

## MONOID OF SELF-EQUIVALENCES AND FREE LOOP SPACES

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ABSTRACT. Let  $M$  be a simply-connected closed oriented  $N$ -dimensional manifold. We prove that for any field of coefficients  $k$  there exists a natural homomorphism of commutative graded algebras  $\Gamma : H_*(\Omega \operatorname{aut}_1 M) \rightarrow \mathbb{H}_*(M^{S^1})$  where  $\mathbb{H}_*(M^{S^1}) = H_{*+N}(M^{S^1})$  is the loop algebra defined by Chas and Sullivan. As usual  $\operatorname{aut}_1 X$  denotes the monoid of self-equivalences homotopic to the identity, and  $\Omega X$  the space of based loops. When  $k$  is of characteristic zero,  $\Gamma$  yields isomorphisms  $H_{(1)}^{n+N}(M^{S^1}) \cong (\pi_n(\Omega \operatorname{aut}_1 M) \otimes k)^\vee$  where  $\bigoplus_{l=1}^\infty H_{(l)}^n(M^{S^1})$  denotes the Hodge decomposition on  $H^*(M^{S^1})$ .

### 1. INTRODUCTION

Let  $M$  be a simply connected  $N$ -dimensional closed oriented manifold with base point  $m_0$ . We denote by  $M^{S^1}$  the space of free loops on  $M$ , by  $\Omega M$  the space of based loops of  $M$  at  $m_0$ , by  $\operatorname{aut} M$  the monoid of (unbased) self equivalences of  $M$ , by  $\operatorname{aut}_1 M$  the connected component of  $\operatorname{Id}_M$  in  $\operatorname{aut} M$ , and by  $H_*(-)$  the singular homology functor with coefficients in the fixed field  $k$ . The composition of loops induce a commutative graded algebra structure on  $H_*(\Omega \operatorname{aut}_1 M)$ .

It is convenient to write

$$\mathbb{H}_*(M) = H_{*+N}(M) \text{ (resp. } \mathbb{H}_*(M^{S^1}) = H_{*+N}(M^{S^1}) \text{)}.$$

Indeed  $\mathbb{H}_*(M)$  becomes a commutative graded algebra with the intersection product, and  $\mathbb{H}_*(M^{S^1})$  a commutative graded algebra with the loop product defined by Chas and Sullivan [1]. The definition of the loop product works as follows: Let  $\alpha : \Delta^n \rightarrow M^{S^1}$  and  $\beta : \Delta^m \rightarrow M^{S^1}$  be simplices of  $M^{S^1}$  and assume that  $q \circ \alpha : \Delta^n \rightarrow M$  and  $q \circ \beta : \Delta^m \rightarrow M$  are transverse in some sense. Then the intersection product  $(q \circ \alpha) \cdot (q \circ \beta)$  makes sense, and at each point  $(s, t) \in \Delta^n \times \Delta^m$  such that  $q\sigma(s) = q\tau(t)$ , the composition of the loops  $\alpha(s)$  and  $\beta(t)$  can be performed. This gives a chain  $\alpha \cdot \beta \in \mathcal{C}_{m+n-N}(M^{S^1})$  and leads to a commutative and associative multiplication ([1], Theorem 3.3):

$$\mathbb{H}_k(M^{S^1}) \otimes \mathbb{H}_l(M^{S^1}) \rightarrow \mathbb{H}_{k+l}(M^{S^1}), \quad a \otimes b \mapsto a \cdot b.$$

Our first result reads:

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**Theorem 1.** *The natural map*

$$g : M \times \Omega \operatorname{aut}_1 M \rightarrow M^{S^1}, \quad g(x, \gamma)(t) = \gamma(t)(x),$$

*induces a morphism of commutative graded algebras*

$$H_*(g) : \mathbb{H}_*(M) \otimes H_*(\Omega \operatorname{aut}_1 M) \rightarrow \mathbb{H}_*(M^{S^1}).$$

Denote by  $\omega$  the fundamental class of  $M$  in homology. Then  $\omega \in \mathbb{H}_0(M) = H_N(M) \cong \mathbb{k}\omega$  is the unit of the algebra  $\mathbb{H}_*(M)$ . The homomorphism  $H_*(g)$  restricts to a morphism of commutative graded algebras

$$\Gamma : H_*(\Omega \operatorname{aut}_1 M) \rightarrow \mathbb{H}_*(M^{S^1}), \quad \Gamma(a) = H(g)(\omega \otimes a).$$

The composition of  $\Gamma$  with the Hurewicz map  $h : \pi_*(\Omega \operatorname{aut}_1 M) \otimes \mathbb{k} \rightarrow H_*(\Omega \operatorname{aut}_1 M)$  is a morphism of graded vector spaces

$$\Gamma_1 = \Gamma \circ h : \pi_*(\Omega \operatorname{aut}_1 M) \otimes \mathbb{k} \rightarrow H_{*+N}(M^{S^1}),$$

which in turn induces the dual morphism

$$\Gamma_1^\vee : H^{*+N}(M^{S^1}) \rightarrow (\pi_*(\Omega \operatorname{aut}_1 M \otimes \mathbb{k}))^\vee.$$

Now recall that  $H^*(M^{S^1})$  is isomorphic as a graded vector space with the Hochschild homology of the cochain algebra  $\mathcal{C}^*(M)$  ([10]):

$$H^*(M^{S^1}) \cong HH_*(\mathcal{C}^*(M); \mathcal{C}^*(M)).$$

Also recall that if  $\mathbb{k}$  is a field of characteristic zero and  $A$  is a commutative graded  $\mathbb{k}$ -algebra, then the Hochschild homology of  $A$ ,  $HH_*(A; A)$ , admits a Hodge decomposition ([8]):

$$\mathbb{H}_*(A; A) = \bigoplus_{l \geq 0}^{\infty} \mathbb{H}_*^{(l)}(A; A).$$

Since  $\mathcal{C}^*(M)$  is quasi-isomorphic to a commutative graded differential algebra  $A$ , we derive from the previous considerations a Hodge decomposition on the free loop space cohomology of  $M$ ,

$$\mathbb{H}^*(M^{S^1}) = \bigoplus_{l \geq 0} \mathbb{H}_{(l)}^*(M^{S^1}).$$

We prove:

**Theorem 2.** *If  $\mathbb{k}$  is a field of characteristic zero, then*

- $\Gamma_1 : \pi_*(\Omega \operatorname{aut}_1 M) \otimes \mathbb{k} \rightarrow \mathbb{H}_*(M^{S^1})$  is injective,
- $\Gamma_1^\vee : \mathbb{H}_{(1)}^n(M^{S^1}) \xrightarrow{\cong} (\pi_n(\Omega \operatorname{aut}_1 M) \otimes \mathbb{k})^\vee$  is an isomorphism for  $n \geq 0$ ,
- $\Gamma_1^\vee$  vanishes on the components  $\mathbb{H}_{(p)}^*(M^{S^1})$  for  $p \geq 2$ .

Theorems 1 and 2 are proved respectively in sections 2 and 3. Section 4 contains examples and final remarks.

## 2. PROOF OF THEOREM 1

We denote by  $q : M^{S^1} \rightarrow M$  the free loop space fibration and by  $\text{Sect}(q)$  the space of sections of  $q$ . The composition of loops makes  $\text{Sect}(q)$  into a monoid with multiplication  $\mu$  defined by

$$\mu(\sigma, \tau)(m)(t) = \begin{cases} \sigma(m)(2t), & t \leq \frac{1}{2}, \\ \tau(m)(2t-1), & t \geq \frac{1}{2}, \end{cases} \quad \sigma, \tau \in \text{Sect}(q), \quad t \in [0, 1], m \in M.$$

Clearly the map  $\psi : \Omega(\text{aut}_1 M, id_M) \rightarrow \text{Sect}(q)$  defined by

$$\psi(f)(m)(t) = f(t)(m)$$

is a homeomorphism of monoids making commutative the diagram

$$\begin{array}{ccc} M \times \text{Sect}(q) & \xrightarrow{ev} & M^{S^1} \\ 1 \times \psi \downarrow & & \parallel \\ M \times \text{aut}_1 M & \xrightarrow{g} & M^{S^1}, \end{array}$$

where  $ev$  denotes the evaluation map.

To prove Theorem 1, it therefore suffices to establish that the evaluation map  $H_*(ev) : \mathbb{H}_*(M) \otimes H_*(\text{Sect}(q)) \rightarrow \mathbb{H}_*(M^{S^1})$  is a morphism of graded algebras.

We first remark that Chas and Sullivan prove that the morphism  $H_*(\sigma_0) : \mathbb{H}(M) \rightarrow \mathbb{H}(M^{S^1})$ , induced by the trivial section  $\sigma_0$ , is a morphism of graded algebras ([1], Proposition 3.4). Therefore the restriction of  $H_*(ev)$  to  $\mathbb{H}_*(M)$  is a morphism of graded algebras.

Recall now that the unit of  $\mathbb{H}_*(M)$  is the fundamental class  $\omega \in \mathbb{H}_0 M = H_N M$ . Therefore for a cycle  $\sum_i n_i \alpha_i$ , with  $\alpha_i : \Delta^r \rightarrow \text{Sect}(q)$ ,  $H_*(ev)(\omega \otimes \alpha)$  is the homology class of the sum  $\sum_i n_i \alpha'_i$  where  $\alpha'_i$  denotes the composition

$$\alpha'_i : M \times \Delta^r \xrightarrow{id \times f} M \times \text{Sect}(q) \xrightarrow{ev} M^{S^1}.$$

Thus let  $\alpha : \Delta^r \rightarrow \text{Sect}(q)$  and  $\beta : \Delta^s \rightarrow \text{Sect}(q)$  be simplices. Since the simplices  $q \circ \alpha'$  and  $q \circ \beta'$  are transverse in  $M$ , the Chas-Sullivan product

$$\alpha' \cdot \beta' : M \times \Delta^r \times \Delta^s \xrightarrow{id \times \alpha \times \beta} M \times \text{Sect}(q) \times \text{Sect}(q) \xrightarrow{(ev, ev)} M^{S^1} \times_M M^{S^1} \xrightarrow{c} M^{S^1}$$

is well defined,  $c$  denoting pointwise composition of loops.

As the multiplication  $\mu$  makes commutative the diagram

$$\begin{array}{ccc} M \times \text{Sect}(q) \times \text{Sect}(q) & \xrightarrow{(ev, ev)} & M^{S^1} \times_M M^{S^1} \\ \downarrow id \times \mu & & \downarrow c \\ M \times \text{Sect}(q) & \xrightarrow{ev} & M^{S^1}, \end{array}$$

the map  $\alpha' \cdot \beta'$  is equal to  $\mu(\alpha, \beta)'$ . Therefore the restriction of  $H_*(ev)$  to the component  $k\omega \otimes H_*(\text{Sect}(q))$  is also a morphism of algebras.

Finally let  $\alpha : \Delta^r \rightarrow M$  and  $\beta : \Delta^s \rightarrow \text{Sect}(q)$ . Then the simplices  $\alpha$  and  $q\beta'$  are transverse and the Chas-Sullivan product  $\alpha \cdot \beta$  is equal to  $ev(\alpha \times \beta)$ . Therefore  $H_*(ev)(\alpha) \cdot H_*(ev)(\beta) = H_*(ev)(\alpha \otimes \beta)$ .  $\square$

## 3. PROOF OF THEOREM 2

Since  $\mathbb{Q} \subset \mathbb{k}$  we may as well suppose that  $\mathbb{k} = \mathbb{Q}$ . Hereafter we will make extensive use of the theory of minimal models in the sense of Sullivan ([12]), for which we refer systematically to [5], §12. We denote by  $(\wedge V, d)$  the minimal model of  $M$ . By [13] a relative minimal model for the fibration  $q : M^{S^1} \rightarrow M$  is given by the extension

$$(\wedge V, d) \hookrightarrow (\wedge V \otimes \wedge sV, D), |sv| = |v| - 1, D(v) = d(v), D(sv) = -s(dv),$$

where  $s : \wedge V \rightarrow \wedge V \otimes \wedge sV$  is the unique derivation defined by  $s(v) = sv$ . The cochain complex  $(\wedge V \otimes \wedge sV, D)$  decomposes into a direct sum of complexes

$$(\wedge V \otimes \wedge sV, D) = \bigoplus_{k \geq 0} (\wedge V \otimes \wedge^k sV, D).$$

This induces a new graduation on  $H^*(M^{S^1})$ ,  $H^*(M^{S^1}) = \bigoplus_k H_{(k)}^*(M^{S^1})$  with

$$H_{(k)}^*(M^{S^1}) = H^*(\wedge V \otimes \wedge^k sV, D).$$

In [14], Vigué proves that this decomposition coincides with the Hodge decomposition of the Hochschild homology  $\mathbb{H}_*((\wedge V, d); (\wedge V, d))$ :

$$H^*(\wedge V \otimes \wedge^k sV, D) \cong \mathbb{H}_*^{(k)}((\wedge V, d); (\wedge V, d)).$$

By the Milnor-Moore Theorem ([11]),  $H_*(\text{Sect}(q); \mathbb{Q})$  is isomorphic as a Hopf algebra to the universal enveloping algebra on the graded homotopy Lie algebra  $\pi_*(\Omega \text{aut}_1 M) \otimes \mathbb{Q}$ . Thus Theorem 2 in the Introduction is a direct consequence of Theorem 3 below.

**Theorem 3.** *Let*

$$\Phi_1 : \pi_*(\text{Sect}(q)) \otimes \mathbb{Q} \rightarrow \mathbb{H}_*(M^{S^1}; \mathbb{Q})$$

*denote the restriction of  $H_*(ev)$  to  $\omega \otimes \pi_*(\text{Sect}(q)) \otimes \mathbb{Q}$ . Then,*

- $\Phi_1$  *is an injective morphism,*
- *the dual map  $\Phi_1^\vee$  vanishes on each  $H_{(p)}^*(M^{S^1}; \mathbb{Q})$ ,  $p \geq 2$ , and induces an isomorphism  $\bigoplus_{q \geq N} H_{(1)}^q(M^{S^1}; \mathbb{Q}) \cong \pi_*(\text{Sect}(q))^\vee$ .*

*Proof.* We first construct a quasi-isomorphism  $\rho : (\wedge V, d) \rightarrow (A, d)$  with  $(A, d)$ , a commutative differential graded algebra satisfying  $A^0 = \mathbb{Q}$ ,  $A^1 = 0$ ,  $A^{>N} = 0$ ,  $A^N = \mathbb{Q}\omega$ , and  $\dim A^i < \infty$  for all  $i$ .

For this we denote

$$Z^k = \text{Ker}(d : (\wedge V)^k \rightarrow (\wedge V)^{k+1}),$$

and we choose a supplement  $S^k$  of  $Z^k$  in  $(\wedge V)^k$ :

$$(\wedge V)^k = Z^k \oplus S^k.$$

The quotient  $(\wedge V)^N / (S^N \oplus dS^{N-1}) \cong H^N(M)$  has dimension one. Since  $V^1 = 0$ , the subcomplex  $I = S^{N-1} \oplus dS^{N-1} \oplus S^N \oplus (\wedge V)^{>N}$  is an acyclic ideal in  $(\wedge V, d)$ . Therefore the natural projection  $\rho : (\wedge V, d) \rightarrow (\wedge V/I, d)$  is a quasi-isomorphism of differential graded algebras. We define  $(A, d) = (\wedge V/I, d)$ .

The homomorphism  $\rho$  extends to a quasi-isomorphism  $\rho \otimes 1 : (\wedge V \otimes \wedge sV, D) \rightarrow (A \otimes \wedge sV, D)$  with  $D(a \otimes sv) = d(a) \otimes sv - (-1)^{|a|} a \cdot (\rho \otimes 1)(Dsv)$ . The complex  $(A \otimes \wedge sV, D)$  also decomposes into the direct sum of the complexes  $(A \otimes \wedge^k sV, D)$ .

Denote by  $(a_i), i = 1, \dots, n$ , a homogeneous linear basis of  $A$  with  $a_n = \omega$ , and by  $(a_i^\vee)$  the dual basis, i.e. the linear basis of  $A^\vee = \text{Hom}(A, \mathbb{Q})$  such that

$$\langle a_i^\vee, a_j \rangle = \delta_{ij}.$$

In [9], Haefliger proved that a model for the evaluation map  $ev : M \times \text{Sect}(q) \rightarrow M^{S^1}$  is given by the morphism

$$\theta : (A \otimes \wedge sV, D) \rightarrow (A, d) \otimes (\wedge(A^\vee \otimes sV), \delta), \quad \theta(a \otimes sv) = \sum_i aa_i \otimes (a_i^\vee \otimes sv).$$

Since  $D(sV) \subset A \otimes sV$  and  $\theta$  is a morphism of differential graded algebras, then  $\delta(A^\vee \otimes sV) \subset A^\vee \otimes sV$ . We now fix some notations:

- $\rho_1 : (\wedge(A^\vee \otimes sV), \delta) \rightarrow (A^\vee \otimes sV, \delta)$  denotes the projection on the complex of indecomposable elements,
- $P : (A, d) \rightarrow (\mathbb{Q}\omega, 0)$  is the homogeneous projection onto the component of degree  $N$ ,
- $\pi_1 : (A \otimes \wedge sV, D) \rightarrow (A \otimes sV, D)$  is the canonical projection on the sub-complex  $(A \otimes sV, D)$ .

The dual of  $\Phi_1$ ,

$$\Phi_1^\vee : H^{*+d}(M^{S^1}; \mathbb{Q}) \rightarrow (\pi_*(\text{Sect}(q)) \otimes \mathbb{Q})^\vee,$$

therefore coincides with  $H^*(P \otimes \rho_1) \circ H^*(\theta)$ :

$$(A \otimes \wedge sV, D) \xrightarrow{\theta} (A, d) \otimes (\wedge(A^\vee \otimes sV), \delta) \xrightarrow{P \otimes \rho_1} \mathbb{Q}\omega \otimes (A^\vee \otimes sV, \delta),$$

and vanishes on  $(A \otimes \wedge^{\geq 2} sV, D)$ .

**Lemma.** *The duality map  $\Delta : A \rightarrow A^\vee$  defined by*

$$\langle \Delta(a), b \rangle = P(ab) \in \mathbb{Q}\omega \cong \mathbb{Q}$$

*extends into a quasi-isomorphism of complexes*

$$\Delta \otimes 1 : (A \otimes sV, D) \rightarrow (A^\vee \otimes sV, \delta).$$

*Proof.* Denote by  $\alpha_{ij}^k$  and  $\beta_i^j$  rational numbers defined by the relations

$$\begin{cases} a_i \cdot a_j = \sum_k \alpha_{ij}^k a_k, \\ d(a_i) = \sum_j \beta_i^j a_j. \end{cases}$$

Recall that  $\{a_i^\vee\}_i$  denotes the dual basis of  $\{a_i\}_i$ . Then straightforward computations show that

- $d(a_i^\vee) = -(-1)^{|a_i|} \sum_j \beta_j^i a_j^\vee$ .
- $\sum_r \alpha_{ij}^r \alpha_{rk}^t = \sum_s \alpha_{jk}^s \alpha_{is}^t$ , for  $i, j, k, t = 1, \dots, n$  (associativity of the multiplication law).
- $\sum_r \alpha_{ij}^r \beta_r^s = \sum_t \beta_i^t \alpha_{tj}^s + (-1)^{|a_i|} \sum_l \beta_j^l \alpha_{il}^s$ , for  $i, j, l = 1, \dots, n$  (compatibility of the differential  $d$  with the multiplication).
- $\delta(a_j^\vee \otimes sv) = (-1)^{|a_j|} \left[ \sum_{i,l} \alpha_{il}^j (a_l^\vee \otimes sv_i) - \sum_r \beta_r^j (a_r^\vee \otimes sv) \right]$ .
- $\Delta(a_i) = \sum_j \alpha_{ij}^n a_j^\vee$ .

The duality morphism has degree  $N$ . A standard computation then shows that

$$\delta \circ (\Delta \otimes 1) = (-1)^N (\Delta \otimes 1) \circ d.$$

Since  $H^*(M)$  is a Poincaré duality algebra and since  $H^*(\Delta) : H^*(M) \rightarrow H_*(M)$  is the Poincaré duality,  $\Delta \otimes 1$  is a quasi-isomorphism.  $\square$

*End of the proof of Theorem 3.* It is easy to check the commutativity of the following diagram of complexes:

$$\begin{array}{ccc} (A \otimes \wedge sV, D) & \xrightarrow{\theta} & (A, d) \otimes (\wedge(A^\vee \otimes sV), \delta) \xrightarrow{P \otimes \rho_1} \mathbb{Q}\omega \otimes (A^\vee \otimes sV, \delta) \\ \pi_1 \downarrow & & \uparrow 1 \otimes (\Delta \otimes 1) \\ (A \otimes sV, D) & \xrightarrow{\sigma} & \mathbb{Q}\omega \otimes (A \otimes sV, D), \end{array}$$

with  $\sigma(a \otimes sv) = \omega \otimes a \otimes sv$ . By the above lemma,  $H_*(1 \otimes \Delta \otimes 1)$  is an isomorphism. Therefore  $H^*((1 \otimes (\Delta \otimes 1)) \circ \sigma \circ \pi_1)$  is surjective and this implies the surjectivity of  $\Phi_1^\vee = H_*(P \otimes \rho_1) \circ H^*(\theta)$ .  $\square$

#### 4. EXAMPLES AND FURTHER COMMENTS

*Remark 1.* The morphism  $\Gamma : H_*(\Omega \operatorname{aut}_1 M) \rightarrow H_*(M^{S^1})$  is not injective in general, as we shall now explain.

Denote by  $ev_0 : \operatorname{aut}_1 M \rightarrow M$  the evaluation at the base point. The image of the morphism  $\pi_n(ev_0) : \pi_n(\operatorname{aut}_1 M) \rightarrow \pi_n M$  is known as the  $n$ -th Gottlieb group of  $M$ ,  $G_n(M)$  ([5]). Since  $\Omega ev_0 : \Omega \operatorname{aut}_1 M \rightarrow \Omega M$  is an H-map,  $H_*(\Omega ev_0; \mathbb{Q}) = U(\pi_*(\Omega ev_0) \otimes \mathbb{Q})$  is the enveloping algebra on  $\pi_*(\Omega ev_0) \otimes \mathbb{Q}$ , whose image is the enveloping algebra on the abelian graded Lie algebra  $\overline{G}_*(X)$  that corresponds by duality to  $G_*(X) \otimes \mathbb{Q}$ .

Denote by  $I : \mathbb{H}_*(M^{S^1}) \rightarrow H_*(\Omega M)$  the intersection morphism defined in ([1], Proposition 3.4), and let  $\psi$  be defined as in the beginning of section 2. The commutativity of the following diagram

$$\begin{array}{ccc} H_*(\Omega \operatorname{aut}_1 M) & \xrightarrow{\pi_*(\psi)} & H_*(\operatorname{Sect}(q)) \\ H_*(\Omega ev_0) \downarrow & & \downarrow H_*(ev)(\omega \otimes -) \\ H_*(\Omega M) & \xleftarrow{I} & \mathbb{H}_*(M^{S^1}) \end{array}$$

shows that the image of  $I \circ \Phi_1$  is the universal enveloping algebra on  $\overline{G}_*(X)$ .

On the other hand, the kernel of  $I$  is a nilpotent ideal with nilpotency index less than or equal to  $N$  ([6]).

Now consider the manifold  $M = S^3 \times S^3 \times S^{11}$ . A simple computation using minimal models shows that  $\pi_5(\operatorname{aut}_1 M) \otimes \mathbb{Q} \neq 0$  and  $G_5(M) \otimes \mathbb{Q} = 0$ . Then denote by  $x$  a nonzero element in  $\pi_4(\Omega \operatorname{aut}_1 M) \otimes \mathbb{Q}$ . Since  $H_*(\Omega \operatorname{aut}_1 M; \mathbb{Q})$  is a free commutative graded algebra, some power of  $x$  belongs in the kernel of  $\Gamma$ .

*Remark 2.* In [2] Cohen and Jones prove that  $\mathbb{H}_*(M^{S^1})$  is isomorphic as an algebra to the Hochschild cohomology  $HH^*(\mathcal{C}^*(M), \mathcal{C}^*(M))$ . On the other hand, in [7], Gatsinzi establishes for any space  $M$  an algebraic isomorphism between  $\pi_*(\operatorname{aut}_1 M) \otimes \mathbb{Q}$  and a sub-vector space of  $HH^*(\mathcal{C}^*(M), \mathcal{C}^*(M))$ . Our Theorem 2 relates these two results.

**Problem.** We would like to know if the homomorphism

$$\Gamma : \mathbb{H}_*(M) \otimes H_*(\Omega \text{aut}_1 M) \rightarrow \mathbb{H}_*(M^{S^1})$$

is surjective. It is true for example when  $M = \mathbb{C}P^{2N}$ . When  $\Gamma$  is surjective there is a strong connection between the behaviour of the sequences of Betti numbers  $\dim H_i(M^{S^1})$  and  $\dim \pi_i(\text{aut } M) \otimes \mathbb{Q}$ .

**Example 1.** Let  $G$  be a Lie group. The minimal model of  $G$  is  $(\wedge V, 0)$  with  $V$  finite dimensional and concentrated in odd degrees ([5], §12(a)). Therefore a model of the free loop space  $G^{S^1}$  is  $(\wedge V \otimes \wedge sV, 0)$  and the Haefliger model for the space  $\text{Sect}(q)$  is  $(\wedge((\wedge V)^\vee \otimes sV), 0)$ . Since the model  $\theta$  of the evaluation map  $ev$  is injective,  $H_*(ev) : H_*(M) \otimes H_*(\text{Sect}(q)) \rightarrow H_*(M^{S^1})$  is surjective. This implies the existence of an isomorphism of graded algebras,

$$\mathbb{H}_*(M^{S^1}) \cong \mathbb{H}_*(M) \otimes H_*(\Omega M).$$

Here the multiplication on the right is the product of the intersection product on  $\mathbb{H}_*(M)$  with the usual Pontryagin product on  $H_*(\Omega M)$ .

**Example 2.** Let us assume that  $M$  is a  $\mathbb{Q}$ -hyperbolic space satisfying either  $(H^+(M))^3 = 0$  or  $(H^+(M))^4 = 0$ , and  $M$  is a coformal space.

Recall that a space  $M$  is  $\mathbb{Q}$ -hyperbolic if  $\dim \pi_*(M) \otimes \mathbb{Q} = \infty$  and is coformal if the differential graded algebras  $\mathcal{C}_*(\Omega M)$  and  $(H_*(\Omega M), 0)$  are quasi-isomorphic. Under the above hypothesis, in [15] Vigué proves that there exist an integer  $n_0$  and some constants  $C_1 \geq C_2 > 1$  such that

$$C_2^n \leq \sum_{i=1}^n \dim H_{(1)}^i(X^{S^1}) \leq C_1^n, \text{ for all } n \geq n_0.$$

We deduce from Theorem 3 that the same relations hold for the sequence of dimensions of  $\pi_i(\text{aut } M) \otimes \mathbb{Q}$ , i.e., in both cases the sequences of Betti numbers have exponential growth.

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