Part 2

Singular problems

Introduction

In Part 1 of this thesis, we emphasize the fact that using the direct method of the Calculus of Variations can be delicate in the absence of compactness. One way to overcome this difficulty was to limit the analysis to particular subspaces of functions.

In this second part, we tackle a class of problems with singularities, which also imply some lack of compactness. Here, we attack these obstructions directly, proving that they do not appear for any minimizing sequence.

A p-Laplacian equation

We study problems of the form

$$-\Delta_p u = \lambda \frac{|u|^{p-2}u}{|x|^p} + \frac{f(x)}{p} \quad \text{in } \Omega \ni 0, \tag{1}$$

with Dirichlet boundary conditions, where λ is a real parameter and $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian, a generalization of the Laplace operator (recovered when $p = 2$).

The solutions of (1) can be seen as stationary states of the corresponding parabolic equation

$$\begin{cases} u_t - \Delta_p u &= \lambda \frac{|u|^{p-2}u}{|x|^p} + \frac{f(x)}{p} & \text{for } x \in \Omega, t \in \mathbf{R}_0^+, \\ u(x,0) &= g(x) & \text{for } x \in \Omega, \\ u(x,t) &= 0 & \text{for } x \in \partial\Omega, t \in \mathbf{R}_0^+, \end{cases}$$

an heat equation with diffusion term given by the p-Laplacian and with a forcing term f.

Let us also mention the work of Fleckinger-Hernández-Takáč-De Thélin [24]. They study a variant of (1) without singularity, i.e.

$$-\Delta_p u = \lambda a(x)\, |u|^{p-2}u + \frac{f(x)}{p} \quad \text{in } \Omega \tag{2}$$

with Dirichlet boundary conditions on a bounded domain and where a, f are bounded functions non identically vanishing, a being positive.

When λ is less or equal to the first eigenvalue associated with the autonomous problem, the direct method implies the existence of solutions as the associated energy functional is weakly lower semi-continuous and coercive (i.e. tends to infinity when $||u||$ tends to infinity).

One can try to handle problem (1) in the same way. The associated functional is

$$J(u) := \int_\Omega L(x, u, \nabla u) dx,$$

where

$$L(x, u, \nabla u) := |\nabla u|^p - \lambda \frac{|u|^p}{|x|^p} - f\, u.$$

Classical results ensure weak lower semi-continuity under hypotheses easy to verify. For example, if $L(x, u, \xi)$ is positive, smooth and convex in ξ, then the functional J is weakly lower semi-continuous (see e.g. [56, Theorem 1.6]). Unfortunately, this is not the case here. To handle the problem, one way is to use other tools than the direct method; another way is to prove lower semi-continuity of the functional by less classical means.

In 1998, García-Peral [25] studied equation (1) on a bounded domain, without using the direct method. They first exhibit an almost critical minimizing sequence (u_n) for the functional, guaranteed by the Ekeland Variational Principle (see e.g. [59, Theorem 2.4]). On this basis, they succeed to obtain strong convergence of (u_n) by means of an appropriate convergence result of Boccardo-Murat [8]. Note this technique yields the ground state solution of the equation without proving the lower semi-continuity of the functional.

Recovering lower semi-continuity

On the other hand one can try to prove weak lower semi-continuity of the functional. This is the idea underlying our work.

As pointed out above, the problem does not satisfy the framework of classical results. To obtain a solution by minimization, we proceed in two steps. The first one consists in exhibiting an adapted inequality ensuring coercivity and regularity of the energy functional. For problem (1), this is the classical Hardy inequality: roughly speaking, on $\mathcal{D}(\Omega)$, we have

$$\int_\Omega \frac{|u|^p}{|x|^p} dx \leqslant C \int_\Omega |\nabla u|^p dx.$$

The second step consists in the study of the possible obstructions to weak lower semi-continuity. Considering a weakly convergent sequence $u_n \rightharpoonup u$ in the relevant space, the lack of compactness can appear by translation, as showed in Example 1 on page 18, or by concentration at the singularity. The latter phenomenon appears as it is possible to construct a sequence of functions (u_n) such that

- (u_n) concentrates at the origin to a kind of Dirac mass, i.e. weakly converges to 0;
- $\int_\Omega |\nabla u_n|^p$ and $\int_\Omega \frac{|u_n|^p}{|x|^p}$ increase at the same rate when $\int_\Omega f u_n$ is constant, implying that $J(u_n)$ is independent of n.

In order to deal with these obstructions, we split the functional in two terms by a decomposition of the domain Ω in two parts. The first one, denoted by Ω_R, encloses the *unpleasant* component of the domain. Namely, Ω_R is the union of a *small* neighborhood of the point of singularity (to include concentration) and of the exterior of a *large ball* (to include the possible translation). The second part is the complementary $\Omega \setminus \Omega_R$ which encloses the *pleasant* component of the domain. The functional is thus decomposed in the following form:

$$J(u) = J_{\Omega_R}(u) + J_{\Omega\setminus\Omega_R}(u),$$

where $J_\omega(u) := \int_\omega L(x, u, \nabla u)dx$. Roughly speaking, lower semi-continuity of the second pleasant part is obtained by classical arguments such as Rellich Theorem and convexity. The first unpleasant part is more difficult to handle. We manage to compare its components in u and ∇u by means of Hardy's inequality, implying its positivity. This is sufficient to obtain the weak lower semi-continuity of the functional. The existence of a minimizer of J follows.

To sum up, the above decomposition technique permits to quantify the lack of compactness. Besides giving a new proof of the result, also valid for unbounded domains, the main advantage of this approach is its versatility: it can be easily adapted to a wide class of problems, including ordinary differential equations, provided one can find an appropriate *Hardy-type inequality.* This is developed in Chapter IV.

In particular, the method perfectly works for solving the forced problem

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right) = \lambda\frac{|u|^{p-2}u}{|x|^{(a+1)p}} + \frac{f(x)}{p} \qquad \text{in } \Omega \ni 0.$$

This completes the work of Abdellaoui-Peral [1,2] which have studied the corresponding unforced problem. They obtained nonexistence results for

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right) = \lambda \frac{u^{\alpha}}{|x|^{(a+1)p}} \qquad \text{in } \Omega \ni 0$$

when $\alpha > p-1$ or when $\alpha = p-1$ and λ is sufficiently large, namely is greater than the Hardy-Sobolev constant $\left(\frac{N-(a+1)p}{p}\right)^p$.

Some problems dealing with singularities linked to the Caffarelli-Kohn-Nirenberg inequality are also studied in the literature; see [23].

Unicity of solutions

After having tackled the question of the existence of a solution in Chapter IV, we study the problem of (non)unicity in Chapter V. In the case $p = 2$, equation (1) is linear. This implies strict convexity of the energy functional. So, as in the finite-dimensional case, the unicity of the critical point is clear.

The case $p \neq 2$ is harder to handle. Fleckinger et al. [24] proved the unicity of the solution of problem (2) when f, a are bounded and positive. They obtained this result using a variational approach due to Díaz-Saa [19], via a nice and elementary inequality involving the p-Laplacian and powers of functions.

On the other hand, del Pino-Elgueta-Manasevich [20] studied a corresponding one-dimensional non-singular problem on a bounded interval:

$$-\left(|u'|^{p-2}u'\right)' = \lambda|u|^{p-2}u + \frac{f}{p} \qquad \text{in } [0,T]$$

with Dirichlet boundary conditions. They proved the existence of two different solutions for a class of suitable forcing terms f when $p > 2$.

So the treatment of uniqueness does not seem obvious. The method of del Pino et al. can be adapted to exhibit two different solutions for problem (1) when $p > 2$. The idea is to define a forcing term f for which one solution of the problem is easy to obtain, but in such a way that this solution is not the minimizer of the functional. So there exists another solution provided by the direct minimization method developed in Chapter IV. The non-minimizing property of the first solution is obtained by computing the second differential of J and showing that it is negative for some direction, so it cannot be a minimizer.

When $1 < p < 2$, the Fréchet derivative is more difficult to handle. We again must pay attention to obtain an adequate forcing term and a direction implying negativity of the second derivative of J. This is done using the optimal constant and the class of optimal functions for the corresponding Hardy inequality.

So we must know exactly the optimal constant to follow this approach; this was not necessary for existence or non-unicity in the previous cases. Because of this constraint, we are unable to treat one of the considered problem, see §V.3.3.

CHAPTER IV

Existence of minimizers of functionals involving Hardy-type inequalities

1. Introduction

Consider the problem

$$
(1.1)\qquad \begin{cases} -\Delta_p u &= \lambda \frac{|u|^{p-2}u}{|x|^p} + \frac{f(x)}{p} & \text{in } \Omega, \\ u &= 0 & \text{in } \partial\Omega, \\ \lim_{|x|\to+\infty} u(x) &= 0. \end{cases}
$$

This is a p-Laplacian equation with an unbounded potential and a forcing term f. We will study the corresponding energy functional to obtain a solution. The classical minimization method cannot be used here because in general the Lagrangian

$$
L(x,u,\nabla u) \;=\; |\nabla u|^p - \lambda \frac{|u|^p}{|x|^p} - f\,u
$$

is not bounded from below and Ω is unbounded. To overcome this difficulty, we use the concentration-compactness argument (Lions [34]) or more precisely the decomposition technique (Bianchi-Chabrowski-Szulkin [7], Smets [53] or Willem [61]). Contrary to the first part of this thesis, we consider here an unconstrained free minimization problem. Another key ingredient to prove the result is the Hardy inequality (see e.g. [31,60]). Remark a lack of compactness is due to the invariance by dilation of the quotient

$$
\frac{\int_{\mathbf{R}^N} |\nabla u|^p dx}{\int_{\mathbf{R}^N} \frac{|u|^p}{|x|^p} dx}.
$$

The idea of the proof is to consider a minimizing sequence (u_n) weakly converging to u. As explained in the introduction to Part 2, we consider Ω_R, a subset of the domain Ω designed to control the lack of compactness, i.e. the components of Ω included in the union of $B(0,\frac{1}{R})$

and $B^c(0,R)$. We decompose the functional in two components

$$J(u_n) = \int_{\Omega\setminus\Omega_R} L(x,u_n,\nabla u_n)dx + \int_{\Omega_R} L(x,u_n,\nabla u_n)dx$$

and we study the limit of each part:

$$\lim_{R\to\infty}\varliminf_{n\to\infty}\int_{\Omega\setminus\Omega_R} L(x,u_n,\nabla u_n)dx \geqslant J(u),$$
$$\lim_{R\to\infty}\varlimsup_{n\to\infty}\int_{\Omega_R} L(x,u_n,\nabla u_n)dx \geqslant 0.$$

These inequalities, obtained using a decomposition lemma, imply that u is a minimizer of the functional.

Problem (1.1) for bounded domains was studied by García-Peral [25]. They proved the existence of a minimizer using a Hardy-type inequality, the Ekeland Variational Principle and a convergence theorem. An improvement obtained here is the validity of the result for unbounded domains Ω.

Moreover, our method is applicable for a larger class of problems, provided an adapted Hardy-type inequality can be found. As Secchi-Smets-Willem [46] proved a cylindrical Hardy inequality, we can study the corresponding equation involving the operator

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|y|^{ap}}\right) - \lambda\frac{|u|^{p-2}u}{|y|^{(a+1)p}},$$

where $x=(y,z)\in\mathbf{R}^k\times\mathbf{R}^{N-k}$. This is carried out in Section 3. The particular case $k=N$, corresponding to a radial singularity, is tackled in Section 4.

Section 5 follows the same approach to study a corresponding ordinary differential equation using the classical one-dimensional Hardy inequality.

Marcus-Mizel-Pinchover [35], Colin [17] and Chabrowski-Willem [15] developed an inequality on exterior domains with a singularity located on the boundary. We can similarly establish the existence of solutions of associated problems. This is done in Section 6.

Remark 1.1. After the completion of this work, we received a manuscript by X. Zhong [62] and B. Pellacci send us [36] and [41]. These papers concern problem (1.1). In [62], the case of a bounded Ω is treated by an elementary argument. In [36, 41], the classical concentration-compactness principle from [34] is used. All of them obtain identical result as our Theorem 2.1 of Section 2, but none of them study the cases tackled here in the other sections.

2. The p-Laplacian case

To solve problem (1.1), we study the associated energy functional

$$J(u) := \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega \frac{|u|^p}{|x|^p} dx - \langle f, u \rangle$$

defined on the closure $\mathcal{D}_0^{1,p}(\Omega)$ of $\mathcal{D}(\Omega)$ with respect to the norm $||\nabla u||_p$, where Ω is a smooth open subset of $\mathbf{R}^N$ with a compact boundary, $0 \notin \partial\Omega$ and

$$\begin{array}{ll} \text{if } 0 \notin \Omega \text{ then } 1 < p \neq N, \\ \text{if } 0 \in \Omega \text{ then } 1 < p < N, \end{array} \tag{2.1}$$

$$f \in \left(\mathcal{D}_0^{1,p}(\Omega)\right)'. \tag{2.2}$$

We consider

$$\inf J := \inf_{u \in \mathcal{D}_0^{1,p}(\Omega)} J(u).$$

The aim of this section is to prove the following result.

Theorem 2.1. *Assume* (2.1), (2.2) *and* $\lambda < \lambda_p := \left|\frac{N-p}{p}\right|^p$. *Then* $\inf J$ *is achieved and problem* (1.1) *has a solution.*

The proof of Theorem 2.1 is given after the two following lemmas. We first recall the classical multidimensional Hardy inequality (see e.g. [31, 60]).

Lemma 2.2 (Hardy-Sobolev Inequality). *Assume* (2.1). *If* $u \in \mathcal{D}_0^{1,p}(\Omega)$, *then*

$$\lambda_p \int_\Omega \frac{|u|^p}{|x|^p} dx \leqslant \int_\Omega |\nabla u|^p dx.$$

This inequality implies the functional J is continuous, G-differentiable and coercive if $\lambda < \lambda_p$.

The key ingredient proving Theorem 2.1 is the following decomposition lemma quantifying the loss of compactness.

Lemma 2.3. *Let* $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$ *be such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}_0^{1,p}(\Omega)$ *and*

$$\begin{aligned} \Omega_R &:= \left\{x \in \Omega : |x| < R^{-1} \text{ or } |x| > R\right\}, \\ \mu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} |\nabla u_n|^p dx, \\ \nu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|x|^p} dx. \end{aligned}$$

Then

$$\overline{\lim_{n\to\infty}}\int_\Omega \frac{|u_n|^p}{|x|^p}dx = \int_\Omega \frac{|u|^p}{|x|^p}dx + \nu_0, \tag{2.3}$$

$$\overline{\lim_{n\to\infty}}\int_\Omega |\nabla u_n|^p dx \geqslant \int_\Omega |\nabla u|^p dx + \mu_0, \tag{2.4}$$

$$\lambda_p\, \nu_0 \leqslant \mu_0. \tag{2.5}$$

Proof of (2.3). We have, for every $R > 0$,

$$\begin{aligned}\overline{\lim_{n\to\infty}}\int_\Omega \frac{|u_n|^p}{|x|^p}dx &= \lim_{n\to\infty}\int_{\Omega\setminus\Omega_R} \frac{|u_n|^p}{|x|^p}dx + \overline{\lim_{n\to\infty}}\int_{\Omega_R} \frac{|u_n|^p}{|x|^p}dx,\\ &= \int_{\Omega\setminus\Omega_R} \frac{|u|^p}{|x|^p}dx + \overline{\lim_{n\to\infty}}\int_{\Omega_R} \frac{|u_n|^p}{|x|^p}dx,\end{aligned}$$

since $u_n \to u$ in $L^p_{loc}(\Omega)$ by the Rellich Theorem. So, taking the limit for $R \to \infty$, we obtain (2.3).

Proof of (2.4) We have

$$\overline{\lim_{n\to\infty}}\int_\Omega |\nabla u_n|^p dx \geqslant \underline{\lim}_{n\to\infty}\int_{\Omega\setminus\Omega_R} |\nabla u_n|^p dx + \overline{\lim_{n\to\infty}}\int_{\Omega_R} |\nabla u_n|^p dx.$$

Hence, as the mapping $v \mapsto \int_{\Omega\setminus\Omega_R} |\nabla v|^p dx$ is convex, we get

$$\overline{\lim_{n\to\infty}}\int_\Omega |\nabla u_n|^p dx \geqslant \int_{\Omega\setminus\Omega_R} |\nabla u|^p dx + \overline{\lim_{n\to\infty}}\int_{\Omega_R} |\nabla u_n|^p dx.$$

Taking the limit for $R \to \infty$, we obtain (2.4) by the Levi Theorem.

Proof of (2.5). Now we introduce the truncation $\psi_R \in \mathcal{C}^\infty(\mathbf{R}^N)$ such that $0 \leqslant \psi_R \leqslant 1$ and

$$\psi_R(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus \Omega_R,\\ 1 & \text{if } x \in \Omega_{R+1}.\end{cases}$$

For each $v \in \mathcal{D}^{1,p}_0(\Omega)$, Lemma 2.2 implies

$$\lambda_p \int_\Omega \frac{|\psi_R v|^p}{|x|^p}dx \leqslant \int_\Omega |\nabla(\psi_R v)|^p dx. \tag{2.6}$$

On the other hand, for each $\varepsilon > 0$, there exists a constant $c(\varepsilon, p) > 0$ such that

$$\big||a+b|^p - |a|^p\big| \leqslant \varepsilon |a|^p + c(\varepsilon, p)\, |b|^p. \tag{2.7}$$

It is then easy to verify that

$$\begin{aligned}\mu_0 &= \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{\Omega_R}|\nabla(u_n-u)|^p dx,\\ \nu_0 &= \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{\Omega_R}\frac{|u_n-u|^p}{|x|^p}dx.\end{aligned}$$

Hence, using the truncation ψ_R,

$$\begin{aligned}\mu_0 &= \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{\Omega}|\nabla(u_n-u)|^p\psi_R^p dx,\\ \nu_0 &= \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{\Omega}\frac{|u_n-u|^p\psi_R^p}{|x|^p}dx.\end{aligned}\tag{2.8}$$

Using once again (2.7) and the compactness of the embedding $\mathcal{D}_0^{1,p}(\Omega)\subset L^p(\Omega)$ for bounded domains, we get

$$\mu_0 = \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{\Omega}|\nabla((u_n-u)\psi_R)|^p dx.\tag{2.9}$$

The inequality (2.5) follows from (2.6), (2.8) and (2.9). □

Proposition 2.4. *Assume* (2.1), (2.2) *and* $\lambda<\lambda_p$. *Then the functional* J *is weakly lower semi-continuous.*

Proof. Let (u_n) be such that $u_n\rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Going if necessary to a subsequence, we can assume $J(u_n)$ is convergent. So, by virtue of Lemma 2.3, we have

$$\begin{aligned}\lim_{n\to\infty}J(u_n) &= \overline{\lim_{n\to\infty}}\int_{\Omega}|\nabla u_n|^p dx-\lambda\,\overline{\lim_{n\to\infty}}\int_{\Omega}\frac{|u_n|^p}{|x|^p}dx-\langle f,u\rangle,\\ &\geqslant J(u)+\mu_0-\lambda\nu_0,\\ &\geqslant J(u)+(\lambda_p-\lambda)\nu_0,\\ &\geqslant J(u).\end{aligned}$$

□

Proof of Theorem 2.1. This is a direct application of Proposition 2.4. As the proof of Proposition 2.4 shows that $\mu_0=\nu_0=0$, the minimizing sequence (u_n) is strongly convergent up to a subsequence. □

3. Cylindrical-weight case

We now study

$$
\text{(3.1)} \qquad \begin{cases} -\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|y|^{ap}}\right) & = & \lambda\frac{|u|^{p-2}u}{|y|^{(a+1)p}} + \frac{f(x)}{p} & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \\ \lim_{|x|\to+\infty} u(x) & = & 0, & \end{cases}
$$

where $x = (y, z) \in \mathbf{R}^k \times \mathbf{R}^{N-k}$. The corresponding functional is

$$
J(u) := \int_\Omega \frac{|\nabla u|^p}{|y|^{ap}}dx - \lambda\int_\Omega \frac{|u|^p}{|y|^{(a+1)p}}dx - \langle f, u\rangle,
$$

defined on the closure $\mathcal{D}^{1,p}_{a,k}(\Omega)$ of $\mathcal{D}(\Omega)$ with respect to the norm

$$
||u|| = \left(\int_\Omega \frac{|\nabla u|^p}{|y|^{ap}}dx\right)^{1/p}.
$$

Moreover

$$
\text{(3.2)} \qquad a \geqslant 0,
$$

$$
\text{(3.3)} \qquad 1 < p < \frac{k}{a+1},
$$

$$
\text{(3.4)} \qquad f \in \left(\mathcal{D}^{1,p}_{a,k}(\Omega)\right)'.
$$

For simplicity we assume Ω is either a ball centered at the origin or the complement of a closed ball. We can consider more general domains, but at any rate we have to avoid cases such as the spiral in Figure 1.

The Hardy-type inequality associated with problem (3.1) is due to Secchi-Smets-Willem [46].

Lemma 3.1 (Secchi-Smets-Willem). *Assume* (3.2) *and* (3.3). *If* $u \in \mathcal{D}^{1,p}_{a,k}(\Omega)$, *then*

$$
\lambda_{kap}\int_\Omega \frac{|u|^p}{|y|^{(a+1)p}}dx \leqslant \int_\Omega \frac{|\nabla u|^p}{|y|^{ap}}dx,
$$

where $\lambda_{kap} := \left(\frac{k-(a+1)p}{p}\right)^p$.

The aim of this section is to prove the following result.

Theorem 3.2. *Assume* (3.2), (3.3), (3.4) *and* $\lambda < \lambda_{kap}$. *Then* $\inf J$ *on* $\mathcal{D}^{1,p}_{a,k}(\Omega)$ *is achieved and problem* (3.1) *has a solution.*

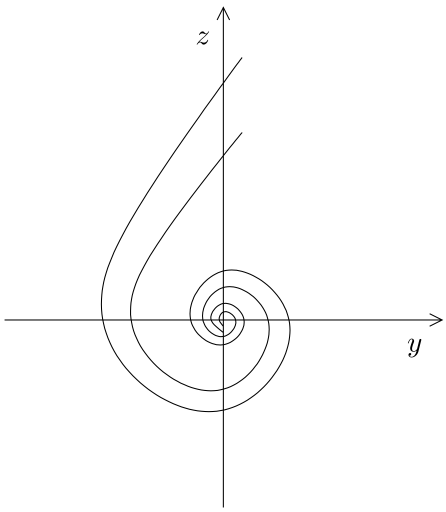

FIGURE 1. A challenging domain Ω: a spiral.

Lemma 3.1 above implies that the functional J is continuous, G-differentiable and coercive if $\lambda < \lambda_{kap}$. Let $(u_n) \subset \mathcal{D}^{1,p}_{a,k}(\Omega)$ a minimizing sequence for J, i.e.

$$J(u_n) \to \inf J \quad \text{as } n \to +\infty,$$

where $\inf J := \inf_{u \in \mathcal{D}^{1,p}_{a,k}(\Omega)} J(u)$. The goal is to prove that this sequence (u_n) contains a subsequence converging to a minimizer of J. As J is coercive, (u_n) is bounded so we can assume $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_{a,k}(\Omega)$. Our first result is the following compactness lemma.

Lemma 3.3. *Assume* (3.2), (3.3) *and* $u_n \rightharpoonup u$ *in* $\mathcal{D}^{1,p}_{a,k}(\Omega)$. *Then* $\frac{u_n}{|y|^a} \to \frac{u}{|y|^a}$ *in* $L^p(A_c)$, *where* $A_c := \{x = (y,z) \in \Omega : \frac{1}{c} < |y|$ *and* $|x| < c\}$ *for every* $c > 1$.

Proof. The thesis is clear as A_c is bounded and as the weight $|y|^{-a}$ is bounded on the considered set. $\square$

We now introduce a decomposition lemma.

Lemma 3.4. *Let* $(u_n) \subset \mathcal{D}^{1,p}_{a,k}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}^{1,p}_{a,k}(\Omega)$. *We define*

$$\begin{aligned}
\Omega_R &:= \Omega \cap \Big(\big(B^k(0,R^{-1}) \times \mathbf{R}^{N-k}\big) \cup \big(\mathbf{R}^N \setminus B^N(0,R)\big)\Big), \\
\mu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}} dx, \\
\nu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}} dx.
\end{aligned}$$

Then we have

$$\overline{\lim_{n\to\infty}} \int_\Omega \frac{|u_n|^p}{|y|^{(a+1)p}} dx = \int_\Omega \frac{|u|^p}{|y|^{(a+1)p}} dx + \nu_0, \tag{3.5}$$

$$\overline{\lim_{n\to\infty}} \int_\Omega \frac{|\nabla u_n|^p}{|y|^{ap}} dx \geqslant \int_\Omega \frac{|\nabla u|^p}{|y|^{ap}} dx + \mu_0, \tag{3.6}$$

$$\lambda_{kap}\, \nu_0 \leqslant \mu_0. \tag{3.7}$$

Proof of (3.5). As $u_n \to u$ in $L^p_{loc}(\Omega)$, we have for every $R > 0$,

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \int_\Omega \frac{|u_n|^p}{|y|^{(a+1)p}} dx &= \lim_{n\to\infty} \int_{\Omega\setminus\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}} dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}} dx, \\
&= \int_{\Omega\setminus\Omega_R} \frac{|u|^p}{|y|^{(a+1)p}} dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}} dx.
\end{aligned}$$

So, taking the limit as $R \to \infty$, we obtain (3.5).

Proof of (3.6). We have

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \int_\Omega \frac{|\nabla u_n|^p}{|y|^{ap}} dx &\geqslant \underline{\lim_{n\to\infty}} \int_{\Omega\setminus\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}} dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}} dx, \\
&\geqslant \int_{\Omega\setminus\Omega_R} \frac{|\nabla u|^p}{|y|^{ap}} dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}} dx,
\end{aligned}$$

as the mapping $v \mapsto \int_{\Omega\setminus\Omega_R} \frac{|\nabla v|^p}{|y|^{ap}} dx$ is convex. Taking now the limit as $R \to \infty$, we obtain (3.6).

Proof of (3.7). We begin finding new formulations for μ_0 and ν_0. Using inequality (2.7), we obtain:

$$\mu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla(u_n - u)|^p}{|y|^{ap}} dx,$$
$$\nu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n - u|^p}{|y|^{(a+1)p}} dx.$$

We introduce the truncation $\psi_R \in \mathcal{C}^\infty(\mathbf{R}^N)$ with $\psi_R(x) = 0$ if $x \in \Omega\backslash\Omega_R$, $\psi_R(x) = 1$ if $x \in \Omega_{R+1}$ and $0 \leqslant \psi_R \leqslant 1$. With this truncation, we get

$$\mu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|\nabla(u_n - u)|^p \psi_R^p}{|y|^{ap}} dx,$$
$$\nu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|u_n - u|^p \psi_R^p}{|y|^{(a+1)p}} dx.$$

Using Lemma 3.3 and (2.7) again, we obtain

$$\mu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|\nabla((u_n - u)\psi_R)|^p}{|y|^{ap}} dx.$$

Applying Lemma 3.1 to the function $(u_n - u)\psi_R$ and taking the limits we get inequality (3.7). □

Proof sketch for Theorem 3.2. Lemma 3.4 is a decomposition lemma which entails the weak lower semi-continuity of J. This allows to prove Theorem 3.2, as done for Theorem 2.1 using Lemma 2.3 in Section 2. □

4. Radial singularity

When $k = N$, the singularity in (3.1) is radial, i.e. the problem is

$$\left\{ \begin{array}{rcll} -\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right) & = & \lambda \frac{|u|^{p-2}u}{|x|^{(a+1)p}} + \frac{f(x)}{p} & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} u(x) & = & 0. & \end{array} \right. \tag{4.1}$$

We can work exactly as in the previous section under the same hypotheses, i.e. considering the space $\mathcal{D}^{1,p}_{a,N}(\Omega)$ when $a \geqslant 0$, $1 < p < \frac{N}{a+1}$, $\lambda < \lambda_{Nap}$ and $f \in \left(\mathcal{D}^{1,p}_{a,N}(\Omega)\right)'$.

But the singularity is localized at a single point, namely the origin. Thus the proof works for every regular domain Ω such that $0 \notin \partial\Omega$: there are no problematic cases to avoid. The result obtained is the following.

Theorem 4.1. *If* $\lambda < \lambda_{kap}$, *then* $\inf J$ *on* $\mathcal{D}^{1,p}_{a,N}(\Omega)$ *is achieved and problem* (4.1) *has a solution.*

5. One-dimensional case

We consider the one-dimensional problem

$$
\text{(5.1)} \qquad \begin{cases} -\left(\frac{|u'|^{p-2}u'}{x^{ap}}\right)' &= \lambda \frac{|u|^{p-2}u}{x^{(a+1)p}} + \frac{f(x)}{p} \quad \text{in } \mathbf{R}_0^+, \\ u(0) &= 0, \\ \lim_{x\to+\infty} u(x) &= 0. \end{cases}
$$

To solve it, we introduce the associated energy functional

$$
J(u) := \int_0^\infty \frac{|u'|^p}{x^{ap}} dx - \lambda \int_0^\infty \frac{|u|^p}{x^{(a+1)p}} dx - \langle f, u \rangle
$$

defined on the closure $\mathcal{D}^{1,p}_a(\mathbf{R}_0^+)$ of $\mathcal{D}(\mathbf{R}_0^+)$ with respect to the norm $||u|| = (\int_0^\infty \frac{|u'|^p}{x^{ap}} dx)^{1/p}$, where

$$
\text{(5.2)} \qquad a \geqslant 0,
$$

$$
\text{(5.3)} \qquad 1 < (a+1)p < \infty,
$$

$$
\text{(5.4)} \qquad f \in \left(\mathcal{D}^{1,p}_a(\mathbf{R}_0^+)\right)'.
$$

We consider

$$
\inf J := \inf_{u \in \mathcal{D}^{1,p}_a(\mathbf{R}_0^+)} J(u).
$$

The aim of this section is to prove the following result.

Theorem 5.1. *Assume* (5.2), (5.3), (5.4) *and* $\lambda < \lambda_{ap} := (\frac{(a+1)p-1}{p})^p$. *Then* $\inf J$ *is achieved and problem* (5.1) *admits a solution.*

In order to prove this theorem, we recall the classical Hardy inequality (see e.g. [31, 60]).

Lemma 5.2 (Hardy Inequality). *Assume* (5.2) *and* (5.3). *If* $u \in \mathcal{D}^{1,p}_a(\mathbf{R}_0^+)$, *then*

$$
\lambda_{ap} \int_0^\infty \frac{|u|^p}{x^{(a+1)p}} dx \leqslant \int_0^\infty \frac{|u'|^p}{x^{ap}} dx.
$$

This inequality implies that the functional J is continuous, G-differentiable and coercive if $\lambda < \lambda_{ap}$. Let $(u_n) \subset \mathcal{D}^{1,p}_a(\mathbf{R}_0^+)$ a minimizing sequence for J, i.e.

$$
J(u_n) \to \inf J \quad \text{as } n \to +\infty.
$$

The following lemma is used in the proof of Lemma 5.4.

Lemma 5.3. *Assume* (5.2) *and* (5.3)*. If* $u_n \rightharpoonup u$ *in* $\mathcal{D}_a^{1,p}(\mathbf{R}_0^+)$*, then* $\frac{u_n}{x^a} \to \frac{u}{x^a}$ *in* $L^p([\frac{1}{c}, c])$ *for every* $c > 1$*.*

Proof. The result is clear as $\frac{1}{x^a}$ is bounded on the interval considered. □

The following decomposition lemma is proved as Lemmas 2.3 and 3.4.

Lemma 5.4. *Let* $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbf{R}_0^+)$ *such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}_a^{1,p}(\mathbf{R}_0^+)$*. We define*

$$\begin{aligned} \Omega_R &:= (0, R^{-1}) \cup (R, +\infty), \\ \mu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n'|^p}{x^{ap}} dx, \\ \nu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{x^{(a+1)p}} dx. \end{aligned}$$

Then we have

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int_0^\infty \frac{|u_n|^p}{x^{(a+1)p}} dx &= \int_0^\infty \frac{|u|^p}{x^{(a+1)p}} dx + \nu_0, \\ \overline{\lim_{n\to\infty}} \int_0^\infty \frac{|u_n'|^p}{x^{ap}} dx &\geqslant \int_0^\infty \frac{|u'|^p}{x^{ap}} dx + \mu_0, \\ \lambda_{ap}\, \nu_0 &\leqslant \mu_0. \end{aligned}$$

Proof sketch for Theorem 5.1. Theorem 5.1 results from Lemmas 5.2 and 5.4, similarly to the proofs of Theorems 2.1 and 3.2; note the similarity between Lemma 5.4 and Lemma 2.3. □

6. Exterior-domain case

Let the problem

$$(6.1) \qquad \left\{ \begin{aligned} -\Delta_p u &= \lambda \tfrac{|u|^{p-2}u}{\delta(x)^p} + \tfrac{f(x)}{p} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} u(x) &= 0, \end{aligned} \right.$$

where Ω is an exterior domain, i.e. an open subset of $\mathbf{R}^N$ with a compact boundary of class $\mathcal{C}^2$ and without bounded component, and where $\delta(x)$ denotes the function

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

We also suppose $N \geqslant 2$, $1 < p \neq N$ and $f \in \left(\mathcal{D}_0^{1,p}(\Omega)\right)'$. To solve problem (6.1), we introduce the associated functional:

$$J(u) := \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega \left|\frac{u}{\delta}\right|^p dx - \langle f, u\rangle,$$

defined on $\mathcal{D}_0^{1,p}(\Omega)$. In this section we use the following Hardy's inequality for exterior domains [15].

Lemma 6.1 (Chabrowski-Willem). *Assume $N \geqslant 2$, $p > 1$ and $p \neq N$. Then there exists $\lambda_{Np} > 0$ such that for every $u \in \mathcal{D}_0^{1,p}(\Omega)$,*

$$\lambda_{Np} \int_\Omega \left|\frac{u}{\delta}\right|^p dx \leqslant \int_\Omega |\nabla u|^p dx.$$

The main result of this section is the following theorem.

Theorem 6.2. *Let $N \geqslant 2$ and $1 < p \neq N$. If $\lambda < \lambda_{Np}$, then* $\inf J$ *on $\mathcal{D}_0^{1,p}(\Omega)$ is achieved and problem* (6.1) *has a solution.*

Proof sketch for Theorem 6.2. Again, the proof is akin to the proofs of Theorems 2.1, 3.2 and 5.1, We only sketch the main ideas.

The Hardy-type inequality implies that the functional J is continuous, G-differentiable and coercive if $\lambda < \lambda_{Np}$. Let $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$ be a minimizing sequence for J. We can assume that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. The end of the proof is a direct consequence of the following decomposition lemma; which is proved as Lemmas 2.3 and 3.4.. $\square$

Lemma 6.3. *Let $N \geqslant 2$, $1 < p \neq N$ and $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. We define*

$$\begin{aligned} \Omega_R &:= \left\{x \in \Omega : \delta(x) < R^{-1} \textit{ or } |x| > R\right\}, \\ \mu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} |\nabla u_n|^p dx, \\ \nu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \left|\frac{u_n}{\delta}\right|^p dx, \end{aligned}$$

then we have

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int_\Omega \left|\frac{u_n}{\delta}\right|^p dx &= \int_\Omega \left|\frac{u}{\delta}\right|^p dx + \nu_0, \\ \overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p dx &\geqslant \int_\Omega |\nabla u|^p dx + \mu_0, \\ \lambda_p \nu_0 &\leqslant \mu_0. \end{aligned}$$

Remark 6.4. Theorem 6.2 is also valid for bounded $\mathcal{C}^2$ domains for any $p > 1$.

CHAPTER V

Non-uniqueness results for equations involving singularities

1. Introduction

In Chapter IV, we studied the existence of solutions of singular problems of the form

$$\begin{cases} -\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right) & = & \lambda\frac{|u|^{p-2}u}{|x|^{(a+1)p}} + \frac{f(x)}{p} & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} u(x) & = & 0 & \end{cases}$$

by minimization. We showed solutions exist by developing suitable concentration-compactness tools to control the lack of compactness at infinity or near the singularity. This study was also done for operators having a singularity localized on an hyperplane, on the boundary of an exterior domain, as well as for the corresponding ordinary differential equation defined on the positive real axis.

When the existence of a solution is ensured, one can try to treat the question of unicity.

For the linear case $p = 2$, the functional corresponding to the problem is strictly convex, so the solution is unique.

A more delicate question is the unicity if $1 < p < 2$ or $p > 2$. We give a class of forcing terms implying the non-uniqueness of solutions for all the problems studied when $p > 2$, and for problems (IV.1.1), (IV.4.1) and (IV.5.1) when $1 < p < 2$.

2. Case $p > 2$.

In general, solutions are not unique. The idea of the counter example is due to del Pino-Elgueta-Manasevich [20]. García-Peral [25] adapted it to problem (IV.1.1) on bounded domains. Their argument is also valid for unbounded Ω. For the sake of completeness, we give here the proof for problem (IV.3.1).

Proposition 2.1. *Let $p > 2$ and Ω a regular domain of $\mathbf{R}^N$. There exists a forcing term f such that problem* (IV.3.1) *has at least two solutions.*

Proof. Consider $B(x_0, 2R) \subset \Omega$ a ball such that $0 \notin B(x_0, 2R)$, and a function $u_0 \in \mathcal{D}(B(x_0, 2R))$ such that $u_0 = 1$ in $B(x_0, R)$. Let

$$f(x) := -p\left(\operatorname{div}\left(\frac{|\nabla u_0|^{p-2}\nabla u_0}{|y|^{ap}}\right) + \lambda\frac{|u_0|^{p-2}u_0}{|y|^{(a+1)p}}\right).$$

So obviously u_0 is a solution of the problem. On the other hand, $f \in \left(\mathcal{D}^{1,p}_{a,k}(\Omega)\right)'$ and then we can find a solution u_1 by minimization of J. We can see that $u_0 \neq u_1$ because u_0 is not a minimum of J: as $p > 2$ and by the choice of u_0, the functional J is twice differentiable in u_0 and

$$\langle J''(u_0)v, v\rangle = p(p-1)\left(\int_\Omega \frac{|\nabla u_0|^{p-2}|\nabla v|^2}{|y|^{ap}}dx - \lambda\int_\Omega \frac{|u_0|^{p-2}v^2}{|y|^{(a+1)p}}dx\right).$$

Hence, for $v \in \mathcal{D}(B(x_0, R))$, we obtain

$$\langle J''(u_0)v, v\rangle = -p(p-1)\lambda\int_\Omega \frac{|u_0|^{p-2}v^2}{|y|^{(a+1)p}}dx,$$

which is negative for $\lambda > 0$, so u_0 is not a minimum of J. □

This approach works too for problems (IV.1.1), (IV.4.1), (IV.5.1) and (IV.6.1).

3. Case $1 < p < 2$.

The main idea in this section is due to X. Zhong [62] who studied problem (IV.1.1). We adapt his technique to problem (IV.4.1). This method works provided $0 \in \Omega$.

3.1. *The radial case*

We consider the particular case $k = N$ and $p = 2$ of the Hardy-Sobolev inequality given at Lemma IV.3.1, and we prove its optimality.

Lemma 3.1. *Let Ω be a regular open domain containing the origin. If $b < N$ and $u \in \mathcal{D}(\Omega)$, then*

$$\int_\Omega \frac{|u|^2}{|x|^b}dx \leqslant C_b\int_\Omega \frac{|\nabla u|^2}{|x|^{b-2}}dx,$$

where $C_b = \left(\frac{2}{N-b}\right)^2$.

Moreover the constant C_b is optimal when Ω is the whole space $\mathbf{R}^N$ or a ball centered at the origin.

Proof. The inequality is a particular case of Lemma IV.3.1. To obtain the optimality of the constant, it suffices to consider, given $\varepsilon > 0$, the radial function

$$u(x) := \begin{cases} C_{b,\varepsilon} & \text{if } 0 \leqslant |x| \leqslant 1, \\ C_{b,\varepsilon}|x|^{-\frac{N-b}{2}-\varepsilon} & \text{if } |x| > 1, \end{cases}$$

where $C_{b,\varepsilon} = \frac{2}{N-b+2\varepsilon}$. A straightforward computation shows the constant C_b is optimal.

The inequality is obviously true on subsets of $\mathbf{R}^N$, and the optimality of the constant is shown by dilation on every ball $B(0,R)$. $\square$

Given Lemma 3.1, we can prove non-uniqueness.

Proposition 3.2. *Let $1 < p < 2$ and Ω a regular domain of $\mathbf{R}^N$ with $0 \in \Omega$. There exists a forcing term f such that problem* (IV.4.1) *has at least two solutions.*

Proof. Consider a ball $B(0,2R) \subset \Omega$ and take a function $\varphi \in \mathcal{D}(B(0,2R))$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $B(0,R)$. Let, for $\varepsilon > 0$ (small),

$$u_0(x) := \frac{2-p}{N-(a+1)p-\varepsilon}|x|^{\frac{N-(a+1)p-\varepsilon}{2-p}}\varphi(x).$$

Such a u_0 is in $\mathcal{D}^{1,p}_a(\Omega)$ for ε sufficiently small.

As in the case $p > 2$, let

$$f(x) := -p\left(\operatorname{div}\left(\frac{|\nabla u_0|^{p-2}\nabla u_0}{|x|^{ap}}\right) + \lambda\frac{|u_0|^{p-2}u_0}{|x|^{(a+1)p}}\right).$$

This function f belongs to $(\mathcal{D}^{1,p}_a(\Omega))'$, so u_0 is a solution of problem (IV.4.1).

On the other hand, we can find a solution u_1 by minimization of J, as seen in Section IV.4. Now, showing that u_0 is not a minimizer of J, we check that $u_0 \neq u_1$. Indeed,

$$\langle J'(u_0), v\rangle = p\left(\int_\Omega \frac{|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla v}{|x|^{ap}}dx - \lambda\int_\Omega \frac{|u_0|^{p-2}u_0 v}{|x|^{(a+1)p}}dx\right) - \langle f, v\rangle,$$

$$\begin{aligned}\langle J''(u_0)v,v\rangle &= p\left((p-2)\int_\Omega \frac{|\nabla u_0|^{p-4}(\nabla u_0\cdot\nabla v)^2}{|x|^{ap}}dx\right.\\ &\qquad\left.+\int_\Omega \frac{|\nabla u_0|^{p-2}|\nabla v|^2}{|x|^{ap}}dx-\lambda(p-1)\int_\Omega \frac{|u_0|^{p-2}v^2}{|x|^{(a+1)p}}dx\right),\\ &\leqslant p\left(\int_\Omega \frac{|\nabla u_0|^{p-2}|\nabla v|^2}{|x|^{ap}}dx-\lambda(p-1)\int_\Omega \frac{|u_0|^{p-2}v^2}{|x|^{(a+1)p}}dx\right).\end{aligned}$$

If $v\in\mathcal{D}(B(0,R))$, we have

$$\begin{aligned}\langle J''(u_0)v,v\rangle \leqslant p&\left(\int_{B(0,R)}\frac{|\nabla v|^2}{|x|^{N-\varepsilon-2}}dx\right.\\ &\left.-\lambda(p-1)\left(\tfrac{2-p}{N-(a+1)p-\varepsilon}\right)^{p-2}\int_{B(0,R)}\frac{v^2}{|x|^{N-\varepsilon}}dx\right).\end{aligned}$$

By Lemma IV.3.1 with $b=N-\varepsilon$, we have for every $v\in\mathcal{D}(B(0,R))$

$$\frac{\int_{B(0,R)}\frac{|\nabla v|^2}{|x|^{N-\varepsilon-2}}dx}{\int_{B(0,R)}\frac{|v|^2}{|x|^{N-\varepsilon}}dx}\geqslant\left(\frac{\varepsilon}{2}\right)^2$$

and the constant is optimal. We can choose a v such that

$$\frac{\int_{B(0,R)}\frac{|\nabla v|^2}{|x|^{N-\varepsilon-2}}dx}{\int_{B(0,R)}\frac{|v|^2}{|x|^{N-\varepsilon}}dx}\leqslant 2\left(\frac{\varepsilon}{2}\right)^2.$$

Hence

$$\begin{aligned}&\langle J''(u_0)v,v\rangle\\ &\quad\leqslant p\left(2\left(\tfrac{\varepsilon}{2}\right)^2-\lambda(p-1)\left(\tfrac{2-p}{N-(a+1)p-\varepsilon}\right)^{p-2}\right)\int_{B(0,R)}\frac{v^2}{|x|^{N-\varepsilon}}dx\end{aligned}$$

and the coefficient is negative for $\lambda>0$ and ε small enough. So u_0 is not a minimizer and we have two different solutions of problem (IV.4.1). □

3.2. *One-dimensional case*

The method works well for the one-dimensional case with some modifications. Consider the Hardy inequality of Lemma IV.5.2 with $p=2$ and $b=ap-2$, i.e. the classical Hardy inequality.

Lemma 3.3. *Assume $b>1$ and $u\in\mathcal{D}(\mathbf{R}_0^+)$. Then*

$$\int_0^\infty\frac{|u|^2}{x^b}dx\leqslant C_b\int_0^\infty\frac{|u'|^2}{x^{b-2}}dx,$$

where $C_b = \left(\frac{2}{b-1}\right)^2$, *and the constant* C_b *is optimal.*

Moreover, the result is true in every interval (R, ∞) *with* $R > 0$.

Proof. See for instance [60]. □

Having that lemma, we prove non-uniqueness.

Proposition 3.4. *Let* $1 < p < 2$. *There exists a forcing term* f *such that problem* (IV.5.1) *has at least two solutions.*

Let $\varphi \in \mathcal{C}^\infty(\frac{R}{2}, +\infty)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $(R, +\infty)$. Let, for $\varepsilon > 0$ (small),

$$u_0(x) := \frac{p-2}{(a+1)p-1-\varepsilon} x^{\frac{(a+1)p-1-\varepsilon}{p-2}} \varphi(x).$$

Such a u_0 is in $\mathcal{D}_a^{1,p}(\mathbf{R}_0^+)$ for ε small enough.

Let

$$f(x) := -p\left(\left(\frac{|u_0'|^{p-2}u_0'}{x^{ap}}\right)' + \lambda\frac{|u_0|^{p-2}u_0}{x^{(a+1)p}}\right),$$

this function belongs to $(\mathcal{D}_a^{1,p}(\mathbf{R}_0^+))'$, so u_0 is a solution of equation (IV.5.1).

On the other hand, we can find a solution u_1 by minimization of J, as seen in Section IV.5. Now, showing that u_0 is not a minimizer of J, we check that $u_0 \neq u_1$. Indeed, if $v \in \mathcal{D}(0, R)$, we have

$$\frac{1}{p}\langle J''(u_0)v, v\rangle \leqslant \int_R^\infty \frac{|v'|^2}{x^{1+\varepsilon-2}}dx - \lambda(p-1)\left|\frac{p-2}{(a+1)p-1-\varepsilon}\right|^{p-2}\int_R^\infty \frac{v^2}{x^{1+\varepsilon}}dx.$$

But, by Lemma IV.3.3 with $b-2 = 1-\varepsilon$, for every $v \in \mathcal{D}(R, \infty)$ we have

$$\frac{\int_R^\infty \frac{|v'|^2}{x^{1+\varepsilon-2}}dx}{\int_R^\infty \frac{|v|^2}{x^{1+\varepsilon}}dx} \geqslant \left(\frac{\varepsilon}{2}\right)^2$$

and the constant is optimal. We can choose a function v such that

$$\frac{\int_R^\infty \frac{|v'|^2}{x^{1+\varepsilon-2}}dx}{\int_R^\infty \frac{|v|^2}{x^{1+\varepsilon}}dx} \leqslant 2\left(\frac{\varepsilon}{2}\right)^2.$$

Hence

$$\langle J''(u_0)v, v\rangle \leqslant p\left(2\left(\frac{\varepsilon}{2}\right)^2 - \lambda(p-1)\left|\frac{p-2}{(a+1)p-1-\varepsilon}\right|^{p-2}\right)\int_R^\infty \frac{v^2}{x^{1-\varepsilon}}dx$$

and the coefficient is negative for $\lambda > 0$ and $\varepsilon > 0$ small enough. So the solution u_0 is not a minimizer of the functional, whereas the solution u_1 is a minimizer.

Remark 3.5. For the one-dimensional case, we considered a function u_0 localized "far away from the origin". This technique can be used for the multidimensional case (i.e. for problem (IV.1.1)) when $0 \notin \Omega$ and Ω unbounded.

3.3. *Exterior domain case*

For the exterior-domain case, the same idea does not work. Considering u_0 localized "far away" from the boundary $u_0(x) := \frac{1}{\alpha}|x|^\alpha \varphi(x)$ where $\varphi \in \mathcal{C}^\infty(B^c(0, \frac{R}{2}))$ and $\alpha < \frac{p-N}{p}$. Doing the computations as before, for $v \in \mathcal{D}(B^c(0,R))$ we have

$$\frac{1}{p}\langle J''(u_0)v, v\rangle \leqslant \int_{B^c} \frac{|\nabla v|^2}{|x|^{(\alpha-1)(2-p)}} dx - \frac{\lambda(p-1)}{|\alpha|^{p-2}} \int_{B^c} \frac{v^2}{\delta^p\, |x|^{\alpha(2-p)}} dx,$$

which is difficult to handle using an Hardy inequality as the weights are of different nature.

To overcome this problem, let $u_0(x) := \frac{1}{C}\delta^\alpha(x)\varphi(x)$ where $\varphi \in \mathcal{C}^\infty(B^c(0, \frac{R}{2}))$ and C is a real constant. For $v \in \mathcal{D}(B^c(0,R))$ we have

$$\frac{1}{p}\langle J''(u_0)v, v\rangle$$
$$\leqslant \int_{B^c} \frac{1}{|C|^{p-2}} \left|\nabla(\delta^\alpha)\right|^{p-2} |\nabla v|^2 dx - \frac{\lambda(p-1)}{|C|^{2-p}} \int_{B^c} \frac{v^2}{\delta^{(1-\alpha)p+2\alpha}} dx,$$

and the expression $\nabla(\delta^\alpha)$ is delicate to treat.

Moreover, if we are able to express $\langle J''(u_0)v, v\rangle$ in terms of v and δ, another problem appears: the optimal constant in the corresponding Hardy inequality is not known. So we are not able to adapt the technique to the exterior-domain problem (IV.6.1).

Bibliography

[1] B. Abdellaoui. *Ecuaciones elípticas y parabólicas relacionadas con las desigualdades de Caffarelli-Kohn-Nirenberg.* Tesis Doctoral, Universidad Autónoma de Madrid, Madrid, 2003.

[2] B. Abdellaoui and I. Peral. Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential. *Ann. Mat. Pura Appl.* 182 (2003), 247–270

[3] A. Ambrosetti and P. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* 14 (1973), 349–381.

[4] T. Bartsch and Z.Q. Wang. Existence and multiplicity results for some superlinear elliptic problems on $\mathbf{R}^N$. *Comm. Partial Diff. Eqs.* 20 (1995), 1725-1741.

[5] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg. Variational methods for indefinite superlinear homogeneous elliptic problems. *NoDEA* 2 (1995), 553–572.

[6] H. Berestycki and P.L. Lions. Nonlinear scalar field equations. *Arch. Rat. Mech. Anal.* 82 (1983), 313-379.

[7] G. Bianchi, J. Chabrowski and A. Szulkin. On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent. *Nonlinear Anal.* 25 (1995), 41–59.

[8] L. Boccardo and F. Murat. Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.* 19 (1992), 581–597.

[9] H. Brezis. *Analyse fonctionnelle. Théorie et application.* Masson, Paris, 1983.

[10] H. Brezis. Symmetry in nonlinear PDE's. In *Proceedings of Symposia in Pure Mathematics* vol. 65. Amer. Math. Soc., Providence, 1999, pp. 1–12.

[11] H. Brezis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* 36 (1983), 437–477.

[12] F. Brock, V. Ferone and B. Kawohl. A symmetry problem in the calculus of variations. *Calc. of Var. and PDE* 4 (1996), 593–599.

[13] J. Byeon. Existence of many nonequivalent nonradial positive solutions of semilinear elliptic equations on three-dimensional annuli. *J. Diff. Eqs.* 136 (1997), 136–165.

[14] L. Caffarelli, R. Kohn and L. Nirenberg. First order interpolation inequalities with weights. *Compos. Math.* 53 (1984), 259–275.

[15] J. Chabrowski and M. Willem. Hardy's inequality on exterior domains. *Proc. AMS*, to appear.

[16] C.V. Coffman. A nonlinear boundary value problem with many positive solutions. *J. Diff. Eqs.* 54 (1984), 429–437.

[17] F. Colin. Hardy's inequality in unbounded domains. *Topol. Meth. Nonlinear Anal.* 17 (2001), 277–284.

[18] D.G. Costa. On a class of elliptic systems in $\mathbf{R}^N$. *Elec. J. Diff. Eqs.* 7 (1994), 1-14.

[19] J. Díaz and J. Saa. Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires. *C. R. Acad. Sci. Paris* 305 (1987), 521–524.

[20] M. del Pino, M. Elgueta and R. Manasevich. A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(x,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. *J. Diff. Eqs.* 80 (1989), 1–13.

[21] W.Y. Ding and W.M. Ni. On the existence of positive entire solutions of a semilinear elliptic equation. *Arch. Rat. Mech. Anal.* 91 (1986), 283-308.

[22] M. J. Esteban. Nonsymmetric ground states of symmetric variational problems. *Comm. Pure Appl. Math.* 44 (1991), 259–274.

[23] V. Felli and M. Schneider. Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type. *J. Diff. Eqs.* 191 (2003), 121–142.

[24] J. Fleckinger, J. Hernández, P. Takáč and F. De Thélin. Uniqueness and positivity for solutions of equations with the p-Laplacian. In *Reaction diffusion systems (Trieste, 1995).* Lecture Notes in Pure and Appl. Math. volume 194. Dekker, New York, 1998, pp. 141–155.

[25] J. García and I. Peral. Hardy inequalities and some critical elliptic and parabolic problems. *J. Diff. Eqs.* 144 (1998), 441–476.

[26] B. Gidas, W.M. Ni and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* 68 (1979), 209–243.

[27] P. Guasoni. *Problemi di ottimizzazione di forma su classi di insiemi convessi.* Laurea Thesis, Università di Pisa, 1996.

[28] O. Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques.* Springer-Verlag, Paris, 1993.

[29] B. Kawohl. Symmetry or not? *Math. Intelligencer* 20 (1998), 16–22.

[30] B. Kawohl. Symmetrization - or how to prove symmetry of solutions to a PDE. In *CRC Res. Notes Math.* vol. 406. Chapman & Hall, Boca Raton, 2000, pp. 214–229.

[31] A. Kufner and B. Opic. *Hardy-type Inequalities.* Pitman Research Notes in Math. vol. 219. Longman, Harlow, 1990.

[32] Y. Y. Li. Existence of many positive solutions of semilinear elliptic equations on annulus. *J. Diff. Eqs.* 83 (1990), 348–367.

[33] P.L. Lions. Symétrie et compacité dans les espaces de Sobolev. *J. Funct. Anal.* 49 (1982), 315–334.

[34] P.L. Lions. The concentration-compactness method in the calculus of variations: the locally compact case. *Ann. Inst. H. Poincaré* 1 (1984), 109–145 and 223–283.

[35] M. Marcus, V. J. Mizel and Y. Pinchover. On the best constant for Hardy's inequality in $\mathbf{R}^N$. *Trans. AMS* 350 (1998), 3237–3255.

[36] E. Montefusco. Lower semicontinuity of functionals via the concentration-compactness principle. *J. Math. Anal. Appl.* 263 (2001), 264–276.

[37] I. Newton. *The Principia, Mathematical Principles of Natural Philosophy.* Translated by I. Cohen and A. Whitman. The University of California Press, Berkeley, 1999.

[38] W.M. Ni. A nonlinear Dirichlet problem on the unit ball and its applications. *Indiana Univ. Math. J.* 31 (1982), 801–807.

[39] W. Omana and M. Willem. Homoclinic orbits for a class of Hamiltonian systems. *Diff. Int. Eqs.* 5 (1992), 1115-1120.

[40] R. Palais. The Principle of Symmetric Criticality. *Comm. Math. Phys.* 69 (1979), 19–30.

[41] B. Pellacci. Multiple critical points for nondifferentiable functionals involving Hardy potentials. *Preprint*.

[42] P.H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*. CBMS Reg. Conf. Series in Math. vol. 65, Amer. Math. Soc., Providence, 1986.

[43] P.H. Rabinowitz. On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* 43 (1992), 270-291.

[44] W. Rother. Some existence results for the equation $-\Delta u + K(x)u^p = 0$. *Comm. Partial Diff. Eqs.* 15 (1990), 1461-1473.

[45] M. Schneider. *Entire Solutions of Semilinear Elliptic Problems with Indefinite Nonlinearities*. Dissertation Dr. Naturwiss., Shaker Verlag, Aachen, 2001.

[46] S. Secchi, D. Smets and M. Willem. Remarks on a Sobolev-Hardy inequality. *C. R. Acad. Sci. Paris* 336 (2003), 811–815.

[47] P. Sintzoff. Symmetry of solutions of a semilinear elliptic equation with unbounded coefficients. *Diff. Int. Eqs.* 16 (2003), 769–786.

[48] P. Sintzoff. Multiplicity of solutions of a semilinear elliptic equation with coefficients on symmetric domains. *Adv. Nonlinear Stud.* 3 (2003), 419–429.

[49] P. Sintzoff. Existence of minimizer of some functionals involving Hardy-type inequalities. *Topol. Meth. Nonlinear Anal.*, to appear.

[50] P. Sintzoff and M. Willem. A semilinear elliptic equation on $\mathbf{R}^N$ with unbounded coefficients. In *Variational and topological methods in the study of nonlinear phenomena*. PNLDE vol. 49, Birkhäuser, Boston, 2001, pp. 107–115.

[51] B. Sirakov. Existence and multiplicity of solutions of semi-linear elliptic equations in $\mathbf{R}^N$. *Calc. of Var. and PDE* 11 (2000), 119–142.

[52] B. Sirakov. *Contributions à l'étude des problèmes elliptiques dans des domaines non bornés*. Thèse, Université de Paris 6, Paris, 2000.

[53] D. Smets. A concentration-compactness lemma with applications to singular eigenvalue problems. *J. Funct. Anal.* 167 (1999), 463–480.

[54] D. Smets, J. Su and M. Willem. Non radial ground states for the Hénon equation. *Commun. Contemp. Math.* 4 (2002), 467–480.

[55] W.A. Strauss. Existence of Solitary Waves in Higher Dimensions. *Comm. Math. Phys.* 55 (1977), 149-162.

[56] M. Struwe. *Variational Methods*. Springer-Verlag, Berlin, 1990.

[57] A. Szulkin and M. Willem. Eigenvalue problems with indefinite weight. *Studia Math.* 135 (1999), 191–201.

[58] Virgil. *The Aeneid*. Translated by R. Fitzgerald. Vintage Classics, New York, 1990.

[59] M. Willem. *Minimax Theorems*. Birkhäuser, Boston, 1996.

[60] M. Willem. *Analyse fonctionelle élémentaire*. Cassini, Paris, 2003.

[61] M. Willem. A decomposition lemma and critical minimization problems. In *Proceedings of the Workshop on Morse Theory, Minimax Theory and Their Applications*. Chinese Academy of Sciences, Beijing, 2003, pp. 247–256.

[62] X. Zhong. A note on quasilinear elliptic equations with critical potential. University of Jyväskylä, *personal communication*.