Convergence of Polynomial Approximations for Infinite-Dimensional Optimization Problems

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1 Introduction

We introduce and analyze polynomial approximation methods for solving infinite-dimensional optimization problems numerically.

More precisely, we consider the following class of convex problems in Lebesgue spaces involving linear equalities and an additional norm constraint:

$$\begin{aligned} F^{q,*} = & \inf_{f \in L^q([\alpha,\beta])} < \gamma, f > \qquad (F^q) \\ & \text{s.t. } < a_i, f >= b_i \quad \forall i = 1, \ldots, L \\ & \text{and } \|f\|_q \leq M \end{aligned}$$

where functions a_i and γ belong to functional space $L^{q'}([\alpha,\beta])$ (with $\frac{1}{q} + \frac{1}{q'} = 1$), $M \in \mathbb{R}_0^+$ and the interval $[\alpha,\beta] \subset \mathbb{R}$ is bounded.

Instead of using a classical discretization technique, we propose a new approach based on the restriction of the problem to polynomial functions: we introduce the family of problems

$$\begin{aligned} P_n^{q,*} = & \inf_{p \in \pi^n([\alpha,\beta])} < \gamma, p > \qquad (P_n^q) \\ & \text{s.t. } < a_i, p >= b_i \quad \forall i = 1, \ldots, L \\ & \text{and } \|p\|_q \leq M \end{aligned}$$

where $\pi^n([\alpha,\beta])$ is the space of polynomials with degree less or equal than n.

Our scheme relies on the resolution of the polynomial problem (P_n^q) in order to obtain a solution admissible for the general problem (F^q) and not too far from an optimal solution (with a guarantee for the quality of the approximation depending on the degree n).

The relevance of this method depends on the following two properties, described in the next sections: these problems can be solved numerically, and it can be guaranteed that the optimal value of the approximate solutions obtained converge to the optimal value of the original infinite-dimensional problem.

2 Solution of the Polynomial Problems

The polynomial problem (P_n^q) can be stated as a finite-dimensional convex optimization problem, choosing as variables the coefficients of the polynomial expressed in some basis.

In particular, in the case $q = 2$, we use the Legendre polynomials and obtain a standard convex quadratic program, while for the case $q = +\infty$ we use a basis of Chebyshev polynomials and obtain a semidefinite programming problem, see e.g. [2]. In both cases, the resulting structured convex optimization problems can be readily solved numerically using well-known techniques (e.g. interior-point methods).

3 Convergence of Optimal Values

Our main result shows that, under some mild assumptions, the optimal values of the family of polynomial problem (P_n^q) converge to the optimal value of the initial problem (F^q). More precisely, we obtain convergence results of the form:

$$F^{q,*} \leq P_n^{q,*} \leq F^{q,*} + \Delta^q(n)$$

where the rate of convergence to zero of the term $\Delta^q(n)$ follows that of $E_n(f^{q,*})_q$, where $f^{q,*}$ is an arbitrary optimal solution of (F^q) and $E_n(f^{q,*})_q$ is its best approximation error in $\pi^n([\alpha,\beta])$ in the norm of L^q.

Furthermore, imposing some additional regularity conditions, convergence results for $E_n(f^{q,*})_q$ in terms of moduli of smoothness [1] allow us to obtain a well-defined convergence rate for the approximation scheme. For instance, if $f^{q,*}$ is r-times continuously differentiable, the order of convergence of our approximation scheme will be $O(\frac{1}{n^{r-1}})$, and even $O(\frac{1}{n^{2r}})$ when $q = 2$.

References

[1] Z. Ditzian and V. Totik, " Moduli of smoothness" Springer-Verlag, 1987.

[2] Y. E. Nesterov, "Squared functional systems and optimization problems" , High Performance Optimization, Kluwer Academic Publishers, 2000.