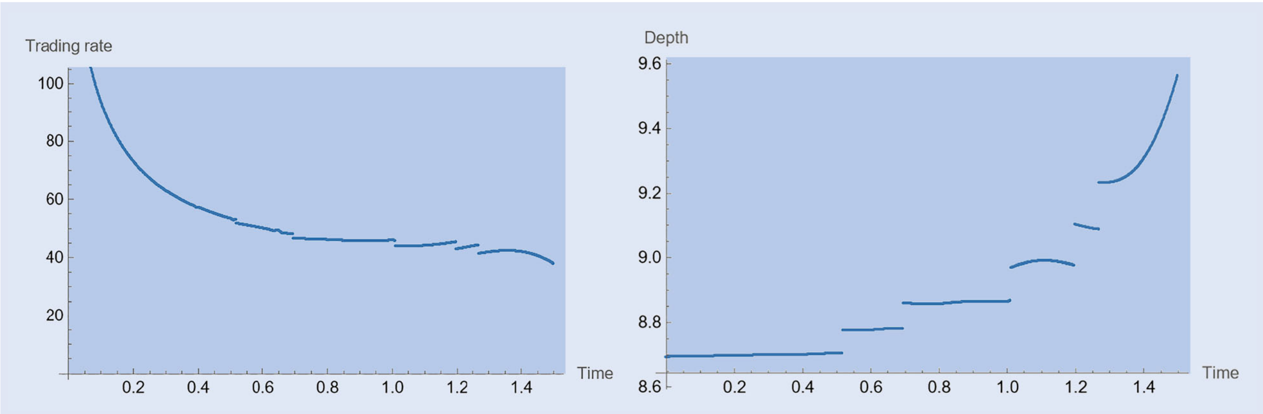

Figure 20. Optimal strategy of trading rate $(\nu_t^*)_t$ (left plot) and depth $(\delta_t^*)_t$ (right) from a temporary price impact model allowing MOs and LOs with $\gamma = 1/2$, $\alpha = 0.5$, $\eta = 0.01$, $\kappa = 0.5$, $\pi = 0.1$, $\phi = 0$, $\Lambda = 1$, $\rho = 0.08$, $\Delta = 1$, $\lambda_0 = 0.5$, $Q = 100$, $T = 1.5$.

$$\left. + \pi\lambda_\tau Q_\tau^2 + \phi \int_t^\tau Q_s^2\,\mathrm{d}s \right]$$
$$= x + S^{(\nu)} q - \widetilde{h}^{\nu,\delta}(t, q, \lambda, Z).$$

In the permanent case 2.2 with $G \equiv 1$, we find for instance

$$\widetilde{h}^{\nu,\delta}(t, q, \lambda, Z) = \mathbb{E}_{t,q,\lambda}\left[\int_t^\tau \kappa Q_s \nu_s^\gamma \lambda_s\,\mathrm{d}s - \int_t^\tau \delta_s \Delta\Lambda\,\mathrm{e}^{-\rho\delta_s}\,\mathrm{d}s \right.$$
$$\left. + \pi\lambda_\tau Q_\tau^2 + \phi \int_t^\tau Q_s^2\,\mathrm{d}s \right].$$

In conclusion, this section extends our indirect price impact model with limit orders. This results in more elaborated and realistic strategies, which are financially consistent and not deterministic anymore.

4. Estimation and limitations

As the estimation methodology will be addressed in future research, we only provide here a brief outline of the procedure. The first step is to estimate the impact parameters κ and γ in (7) such as to reproduce the power-law behavior observed in figures 1 and 2. These parameters are thus chosen to minimize the distance between the power-law function $\kappa\nu^\gamma$ and the black (wider) line of figure 2, obtained from the time series of AMZN's LOB data on a sufficiently refined time-grid $0 = t_0, \ldots, t_n = T$. In the second step, we need to disentangle the variations in the execution price $(S_t^{(\nu)})_t$ in (6) caused by the Brownian motion $(\sigma W_t)_t$, from those due to the jump process $(L_t)_t$. This can be done straightforwardly using the version of the Peaks-Over-Threshold (POT) procedure described in Hainaut and Moraux (2018) or Dupret and Hainaut (2022). This POT procedure provides an estimate of σ and detects the jump times $\{\tau_1, \ldots, \tau_m\}$ of the process $(L_t)_t$. Note that the definition of a jump will depend on the chosen propagator. For instance, in the temporary setting 2.1, a variation of the execution price at time t_j is considered as a jump if $S_{t_j}^{(\nu)} - S_{t_{j-1}}^{(\nu)} < g(p)$ and if $|S_{t_{j+1}}^{(\nu)} - S_{t_{j-1}}^{(\nu)}| < \varepsilon$, for some $\varepsilon \geq 0$ and for $g(p)$ the POT threshold defined in Hainaut and Moraux (2018) as a function of the p-percentile of the Brownian motion $(W_t)_t$. In the permanent case 2.2, a jump is instead simply defined if $S_{t_j}^{(\nu)} - S_{t_{j-1}}^{(\nu)} < g(p)$. In the final step, we then want to estimate the parameters (α, η) of the intensity process (8) (or just η for the geometric intensity (19)). We then use the fact that, given a trading strategy $(\nu_t)_t$, the counting process $(N_t)_t$ is a non-homogenous Poisson process with log-likelihood function

$$\ell(\{\tau_1,\dots,\tau_m\}\,|\,\alpha,\eta) = \sum_{i=1}^{m} \log \lambda(\tau_i\,|\,\alpha,\eta) - \int_0^T \lambda(s\,|\,\alpha,\eta)\,\mathrm{d}s.$$

The maximum likelihood estimate of α, η can then be found as

$$\hat{\alpha},\hat{\eta} = \arg\max_{\alpha,\eta} \ell(\{\tau_1,\dots,\tau_m\}\,|\,\alpha,\eta).$$

Therefore, given a past implemented strategy $(\nu_{t_j})_{j=1,\dots,n}$ by the liquidating trader on the considered time horizon $[0,T]$ with observed execution prices $(S^{(\nu)}_{t_j})_{j=1,\dots,n}$, we can estimate all of the model parameters.

However, our indirect price impact model still suffers from several limitations. First, our estimation procedure should be detailed and applied based on real LOB data, in order to determine which kernel G (and which exponent γ) best fit the data, provided that high-frequency price data of sufficient quality can be obtained. Back-testing our optimal execution strategies on real market data is also necessary in order to evaluate the performance of our model relative to standard industry models, see Almgren and Chriss (2001), Almgren (2003). Secondly, we should incorporate a short-term alpha signal with discretized MOs' trading rates into our dynamic execution schedule in order to obtain more realistic and implementable liquidation strategies for real-time decision making, see Fouque *et al.* (2022). In our setting, we could make the intensity $(\lambda_t)_t$ depend on the volume posted on both sides of the LOB. Indeed, Cartea *et al.* (2018) build such a measure of volume imbalance in the LOB and show that it is a good predictor of buying and selling pressures in the market. As the intensity $(\lambda_t)_t$ precisely models this pressure from net order flow imbalance, making it depend on the volume imbalance measure of Cartea *et al.* (2018) should allow us to easily incorporate signals into our strategies. Moreover, deep learning methods should be investigated and applied for solving high-dimensional HJB equations without explicit solutions, as in (34) or (A5). Finally, price manipulation with the no-dynamic-arbitrage properties of the introduced indirect price impact model should also be assessed for the various propagators G studied in this paper.

5. Conclusion

This paper introduces a new jump-dependent price impact model with propagator allowing to best reproduce the empirical direct and indirect effects of market orders on the transaction price, while including a stochastic liquidity factor in the form of a trade-driven intensity process. More precisely, the jump process $(N_t)_t$ is used in our framework to model the occurrence of the (temporary/permanent/transient) price impacts arising from MOs, since all trades do not lead to a direct negative impact on the transaction price (even if they systematically exert an indirect downward pressure on this price). This indirect downward price effect from order flow imbalance is captured through an increased mean-reverting intensity $(\lambda_t)_t$ of this jump process $(N_t)_t$ with negative jump size, which then scales as a power-law function of the trading rate. Moreover, as the direct impact of market orders depends on the propagator G in (6), we discuss for each choice of G the most relevant optimal liquidation problem faced by the agent, derive the related HJB equation and solve it numerically. In the temporary case $G \equiv \delta$, we find that a square-root impact provides the most consistent results. In this scenario, we obtain a deterministic decreasing optimal trading strategy that can be set up at time $t = 0$. Moreover, when choosing a geometric intensity (19), we show that the dependence of this optimal strategy on the intensity disappears. Secondly, in the permanent setting (24) with $G \equiv 1$ and exponent $\gamma = 3/2$, we again find a deterministic optimal trading strategy whose trend strongly varies according to the model parameters and which does not depend on λ when considering the geometric intensity (19). At last, in the transient case, alternative assumptions on the direct price impact and on the terminal penalty have to be assumed in order to obtain the optimal trading rate explicitly (and deterministically). In both the permanent and the transient settings, we obtain exploding training rates at the time of market drop-out $q = 0$. Finally, these three scenarios based solely on MOs can be expanded by allowing the agent to post limit orders in the LOB at a depth δ, which is also a control process to be optimized. Optimal strategies of trading rates $(\nu^*_t)_t$ and depths $(\delta^*_t)_t$ are again obtained explicitly in this enhanced setting, leading to financially sound results. Our indirect price impact model with propagator therefore offers sufficient tractability and flexibility to best reproduce the observed underlying dynamic of the LOB, thereby bridging the gap with high-frequency price models.

Acknowledgments

The authors thank the two anonymous referees for helpful comments related to this work.

Data availability statement

The datasets and code generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by the FNRS, Belgium [grant number 33658713].

ORCID

Jean-Loup Dupret http://orcid.org/0000-0002-3716-8945

References

Abergel, F. and Jedidi, A., A mathematical approach to order book modeling. *Int. J. Theor. Appl. Finan.*, 2013, **16**(5), 1350025.
Abi Jaber, E. and Neuman, E., Optimal liquidation with signals: The general propagator case. Available at SSRN, 2022.
Alfonsi, A. and Blanc, P., Dynamic optimal execution in a mixed-market-impact hawkes price model. *Finance Stoch.*, 2016, **20**(1), 183–218.
Alfonsi, A., Fruth, A. and Schied, A., Optimal execution strategies in limit order books with general shape functions. *Quant. Finance*, 2010, **10**(2), 143–157.
Almgren, R., Optimal execution with nonlinear impact functions and trading-enhanced risk. *Appl. Math. Finance*, 2003, **10**(1), 1–18.
Almgren, R. and Chriss, N., Value under liquidation. *Risk*, 1999, **12**(12), 61–63.
Almgren, R. and Chriss, N., Optimal execution of portfolio transactions. *J. Risk*, 2001, **3**(2), 5–39.
Almgren, R., Thum, C., Hauptmann, E. and Li, H., Direct estimation of equity market impact. *Risk*, 2005, **18**(7), 58–62.
Ankirchner, S., Jeanblanc, M. and Kruse, T., Bsdes with singular terminal condition and a control problem with constraints. *SIAM J. Control. Optim.*, 2014, **52**(2), 893–913.
Bacry, E. and Muzy, J.-F., Hawkes model for price and trades high-frequency dynamics. *Quant. Finance*, 2014, **14**(7), 1147–1166.
Bayraktar, E. and Ludkovski, M., Liquidation in limit order books with controlled intensity. *Math. Finance*, 2014, **24**(4), 627–650.
Bertsimas, D. and Lo, A.W., Optimal control of execution costs. *J. Financ. Mark.*, 1998, **1**(1), 1–50.
Bouchard, B., Introduction to stochastic control of mixed diffusion processes, viscosity solutions and applications in finance and insurance. *Lecture Notes Preprint*, 2007.
Bouchard, B. and Touzi, N., Weak dynamic programming principle for viscosity solutions. *SIAM J. Control Optim.*, 2011, **49**(3), 948–962.
Bouchaud, J.-P., Gefen, Y., Potters, M. and Wyart, M., Fluctuations and response in financial markets: The subtle nature of 'random' price changes. *Quant. Finance*, 2003, **4**(2), 176–190.
Bouchaud, J.-P., Bonart, J., Donier, J. and Gould, M., *Trades, Quotes and Prices: Financial Markets under the Microscope*, 2018 (Cambridge University Press: Cambridge).
Cartea, Á. and Jaimungal, S., Incorporating order-flow into optimal execution. *Math. Financ. Econ.*, 2016, **10**, 339–364.
Cartea, Á., Jaimungal, S. and Ricci, J., Buy low, sell high: A high frequency trading perspective. *SIAM J. Financ. Math.*, 2014, **5**(1), 415–444.
Cartea, Á., Jaimungal, S. and Penalva, J., *Algorithmic and High-Frequency Trading*, 2015 (Cambridge University Press: Cambridge).
Cartea, Á., Donnelly, R. and Jaimungal, S., Algorithmic trading with model uncertainty. *SIAM J. Financ. Math.*, 2017, **8**(1), 635–671.
Cartea, A., Donnelly, R. and Jaimungal, S., Enhancing trading strategies with order book signals. *Appl. Math. Finance*, 2018, **25**(1), 1–35.
Cont, R. and De Larrard, A., Price dynamics in a Markovian limit order market. *SIAM J. Financ. Math.*, 2013, **4**(1), 1–25.
Cont, R., Kukanov, A. and Stoikov, S., The price impact of order book events. *J. Financ. Economet.*, 2014, **12**(1), 47–88.
Curato, G., Gatheral, J. and Lillo, F., Optimal execution with non-linear transient market impact. *Quant. Finance*, 2017, **17**(1), 41–54.
Da Fonseca, J. and Malevergne, Y., A simple microstructure model based on the cox-besq process with application to optimal execution policy. *J. Econ. Dyn. Control*, 2021, **128**, 104137.
Donnelly, R., Optimal execution: A review. *Appl. Math. Finance*, 2022, **29**(3), 181–212.
Dupret, J.-L. and Hainaut, D., A fractional hawkes process for illiquidity modeling. Available at SSRN 4162342, 2022.
Forsyth, P.A., Kennedy, J.S., Tse, S.T. and Windcliff, H., Optimal trade execution: A mean quadratic variation approach. *J. Econ. Dyn. Control*, 2012, **36**(12), 1971–1991.
Fouque, J.-P., Jaimungal, S. and Saporito, Y.F., Optimal trading with signals and stochastic price impact. *SIAM J. Financ. Math.*, 2022, **13**(3), 944–968.
Fruth, A., Schöneborn, T. and Urusov, M., Optimal trade execution and price manipulation in order books with time-varying liquidity. *Math. Finance*, 2014, **24**(4), 651–695.
Fruth, A., Schöneborn, T. and Urusov, M., Optimal trade execution in order books with stochastic liquidity. *Math. Finance*, 2019, **29**(2), 507–541.
Gareche, A., Disdier, G., Kockelkoren, J. and Bouchaud, J.-P., Fokker-planck description for the queue dynamics of large tick stocks. *Phys. Rev. E*, 2013, **88**(3), 032809.
Gatheral, J., No-dynamic-arbitrage and market impact. *Quant. Finance*, 2010, **10**(7), 749–759.
Gatheral, J. and Schied, A., *Dynamical Models of Market Impact and Algorithms for Order Execution*, pp. 579–599, 2013 (Cambridge University Press: Cambridge).
Gökay, S., Roch, A.F. and Soner, H.M., *Liquidity Models in Continuous and Discrete Time*, pp. 333–365, 2011 (Springer: Berlin Heidelberg).
Graewe, P. and Horst, U., Optimal trade execution with instantaneous price impact and stochastic resilience. *SIAM J. Control Optim.*, 2017, **55**(6), 3707–3725.
Guéant, O., *The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making*, vol. 33, 2016 (CRC Press: Boca Raton, FL).
Guéant, O. and Lehalle, C.-A., General intensity shapes in optimal liquidation. *Math. Finance*, 2015, **25**(3), 457–495.
Hainaut, D. and Moraux, F., Hedging of options in the presence of jump clustering. *J. Comput. Finance*, 2018, **22**(3), 1–35.
Huang, W., Lehalle, C.-A. and Rosenbaum, M., Simulating and analyzing order book data: The queue-reactive model. *J. Am. Stat. Assoc.*, 2015, **110**(509), 107–122.
Huberman, G. and Stanzl, W., Price manipulation and quasi-arbitrage. *Econometrica*, 2004, **72**(4), 1247–1275.
Kenyon, C. and Green, A., Dirac processes and default risk. *arXiv preprint arXiv:1504.04581*, 2015.
Lu, B., Optimal selling of an asset with jumps under incomplete information. *Appl. Math. Finance*, 2013, **20**(6), 599–610.
Obizhaeva, A.A. and Wang, J., Optimal trading strategy and supply/demand dynamics. *J. Financ. Mark.*, 2013, **16**(1), 1–32.
Sadoghi, A. and Vecer, J., Optimal liquidation problem in illiquid markets. *Eur. J. Oper. Res.*, 2022, **296**(3), 1050–1066.
Schwartz, L., *Théorie des distributions*. Publ. de l'Inst. de Math. de l'Univ. de Strasbourg. Hermann, 1966.
Stein, H.J., Cohen, A. and Costanzino, N., A unified framework for default modeling. Available at SSRN, 2022.
Torre, N.*Barra Market Impact Model Handbook*, 1997 (BARRA Inc.: Berkeley, 208).

Appendix 1. Dirac's delta propagator

This shows the equivalence in the temporary case between the execution price (6) with Dirac propagator $G(t-s) = \delta(t-s)$ and the wealth process introduced in (10). We can indeed write formally in this case the execution price

$$S_t^{(\nu)} = S_t + \int_0^t G(t-s)\,\mathrm{d}L_s = S_t - \kappa \int_0^t G(t-s)\nu_s^{\gamma}\,\mathrm{d}N_s$$

$$= S_t - \kappa \sum_{j=1}^{N_\tau} \nu_{t_j}^\gamma \delta(t - t_j)$$

$$= S_t + \mathrm{d}L_t/\mathrm{d}t, \quad \text{(A1)}$$

where $(t_j)_{j=1,\ldots,N_\tau^\nu}$ are the jump times of $(N_t)_t$ over $[0, \tau]$. This indeed leads to the expression (10) for the wealth process $X_\tau = \int_0^\tau S_s \nu_s ds + \int_0^\tau \frac{\mathrm{d}L_s}{\mathrm{d}s} \nu_s \,\mathrm{d}s = \int_0^\tau S_s \nu_s \,\mathrm{d}s - \int_0^\tau \kappa \nu_s^{\gamma+1} \,\mathrm{d}N_s$. Such process $(S_t^{(\nu)})_t$ for the execution price (A1) is sometimes called a '*Dirac process*' and has already been investigated in Kenyon and Green (2015) and Stein *et al.* (2022) as arising from the time derivative of a Poisson process (in a similar way as for impulse control problems). This form of the execution price $S_t^{(\nu)}$ is not interesting per se and cannot be used to define our model: just as a Brownian motion alone is not particularly useful, a Dirac process is only useful for what can be done with it. However, this price impact model (A1) is justified by the fact that it is consistent with the definition of the wealth process (10), which is the only reasonable form for $(X_t)_t$ in the temporary case and which is the process of interest in the optimization program (13) (the state variables in the temporary setting are indeed S and x, but never $S^{(\nu)}$).

We also derive the terminal penalty $Q_T(S_T - \pi\lambda_T Q_T)$ used in the optimization problem (13). In accordance with the existing literature where the temporary price impact $\kappa \nu_t^\gamma$ is replaced by a penalty πQ_T at terminal time T (see Cartea *et al.* 2015, Donnelly 2022), we introduce our penalized price $S_T^{(\pi)}$ at T as a sum of Dirac's delta according to the execution price (A1), but with jump size $-\pi Q_T$ instead of $-\kappa \nu_t^\gamma$. Formally, this penalized price is defined as

$$S_T^{(\pi)} := S_T + \mathrm{d}L_T^{(\pi)}/\mathrm{d}t = S_T - \pi Q_T \sum_{j=1}^{N_T} \delta(T - t_j), \quad \text{(A2)}$$

where $L_T^{(\pi)} := -\pi \int_0^T Q_s \,\mathrm{d}N_s$ is a new random variable for the terminal penalty at time T and with $(N_t)_t$ the counting process (8). Note that even when the direct price impact is not temporary but permanent or transient (see sections 2.2 and 2.3 with other kernels G), the terminal penalty remains a temporary direct impact with Dirac delta $\delta(T - t_j)$, since it only affects the price of the transaction at final time T and not the other ones. With a penalized price of the form (A2), we retrieve the terminal penalty

$$\mathbb{E}_{t,x,S,q,\lambda}[Q_T S_T^{(\pi)}] = \mathbb{E}_{t,x,S,q,\lambda}[Q_T(S_T - \pi\lambda_T Q_T)]$$
$$= \mathbb{E}_{t,x,S,q,\lambda}[Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau)]. \quad \text{(A3)}$$

Again, the expression (A2) cannot be used to define our model but is induced and justified by the chosen form of the terminal penalty in the optimization (13).

Appendix 2. Alternative risk-aversion penalty

As already explained in section 2, the term $\int_t^\tau Q_u^2 \,\mathrm{d}u$ in (13) can be seen as a form of risk-aversion (urgency to trade) of the agent. We now introduce an alternative form of the agent's risk aversion, based on an expected shortfall penalty, which limits the potential loss on the total revenue of the liquidating agent by the end of the trading τ. For simplicity, we focus on the temporary setting from section 2.1.1, although the following results can be directly extended to other propagators G. We hence suppose the following performance criterion

$$H^\nu(t, x, S, q, \lambda) = \mathbb{E}_{t,x,S,q,\lambda}[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau)$$
$$- \phi[K' - X_\tau]^+]. \quad \text{(A4)}$$

We hence penalize the agent only when the revenue X_τ at the last trading time τ is below a threshold K'. This penalization term $-\phi[K' - x]^+$ is Lipschitz continuous and concave in x. It can then be shown that the bounded value function $h(t, x, S, q, \lambda) = \sup_{(\nu_t)_t \in \mathcal{A}} H^\nu(t, x, S, q, \lambda)$ must be a viscosity solution of the following HJB equation for all $(t, q, \lambda, x, S) \in \mathcal{S} \times \mathbb{R} \times \mathbb{R}$:

$$\partial_t h + \tfrac{1}{2}\sigma^2 \partial_{SS}^2 h + \sup_{\nu \in \Gamma}\{(\eta\nu - \alpha\lambda)\partial_\lambda h - \nu\partial_q h$$
$$+ \nu S \partial_x h + \lambda[h(t, x - \kappa\nu^{\gamma+1}, S, q, \lambda) - h(t, x, S, q, \lambda)]\} = 0, \quad \text{(A5)}$$

with $h(T, x, S, q, \lambda) = x - \phi[K' - x]^+ + qS - \pi\lambda q^2$. This 5-dimensional HJB equation is however hard to solve numerically since an explicit solution for $(\nu_t^*)_t$ is not available. We therefore consider instead the following performance criterion

$$H^\nu(t, x, S, q, \lambda) = \mathbb{E}_{t,x,S,q,\lambda}\left[X_\tau + Q_\tau(S_\tau - \pi\lambda_\tau Q_\tau)\right.$$
$$\left. - \phi \int_t^\tau \nu_u [K - S_u]^+ \mathrm{d}u\right], \quad \text{(A6)}$$

with $h(t, x, S, q, \lambda) = \sup_{(\nu_t)_t \in \mathcal{A}} H^\nu(t, x, S, q, \lambda)$. Indeed, this running penalty implies that if $(K' - X_\tau) < \int_t^\tau \nu_u(K - S_u)\,\mathrm{d}u$, then we have

$$(K' - X_\tau)^+ \le \left(\int_t^\tau \nu_u(K - S_u)\,\mathrm{d}u\right)^+ \le \int_t^\tau \nu_u(K - S_u)^+ \mathrm{d}u,$$

and thus that

$$\phi(K' - X_\tau)^+ \le \phi \int_t^\tau \nu_u(K - S_u)^+ \mathrm{d}u,$$

which ensures that the penalty in (A6) is at least larger than the shortfall penalty (A4). This shows that the optimization (A6) also limits the loss on the revenue X_τ at the final trading time τ. We therefore need to impose

$$K' - \int_t^\tau S_u \nu_u \,\mathrm{d}u + \int_t^\tau \kappa \nu_u^{\gamma+1} \,\mathrm{d}N_u < \int_t^\tau \nu_u(K - S_u)\,\mathrm{d}u$$
$$\Leftrightarrow K' < \int_t^\tau K\nu_u \,\mathrm{d}u - \int_t^\tau \kappa\nu_u^{\gamma+1} \,\mathrm{d}N_u,$$

which is verified when assuming

$$K > \frac{K' + \kappa\left(\max_{s \in [t,\tau]} \nu_s^{\gamma+1}\right)(N_\tau - N_t)}{Q_t - Q_\tau}, \quad \text{(A7)}$$

since $(\nu_t)_t \in \mathcal{A}$ is non-negative and bounded on $[0, \tau)$, $\gamma > 0$. The condition (A7) is systematically satisfied empirically when choosing $K > K'/Q_t$ as the agent aims to liquidate a large number of shares in a short interval of time T. As in section 2.1, we can rewrite the value function (A6) by $h(t, x, S, q, \lambda) = x + qS + \widetilde{h}(t, S, q, \lambda)$ and show that $\widetilde{h}$ is a viscosity solution of the HJB PDE for all $(t, q, \lambda, S) \in \mathcal{S} \times \mathbb{R}$:

$$\partial_t \widetilde{h} + \tfrac{1}{2}\sigma^2 \partial_{SS}^2 \widetilde{h} + \sup_\nu\{-\phi\nu[K - S]^+ + (\eta\nu - \alpha\lambda)\partial_\lambda \widetilde{h} - \nu\partial_q \widetilde{h}$$
$$- \lambda\kappa\nu^{\gamma+1}\} = 0,$$

with terminal/boundary conditions $\widetilde{h}(T, S, q, \lambda) = -\pi\lambda q^2, \widetilde{h}(t, S, 0, \lambda) = 0$. Then, for $\gamma > 0$, we find for the optimal trading rate the explicit representation

$$\nu^*(t, S, q, \lambda) = \left(\frac{-\phi[K - S]^+ + \eta\partial_\lambda \widetilde{h}(t, S, q, \lambda) - \partial_q \widetilde{h}(t, S, q, \lambda)}{(\gamma + 1)\lambda\kappa}\right)^{1/\gamma},$$

while $\widetilde{h}$ is a viscosity solution of the explicit PDE

$$\partial_t \widetilde{h} + \frac{1}{2}\sigma^2 \partial_{SS}^2 \widetilde{h} + \frac{(-\phi[K - S]^+ + \eta\partial_\lambda \widetilde{h} - \partial_q \widetilde{h})^{(1+\gamma)/\gamma}}{((\gamma + 1)\lambda\kappa)^{1/\gamma}}$$
$$\times \left(\frac{\gamma}{\gamma + 1}\right) - \alpha\lambda\partial_\lambda \widetilde{h} = 0.$$

This explicit HJB equation can be numerically solved much more easily than the implicit equation (A5) based on the expected shortfall penalty (A4). Moreover, the optimal trading rate $(\nu_t^*)_t$ is not deterministic anymore in this setting since it depends on $(S_t)_t$.

Appendix 3. Comparison with the Almgren-Chriss model

Figure A1 depicts the comparison of our indirect price impact model with temporary kernel $G \equiv \delta$ to the power-law version of the Almgren-Chriss model, as described in Almgren (2003). Their temporary impact function, denoted $h(\nu)$ in Almgren (2003), is chosen $h(\nu) = \kappa\nu^{\gamma}$ to match our framework. For this first plot, their linear permanent impact function $f(\nu)$ is then set to 0, assuming no permanent impact. The performance criterion to maximize is still given by (12), but without intensity variable ($\lambda_\tau = 1$).

The second plot now depicts the comparison of our indirect price impact model with permanent kernel $G \equiv 1$ to the power-law (permanent) version of the Almgren-Chriss model with $h(\nu) = \kappa\nu^{\gamma}$ and $f(\nu) = b\nu$, see again Almgren (2003).

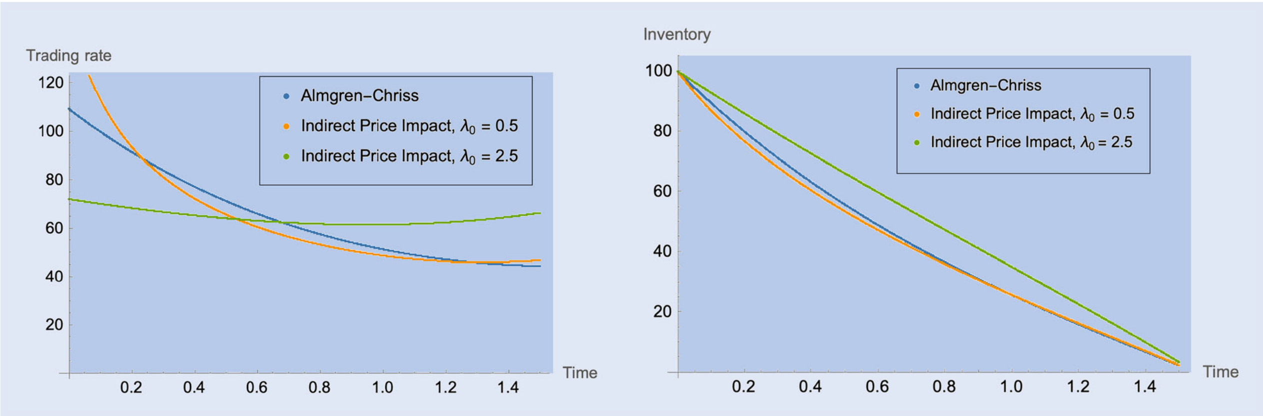

Figure A1. Optimal trading rate $(\nu_t^*)_t$ (left plot) and inventory $(Q_t^{(\nu^*)})_t$ (right plot) from a temporary (direct) price impact model with $\lambda_0 = 0.5$ (orange) and $\lambda_0 = 2.5$ (green) using $\alpha = 0.5$, $\eta = 0.1$. In blue, you have the power-law Almgren-Chriss model (without permanent impact), see Almgren (2003). The corresponding parameters $\gamma = 1/2$, $\pi = 1$, $\phi = 1$, $\kappa = 1$, $Q = 100$ and $T = 1.5$ are chosen to be consistent between the two models.

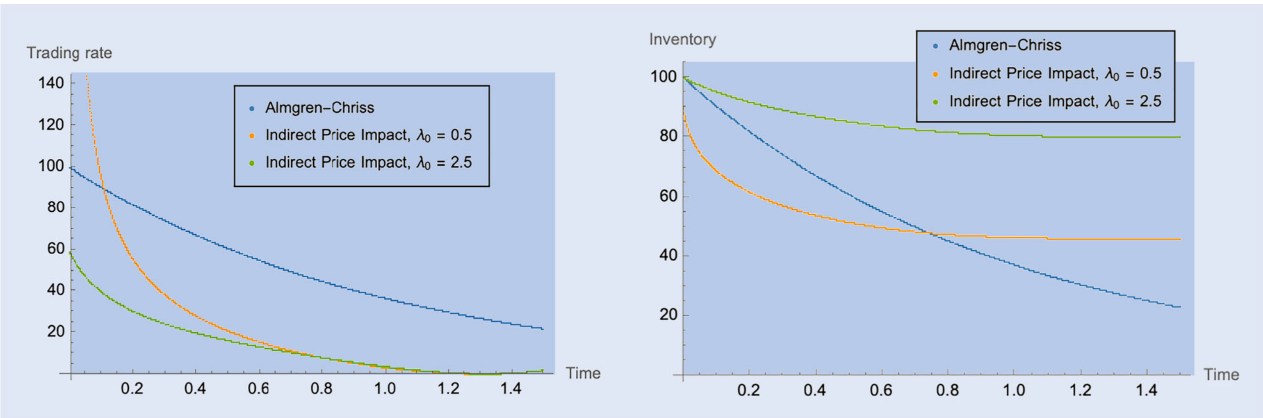

Figure A2. Optimal trading rate $(\nu_t^*)_t$ (left plot) and inventory $(Q_t^{(\nu^*)})_t$ (right plot) from a permanent (direct) price impact model with $\lambda_0 = 0.5$ (orange) and $\lambda_0 = 2.5$ (green) using $\alpha = 0.5$, $\eta = 0.1$. In blue, you have the power-law Almgren-Chriss model (with permanent linear impact using $b = 0.1$), see Almgren (2003). The corresponding parameters $\gamma = 1$, $\pi = 1$, $\phi = 1$, $\kappa = 1$, $Q = 100$ and $T = 1.5$ are chosen the same between the two models.