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Multiplicity, instability and sunspots in games

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Abstract

This paper considers games with two players for which it provides a sufficient condition on the responsiveness of the players' best replies around a Nash equilibrium that implies (i) a multiplicity of Nash equilibria; (ii) the non-isolatedness of this Nash equilibrium as rationalizable strategies; and (iii) the existence of non-trivial correlated equilibria arbitrarily close to it. This simultaneity of multiplicity, instability and vulnerability to sunspots parallels the same pattern observed in overlapping generations economies and finite economies with asymmetric information, and hints at some underlying relation between different avatars of the indeterminacy of economies and games that goes beyond the boundaries of any specific framework. Global links between multiplicity, instability and sunspots are also provided.

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1. Introduction

It is a well-known fact that the simultaneous and independent decision-making of several agents may not lead to a determinate outcome. This is the case in, for instance, a competitive exchange economy with two agents with standard preferences over two commodities¹ if it has a competitive equilibrium allocation c where the two agents' offer curves cross in the specific way shown in Fig. 1. In effect, as a consequence of such crossing there exist at least

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¹ That is to say, representable by a continuous utility function u defined in $\mathbb{R}_{++}^2$, differentiable in $\mathbb{R}_{++}^2$, and such that, for all $c \in \mathbb{R}_{++}^2$, $Du(c) \in \mathbb{R}_{++}^2$ and $D^2u(c)$ is negative definite in the space orthogonal to $Du(c)$, and moreover is "well-behaved" in the boundary, in the sense that for all $i = 1, 2$, $Du(c) \cdot e_i \rightarrow 0$ as $c \rightarrow \bar{c}$, where $\bar{c}$ is any point of the axis of the i th good distinct from the origin and e_i is the i th vector of the canonical basis.

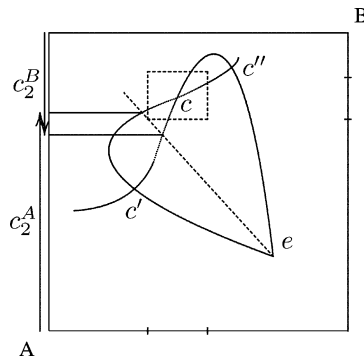


Fig. 1.

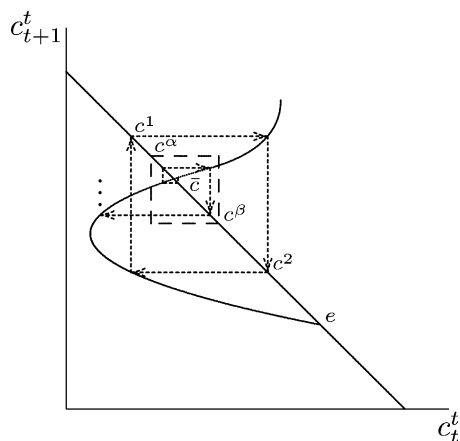
two other competitive equilibrium allocations c' and c'' . Moreover, the competitive equilibrium c is unstable under the tâtonnement process. And finally, there exist arbitrarily close to this equilibrium allocation c other equilibrium allocations in which the agents' choices are contingent to a sunspot on which they have asymmetric information or, equivalently, contingent to two correlated private sunspots against which they cannot insure themselves.² Thus, a multiplicity of outcomes, the instability of some of these outcomes with respect to some process, and the vulnerability to sunspots of the outcome come hand in hand in this framework.

Another instance of this fact is provided by the simple overlapping generations economies. In effect, whenever the slope of the offer curve of the representative agent at the steady state is smaller than 1 in absolute value, there is at least another stationary perfect foresight equilibrium besides the steady state, namely a cycle of period 2 (see Fig. 2). Moreover, the steady state is also indeterminate (i.e. unstable in the backward perfect foresight dynamics) and, finally, there exist local sunspot equilibria around it.³ Thus, in this set-up multiplicity, instability, and vulnerability to sunspots come hand in hand as well.

This paper shows the existence of a similar connection in a game-theoretic set-up between the issues of uniqueness/multiplicity, stability/instability, and vulnerability to sunspots/sunspot-proofness. In effect, in a game with two players a high enough responsiveness of each player's best reply to the other player's strategy around a Nash equilibrium in pure strategies implies (i) the existence of at least two other Nash equilibria, (ii) the existence

² A sunspot is a signal conveying no information about the fundamentals, i.e. a source of extrinsic uncertainty (see Shell, 1977; Cass and Shell, 1983). For the existence of correlated equilibria in this set-up see Maskin and Tirole (1987) (for their relation to sunspot equilibria in OLG economies see Dávila, 1999). In Fig. 1 it can be seen the instability of the tâtonnement around $\bar{c}$, and the support of a correlated equilibrium (the corners of the small box) in which agent A (resp. B) supplies to the market two possible amounts of good 1 (resp. 2) according to the realization of correlated private sunspots.

³ See Guesnerie (1986), Azariadis and Guesnerie (1986) and Woodford (1986). In Fig. 2 it can be seen that (i) a cycle of period 2 in which the global resources $e_1 + e_2$ of the economy are split between young and old agents at either c^1 or c^2 every other period; (ii) the instability of the steady state in the backward perfect foresight dynamics; and (iii) the support of a sunspot equilibrium in which the global resources are split at either c^a or c^b randomly according to a two-state, first-order Markov chain.



of non-trivial correlated equilibria arbitrarily close to the given Nash equilibrium, and (iii) the instability of the eductive dynamics of beliefs that would make the players converge to some Nash equilibrium as the only rationalizable outcome, i.e. the lack of convergence of the iterated elimination of dominated strategies to a single profile, and hence the non-local uniqueness of the Nash equilibrium as a profile of rationalizable strategies (see [Theorem 1](#) in [Section 3](#)). While (i) and (iii) above share the flavor of existing results, (ii) establishes a link between non-trivial correlated equilibria *arbitrarily close* to a Nash equilibrium in pure strategies that is, to the best of my knowledge, new.

Interestingly enough, the previous instances of links between multiplicity, instability, and vulnerability to sunspots of the outcomes of simultaneous and decentralized decisions in different set-ups hint at the existence of a general relation between the different manifestations of the indeterminacy that plagues economies and games alike. Finally, [Theorem 2](#) in [Section 3](#) as well establishes a condition for the iterative elimination of dominated strategies to converge locally to a Nash equilibrium in pure strategies even when it does not converge globally. Global connections are provided also in [Theorems 3 and 4](#) in [Section 4](#).

Two issues are closely related to the results presented in [Theorem 1](#). On the one hand, it was established in [Milgrom and Roberts \(1990\)](#) that in supermodular games there is a largest and a smallest profiles of serially undominated strategies that, moreover, are Nash equilibria in pure strategies. Since the set of profiles of serially undominated strategies contains the supports of the Nash equilibria in pure strategies, the correlated equilibria (either subjective or objective), and the profiles of rationalizable strategies, then the uniqueness of a pure Nash equilibrium would imply the coincidence of the largest and the smallest profiles of serially undominated strategies and, hence, the non-existence of non-trivial correlated equilibria, as well as the local uniqueness of the Nash equilibrium as profile of rationalizable strategies. Therefore, Milgrom and Roberts' result points implicitly also to the fact that multiplicity, vulnerability to sunspots and lack of convergence of the iterative elimination of dominated strategies must come hand in hand in some games, namely in supermodular games.

Nevertheless, the games that I consider in this paper are not necessarily supermodular. More specifically, the increasing differences hypothesis will not necessarily be satisfied globally. In the games studied in this paper, a differentiable version of the increasing or decreasing differences property is satisfied *locally* at every generic profile of strategies on the graph of the best reply function of any player. Hence, both payoff functions will exhibit either increasing or decreasing differences in a neighborhood of any Nash equilibrium in pure strategies. The local condition guaranteeing the connection between the different forms of indeterminacy will be satisfied whenever both payoff functions exhibit strong enough (jointly) increasing differences (or both of them exhibit strong enough decreasing differences) *locally* around a Nash equilibrium. This qualifies the need for supermodularity in [Milgrom and Roberts \(1990\)](#) in order to obtain this link and shows that a local version of the property suffices actually.

On the other hand, while investigating an eductive justification of the rational expectations equilibrium to simple overlapping generations economies, [Guesnerie \(1993\)](#) finds an equivalence between the determinacy of the steady state, the non-existence of local sunspot equilibria around it, and the steady state being strongly rational (i.e. a locally unique rationalizable equilibrium).⁴ Some of the results of this paper ([Theorem 1](#) specifically), can be seen as counterparts of Guesnerie's results in a simple class of games.

The structure of the remainder paper is as follows. [Section 2](#) presents the class of games considered, their Nash equilibria in pure strategies, and a class of simple non-trivial correlated equilibria of these games (they have a finite support of profiles of strategies). [Section 3](#) establishes the local results in [Theorems 1 and 2](#). [Section 4](#) presents global results in [Theorems 3 and 4](#) relating the three previous issues from a global viewpoint, as opposed to the local nature of the previous theorems. [Section 5](#) concludes and [Appendix A](#) collects proofs and auxiliary propositions.

2. The game and its equilibria

Consider a game $\Gamma = \{I, \{X_i, f_i\}_{i \in I}\}$, whose set of players is $I = \{-1, 1\}$, and each player's $i \in I$ strategy set X_i and payoff function f_i are, respectively, the interval $[0, 1]$ without loss of generality, and a C^2 function $f_i : X_i \times X_{-i} \rightarrow \mathbb{R}$ that is strictly unimodal in its first argument, for every value of the second argument.⁵ For every player $i \in I$, let $r_i(x_{-i})$ be the best reply of i to $-i$'s strategy x_{-i} .⁶ Thus, each r_i is a continuous function⁷ that takes values in $(0, 1)$. Let r_i^+ and r_i^- denote the sets $r_i^+ = \{(x_{-i}, x_i) \in X_{-i} \times X_i | x_i \geq r_i(x_{-i})\}$ and $r_i^- = \{(x_{-i}, x_i) \in X_{-i} \times X_i | x_i \leq r_i(x_{-i})\}$ (see [Fig. 3](#)).

In what follows we are going to establish links between the Nash equilibria in pure strategies and the correlated equilibria of this game.⁸

⁴ As well as equivalent to the convergence of any "reasonable" learning process. See also [Guesnerie \(2002\)](#).

⁵ More precisely, for each player $i \in I$ and all $x_{-i} \in [0, 1]$, there exists an $\bar{x}_i \in (0, 1)$ such that $f_i(\cdot, x_{-i})$ is strictly increasing in the interval $(0, \bar{x}_i)$ and strictly decreasing in the interval $(\bar{x}_i, 1)$.

⁶ Under the assumption of unimodality of the payoff functions the best reply is always in $(0, 1)$ and unique.

⁷ By Berge's Theorem of the Maximum.

⁸ Strictly speaking every Nash equilibrium in pure strategies can be identified to a trivial correlated equilibrium, therefore by correlated equilibria I refer to the non-trivial ones.

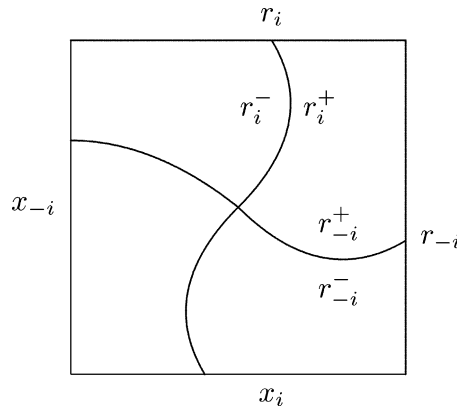


Fig. 3.

Definition 1. A Nash equilibrium⁹ in pure strategies of Γ is a $x \in \times_{i \in I} X_i$ such that, for all $i \in I$, and all $x'_i \in X_i$,

$$f_i(x'_i, x_{-i}) \leq f_i(x_i, x_{-i}). \quad (1)$$

Definition 2. A finite support correlated equilibrium $\{p_i, S_i, x_i\}_{i \in I}$ of Γ consists of

- (1) for each player $i \in I$, a joint probability distribution p_i over two random signals s_i and s_{-i} , privately observed by i and $-i$, respectively, taking values in $S_i = \{1, 2, \dots, S_i\}$ and $S_{-i} = \{1, 2, \dots, S_{-i}\}$ with $S_i, S_{-i} \geq 2$,¹⁰ and such that $p_i^{s_i s_{-i}}$ is the probability i attaches to s_i being his observation and s_{-i} being the other agent's observation, with $p_i^{s_i s_{-i}} = p_{-i}^{s_{-i} s_i}$ for all $s_i \in S_i, s_{-i} \in S_{-i}$,¹¹ and
- (2) two increasing functions $x_i \in X_i^{S_i}$,¹² for all $i \in I$,

⁹ Nash's theorem guarantees the existence of a Nash equilibrium of this game, since for all $i \in I$, X_i is a non-empty, compact, convex subset of the real line, and f_i is continuous and quasi-concave in x_i . Nevertheless, this result is proved in Proposition 1 in Appendix A by means of an argument that requires little more than the intermediate value theorem and, more interestingly, will reappear in the proof of Proposition 5 in Appendix A, which in turn will be needed in Section 4. More specifically, a Nash equilibrium of this game is shown in Proposition 1 to be associated to a zero of an adequately defined real-valued function on the real line. Note incidentally that the argument does not imply the uniqueness of the Nash equilibrium in pure strategies, since that function may have several zeros. Examples of best reply functions fitting in this framework and with multiple Nash equilibria are $r_i(x_{-i}) = 1/2 + 1/\pi \sin(2\pi(x_{-i} - 1/2))$, for all $i \in I$, as well as $r_i(x_{-i}) = 5(x - (1/2))(x^2 - x) + (1/2)$, for all $i \in I$, although an economic example in terms of payoff functions remains to be produced. In particular, the last example provided above has three Nash equilibria in pure strategies: namely $((1/2) - (\sqrt{5}/10), (1/2) + (\sqrt{5}/10))$, $(1/2, 1/2)$, and $((1/2) + (\sqrt{5}/10), (1/2) - (\sqrt{5}/10))$. Note that the games generating such best reply functions cannot be supermodular.

¹⁰ Thus, abusing notation only slightly, for all $i \in I$, S_i denotes both the set of values taken by s_i and the finite number of elements in this set.

¹¹ Therefore, these are *objective* correlated equilibria.

¹² That is to say, such that if $s_i < s'_i$, then $x_i^{s_i} < x_i^{s'_i}$. Note that, therefore, these correlated equilibria are actually non-trivial ones, i.e. those for which the players truly randomize their choices.

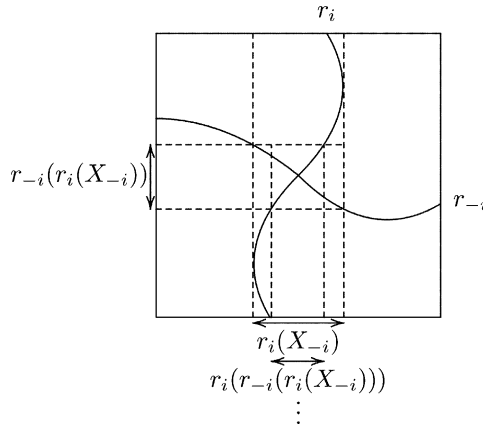


Fig. 4.

such that, for all $i \in I$ and all $x'_i \in X_i^{S_i}$,

$$\sum_{\substack{s_i \in S_i \\ s_{-i} \in S_{-i}}} p_i^{s_i s_{-i}} f_i(x'_i, x_{-i}^{s_{-i}}) \leq \sum_{\substack{s_i \in S_i \\ s_{-i} \in S_{-i}}} p_i^{s_i s_{-i}} f_i(x_i^{s_i}, x_{-i}^{s_{-i}}). \quad (2)$$

Whenever a multiplicity of Nash equilibria may arise, a stability test (very much like the stability of the tâtonnement in the finite economy or that of the perfect foresight dynamics in the overlapping generations economies) can be carried out on a given equilibrium to check whether a spontaneous coordination on it can be claimed on the grounds of such stability. Thus, a stability criterion that can be applied to a Nash equilibrium in pure strategies is its local uniqueness as a profile of rationalizable strategies, i.e. whether it is the unambiguous outcome of the iterative elimination of strategies that are not best responses to any nearby strategy. More specifically, given the best reply functions r_i , for each $i \in I$, x_i is a rationalizable strategy for player i if and only if for all $n \in \mathbb{N}$,

$$x_i \in (r_i \circ r_{-i})^n \circ r_i(X_{-i}). \quad (3)$$

I will denote by R_i the set $\bigcap_{n \in \mathbb{N}} (r_i \circ r_{-i})^n \circ r_i(X_{-i})$ of rationalizable strategies of player i (see Fig. 4 for a few iterations).

3. Local results

The next proposition establishes a sufficient condition on a Nash equilibrium x in pure strategies for the existence of other Nash equilibria in pure strategies, for the existence of non-trivial correlated equilibria arbitrarily close to x , and moreover for x to be a non-isolated profile of rationalizable strategies.

Theorem 1. If $x \in \times_{i \in I} X_i$ is a Nash equilibrium in pure strategies of Γ and

$$D_{11} f_1(x_1, x_{-1}) D_{11} f_{-1}(x_{-1}, x_1) < D_{12} f_1(x_1, x_{-1}) D_{12} f_{-1}(x_{-1}, x_1), \quad (4)$$

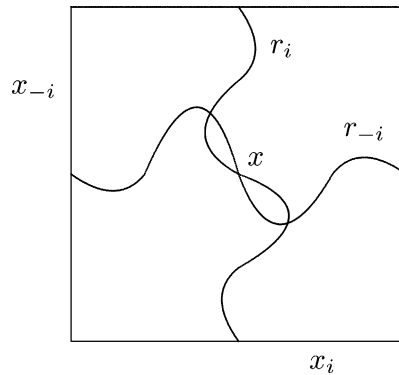


Fig. 5.

or, equivalently,¹³

$$\prod_{i \in I} r'_i(x_{-i}) > 1, \quad (5)$$

then

- (1) *there exist at least two other Nash equilibria in pure strategies,*
- (2) *there exist non-trivial correlated equilibria of Γ with support arbitrarily close to x ,*
and
- (3) *x is an accumulation point of the set of profiles of rationalizable strategies.*

Among the claims made in [Theorem 1](#) it is claim (2) the one that requires a more involved argument and its proof can be found in [Appendix A](#). With respect to claim (1), in order to have the right crossing of the best reply curves for a multiplicity of crossings to appear, the slope of r_i , for any given i , at the Nash equilibrium in pure strategies x must be bigger in absolute value than the reciprocal of the slope of r_{-i} at the same point, and both must have the same sign (see [Fig. 5](#)).

A proof of this intuitive proposition using the intermediate value theorem can be found in [Appendix A](#). In general, this is a consequence of Poincaré–Hopf's index theorem. In effect, the Nash equilibria in pure strategies of Γ happen to be zeros of an outward-pointing vector field $F(x_i, x_{-i}) = (x_i - r_i(x_{-i}), x_{-i} - r_{-i}(x_i))$ defined on the unit square by the equilibrium equations. The sufficient condition for the existence of another Nash equilibrium stated in [Theorem 1](#), is nothing else than the condition on the index of the Nash equilibrium x to be distinct from the Euler characteristic of the unit square. Since the sum of the indices of all the zeros of the vector field must coincide with the Euler characteristic, hence the necessary existence of at least two other zeros of the vector field, i.e. two other Nash equilibria in pure strategies.

¹³ Recall that the payoff functions are assumed to be C^2 and hence the best replies are generically differentiable.

Claim (3) of [Theorem 1](#) follows from the following fact. Not surprisingly, every Nash equilibrium is a profile of rationalizable strategies.¹⁴ Moreover, the convexity of the set of rationalizable strategies follows from the intermediate value theorem.¹⁵ Thus, given a multiplicity of Nash equilibria, the convexity of the sets of rationalizable strategies implies that every Nash equilibrium is an accumulation point of the set of profiles of rationalizable strategies.

Since $D_{11}f_i(x_i, x_{-i}) \leq 0$ for all $i \in I$ at a Nash equilibrium, and hence the left-hand side of condition (4) in [Theorem 1](#) is positive, then condition (4) implies that the two payoff functions must satisfy both either the increasing differences property locally around the Nash equilibrium, or the decreasing differences property both of them as well. Moreover, the product of the cross-partial derivatives must be bounded away from 0, i.e. these properties must hold strongly enough jointly, in such a way that if verified weakly by some agent it must be satisfied strongly enough by the other in order to offset that.

It is worth to note that although a multiplicity of Nash equilibria in Γ prevents any profile of strategies to be an isolated point of the set of profiles of rationalizable strategies, there may still be nonetheless a Nash equilibrium for which the iterative elimination of dominated strategies, if started within a small enough neighborhood of the equilibrium, does succeed to converge to it.¹⁶

Theorem 2. *If x is a Nash equilibrium of Γ such that*

$$\prod_{i \in I} |r_i'(x_{-i})| < 1, \quad (6)$$

then the iterative elimination of dominated strategies converges locally to x if started within a small enough neighborhood of x .

4. Global results

Besides the local links connecting Nash equilibria in pure strategies, correlated equilibria and rationalizability provided in [Theorem 1](#), the existence of a multiplicity of Nash equilibria in pure strategy implies the existence of correlated equilibria that are not a randomization over Nash equilibria in pure strategies. Similarly, a multiplicity of Nash equilibria in pure strategies implies that the set of rationalizable profiles is an interval of $\mathbb{R}^2$ with non-empty interior and, hence, the lack of convergence of the iterated elimination of dominated strategies. These results are collected in the following theorem.

Theorem 3. *If there exist multiple Nash equilibria in pure strategies of the game Γ , then*

- (1) *there exists a continuum of non-trivial correlated equilibria arbitrarily close to some Nash equilibrium in pure strategies also,¹⁷ and*
- (2) *the set of rationalizable profiles of strategies has no isolated point.*

¹⁴ See [Proposition 2](#) in [Appendix A](#).

¹⁵ See [Proposition 3](#) in [Appendix A](#).

¹⁶ In his search for an eductive justification of the rational expectations equilibria as sensible outcomes of an economy, [Guesnerie \(1992\)](#) calls such equilibria *locally strongly rational*.

¹⁷ Note that these correlated equilibria are not trivial lotteries on the multiple Nash equilibria in pure strategies.

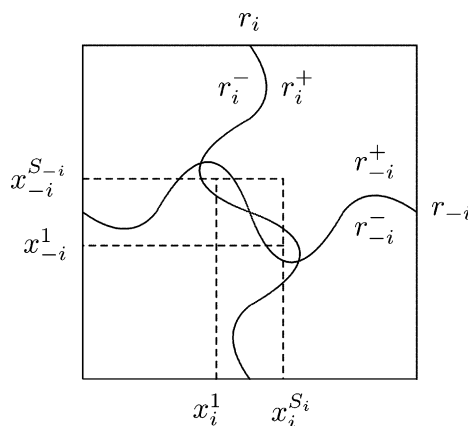


Fig. 6.

The converse statements are far from being true. For instance, if for all $i \in I$, $r_i(x_{-i}) = (1/2) + (1/4)i \sin 2\pi(x_{-i} - (1/2))$, then the only Nash equilibrium in pure strategies of this game is $x_i = 1/2$ for all $i \in I$, while every strategy $x_i \in [1/4, 3/4]$ is rationalizable, for all $i \in I$. Also, if for all $i \in I$, $r_i(x_{-i}) = (1/2) + (1/2)i(x_{-i} - (1/2))^{1/3}$, then the only Nash equilibrium in pure strategies of this game is $x_i = 1/2$ for all $i \in I$, while the game can be shown to have correlated equilibria.¹⁸ Nevertheless, a weaker version of the converse statement linking correlated equilibria to multiple Nash equilibria holds. It is a consequence of the following properties of the supports of both correlated and Nash equilibria in pure strategies.

On the one hand, at a correlated equilibrium no player will play with positive probability a strategy that may lead only to outcomes, so to speak, laying on the same side of his or her best reply function.¹⁹ As a consequence, the best reply curve of every player has to enter and exit the convex hull of the support of any correlated equilibrium at points where he plays either the minimum or the maximum of the strategies played with positive probability (see Fig. 6).

On the other hand, it can be shown that there exists a Nash equilibrium in pure strategies within the convex hull of the support of any non-trivial correlated equilibrium with finite support.²⁰

The two previous properties allow to establish the existence of multiple Nash equilibria whenever there exists a correlated equilibrium (not necessarily local to any Nash equilibrium

¹⁸ By the same argument used to prove claim (2) of Theorem 1. As a matter of fact, by that argument there exist correlated equilibria arbitrarily close to a Nash equilibrium in pure strategies of this game as soon as $\prod_{i \in I} |r'_i(x_{-i})| > 1$, which is implied by, but does not imply $\prod_{i \in I} r'_i(x_{-i}) > 1$.

¹⁹ The rationale for this is quite intuitive: should a player play with some probability a pure strategy such that, no matter what the other player does, the outcome is for sure below the first player's best reply function, then his payoff from every outcome would be bigger for a slightly higher strategy, and hence his expected payoff as well. See Proposition 4 in Appendix A.

²⁰ See Proposition 5 in Appendix A.

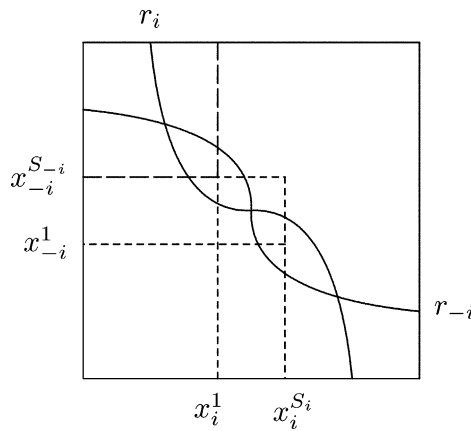


Fig. 7.

in pure strategies) and a Nash equilibrium not contained in the convex hull of the support of the correlated equilibrium is guaranteed to exist, as it is the case under the conditions provided by the following theorem.

Theorem 4. *If there exists a non-trivial correlated equilibrium of Γ and the best reply functions r_i are both non-decreasing or both non-increasing, then there exist multiple Nash equilibria in pure strategies.*

In effect, if $x_i \in X_i^{S_i}$ is player i 's strategy in the correlated equilibrium, and the best replies are, say, both non-increasing (see Fig. 7), then $r_i(x_{-i}^{S_i}) < x_i^1$ and $x_{-i}^{S_i} < r_{-i}(x_i^1)$ necessarily. Therefore, the restriction of the continuous vector field of equilibrium equations $F(x_i, x_{-i}) = (x_i - r_i(x_{-i}), x_{-i} - r_{-i}(x_i))$ to the subset $[0, x_i^1] \times [x_{-i}^{S_i}, 1]$ is still outward-pointing and hence must have at least one zero according to Poincaré–Höpf's index theorem, i.e. a Nash equilibrium in pure strategies of the game Γ . Since, according to Proposition 5 in Appendix A, there is at least another Nash equilibrium in pure strategies in the convex hull of the support of the correlated equilibrium, then there is a multiplicity of these equilibria (a similar argument can be developed for the case of non-decreasing best replies).²¹

Finally, as for the global relations between correlated equilibria and rationalizability in this set-up, the existence of non-trivial correlated equilibria implies trivially that the set of profiles of rationalizable strategies is an interval of $\mathbb{R}^2$ with non-empty interior,²² while the converse does not hold in general, as the following example shows. Let, for all $i \in I$,

²¹ In Appendix A can be found an alternative proof that does not resort to the use of the Poincaré–Höpf theorem. More generally, Proposition 6 in Appendix A gives a version of this result that does not require the monotonicity of the best reply functions.

²² Since the set of profiles of rationalizable strategies contains the support of every correlated equilibrium and, moreover, the set of rationalizable strategies of each player is convex (see Proposition 3 in Appendix A).

$$r_{-i}(x_i) = \begin{cases} x_i + \frac{1}{2} & x_i < \frac{1}{4}, \\ -x_i + 1 & \frac{1}{4} \leq x_i < \frac{3}{4}, \\ x_i - \frac{1}{2} & \frac{3}{4} \leq x_i. \end{cases} \quad (7)$$

The set of rationalizable strategies of player i , for all $i \in I$, is in this game $[1/4, 3/4]$, while there is no non-trivial correlated equilibrium.²³ Nonetheless, the very knife-edge nature of this example conveys the intuition that a generic converse may very likely still hold.

5. Discussion

The previous sections have established, in a specific class of games, both local and global links between multiple Nash equilibria, non-trivial correlated equilibria and the lack of convergence of the iterative elimination of dominated strategies. The local link parallels in part similar ones between the notions of multiplicity, instability, and vulnerability to sunspots in other seemingly completely unrelated frameworks as, for instance, overlapping generations economies and finite economies with asymmetric information.²⁴ Such a link between different forms of indeterminacy across economies and games, hints at a general phenomenon that may be common to every set-up consisting of several optimizers that must, simultaneously and independently, make a decision as they do in economies and games, independently of whether they behave strategically or not. Interestingly enough, a continuity argument shows these results to hold in games with three players (see a discussion on this in [Appendix A](#)), and hence with all likelihood to games with any finite number of players, although a precise proof of this fact is left for further research.

Finally, it may be worth to clarify the extent to which the results presented in this paper may be related to other seemingly close results. Firstly, I will comment about how the results presented here relate to the connection between Nash equilibria in pure strategies and correlated equilibria in [Peck and Shell \(1991\)](#) within the framework of market games. [Peck and Shell \(1991\)](#) study a market game in which there may or may not be the possibility of trading securities contingent to the realization of an extrinsic uncertainty on which the agents have an asymmetric information. For that game they prove that (Proposition 4.12) every allocation of a correlated equilibrium of the market game without securities is the allocation of a Nash equilibrium in pure strategies of the market game with securities contingent to the extrinsic uncertainty of the correlation device. Thus, Proposition 4.12 in [Peck and Shell \(1991\)](#) establishes a link between the correlated equilibria of the market game *without securities* and the Nash equilibria in pure strategies of the market game *with*

²³ There is no support that satisfies the necessary condition for it to be that of a correlated equilibrium established in [Proposition 5](#) in [Appendix A](#).

²⁴ In effect, although the same condition is sufficient for multiplicity, instability, and dependence on sunspots to appear, it is not a necessary condition for the existence of local correlated equilibria around a Nash equilibrium. As a matter of fact, local correlated equilibria around a Nash equilibrium can exist without the existence of a multiplicity of Nash equilibria, as stated in footnote 18.

securities.²⁵ Leaving aside the differences in the set-ups considered, it is interesting to note that the result established in Proposition 4.12 in Peck and Shell (1991) is quite different in nature from the link established in Theorem 1 (2) above. In effect, note that Peck and Shell (1991) link the correlated and pure Nash equilibria of two related but essentially *distinct* games, while Theorem 1 (2) establishes such a link for the *same* game.

Secondly, the kind of multiplicity of pure Nash equilibria that follows from Theorem 1 (1) is also quite different from the one following from the manifold structure of the set of pure Nash equilibria of the market games studied in Peck et al. (1992), if only because (among other things) the one-dimensional nature of the strategy sets of the players of the game considered in Theorem 1 above prevents it to be transformed into any kind of market game. At any rate, the indeterminacy result in Peck et al. (1992) is a consequence of having too few equations to pin down all the variables determining the Nash equilibrium of a market game, i.e. of the existence of degrees of freedom that make the set of pure Nash equilibria of a market game have the structure of a manifold with positive dimension and, hence, prevents any pure Nash equilibrium from being locally unique. Quite on the contrary, the multiplicity shown in Theorem 1 (1) above for non-market games results from Poincaré–Hopf’s index theorem, which entails the local uniqueness of the multiple Nash equilibria.

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Appendix A

Proposition 1. *There exists at least one Nash equilibrium in pure strategies of the game Γ .*

Proof. For any $i \in I$, let ϕ_i be the function from $[0, 1]$ to itself mapping each $x_{-i} \in X_{-i}$ to $\phi_i(x_{-i}) = x_{-i} - r_{-i}(r_i(x_{-i}))$. The function ϕ_i is continuous and takes positive and negative values, specifically $\phi_i(0) = -r_{-i}(r_i(0)) \in (-1, 0)$ while $\phi_i(1) = 1 - r_{-i}(r_i(1)) \in (0, 1)$ (see Fig. 8).

²⁵ Still, Lemma 5.3 in Peck and Shell (1991) makes clear that the market game with securities depending on an extrinsic uncertainty may have Nash equilibria in pure strategies that need not correspond to correlated equilibria of the original market game. Thus, the set of correlated equilibrium allocations is a proper subset of the set of allocations of the Nash equilibria of the market game with securities. As a consequence, the Nash equilibrium in pure strategies of the game with securities shown to exist (under some condition) for market games whose initial allocation is inefficient needs not be a correlated equilibrium of the original market game.

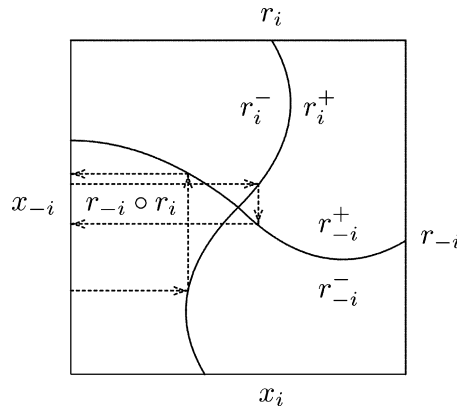


Fig. 8.

Therefore, necessarily there exists an $x_{-i} \in X_{-i}$ such that $\phi_i(x_{-i}) = 0$, i.e. $x_{-i} = r_{-i}(r_i(x_{-i}))$ or, equivalently, $(r_i(x_{-i}), x_{-i}) \in r_{-i}$. Letting x_i be $r_i(x_{-i})$, then $(x_{-i}, x_i) \in r_i \cap r_{-i}^{-1}$. $\square$

Proposition 2. *The strategies of every Nash equilibrium of the game Γ are rationalizable, i.e. for all $i \in I$,*

$$r_i \cap r_{-i}^{-1} \subset R_{-i} \times R_i. \quad (\text{A.1})$$

Proof. Assume that there exists $i \in I$ such that $r_i \cap r_{-i}^{-1} \not\subset R_{-i} \times R_i$. Then there exists $(x_{-i}, x_i) \in r_i \cap r_{-i}^{-1}$ such that $(x_{-i}, x_i) \notin R_{-i} \times R_i$. Thus, there exists $i \in I$ such that $x_i \notin R_i$, i.e. for which there exists $n \in \mathbb{N}$ such that

$$x_i \notin (r_i \circ r_{-i})^n \circ r_i(X_{-i}) = (r_i \circ r_{-i}) \circ (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i}). \quad (\text{A.2})$$

Hence, there does not exist $x'_i \in (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i})$ such that $(x'_i, x_i) \in r_i \circ r_{-i}$ or, equivalently, for all $x'_i \in (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i})$, $(x'_i, x_i) \notin r_i \circ r_{-i}$. Therefore, for all $x'_{-i} \in X_{-i}$, either $(x'_i, x'_{-i}) \notin r_{-i}$ or $(x'_{-i}, x_i) \notin r_i$. Since the domain of r_{-i} is X_i and $(r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i}) \subset X_i$, it cannot be true that, for all $x'_i \in (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i})$ and all $x'_{-i} \in X_{-i}$, $(x'_i, x'_{-i}) \notin r_{-i}$. Thus, necessarily, for all $x'_{-i} \in X_{-i}$, $(x'_{-i}, x_i) \notin r_i$ must hold. But $x_{-i} \in X_{-i}$ and $(x_{-i}, x_i) \in r_i$! Therefore, it must be true that, for all $i \in I$, $r_i \cap r_{-i}^{-1} \subset R_{-i} \times R_i$. $\square$

Proposition 3. *For each player $i \in I$ of the game Γ , the set R_i of rationalizable strategies is non-empty and convex.*

Proof. Since, for all $i \in I$, r_i is continuous and X_i is a closed interval, then $r_i(X_i)$ is a closed interval, because of the intermediate value theorem, and hence convex. Moreover, $r_i \circ r_{-i}$ is continuous as well, and hence, for all $N \in \mathbb{N}$, $\bigcap_{n=1}^N (r_i \circ r_{-i})^n \circ r_i(X_{-i})$ is convex. Assume

$\cap_{n \in \mathbb{N}} (r_i \circ r_{-i})^n \circ r_i(X_{-i})$ is not convex. Then there exist $x, x' \in \cap_{n \in \mathbb{N}} (r_i \circ r_{-i})^n \circ r_i(X_{-i})$ and $x'' \in (x, x')$ such that $x'' \notin \cap_{n \in \mathbb{N}} (r_i \circ r_{-i})^n \circ r_i(X_{-i})$. Then there exists $N \in \mathbb{N}$ such that $x'' \notin (r_i \circ r_{-i})^N \circ r_i(X_{-i})$ while $x, x' \in (r_i \circ r_{-i})^N \circ r_i(X_{-i})$, and hence $x'' \notin \cap_{n=1}^N (r_i \circ r_{-i})^n \circ r_i(X_{-i})$ while $x, x' \in \cap_{n=1}^N (r_i \circ r_{-i})^n \circ r_i(X_{-i})$, which contradicts that, for all $N \in \mathbb{N}$, $\cap_{n=1}^N (r_i \circ r_{-i})^n \circ r_i(X_{-i})$ is convex. Therefore, $\cap_{n \in \mathbb{N}} (r_i \circ r_{-i})^n \circ r_i(X_{-i})$, i.e. R_i is convex. $\square$

Proof of Theorem 1.

- (1) Since a Nash equilibrium is a zero of the function $f(x_1) = r_1(r_{-1}(x_1)) - x_1$, and $f(0) > 0$ and $f(1) < 0$, by the intermediate value theorem, if there exists $x_1 \in (0, 1)$ such that $f'(x_1) > 0$, i.e. $\prod_{i \in I} r'_i(x_1) > 1$ then there exists at least two other distinct zeros x'_1 and x''_1 of f corresponding to two other Nash equilibria.
- (2) In effect, if $\prod_{i \in I} r'_i(x_{-i}) > 1$, that is to say

$$\prod_{i \in I} D_{12} f_i(x_i, x_{-i}) > \prod_{i \in I} D_{11} f_i(x_i, x_{-i}), \quad (\text{A.3})$$

then, since $\prod_{i \in I} D_{11} f_i(x_i, x_{-i}) > 0$,²⁶ necessarily $\prod_{i \in I} D_{12} f_i(x_i, x_{-i}) > 0$ holds as well. Therefore, either, for all $i \in I$, $D_{12} f_i(x_i, x_{-i}) > 0$, or for all $i \in I$, $D_{12} f_i(x_i, x_{-i}) < 0$.

If, for all $i \in I$, $D_{12} f_i(x_i, x_{-i}) > 0$, then

$$0 < -\frac{D_{11} f_1(x_1, x_{-1})}{D_{12} f_1(x_1, x_{-1})} < -\frac{D_{12} f_{-1}(x_{-1}, x_1)}{D_{11} f_{-1}(x_{-1}, x_1)}, \quad (\text{A.4})$$

and, for all $i \in I$, there exist $\epsilon_i > 0$ such that

$$-\frac{D_{11} f_1(x_1, x_{-1})}{D_{12} f_1(x_1, x_{-1})} < \frac{\epsilon_{-1}}{\epsilon_1} < -\frac{D_{12} f_{-1}(x_{-1}, x_1)}{D_{11} f_{-1}(x_{-1}, x_1)}, \quad (\text{A.5})$$

and $\lambda > 0$ small enough, such that for all $i \in I$, $(x_{-i} - \lambda \epsilon_{-i}, x_i - \lambda \epsilon_i)$, $(x_{-i} - \lambda \epsilon_{-i}, x_i + \lambda \epsilon_i) \in r_i^+$ while $(x_{-i} + \lambda \epsilon_{-i}, x_i - \lambda \epsilon_i)$, $(x_{-i} + \lambda \epsilon_{-i}, x_i + \lambda \epsilon_i) \in r_i^-$. Thus, letting, for all $i \in I$,

$$\begin{aligned} x_i^1 &= x_i - \lambda \epsilon_i, \\ x_i^2 &\in r_{-i}^{-1}(x_{-i}^1) \cap (x_i^1, x_i^3), \\ x_i^3 &= x_i + \lambda \epsilon_i, \end{aligned} \quad (\text{A.6})$$

for it to be the support of a correlated equilibrium with a joint distribution p_i , it should satisfy the equations

$$\sum_{s_{-i}=1}^3 p_i^{s_i s_{-i}} D_{11} f_i(x_i^{s_i}, x_{-i}^{s_{-i}}) = 0, \quad (\text{A.7})$$

²⁶ Recall that, for all $i \in I$, x_i maximizes $f_i(x'_i, x_{-i})$, and f_i is differentiable and strictly quasi-concave. Hence, $D_{11} f_i(x_i, x_{-i}) < 0$ for all $i \in I$.

for all $i \in I$, and all $s_i \in \{1, 2, 3\}$. The existence of such a joint distribution p_i is guaranteed for this support. In effect, the matrix of coefficients of the previous system of linear equations is

$$\begin{pmatrix} D_1^{11} & D_1^{12} & D_1^{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1^{21} & D_1^{22} & D_1^{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_1^{31} & D_1^{32} & D_1^{33} \\ D_{-1}^{11} & 0 & 0 & D_{-1}^{12} & 0 & 0 & D_{-1}^{13} & 0 & 0 \\ 0 & D_{-1}^{21} & 0 & 0 & D_{-1}^{22} & 0 & 0 & D_{-1}^{23} & 0 \\ 0 & 0 & D_{-1}^{31} & 0 & 0 & D_{-1}^{32} & 0 & 0 & D_{-1}^{33} \end{pmatrix}, \quad (\text{A.8})$$

where $D_i^{s_i, s_{-i}} = D_1 f_i(x_i^{s_i}, x_{-i}^{s_{-i}})$, for all $i \in I$, $s_i, s_{-i} \in \{1, 2, 3\}$. For each of the first three equations two positive probabilities can be fixed in such a way that they completely determine the third probability corresponding to a non-zero entry in the equation. Nonetheless these probabilities cannot be fixed arbitrarily since they still have to satisfy the last three equations. Hence, the necessity of a consistency condition that expressed in terms of, for instance, p_1^{11} , p_1^{12} , p_1^{21} , and p_1^{22} takes the form

$$\begin{aligned} & \left(\frac{D_{-1}^{11}}{D_{-1}^{13}} - \frac{D_1^{33}}{D_1^{31}} \frac{D_{-1}^{31}}{D_{-1}^{33}} \frac{D_1^{11}}{D_1^{13}} \right) p_1^{11} + \left(\frac{D_1^{32}}{D_1^{31}} \frac{D_{-1}^{21}}{D_{-1}^{23}} - \frac{D_1^{33}}{D_1^{31}} \frac{D_{-1}^{31}}{D_{-1}^{33}} \frac{D_1^{12}}{D_1^{13}} \right) p_1^{12} \\ & + \left(\frac{D_{-1}^{12}}{D_{-1}^{13}} - \frac{D_1^{33}}{D_1^{31}} \frac{D_{-1}^{32}}{D_{-1}^{33}} \frac{D_1^{21}}{D_1^{23}} \right) p_1^{21} + \left(\frac{D_1^{32}}{D_1^{31}} \frac{D_{-1}^{22}}{D_{-1}^{23}} - \frac{D_1^{33}}{D_1^{31}} \frac{D_{-1}^{32}}{D_{-1}^{33}} \frac{D_1^{22}}{D_1^{23}} \right) p_1^{22} = 0. \end{aligned} \quad (\text{A.9})$$

For this equation to be satisfied by positive probabilities, i.e. for its hyperplane to intersect the simplex, there must be two coefficients with opposite signs, which is what happens to the coefficient of p_1^{12} (which becomes positive) and the coefficient of p_1^{21} (which becomes negative) if, for instance, for all $i \in I$, x_i^2 is such that $D_i^{12} = 0$. This is indeed the case given that $x_i^2 \in r_{-i}^{-1}(x_i^1) \cap (x_i^1, x_i^3)$ (see Fig. 9). Such a support is then the support of a continuum of correlated equilibria.

- (3) Since $\prod_{i \in I} r'_i(x_{-i}) > 1$, then by Theorem 1 there exist two more Nash equilibria whose strategies are one at each side of the original Nash equilibrium strategies. Since every Nash equilibrium is rationalizable and the set of rationalizable strategies of each player is convex, then it is an interval with the original Nash equilibrium in its interior. $\square$

A.1. A discussion on Theorem 1 for three or more players

In order to see why Theorem 1 can be reasonably expected to hold for any finite number of players let us consider first a trivial case in which it holds for a game with three players. In this case there is a third player, labeled player 3, whose payoffs are actually independent of the other players, labelled now players 1 and 2.

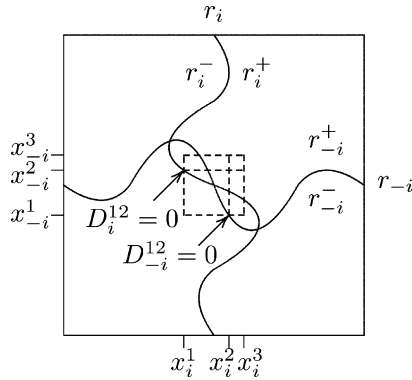


Fig. 9.

Note first that in any three-player set-up, the existence of a multiplicity of Nash equilibria in pure strategies is guaranteed by the fulfillment by a Nash equilibrium (x_1, x_2, x_3) of a sufficient condition analogous to condition (4) following also from Poincaré–Hopf's index theorem:

$$\begin{vmatrix} D_{11}f_1(x_1, x_2, x_3) & D_{12}f_1(x_1, x_2, x_3) & D_{13}f_1(x_1, x_2, x_3) \\ D_{21}f_2(x_1, x_2, x_3) & D_{22}f_2(x_1, x_2, x_3) & D_{23}f_2(x_1, x_2, x_3) \\ D_{31}f_3(x_1, x_2, x_3) & D_{32}f_3(x_1, x_2, x_3) & D_{33}f_3(x_1, x_2, x_3) \end{vmatrix} > 0, \quad (\text{A.10})$$

where f_i denotes now the payoff to player i .²⁷ Note also that if f_3 does not depend on the strategies of players 1 and 2, the condition (A.10) above reduces to condition (5) and hence Theorem 1 guarantees the existence of probabilities p^{ij} and strategies $x_1^{s_1}$ and $x_2^{s_2}$ for players 1 and 2 such that

$$\begin{aligned} \sum_{s_1=1}^3 \sum_{s_2=1}^3 p^{s_1 s_2} D_1 f_1(x_1^{s_1}, x_2^{s_2}, x_3) &= 0, \\ \sum_{s_1=1}^3 \sum_{s_2=1}^3 p^{s_1 s_2} D_2 f_2(x_1^{s_1}, x_2^{s_2}, x_3) &= 0, \end{aligned} \quad (\text{A.11})$$

where x_1, x_2, x_3 are such that

$$D_3 f_3(x_1, x_2, x_3) = 0, \quad (\text{A.12})$$

Note also that, trivially,

$$\sum_{s_1=1}^3 \sum_{s_2=1}^3 p^{s_1 s_2} D_3 f_3(x_1^{s_1}, x_2^{s_2}, x_3) = 0, \quad (\text{A.13})$$

²⁷ For notational convenience, the arguments of the payoff functions are not permuted with the identity of the player now.

and then this results into a trivial correlated equilibrium in which player 3 does not randomize actually and plays a pure strategy.

Nevertheless, this equilibrium can be easily made into a non-trivial correlated equilibrium in which *all* the players randomize their strategies if now player 3's payoff is supposed to depend non-trivially on the other players' strategies in a neighborhood of the Nash equilibrium in pure strategies satisfying condition (A.10), but f_3 still has a critical point at (x_1, x_2, x_3) . In effect, under this conditions, if the trivial correlated equilibrium found above is close enough to the Nash equilibrium in pure strategies, then every $D_3 f_3(x_1^{s_1}, x_2^{s_2}, x_3)$ will still be close to 0, in such a way that

(1) there exist $x_3^{s_3}$, $s_3 = 1, 2, 3$, around x_3 such that

$$\prod_{s_3=1}^3 D_3 f_3(x_1^{s_1}, x_2^{s_2}, x_3^{s_3}) < 0, \quad (\text{A.14})$$

for all $s_1, s_2 = 1, 2, 3$, and

(2) condition (A.10) still holds.

As a consequence, there exist probabilities $p^{s_1 s_2 s_3}$ such that

$$\begin{aligned} \sum_{s_1=1}^3 \sum_{s_2=1}^3 \sum_{s_3=1}^3 p^{s_1 s_2 s_3} D_1 f_1(x_1^{s_1}, x_2^{s_2}, x_3^{s_3}) &= 0, \\ \sum_{s_1=1}^3 \sum_{s_2=1}^3 \sum_{s_3=1}^3 p^{s_1 s_2 s_3} D_2 f_2(x_1^{s_1}, x_2^{s_2}, x_3^{s_3}) &= 0, \\ \sum_{s_1=1}^3 \sum_{s_2=1}^3 \sum_{s_3=1}^3 p^{s_1 s_2 s_3} D_3 f_3(x_1^{s_1}, x_2^{s_2}, x_3^{s_3}) &= 0, \end{aligned} \quad (\text{A.15})$$

and moreover

$$p^{s_1 s_2 1} + p^{s_1 s_2 2} + p^{s_1 s_2 3} = p^{s_1 s_2},$$

for all $s_1, s_2 = 1, 2, 3$. The resulting support $\{x_1^{s_1}, x_2^{s_2}, x_3^{s_3}\}_{s_1, s_2, s_3=1,2,3}$ and joint distribution $\{p^{s_1 s_2 s_3}\}_{s_1, s_2, s_3=1,2,3}$ corresponds then to a non-trivial correlated equilibrium.

A similar argument can be used to construct a correlated equilibrium of a game with n players from a correlated equilibrium of a game with $n - 1$ players.

Proof of Theorem 2. The result is proved for the case $0 < r'_i(x_{-i})$ for all $i \in I$. The extension to the other cases is straightforward.

If $0 < \prod_{i \in I} r'_i(x_{-i}) < 1$, assume with no loss of generality that

$$0 < r'_{-i}(x_i) < \frac{1}{r'_i(x_{-i})}. \quad (\text{A.16})$$

Then, for all $i \in I$, there exist $\lambda_i > 0$ small enough such that,

$$r'_{-i}(x_i) < \frac{\lambda_{-i}}{\lambda_i} < \frac{1}{r'_i(x_{-i})}, \quad (\text{A.17})$$

and hence, for all $i \in I$, the image of $[x_{-i} - \lambda_{-i}, x_{-i} + \lambda_{-i}]$ by r_i is a proper subset of $[x_i - \lambda_i, x_i + \lambda_i]$. The convergence to x_i of the iterative elimination of dominated strategies, if constrained to the interval $[x_i - \lambda_i, x_i + \lambda_i] \times [x_{-i} - \lambda_{-i}, x_{-i} + \lambda_{-i}]$, follows immediately. $\square$

Proof of Theorem 3.

- (1) Assume there exist two distinct Nash equilibria in pure strategies x and x' , and assume both $\prod_{i \in I} r'_i(x_{-i}) < 1$ and $\prod_{i \in I} r'_i(x'_{-i}) < 1$ (otherwise, Theorem 1 guarantees the existence of correlated equilibria already). Applying the Poincaré–Hopf theorem again, there must exist another zero of the vector field determined by the equilibrium equations with an index opposite to those of x and x' , i.e. another Nash equilibrium in pure strategies x'' such that $\prod_{i \in I} r'_i(x''_{-i}) > 1$. Therefore, according to Theorem 1, there must exist non-trivial correlated equilibria in every neighborhood of x'' .
- (2) It follows immediately from the facts that, on the one hand, every Nash equilibrium is a profile of rationalizable strategies (see Proposition 2) and, on the other hand, the set of rationalizable strategies of every agent is convex (see Proposition 3). Therefore, the multiplicity of Nash equilibria in pure strategies implies immediately that the set of rationalizable profiles is a Cartesian product of intervals with non-empty interior and, hence, has no isolated point. $\square$

Proposition 4. If $\{p_i, S_i, x_i\}_{i \in I}$ is a finite support correlated equilibrium of the game Γ , then, for all $i \in I$ and all $s_i \in S_i$,

$$\{(x_{-i}^{s_{-i}}, x_i^{s_i})\}_{s_{-i} \in S_{-i}} \not\subset r_i^+ \quad \text{and} \quad \{(x_{-i}^{s_{-i}}, x_i^{s_i})\}_{s_{-i} \in S_{-i}} \not\subset r_i^-. \quad (\text{A.18})$$

Proof. Since $\{p_i, S_i, x_i\}_{i \in I}$ is a finite support correlated equilibrium of Γ , then, for all $i \in I$, $x_i \in X_i^{S_i}$ solves

$$\max_{x'_i \in X_i^{S_i}} \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} p_i^{s_i s_{-i}} f_i(x_i^{s_i}, x_{-i}^{s_{-i}}), \quad (\text{A.19})$$

and hence, for all $s_i = 1, 2, \dots, S_i$, $x_i^{s_i}$ solves

$$\max_{x_i^{s_i} \in (0,1)} \sum_{s_{-i} \in S_{-i}} p_i^{s_i s_{-i}} f_i(x_i^{s_i}, x_{-i}^{s_{-i}}). \quad (\text{A.20})$$

Therefore, since f_i is differentiable and strictly unimodal, $x_i^{s_i}$ is a critical point of $\sum_{s_{-i}=1}^{S_i} p_i^{s_i s_{-i}} f_i(x_i^{s_i}, x_{-i}^{s_{-i}})$, i.e.

$$\sum_{s_{-i} \in S_{-i}} p_i^{s_i s_{-i}} D_1 f_i(x_i^{s_i}, x_{-i}^{s_{-i}}) = 0. \quad (\text{A.21})$$

Assume, without loss of generality, that $\{(x_{-i}^{s_{-i}}, x_i^{s_i})\}_{s_{-i} \in S_{-i}} \subset r_i^+$, then, for all $s_i \in S_i$,

$$D_1 f_i(x_i^{s_i}, x_{-i}^{s_{-i}}) \leq 0, \quad (\text{A.22})$$

and hence, for all $s_i \in S_i$,

$$D_1 f_i(x_i^{S_i}, x_{-i}^{S_{-i}}) = 0. \quad (\text{A.23})$$

Therefore, because of the strict unimodality of f_i , for all $s_i, s'_i \in S_i$, $x_i^{S_i} = x_i^{S'_i}$, i.e. x would be a Nash equilibrium in pure strategies actually. Therefore, if $\{p_i, S_i, x_i\}_{i \in I}$ is a non-trivial correlated equilibrium, then

$$\{(x_{-i}^{S_{-i}}, x_i^{S_i})\}_{s_{-i} \in S_{-i}} \not\subset r_i^+. \quad (\text{A.24})$$

Similarly, it can easily be established that $\{(x_{-i}^{S_{-i}}, x_i^{S_i})\}_{s_{-i} \in S_{-i}} \not\subset r_i^-$. $\square$

Proposition 5. *If there exists a finite support correlated equilibrium of the game Γ , then there exists a Nash equilibrium in pure strategies whose profile of strategies is in the convex hull of the support of the correlated equilibrium.*

Proof. Let $\{p_i, S_i, x_i\}_{i \in I}$ be a finite support correlated equilibrium. Assume, without loss of generality, that $x_i^1 < \dots < x_i^{S_i}$, for each $i \in I$. Consider, for each $i \in I$ as well, the function

$$\tilde{r}_i(x_{-i}) = \min\{\max\{r_i(x_{-i}), x_i^1\}, x_i^{S_i}\}, \quad (\text{A.25})$$

for all $x_{-i} \in [x_{-i}^1, x_{-i}^{S_{-i}}]$, and for any $i \in I$ consider the function $\tilde{\phi}_i$ from $[x_{-i}^1, x_{-i}^{S_{-i}}]$ to itself mapping each x_{-i} to $\tilde{\phi}_i(x_{-i}) = x_{-i} - \tilde{r}_i(\tilde{r}_i(x_{-i}))$.

According to Proposition 4, for each $i \in I$, either $\tilde{r}_i(x_{-i}^1) = x_i^1$ or $\tilde{r}_i(x_{-i}^1) = x_i^{S_i}$ holds. If $\tilde{r}_i(x_{-i}^1) = x_i^1$ holds for both $i \in I$,²⁸ then for any $x'_{-i} > x_{-i}^1$ but close enough to x_{-i}^1 , it still holds $\tilde{r}_i(x_{-i}) = x_i^1$ and hence $\tilde{\phi}_i(x'_{-i}) > 0$. Moreover, there exists some x''_{-i} in the interval $[x_{-i}^1, x_{-i}^{S_{-i}}]$ such that $\tilde{\phi}_i(x''_{-i}) < 0$, i.e. such that $x''_{-i} < \tilde{r}_i(\tilde{r}_i(x''_{-i}))$. In effect, on the one hand, such an x''_{-i} exists whenever there is some x_i in $[x_i^1, x_i^{S_i}]$ such that $\tilde{r}_{-i}(x_i) > \inf \tilde{r}_i^{-1}(x_i)$, since then there exists some $x_{-i} \in \tilde{r}_i^{-1}(x_i)$, satisfying $\tilde{r}_{-i}(x_i) > x_{-i}$ and $\tilde{r}_i(x_{-i}) = x_i$,²⁹ i.e. such that $\tilde{r}_{-i}(\tilde{r}_i(x_{-i})) > x_{-i}$. On the other hand, should there be no x_i in $[x_i^1, x_i^{S_i}]$ such that $\tilde{r}_{-i}(x_i) > \inf \tilde{r}_i^{-1}(x_i)$, then it would hold that, for all x_i in $[x_i^1, x_i^{S_i}]$, $\tilde{r}_{-i}(x_i) \leq \inf \tilde{r}_i^{-1}(x_i)$, but since $\tilde{r}_i^{-1}(x_i) \subset [x_{-i}^1, x_{-i}^{S_{-i}}]$, certainly $\inf \tilde{r}_i^{-1}(x_i) \leq x_{-i}^{S_{-i}}$ would hold as well, and hence so would $\tilde{r}_{-i}(x_i) \leq x_{-i}^{S_{-i}}$ for all x_i in the interval $[x_i^1, x_i^{S_i}]$. Thus, $\{(x_i^{S_i}, x_{-i}^{S_{-i}})\}_{s_i \in S_i} \subset r_{-i}^+$ would have to be true, which contradicts Proposition 4.

Therefore, for some x_{-i} in (x'_{-i}, x''_{-i}) , $\tilde{\phi}_i(x_{-i}) = 0$ holds, that is to say $x_{-i} = \tilde{r}_i(\tilde{r}_i(x_{-i}))$, and letting x_i be $\tilde{r}_i(x_{-i})$, then $(x_{-i}, x_i) \in \tilde{r}_i$ for all $i \in I$. Now, should it be that case that $x_i \in \{x_i^1, x_i^{S_i}\}$, then for $(x_{-i}, x_i) \in \tilde{r}_i$ to hold for all $i \in I$, necessarily it would have to be true that $x_{-i} \in \{x_{-i}^1, x_{-i}^{S_{-i}}\}$ as well. Therefore, since $x_{-i} \in (x'_{-i}, x''_{-i})$, with $x'_{-i} \in (x_{-i}^1, x_{-i}^{S_{-i}})$ and $x''_{-i} \in [x_{-i}^1, x_{-i}^{S_{-i}}]$, then $x_{-i} \notin \{x_{-i}^1, x_{-i}^{S_{-i}}\}$, and hence $x_i \notin \{x_i^1, x_i^{S_i}\}$ neither. Thus, $x_i \in (x_i^1, x_i^{S_i})$ for all $i \in I$ and hence $\tilde{r}_i(x_{-i}) = r_i(x_{-i})$ and $\tilde{r}_{-i}(r_i(x_{-i})) = r_{-i}(r_i(x_{-i}))$

²⁸ A similar argument can be easily developed for each of the three other possible cases.

²⁹ For all x_i in $[x_i^1, x_i^{S_i}]$, $\tilde{r}_i^{-1}(x_i)$ contains its greatest lower bound because of the continuity of $\tilde{r}_i$.

for any $i \in I$. As a consequence, $x_{-i} = r_{-i}(r_i(x_{-i}))$ as well, and then $(x_{-i}, x_i) \in r_i$ for all $i \in I$, i.e. $\{x_i\}_{i \in I}$ is a Nash equilibrium. $\square$

Proof of Theorem 4. Let $\{p_i, S_i, x_i\}_{i \in I}$ be a finite support correlated equilibrium. Then Proposition 3 guarantees that, for all $i \in I$, there exists a $x_{-i}^1 \in X_{-i}$ such that x_i^1 is the best reply to x_{-i}^1 , i.e. $r_i(x_{-i}^1) = x_i^1$, and x_{-i}^1 is in the interior of the convex hull of the range of x_{-i} ; there exists also $x_{-i}^{S_i} \in X_{-i}$ such that $x_i^{S_i}$ is the best reply to $x_{-i}^{S_i}$, i.e. $r_i(x_{-i}^{S_i}) = x_i^{S_i}$, and $x_{-i}^{S_i}$ is in the interior of the convex hull of the range of x_{-i} . For each $i \in I$, let $r_i^\wedge$ be such that $r_i^\wedge(x_{-i}) = \min\{r_i(x_{-i}), r_i(x_{-i}^1)\}$, and $r_i^\vee$ be such that $r_i^\vee(x_{-i}) = \max\{r_i(x_{-i}), r_i(x_{-i}^{S_i})\}$, that is to say, in the case both r_i are strictly decreasing functions

$$r_i^\wedge(x_{-i}) = \begin{cases} r_i(x_{-i}^1) = x_i^1, & \forall x_{-i} \in [0, x_{-i}^1], \\ r_i(x_{-i}), & \forall x_{-i} \in [x_{-i}^1, 1], \end{cases} \quad (\text{A.26})$$

and

$$r_i^\vee(x_{-i}) = \begin{cases} r_i(x_{-i}), & \forall x_{-i} \in [0, x_{-i}^{S_i}], \\ r_i(x_{-i}^{S_i}) = x_i^{S_i}, & \forall x_{-i} \in [x_{-i}^{S_i}, 1]. \end{cases} \quad (\text{A.27})$$

Consider the continuous function $\phi_i^\wedge(x_{-i}) = x_{-i} - r_{-i}^\vee(r_i^\wedge(x_{-i}))$ from $[0, 1]$ to itself. Then $\phi_i^\wedge(0) = -r_{-i}^\vee(r_i^\wedge(0)) \in (-1, 0)$ while $\phi_i^\wedge(1) = 1 - r_{-i}^\vee(r_i^\wedge(1)) \in (0, 1)$. Thus, there exists $\hat{x}_{-i} \in X_{-i}$ such that $\hat{x}_{-i} = r_{-i}^\vee(r_i^\wedge(\hat{x}_{-i}))$, i.e. letting $\hat{x}_i$ be $r_i^\wedge(\hat{x}_{-i})$, it holds that $(\hat{x}_{-i}, \hat{x}_i) \in r_i^\wedge$ and $(\hat{x}_i, \hat{x}_{-i}) \in r_{-i}^\vee$. Now, since both r_i and r_{-i} are strictly decreasing, $r_i^\wedge$ does not meet $[0, x_{-i}^{S_i}] \times (x_i^1, 1]$, nor does $r_{-i}^\vee$ meet $(x_i^1, 1] \times [0, x_{-i}^{S_i}]$. Then, necessarily, $\hat{x}_{-i} \in [x_{-i}^{S_i}, 1] \subset [x_{-i}^1, 1]$ and $\hat{x}_i \in [0, x_i^1] \subset [0, (x')_i(S_{-i})]$, and therefore $r_i^\wedge(\hat{x}_{-i}) = r_i(\hat{x}_{-i})$ and $r_{-i}^\vee(\hat{x}_i) = r_{-i}(\hat{x}_i)$, that is to say, $(\hat{x}_{-i}, \hat{x}_i) \in r_i$ and $(\hat{x}_i, \hat{x}_{-i}) \in r_{-i}$. Thus, $\{\hat{x}_i\}_{i \in I}$ is a Nash equilibrium of the game not contained in the convex hull of the support of the correlated equilibrium, and hence distinct from the one contained in it.

Similarly it can be proved that there is another Nash equilibrium $\hat{x}'_i \in [x_i^{S_i}, 1]$ and $\hat{x}'_{-i} \in [0, x_{-i}^1]$ by means of the function $\phi_i^\vee(x_{-i}) = x_{-i} - r_{-i}^\wedge(r_i^\vee(x_{-i}))$. Finally, an analogous argument shows the existence of two Nash equilibria outside the convex hull of the support of the correlated equilibrium in the case that both best reply function are strictly increasing. $\square$

Proposition 6. *If there exists a finite support correlated equilibrium of the game Γ and either,*

- (1) *for all $i \in I$, $x_i^{S_i} < r_i(x_{-i}^1)$ and $r_i(x_{-i}^{S_i}) < x_i^1$, or*
- (2) *for all $i \in I$, $x_i^{S_i} < r_i(x_{-i}^{S_i})$ and $r_i(x_{-i}^1) < x_i^1$,*

then there exist at least three Nash equilibria.

Proof. In effect, the existence of a correlated equilibrium guarantees the existence of a Nash equilibrium in pure strategies (x_i, x_{-i}) in the convex hull of its support. Moreover,

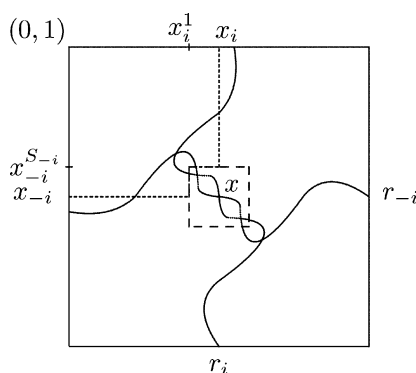


Fig. 10.

Poincaré–Höpf theorem applies necessarily to a continuous deformation of the set of profiles of strategies, e.g. in the case (1), the set circumscribed by the polygon formed by the segments joining the points $(0, 1)$, $(0, x_{-i})$, (x_i^1, x_{-i}) , (x_i^1, x_{-i}^{S-i}) , (x_i, x_{-i}^{S-i}) , $(x_i, 1)$, and $(0, 1)$ again (see Fig. 10).³⁰ Clearly, using the same argument, another Nash equilibrium must exist also in this case to the southeast of x . $\square$

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³⁰ Consider the same continuous transformation as in the proof of Theorem 1. Notice also that the call for Lemma 1 is essential here: a restriction of the vector field defined by the best reply functions that includes the Nash equilibrium within the convex hull of the support of the correlated equilibrium, needs not be inward-pointing, and thus no general argument based on the Poincaré–Höpf theorem can be made.