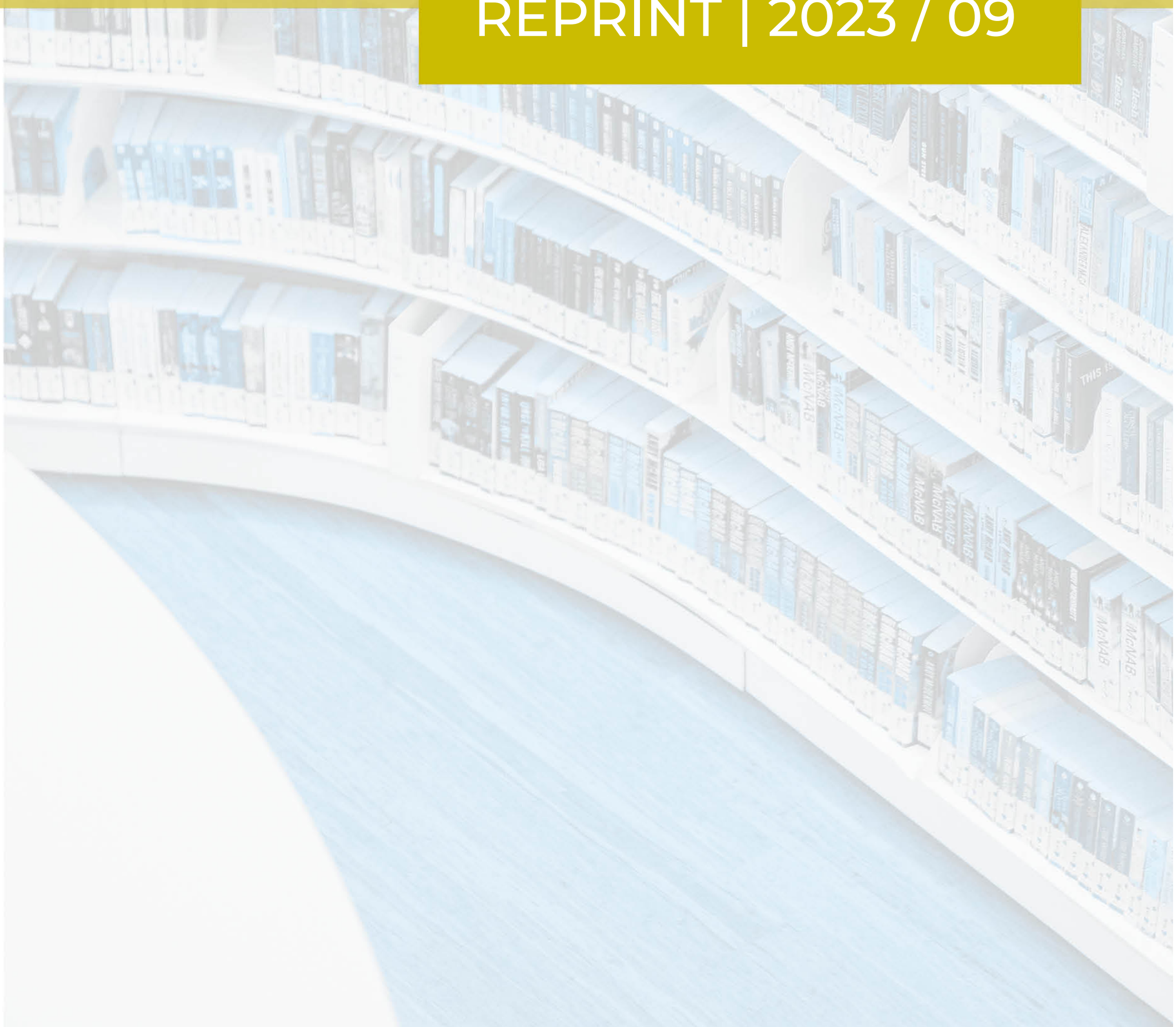
ACCOUNTING FOR PD-LGD DEPENDENCY: A TRACTABLE EXTENSION TO THE BASEL ASRF FRAMEWORK

Matteo Barbagli, Frédéric Vrins

REPRINT | 2023 / 09



UCLouvain

LIDAM
LFIN | Louvain Institute of Data Analysis and Modeling in economics and statistics

LFIN

Voie du Roman Pays 34, L1.03.01 B-1348 Louvain-la-Neuve

Tel (32 10) 47 43 04

Email: lidam-library@uclouvain.be

https://uclouvain.be/en/research-institutes/lidam/lfin/publications.html

Economic Modelling 125 (2023) 106321

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.journals.elsevier.com/economic-modelling

Accounting for PD-LGD dependency: A tractable extension to the Basel ASRF framework

Matteo Barbagli [a,*], Frédéric Vrins [a,b]

[a] LFIN/LIDAM, UCLouvain, 1348 Louvain-la-Neuve, Belgium
[b] HEC Montréal, Department of Decision Sciences, H3T 2A7 Montréal, Canada

ARTICLE INFO

JEL classification:
G21
G28
G32

Keywords:
Credit risk
Factor model
Value-at-risk
Basel regulation
Downturn LGD

ABSTRACT

We extend the asymptotic single risk factor (ASRF) model used in the Basel regulations to accommodate the dependence between the probabilities of default (PD) and the losses given default (LGD) with arbitrary marginal distributions. The PD-LGD link is introduced via the single systematic risk factor and its strength is controlled via a dedicated parameter, in line with the treatment of the default dependence in the current Basel framework. We derive the explicit form of the mapping function translating unconditional LGDs into conditional ones, compute the portfolio-invariant semi-analytical formula of value-at-risk and propose a calibration method. An empirical study featuring defaulted corporate bonds confirms the validity of the calibration procedure and delivers realistic risk metrics. The proposed approach is easy to implement and is useful to design future guidelines for capital requirements, to perform sensitivity analysis or to compute implied downturn LGDs. Compared to the guidelines of the European Banking Authority, our approach delivers more conservative figures on the considered data.

1. Introduction

Since 2008, the banking sector is facing an almost uninterrupted sequence of systemic shocks. From the global financial crisis to the recent spikes in inflation and energy prices, passing by the unprecedented negative rates period, the Covid-19's world pandemia and restrictions on international trading resulting from economic sanctions. Combined with the growing uncertainty about the impact of climate change on the global economy, this new normality puts the most solid financial institutions at risk. In such a context, capital requirements have become an increasing concern for regulators and supervisors who seek to strengthen the stability and resilience of the banking sector worldwide (Mariathasan and Merrouche, 2014; Juelsrud and Wold, 2020; Thakor, 2021). Credit value-at-risk (VaR) portfolio models and their regulatory implementations are the tools by which large internationally active banks compute the amount of capital needed to absorb credit losses at a given confidence level (Puzanova et al., 2009; Tsai and Chen, 2011; Rossignolo et al., 2012). These models aim to provide a quantile of the cumulated credit losses encountered in the considered banking book portfolio. As the VaR crucially depends on the joint distribution of the default events and realised losses, a mathematical model is needed to depict their dependence structure. As far as regulatory frameworks are concerned, the asymptotic single risk factor (ASRF) model is certainly the most widespread. The key idea underlying the ASRF model is to postulate the existence of a latent systematic risk factor, say Y, governing the dependence structure of the default events and the realised losses within and across obligors (Gordy, 2003). This factor can be interpreted as reflecting the 'global state of the economy'.

Since Basel II, the Basel Committee on Banking Supervision (BCBS) has chosen the ASRF as the underlying model for its risk-weighted assets (RWA) formulas featured in the internal ratings-based (IRB) approach (BCBS, 2005, p. 4–5). Notwithstanding its well-known shortcomings,[1] it remains at the core of the current Basel machinery to measure credit risk due to its appealing regulatory features (BCBS, 2023c). The ASRF has the ability to capture the effects of systemic shocks in a simple and tractable way: a closed form expression for the portfolio VaR is available and the portfolio VaR can be decomposed as the sum of *individual risk contributions*, a property known as *portfolio invariance*. It can be shown that the contribution to the portfolio VaR of the loss L_i resulting from the default of obligor i depends on its probability of default (PD), the exposure at default (EAD) and the loss given default (LGD). Due to the specific dependence structure underlying the ASRF, the contribution of L_i in a large portfolio collapses to

* Correspondence to: Université catholique de Louvain, Place des Doyens 1, Boite L2.01.01, Office b 208, 1348 Louvain-la-Neuve, Belgium.
E-mail address: matteo.barbagli@uclouvain.be (M. Barbagli).

[1] For instance, in the implementation used for regulatory purposes, the dependence structure between the default events in all the regulatory asset classes is controlled by a Gaussian copula.

https://doi.org/10.1016/j.econmod.2023.106321
Received 30 September 2022; Received in revised form 1 March 2023; Accepted 13 April 2023
Available online 20 April 2023
0264-9993/

its expectation conditional upon Y, which can be decomposed as the product of its conditional PD, EAD and LGD:

$$\mathbb{E}[L_i|Y] = \mathrm{PD}_i(Y) \times \mathrm{EAD}_i(Y) \times \mathrm{LGD}_i(Y) .$$

In the above expression, each element can be regarded as a function that maps a 'through-the-cycle (TTC)' risk-parameter[2] into a conditional version, given a specific value of the systematic risk factor Y (BCBS, 2005, p. 5). In its most general form, the ASRF model underlying the IRB assumes that $\mathrm{EAD}(Y) = \mathrm{EAD}$ is considered as deterministic,[3] Y follows a standard normal law and the confidence level is set to 99.9%. Following the portfolio invariance property, the overall credit VaR at 99.9% is obtained by adding up the *individual risk contributions*, i.e., the above conditional expectations evaluated at $Y = -z_{0.999}$, where z_p is the pth quantile of standard normal distribution (BCBS, 2005, p. 4–6). The analytical expression of the conditional default probability function $y \mapsto \mathrm{PD}(y)$ derives from the Gaussian copula underlying the dependence between the default events, originally found in section 4.2 of BCBS (2005) and still present in §31.5, 31.14-31.16 of BCBS (2023c). However, neither the foundation IRB (F-IRB) nor the advanced IRB (A-IRB) provide the dependence structure between the LGDs of various obligors, or between the LGD and the default event of a same obligor: in the Basel regulatory formula, the LGD term is represented by a parameter, noted λ^* hereafter (see Section 4.3 of BCBS (2005)).

It is now widely admitted that default rates (DRs) and LGDs are positively related, a phenomenon known as 'PD-LGD dependency'. First established by Altman et al. (2005) in the context of defaulted corporate bonds, this relationship has been corroborated by numerous studies conducted by academia (Caselli et al., 2008; Khieu et al., 2012; Jokivuolle and Virén, 2013; Yao et al., 2017), industry participants (Perraudin and Hu, 2006; Hu, 2007), central banks (Düllmann and Gehde-Trapp, 2004; Mora, 2012) and rating agencies (Varma and Cantor, 2005; Cantor et al., 2007; Meng et al., 2010). Consequently, considering the LGD term λ^* as a TTC average LGD would amount to disregard the relationship between DRs and LGDs, hence, would lead to a serious underestimation of the capital requirements (Altman et al., 2005; Schäfer and Koivusalo, 2013; Han, 2017; García-Céspedes and Moreno, 2020).

Obviously, PD-LGD dependency is accounted for in the Basel regulations by treating λ^* as a stressed estimate. This interpretation is highlighted in the name of this parameter, called *downturn LGD* (DLGD) in Basel's documents, as well as in the general directives related to its computation. Nevertheless, despite its primary importance on the portfolio VaR, Basel's official documentation contains only a few and very general guidelines on how to estimate λ^* operationally (see §36.83-36.88 of BCBS (2023a)): (i) 'A bank must estimate an LGD for each facility that aims to reflect economic downturn conditions' and (ii) 'This LGD cannot be less than the long-run default-weighted average loss rate given default calculated based on the average economic loss [...]'. The overarching idea is to allow banks eligible to use the IRB approach to compute this parameter based on their own internal models, following guidelines dictated by national competent authorities: 'Supervisors will continue to monitor and encourage the development of appropriate approaches to this issue' (BCBS, 2023a). As explained in Calabrese (2014), this has led to different methodologies across regulatory landscapes, hurting the comparability of the estimated DLGD across banks. This observation has been confirmed by the latest work of the European Banking Authority (EBA) on the topic: 'the lack of explicit guidance and limited supervisory and industry consensus on how to incorporate economic downturn component in model estimation has led to significant differences in practices and [...] variability in risk-weighted exposure amounts [...]' (EBA, 2019). This is problematic because banks facing similar risks but adopting different estimation procedures may report very different regulatory capital figures.

With the aim of reducing unwarranted RWA variability across the European Union, the EBA has developed the most complete DLGD estimation guidelines across jurisdictions so far, organised around three key aspects. First, in EBA (2019) the λ^* for a pool is understood as the worst average LGD over a period of economic downturn. Second, the downturn period is identified according to EBA (2018) as a period of at least 12 months, during the past 20 years, where the most severe historical values of a list of macroeconomic and credit-related factors are simultaneously observed. Third, once the period of stress has been identified, no less than three methodologies exist for quantifying the worst average LGD within the considered period (EBA, 2019, p. 9–10). Nevertheless, none of these methodologies prescribes a specific formula for $\mathrm{LGD}(y)$. Hence, the compliance between the final DLGD and the prescribed guidelines has to be performed bank-by-bank and model-by-model. This can lead to intensive and cumbersome back-testing exercises, both for banks and regulators, similar as to the targeted review of internal models (ECB, 2021) conducted by the European Central Bank.

Making a parallel with the ASRF formulae, it is clear that λ^* aims to play the role of $\mathrm{LGD}(-z_{0.999})$. Indeed, as explained in Miu and Ozdemir (2006) and Meng et al. (2010), it is true that computing λ^* as an expected LGD *in economic downturn conditions* (during which the DR is likely to be high) is a possible way to account for a PD-LGD dependency. However, this methodology suffers from several drawbacks. First, the treatment of the conditional LGD is inconsistent with that of the unconditional PD: whereas the latter is handled via a dedicated mapping function $y \mapsto \mathrm{PD}(y)$ implied by the ASRF model, no corresponding mapping function $y \mapsto \mathrm{LGD}(y)$ is given for the LGD. The impact of the systemic factor on the LGD is embedded in the estimation procedure of the λ^* parameter and is not directly related to the dependence structure inherent to the underlying credit loss model. Second, treating the PD using a similar 'downturn procedure', i.e., replacing $\mathrm{PD}(-z_{0.999})$ by a value p^* chosen close to the largest observed default rate, would lead to a severe underestimation of the capital requirements. This shows that using the 'downturn estimate' p^* instead of the model-driven value $\mathrm{PD}(-z_{0.999})$ for computing capital requirements can be dangerous. By analogy, the same conclusion may apply to the LGD: there is a high risk that using a downturn LGD λ^* chosen close to the largest observed average LGD will underestimate the capital requirements compared to using $\mathrm{LGD}(-z_{0.999})$ found from the mapping function implied by a meaningful model. Third, selecting macroeconomic indicators identifying an economic downturn can lead to arbitrary choices across regulatory landscapes. Fourth, from a modelling perspective, Koopman et al. (2011) show that when needing to estimate and forecast DRs, not even 100 macroeconomic covariates are sufficient to replace the need for latent factors. Additionally, Betz et al. (2018) show that macroeconomic variables are, in general, not suitable to capture the true systematic effects when modelling LGDs. Finally, using a large dataset of more than 186,000 defaulted credit exposures covering a time span of 18 years, Hartmann-Wendels and Imanto (2021) reveals that the EBA macroeconomic approach to estimate DLGDs would only enable banks to absorb losses at a 81% confidence level instead of the desired 99.9%.

This suggests that the DLGD methodology is perhaps not the best approach to deal with the PD-LGD dependency. Relying on a mapping function implied by the ASRF model could be an appealing alternative, able to capture the important systematic effects in a tractable and transparent way, in the same vein as what is done for the PDs. To

[2] The TTC risk parameters refer to average PD, EAD and LGD values observed under 'normal business conditions' (BCBS, 2005, p. 5). The procedure to compute those parameters can be found in both the original Basel II documents (see §446-499 of BCBS (2006)) and has essentially remained unchanged in the latest Basel framework. For instance, the PD must be estimated as the long-run average of one-year default rates for borrowers in the grade (BCBS, 2023a, §36.63).

[3] See Gordy (2003), Tarashev and Zhu (2008) and Frontczak and Rostek (2015), we follow the same EAD assumption here.

be compliant with the latest BCBS guidelines, the latter should be built around a simple and transparent extension to the current ASRF featuring a single additional parameter dedicated to controlling the PD-LGD link while allowing for full flexibility over the choice of the LGD marginal distributions. The latter would contribute to fostering the comparability of the models' output, without relying on identifying economic downturn periods based on macroeconomic indicators that could result in arbitrary choices across jurisdictions.

In this paper, we introduce a novel approach to account for the PD-LGD dependency within the Basel ASRF setup, complying with the latest guidelines of the BCBS to develop a regulatory framework promoting simultaneously greater *simplicity, comparability,* and *risk-sensitivity* of the capital requirements (BCBS, 2013). This differs from existing approaches insofar as they do not explicitly account for the modelling constraints prescribed by the Basel Committee. We make the following contributions to the literature on factor models for regulatory capital requirements.

First, we introduce and study two extensions to the current Basel ASRF framework providing an explicit mapping function LGD(y). Both mappings allow to control the strength of the PD-LGD dependency with a single parameter, grant full freedom about the choice of the LGD marginal distributions and preserve tractability and portfolio invariance, two key features of the Basel machinery. Hence, these extensions improve *risk-sensitivity* while preserving *simplicity*. We identify the extension that proves to be superior in terms of coupling scheme and study its behaviour in more details. Second, we propose a procedure to calibrate the parameter controlling the PD-LGD link. Given the Gaussian framework underlying the ASRF and following Altman et al. (2005), we focus on criteria quantifying the linear dependence between the annual observed DRs and average LGDs. Surprisingly, we find that the empirical correlation between those variables (hereafter, *empirical correlation*) is not the most appropriate. For large portfolios, the *empirical covariance rescaled by the standard deviation of the DRs* (hereafter, the *empirical normalised covariance*) has to be preferred. As internationally active banks eligible to use IRB approaches display extremely large credit portfolios and that the portfolio invariance feature of the Basel ASRF stems from the large pool approximation, we calibrate the PD-LGD link on an asymptotic portfolio; hence, choosing the *empirical normalised covariance* as a calibration criterion. Third, our ASRF extension provides a framework to compute DLGDs in a simple and transparent way. This is achieved by computing the *implied DLGD*, defined as the value of λ^* one should plug in the current Basel RWA formula in order to replicate the unexpected loss (UL) provided by our calibrated model where, as a reminder, the UL is the difference between the VaR and the expected loss (EL) (BCBS, 2005, p. 5). Generating the PD-LGD dependency via an explicit factor model whose parameters can be connected to observable metrics further enhances both transparency and explainability, whilst contributing to improve the models' *comparability*. Fourth, numerical experiments are used to validate our formulae, to conduct a sensitivity analysis, to understand the impact of the PD-LGD dependency parameter on the VaR and UL and to illustrate the convergence of the actual VaR to our large pool approximation. Finally, we perform an empirical study using the Moody's Analytics Default and Recovery Database featuring the observed time series of the DRs and average realised LGDs (ARLGDs) collected by Moody's on their corporate world bond portfolio for the period 1982–2014. The latter are used to build a realistic prototypical credit portfolio of bonds and to calibrate our model. We compare VaR, UL and implied DLGDs values to those found using various 'schemes' compatible with the Basel IRB approaches. Our results show that the Basel figures tend to underestimate the capital requirements.

The paper is structured as follows. Section 2 introduces the literature review. In Section 3, we provide a brief overview of the credit VaR portfolio model, underlying the RWA formulas available under the IRB approach of Basel III. In Section 4, we introduce two extensions to the Gaussian ASRF model allowing to control the strength of the PD-LGD dependency, and compare the resulting LGD mapping functions. The formulae connecting the model parameter controlling the PD-LGD dependency to linear risk metrics inspired by Altman et al. (2005) are given in Section 5. In Section 6, we present and discuss the choice of the calibration criterion for PD-LGD dependency. The empirical analysis on a Moody's dataset is performed in Section 7. Section 8 concludes.

2. Literature review

Several approaches accounting for the PD-LGD dependency using a dedicated LGD stochastic model can be found in the literature, which are all based on the seminal work of Merton (1974). Frye (2000) defines the RR as a floored normal variable representing the collateral value, modelled as a linear transform of the systematic risk factor. Chabaane et al. (2004) and Pykhtin (2003) extend the work of Frye (2000) and feature either the systematic factor only, or both the systematic and idiosyncratic factors driving default events, respectively. Unfortunately, no guidance is provided regarding how to calibrate the additional parameters involved. Moreover, both approaches postulate a specific shape for the marginal distributions of the LGD, whose parameters are controlled indirectly, via lognormal collateral values. In Tasche (2004), the author directly models the loss L_i as a variable with a continuous distribution on $(0, 1)$ and a probability mass at 0 (to include the no-default case). The correlation parameter introduced in the Basel ASRF model to capture the dependency between the default events (see Section 4.5 of BCBS (2005) and BCBS (2023c)) is then transposed in the model to correlate the losses (not the LGD). Although this approach allows making the economy of extra dependence parameters, it implicitly assumes that the coupling scheme between default indicators coincides with that between the realised losses, and it is not clear why this should be the case.[4] (Giese, 2005) models the PD-LGD dependency via a re-parametrisation of the expected LGD conditional upon the systematic factor using the PD : LGD(Y(PD)). This function is obtained by regression. However, the PD-LGD dependency is controlled by a functional parameter (the regression function), and is not explained by a latent factor model. Miu and Ozdemir (2006) and Meng et al. (2010) focus on determining the extent to which the average LGD for a pool should be increased to account for the correlation between PD and LGD, compared to models that do not explicitly model this correlation. However, their stochastic LGD models result in significant modifications to the ASRF framework used in the Basel IRB approach. Miu and Ozdemir (2006) introduces and estimates five new risk parameters, which would require identification and extraction of the unobservable risk factor. Meng et al. (2010) proposes two latent factor models driving the defaults and LGDs separately, featuring two distinct systematic risk factors which subsequently lose the analytical tractability of value-at-risk. In Schäfer and Koivusalo (2013) the authors consider several structural models where the LGD is set to 1 minus the ratio of the residual asset value at maturity over the debt's face value. This framework gives birth to a positive relationship between PD and LGD, which can be computed analytically in the diffusion case. By specifying the LGDs using the parameters governing the default events, the model is parsimonious, in the sense that it does not feature any new parameter compared to the fixed-LGD equivalent. This, however, also implies that there is no flexibility to control the strength of the PD-LGD relationship, independently of the individual PD or from the default correlation. Han (2017) proposes a more complex modelling framework, based on CreditRisk+ that is workable using appropriate

[4] In the Basel framework, for instance, the correlation parameter connecting the default indicators is a function of the PDs only. It is not clear why the same expression should be used to control the dependence between the losses because the latter features both default indicators and LGDs, hence, should probably embed evidences related to realised LGDs.

numerical methods. This comes at the cost of analytical tractability and portfolio invariance feature, which are then lost.

More recently, García-Céspedes and Moreno (2020) extended the multi-factor approach of Pykhtin (2004) to include stochastic and correlated LGDs. The authors can handle various distributions for the marginal losses, which are coupled using two systematic factors driving the default event and the LGD, respectively. Nevertheless, this comes at the expense of simplicity of the regulatory architecture. Given the multi-factor nature of their model, reaching an analytical approximation for the quantile of the portfolio loss requires computing all the multi-factor adjustment formulas term together with a new series of model parameters. Moreover, in their regulatory application, the LGD is not a 'genuine' LGD, as it is a random variable that can be observed independently of the default event.

Closer to our work, Van Damme (2011) allows modelling the LGD as a function of the latent creditworthiness variable governing the default event of the corresponding obligor, and is a 'true loss-given-default', as the LGD variables are only defined *conditional upon default*. This provides a clean mathematical setup, maintaining full flexibility with respect to the choice of the marginal LGD distributions in a homogeneous pool environment. Unfortunately, there is no control parameter available to tune the strength of the PD-LGD dependency (the same issue arises in Frye (2014), where the form of the conditional loss rate is set identical to that of the conditional default probability). Moreover, as we show in the paper, the strength of this link is capped by the default dependence, and does not comply with empirical evidence.

3. Capital requirements in Basel regulations

Under the Basel II Accords (IRB approach), capital requirements on the banking book are defined through a one-year VaR paradigm. For a given credit portfolio, it is defined as the amount K_α of capital a financial institution needs to hold in order to absorb the UL on the portfolio over a one-year horizon at confidence level α:

$$K_\alpha := \mathbb{Q}_\alpha[L] - \mathbb{E}[L] = \text{UL} . \tag{1}$$

In this expression, L represents the loss on the portfolio, $\mathbb{E}[L]$ stands for the expected loss and $\mathbb{Q}_\alpha[L]$ is the loss quantile at confidence level α, that is, the value-at-risk.

The expectation in (1) only depends on the marginal characteristics of the portfolio constituents. By contrast, the quantile is impacted by the dependence structure underlying the individual losses. The Basel IRB approach relies on an ASRF model postulating a one-factor Gaussian copula structure between the defaults. Since Basel II (BCBS, 2006), this has remained the cornerstone of the RWA formulas found under the current IRB approach (BCBS, 2017, 2023a,c).

3.1. One-factor Gaussian default model

We consider a portfolio made of I credit exposures.[5] We note L_i the loss resulting from the default of obligor i and w_i the weight of exposure i, defined as the EAD of exposure i relative to the total EAD of the portfolio. The default indicator of obligor i is B_i. In case of default, the loss severity on exposure i is modelled as a random variable Λ_i with support in $[0, 1]$. Mathematically, the loss on the pool (relative to the total EAD) can be modelled as the weighted sum

$$L = \sum_{i=1}^{I} w_i L_i \quad \text{where} \quad L_i := \begin{cases} \Lambda_i & \text{if} \quad B_i = 1 \\ 0 & \text{if} \quad B_i = 0 \end{cases}. \tag{2}$$

[5] In the Basel Accords, credit exposures are defined as any types of loans, bonds, or debt instruments subject to credit risk in the banking book (BCBS, 2000). As in Gordy (2003), we assume that the portfolio contains a single exposure per obligor.

Clearly, $B_i \sim Ber(p_i)$ is a Bernoulli random variable whose parameter p_i corresponds to the default probability associated with exposure i and Λ_i is the corresponding LGD. Note that, as suggested by the name, Λ_i is only defined *given default*, that is, on the outcomes for which $B_i = 1$. The expectation and variance of the LGDs are defined as

$$\lambda_i := \mathbb{E}[\Lambda_i | B_i = 1] \quad \text{and} \quad \sigma_i^2 := \mathbb{V}[\Lambda_i | B_i = 1] .$$

The EL of the portfolio only depends on the expected value of the individual losses,

$$\mathbb{E}[L] = \sum_{i=1}^{I} w_i \mathbb{E}[L_i] .$$

From the law of iterated expectations, $\mathbb{E}[L_i] = \mathbb{E}\big[\mathbb{E}[L_i|B_i]\big]$. Using that $L_i = \Lambda_i$ on the set $\{B_i = 1\}$ and $L_i = 0$ otherwise, one gets

$$\mathbb{E}[L] = \sum_{i=1}^{I} w_i \mathbb{E}[L_i] = \sum_{i=1}^{I} w_i \mathbb{E}[\Lambda_i | B_i = 1] \mathbb{P}(B_i = 1) = \sum_{i=1}^{I} w_i \lambda_i p_i . \tag{3}$$

The computation of $\mathbb{Q}_\alpha[L]$ is more involved because the quantile is not a linear operator, hence, some assumptions must be made about the joint distribution of the individual losses. Complying with our regulatory context, we follow the IRB approach and postulate the Gaussian one-factor model first introduced by Vasicek (1991, 2002) for homogeneous portfolios, and later extended by Gordy (2003) to fit heterogeneous portfolios. Precisely, the default events are modelled using correlated latent creditworthiness variables $X_1, \ldots, X_I$:

$$B_i = 1_{\{X_i \le k_i\}} \quad \text{where} \quad X_i := \sqrt{\rho_i} Y + \sqrt{1-\rho_i} Z_i , \tag{4}$$

where $1_{\{A\}}$ is the indicator function (taking the value 1 if A is true and 0 otherwise) and $Y, Z_1, \ldots, Z_I$ are i.i.d. standard normal random variables. The dependence between the default events results from the common systematic risk factor Y and is controlled by the factor loadings $\sqrt{\rho_i}$'s. Noting $\Phi(\cdot)$ the cumulative distribution function (cdf) of the standard normal variable, the default threshold defined as $k_i := \Phi^{-1}(p_i)$ ensures that

$$\mathbb{P}(B_i = 1) = \mathbb{P}(X_i \le k_i) = \Phi(k_i) = p_i .$$

Even in this simple model, the distribution of the portfolio loss L (hence, of its quantile) is a rather complicated function of the model parameters. However, a simple expression can be derived for the portfolio loss quantile asymptotically, as $I \to \infty$; this is the *large pool* (LP) approximation.

3.2. Large pool approximation

Following Gordy (2003), an analytical expression of the portfolio loss quantiles can be found based on two core assumptions: (i) the portfolio is 'asymptotically fine-grained', meaning that $\sum_i EAD_i = \infty$ and $\sum_i w_i^2 < \infty$, when $I \to \infty$ and (ii) the L_is are independent conditional upon Y. Under these two conditions, it follows from the strong law of large numbers that the idiosyncratic risks Z_i's can be fully diversified away, and the distribution of the portfolio loss degenerates to its conditional expectation given Y: $L \to \mathbb{E}[L|Y]$ as $I \to \infty$. In particular,

$$\mathbb{Q}_\alpha[L] \xrightarrow{I\to\infty} \mathbb{Q}_\alpha\big[\mathbb{E}[L|Y]\big] = \mathbb{Q}_\alpha\left[\sum_{i=1}^{I} w_i \mathbb{E}[L_i|Y]\right] .$$

Using the same trick as in (3), the conditional expectation of L_i becomes

$$\mathbb{E}[L_i|Y] = \mathbb{E}[\Lambda_i|Y, B_i = 1] \times \mathbb{P}(B_i = 1|Y) = \lambda_i(Y) \times p_i(Y) \tag{5}$$

where

$$\lambda_i(Y) := \mathbb{E}[\Lambda_i | B_i = 1, Y]$$

and

$$p_i(Y) := \mathbb{P}(B_i = 1|Y) = \mathbb{P}(X_i \le k_i|Y) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}\right) \tag{6}$$

are the conditional expected LGD and conditional PD associated with exposure i, respectively. Note that the mapping functions $p_i(y)$ and $\lambda_i(y)$ convert unconditional PD (resp. LGD) into PD (resp. LGD) conditional upon Y.

An immediate consequence of Proposition 4 in Gordy (2003) is that if the λ_i's are decreasing functions, then

$$\mathbb{Q}_\alpha\left[\mathbb{E}[L|Y]\right] = \sum_{i=1}^{I} w_i \lambda_i(\mathbb{Q}_{1-\alpha}[Y]) p_i(\mathbb{Q}_{1-\alpha}[Y]) . \tag{7}$$

Using $Y \sim \mathcal{N}(0,1)$ leads to $\mathbb{Q}_{1-\alpha}[Y] = -z_\alpha$ where $z_\alpha := \Phi^{-1}(\alpha)$ is the α-quantile of the standard normal variable.

Interestingly, the quantile of $\mathbb{E}[L|Y]$ in (7) takes the same form as the expectation in (3), in the sense that it is a simple sum of terms associated with the individual exposures, each of which being the product of three factors: the relative exposure, a (conditional) LGD and a (conditional) PD. As a result, we have that in a Gaussian one-factor model and under the LP approximation, the UL on the pool can be decomposed as the sum of the UL on each constituent:

$$K_\alpha \xrightarrow{I\to\infty} \mathbb{Q}_\alpha\left[\mathbb{E}[L|Y]\right] - \mathbb{E}\left[\mathbb{E}[L|Y]\right] = \sum_{i=1}^{I} w_i \Big(\lambda_i(-z_\alpha)p_i(-z_\alpha) - \lambda_i p_i\Big) . \tag{8}$$

3.3. Basel IRB formula

In the Basel regulations, the formula to compute the minimum capital requirements under the IRB approach can be found in §31.5 of BCBS (2023c).[6] Adapted to our notation system, it reads as

$$\begin{aligned} K &= \sum_{i=1}^{I} w_i \lambda_i^* \left(\Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho_i} z_{0.999}}{\sqrt{1-\rho_i}}\right) - p_i\right) \\ &= \sum_{i=1}^{I} w_i \left(\lambda_i^* p_i(-z_{0.999}) - \lambda_i^* p_i\right) , \end{aligned} \tag{9}$$

where the factor loading ρ_i is determined by p_i and the type of exposure i. In the case of a pool made of credit exposures classified as corporate, sovereign and bank exposures, the Basel's expression to consider is (see section 31.5 of BCBS (2023c))

$$\rho_i = \rho(p_i) := 0.12 \times \frac{(1-e^{-50\times p_i})}{(1-e^{-50})} + 0.24 \times \left(1 - \frac{(1-e^{-50\times p_i})}{(1-e^{-50})}\right) . \tag{10}$$

In the sequel, we always assume that ρ is given by $\rho(p)$ in (10).[7]

Looking at (9), it is clear that all parameters have been previously defined and have the same interpretation in the Basel formula and the ASRF model, except λ_i^* which stands for a *downturn LGD* of obligor i. Evidently, formula (9) corresponds to formula (8) with $\alpha = 99.9\%$ and replacing both $\lambda_i(-z_\alpha)$ and λ_i by λ_i^* (BCBS, 2005, p. 7–8). It is important to note that the same Gaussian coupling scheme is behind the RWA formulas presented for the various regulatory asset classes, which are divided into two main categories in BCBS (2023c): corporate, sovereign and bank exposures (§31.4 onwards) and retail exposures (§31.13 onwards). Formula (8) and (9) exhibit features that make them practical for regulatory purposes. First, they are additive: the capital requirement of the pool is decomposed as a sum of the marginal contributions of each constituent, a property known as *portfolio invariance*. The capital charge for exposure i only depends on its own risk-profile, and not on the risk-profile of the portfolio it is added to. Second, the LP approximation works quite well even for relatively small portfolio sizes compared to actual pools. Finally, all the terms have a clear interpretation, except possibly the factor loadings ρ_i controlling the dependence between the (unobservable) creditworthiness variables.

The specific shape of the conditional PD function $p_i(y)$ in (6) follows from the Gaussian one-factor model. By contrast, there is no specific model proposed for the conditional LGD $\lambda_i(y)$ (BCBS, 2005, 2006), only the term 'LGD' is encountered in the risk-weighted assets formulas (BCBS, 2023c). Note that the IRB formula (9) is compatible with the ASRF formula (8) provided that $\lambda_i(-z_{0.999}) = \lambda_i = \lambda^*$, essentially meaning that the LGD is either (i) deterministic or (ii) random but independent from both the idiosyncratic and systematic risk factor driving the default event (Tarashev and Zhu, 2008). Looking into the Basel documents, the F-IRB approach prescribes to treat λ^* as a fixed value depending on the regulatory asset class (BCBS, 2023b, §32.6-32.14) while under the A-IRB approach, banks are allowed to estimate this parameter internally as an expected LGD in an *adverse macroeconomic scenario* (see §32.15-32.19 of BCBS (2023b) and §36.83-36.88 of BCBS (2023a)).

Nevertheless, in line with the seminal works of Altman et al. (2005), numerous studies from academia (Caselli et al., 2008; Khieu et al., 2012; Jokivuolle and Virén, 2013; Yao et al., 2017), industry participants (Perraudin and Hu, 2006; Hu, 2007), central banks (Düllmann and Gehde-Trapp, 2004; Mora, 2012) and rating agencies (Varma and Cantor, 2005; Cantor et al., 2007; Meng et al., 2010) provide empirical evidence of a negative correlation existing between RRs and DRs. It is widely admitted that failing to include in the credit risk model the above PD-LGD dependency leads to underestimate regulatory capital requirements (Bade et al., 2011; Saurina and Trucharte, 2007; Altman, 2011).

4. Extended Gaussian frameworks with stochastic and correlated LGD

As explained above, replacing both $\lambda_i(-z_{0.999})$ and λ_i in (8) by the same constant λ_i^* amounts to assuming that the conditional expectation of the LGD does not depend on the systemic risk factor in the ASRF model. This leads to assume that the realised losses are independent from the default rates in large pools and to interpret the above constant as the expected LGD of exposure i. This is not acceptable for regulatory purposes because there are evidences that default rates and realised losses tend to move together, and it is known that this impacts the tail of the loss distribution. Discarding this reality would lead to underestimate the capital charges.

The Basel IRB framework circumvents this issue by relying on the notion of DLGD: the constants λ_i^* in (9) are not expected LGDs but are stressed LGD estimates, computed by focusing on realised losses in adverse scenarios. Therefore, the Basel regulations implicitly acknowledge a dependence between the default likelihood and the loss severity, but without explicitly modelling this relationship. By comparing the first terms of (8) and (9), it is clear that the downturn LGD λ_i^* could be interpreted as the worst-case LGD at confidence level 99.9%, $\lambda_i(-z_{0.999})$, thereby justifying the 'downturn LGD' procedure. However, in contrast to the conditional PD, no functional form is provided for the conditional LGD, and this way to proceed remains obscure for at least two reasons. First, the concept of DLGD is related to but is also different from the expected LGD in the 99.9% worst-case scenario for the systematic factor. This creates a discrepancy between the ASRF and the regulatory frameworks. Second, the comparison of (8) and (9) reveals that the DLGD is used in both the quantile and the EL terms of each exposure (see section 4.4 of BCBS (2005)). This seems questionable because replacing λ_i by λ_i^* in the EL term suggests that λ_i^* is the expected LGD of exposure i, which is not compatible with the concept of DLGD. Moreover, replacing the EL λ_i by a stressed estimate λ_i^* in the second term is not conservative: it tends to increase the EL, hence, to decrease

[6] We ignore the maturity adjustment (MA) factor within the scope of this analysis because, the latter is simply added 'on top' of the resulting analytical formula coming from (8).

[7] By doing so, we implicitly exclude all sub-categories of this regulatory asset class as detailed in §31.7-31.12 of BCBS (2023c). For exposures classified in these sub-categories, the formula of $\rho(p)$ needs to be adjusted.

the capital requirements.[8] Overall, it seems that making the economy of specifying the mapping functions $\lambda_i(y)$'s comes at a cost of a PD-LGD dependency scheme that is somewhat inconsistent with the underlying loss model and hard to grasp, which is not desirable for regulatory purposes.

In the sequel, we discuss two different ways to include, within the Basel framework, LGDs correlated to the systematic factor. Precisely, we include a PD-LGD link by proposing two functional forms for the mapping function $\lambda_i(y)$ which, by analogy with $p_i(y)$, derive from the underlying loss model. These two approaches lead to different expressions for $\lambda_i(y)$, but are similar in the sense that both:

(i) depict the same conditional independence property: conditionally upon Y, not only $B_i \perp B_{j\neq i}$, but also $L_i \perp L_{j\neq i}$ and $\Lambda_i \perp B_{j\neq i}$, for each pair of obligors (i,j);
(ii) can accommodate marginal distributions for the LGDs that could be given exogenously, and lead to decreasing mapping functions, in compliance with the assumption underlying (7).

For each dependency scheme, we compute the conditional moments $\mathbb{E}[\Lambda_i^m|B_i=1,Y]$. We note the conditional mean and conditional variance as $\lambda_i(Y)$ and $v_i(Y)$:

$$\lambda_i(Y) := \mathbb{E}[\Lambda_i|B_i=1,Y], \quad \gamma_i(Y) := \mathbb{E}[\Lambda_i^2|B_i=1,Y] \quad \text{and}$$
$$v_i(Y) := \gamma_i(Y) - \lambda_i(Y)^2.$$

Those expressions play a central role in the models presented below.

4.1. Approach 1: dependency through X_i

In Van Damme (2011), the author develops a generic framework to include correlated LGDs into structural models for credit risk, defining

$$\Lambda_i = F_i^{-1}\left(1-\frac{\Phi(X_i)}{p_i}\right), \tag{11}$$

where X_i is the random variable in (4) driving the default indicator and F_i represents the cdf of Λ_i (whose support is assumed to be a subset of $[0,1]$). It is easy to verify that the cdf of L_i upon default is indeed F_i (see Appendix A.1).

Two comments are worth being made. First, the author actually considers a homogeneous pool of credit exposures in its paper, both in terms of the default risk profile ($p_i = p, \rho_i = \rho$) and LGD risk profile ($F_i = F$). Second, this model does not feature any extra parameter compared to the case where Λ_i and B_i are independent. Consequently, there is no degree of freedom to control the strength of the PD-LGD relation.

In order to allow for heterogeneity and, more importantly, to provide control over the PD-LGD link, we consider instead

$$\Lambda_i = F_i^{-1}\left(\Phi\left(\hat{X}_i\right)\right) \quad \text{where} \quad \hat{X}_i := \sqrt{\rho_i^{LGD}}\,\Phi^{-1}\left(1-\frac{\Phi(X_i)}{p_i}\right) + \sqrt{1-\rho_i^{LGD}}\,\hat{Z}_i \tag{12}$$

and $\hat{Z}_1,\dots,\hat{Z}_I$ are i.i.d. standard normal random variables independent from $Y, Z_1,\dots,Z_I$.

Eq. (12) can be viewed as a natural extension of the model in Van Damme (2011, p. 2526, eq. 9) allowing for pool heterogeneity as well as control over the PD-LGD dependency. In particular, (11) is recovered by imposing $\rho_i^{LGD}=1$ in (12). Moreover, the cdf of L_i conditional upon default is F_i independently of ρ_i^{LGD}, as shown in the next lemma.

Lemma 1. *The cdf of Λ_i in (12) conditional upon $\{B_i=1\}$ is F_i for all values of ρ_i^{LGD}, i.e.,*

$$\mathbb{P}(L_i \le l|B_i=1) := \mathbb{P}(\Lambda_i \le l|B_i=1) = F_i(l).$$

The proof is provided in Appendix A.2 and consists in using the probability integral transform and showing that, conditional upon $\{B_i=1\}$, $\hat{X}_i \sim \mathcal{N}(0,1)$.

Within this model setup, the dependence between the default event and the LGD of exposure i is driven by the creditworthiness variable X_i, that is, *both* by the systematic factor Y and the idiosyncratic factor Z_i. Consequently, the independence between B_i and LGD_i is achieved when conditioning upon the pair (Y,Z_i).

Lemma 2. *The m-th conditional moment of Λ_i is given by*

$$\mathbb{E}[\Lambda_i^m|B_i=1,Y] := \frac{1}{p_i(Y)}\times \int_{-\infty}^{\Phi^{-1}(p_i(y))}\int_{-\infty}^{\infty}\left(F_i^{-1}\left[\Phi\left(\sqrt{\rho_i^{LGD}}\,\Phi^{-1}\left(1-\frac{\Phi(\sqrt{\rho_i}Y+\sqrt{1-\rho_i}z)}{p_i}\right) + \sqrt{1-\rho_i^{LGD}}\,\hat{z}\right)\right]\right)^m \phi(\hat{z})\phi(z)\,d\hat{z}\,dz, \tag{13}$$

with ϕ is the density of the standard normal variable.

The proof is given in Appendix A.3.

The assumptions underlying (7) are satisfied provided that the mapping functions $\lambda_i(y)$ associated with (13) are decreasing. This is the case in the above model, as shown in Appendix C.

4.2. Approach 2: dependency through Y

Similarly to the X_i's used in *Approach 1*, we introduce latent standard normal variables $\tilde{X}_i$'s taking the form of a linear combination of the systematic factor and i.i.d. normal variables. Precisely,

$$\Lambda_i = F_i^{-1}\left(H_i(\tilde{X}_i)\right) \quad \text{where} \quad \tilde{X}_i := -\sqrt{\rho_i^{LGD}}\,Y + \sqrt{1-\rho_i^{LGD}}\,\tilde{Z}_i \tag{14}$$

where $\tilde{Z}_1,\dots\tilde{Z}_I$ are i.i.d. standard normal random variables independent from $Y, Z_1,\dots,Z_I$.

The variable $\tilde{X}_i$ in (14) plays a similar role as $\hat{X}_i$ in (12). However, in contrast with the latter, the coupling scheme between B_i and LGD_i is controlled solely by the systematic factor Y, so that their independence is achieved when conditioning with respect to Y only. Similarly to Lemma 1 in *Approach 1*, one can show that the distribution of Λ_i conditional upon $\{B_i=1\}$ is proven to be F_i, provided that H_i is the cdf of $\tilde{X}_i$ given default.

Lemma 3. *Define the function*

$$H_i(x) := \Phi_2\left(x, \Phi^{-1}(p_i), -\sqrt{\rho_i^{LGD}\rho_i}\right)/p_i, \tag{15}$$

where $\Phi_2(x,y,r)$ is the cumulative distribution function of the bivariate standard normal with correlation coefficient r. The cdf of Λ_i in (14) conditional upon $\{B_i=1\}$ is F_i for all values of ρ_i^{LGD}, i.e.,

$$\mathbb{P}(L_i \le l|B_i=1) := \mathbb{P}(\Lambda_i \le l|B_i=1) = F_i(l).$$

The proof is given in Appendix A.4.

Both in *Approach 1* and *Approach 2*, the unconditional cdf of L_i for $l \in [0,1]$ is the mixed distribution $\mathbb{P}(L_i \le l) = (1-p_i) + p_i \times F_i(l)$ for every value of ρ^{LGD}.

The negative sign of the factor loading $-\sqrt{\rho_i^{LGD}}$ in (14) guarantees that fewer defaults and lower LGD values are observed for higher values of Y, on average. The correlation between X_i and $\tilde{X}_i$ is

$$\mathbb{C}\mathrm{orr}\left(X_i,\tilde{X}_i\right) = -\sqrt{\rho_i^{LGD}\rho_i},$$

[8] To the best of our knowledge, there is no justification in the official Basel literature for using the DLGD in both the quantile and EL term of the capital requirements formula.

and it is clear from (4) and (14) that this setup triggers a dependence between the conditional PD and the loss given default. However, because

$$\mathbb{C}\mathrm{orr}\left(\tilde{X}_i, \tilde{X}_j\right) = \sqrt{\rho_i^{LGD}\rho_j^{LGD}}\,,$$

this setup also generates a dependence between the stochastic LGDs themselves. These simple expressions result from the fact that the variable $\tilde{X}_i$ is a linear function of the common factor Y in *Approach 2*. This is not the case in *Approach 1*, where the variable $\hat{X}_i$ is a non-linear function of X_i (hence, of Y). Consequently, the expressions of $\mathbb{C}\mathrm{orr}(X_i, \hat{X}_i)$ and $\mathbb{C}\mathrm{orr}(\hat{X}_i, \hat{X}_j)$ are much more involved.

Similarly to Lemma 2 in *Approach 1*, the above results lead to the following expression for the conditional expectation of Λ_i^m in *Approach 2*.

Lemma 4. *The m-th conditional moment of the LGD on exposure i is given by*

$$\mathbb{E}[\Lambda_i^m | B_i = 1, Y] = \int_{-\infty}^{\infty}\left(F_i^{-1}\left[H_i\left(-\sqrt{\rho_i^{LGD}}Y + \sqrt{1-\rho_i^{LGD}}\,\tilde{z}\right)\right]\right)^m \phi(\tilde{z})d\tilde{z} \tag{16}$$

where H_i is given in Lemma 3.

The conditional mean $\lambda_i(y)$ in *Approach 2* is obtained by setting $m = 1$ in (16). As in *Approach 1*, it is a monotonic decreasing function of y (both the cdf H_i and the inverse cdf F_i^{-1} are monotonic increasing functions of their arguments). Therefore, the assumptions associated with (7) hold and $\mathbb{E}[L_i|Y]$ remains a monotonic decreasing function of Y, too.

Remark 1. Throughout the paper, integrals such as (13) and (16) are computed numerically using `integrate` (R Core Team, 2022), a built-in function available in R. Other powerful methods are tailored for this purpose, such as quadrature schemes. The reason for choosing `integrate` in the paper is because it is an adaptive scheme allowing to control the tolerance on the error. The bivariate normal cdf Φ_2 is computed via the `pbivnorm` function available in the R package of the same name (Genz et al., 2015). Appendix B provides guidelines about how to perform integration and compute Φ_2 in softwares such as Excel.

4.3. Comparison between Approach 1 & Approach 2

Compared to the original Basel framework, it is clear from (7) that the only difference in the VaR formula when using *Approach 1* or *Approach 2* are the mapping functions $\lambda_i(y)$. The latter are obtained by setting $m = 1$ in (13) and (16), respectively. Fig. 1 shows the conditional LGDs in either approaches for a typical homogeneous credit portfolio characterised by a default probability of $p_i = p = 1\%$ and $\rho_i = \rho(p) = 19.28\%$ complying with Basel regulations.[9] Although both approaches can handle arbitrary distributions, we follow the seminal work of Pykhtin (2004) and choose, in guise of illustration, a beta distribution with mean $\lambda = 40\%$ and standard deviation $\sigma = 20\%$, i.e., $\Lambda_i \sim Beta(2,3)$. As discussed in Meng et al. (2010), this is indeed a common choice made by rating agencies.

It is clear from Fig. 1 that the image of the mapping function $\lambda_i(y)$ associated with *Approach 2* is wider than that of *Approach 1*. This difference persists across all possible values of the parameter ρ^{LGD} controlling the strength of the PD-LGD link. It is important to note that, from Lemmas 1 and 3, all these curves correspond to the same unconditional distribution $Beta(2,3)$ for the LGDs.

[9] The value of p proposed here is frequently observed, at the banking book level, by banking professionals and meets the prescribed floor of 5 basis points (BCBS, 2017, p. 65, §68).

The difference in the shapes of $\lambda_i(y)$ should not come as a surprise, as it is a direct consequence of the fact that in *Approach 1*, the Λ_is are coupled via the latent variable X_is driving the default events, hence, the strength of the PD-LGD dependence is capped by that of the default events, controlled by $\rho_i = \rho$. On the one hand, it is clear from (12) that all similar extensions of Van Damme (2011) will suffer from this limitation of *Approach 1* because the problem is inherent to (11). On the other hand, *Approach 2* appears to be a more flexible setup where the PD-LGD dependence is not constrained by the default dependence because the Λ_is do not depend on ρ. This shows the superiority of *Approach 2* over *Approach 1* and explains why, in the sequel, we focus on *Approach 2*. The corresponding results for *Approach 1* are provided in Appendix G, for the sake of completeness.

5. Empirical vs. copula correlation

One of the key features of our proposed ASRF extensions is to comply with the empirical evidences provided by Altman et al. (2005) via an explicit modelling of the PD-LGD relationship, where the strength of the latter is controlled via a dedicated parameter in a similar way to what is done for the default events. Nevertheless, the choice of ρ_i^{LGD} is not trivial because they control the dependency between *latent* variables, which cannot be directly observed from credit data.

Following Altman et al. (2005), relevant data to estimate the strength of the PD-LGD link are the DRs and the averaged LGDs. The default rate $\hat{N}$ is defined as the number N of defaults observed in a given pool (up to time T) relative to the size of the pool,

$$\hat{N} := \frac{N}{I} \quad \text{where} \quad N := \sum_{i=1}^{I} B_i\,. \tag{17}$$

Similarly, assuming $\{N > 0\}$, one observes $\hat{L}$, the average value of the realised LGDs:

$$\hat{L} := \frac{1}{N}\sum_{i=1}^{I} L_i \quad \text{assuming} \quad N > 0\,. \tag{18}$$

Observe that the denominator in (18) is the number of defaulted obligors (N), not the total number of observed obligors (I). We define the *empirical correlation* and the *empirical covariance* as the correlation and covariance between $\hat{N}$ and $\hat{L}$, respectively.

Recall that the existence of a PD-LGD link has been emphasised by showing that there is a positive correlation between historical time series of observed DRs and average LGDs. This suggests a natural calibration procedure: set the model parameter in order to match a given level of empirical correlation or empirical covariance found in a relevant dataset. This requires computing the theoretical expressions of the empirical correlation and empirical covariance in our extended ASRF models. For analytical tractability purposes, we consider the case of a homogeneous portfolio, recall that the latter is defined as: $p_i = p, \rho_i = \rho, w_i = w = 1/I, F_i = F, \rho_i^{LGD} = \rho^{LGD}, \forall i \in \{1,\dots,I\}$.

Theorem 1. *Consider a homogeneous portfolio of size I and define $q(y) := 1 - p(y)$.*

The empirical covariance between the default rate and the average realised LGDs is

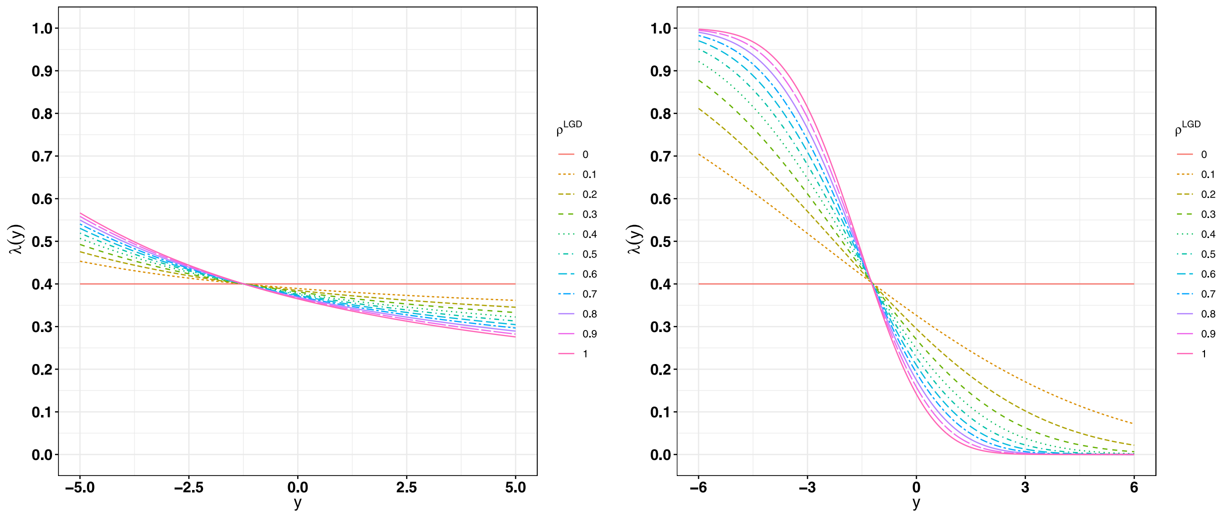
$$c_I := \mathbb{C}\mathrm{ov}\left(\hat{N}, \hat{L}|N > 0\right) = \frac{\left(\lambda - \mathbb{E}[\lambda(Y)|N > 0]\right) \times p}{\mathbb{P}(N > 0)}\,,$$

where

$$f_{Y|N>0}(y) = \frac{\left(1 - q(y)^I\right)\phi(y)}{\mathbb{P}(N > 0)} \quad \text{and} \quad \mathbb{P}(N > 0) = 1 - \int_{-\infty}^{+\infty} q(x)^I \phi(x)dx\,.$$

The empirical correlation between the default rate and the average realised LGDs is

$$r_I := \frac{c_I}{\sqrt{\mathbb{V}\left[\hat{N}|N > 0\right]\mathbb{V}\left[\hat{L}|N > 0\right]}}$$



(a) Approach 1 (b) Approach 2

Fig. 1. Impact of ρ^{LGD} on the function $\lambda(y)$ in Approach 1 and Approach 2 for a homogeneous credit portfolio with parameters $p = 1\%$, $\rho = \rho(1\%) = 19.28\%$ and marginal LGDs distributed according to a $Beta(2,3)$, leading to $(\lambda, \sigma) = (40\%, 20\%)$.

where the conditional variances are given by

$$\mathbb{V}\left[\hat{N}|N>0\right] = \frac{1}{I\mathbb{P}(N>0)} \times \left(p - \frac{Ip^2}{\mathbb{P}(N>0)} + (I-1)\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \rho\right)\right)$$

$$\mathbb{V}\left[\hat{L}|N>0\right] = \frac{1}{\mathbb{P}(N>0)} \times \mathbb{E}\left[v(Y)\Psi(Y)\right] + \mathbb{V}\left[\lambda(Y)|N>0\right]$$

with $v(Y) = v_i(Y)$ the conditional variance of the LGD and

$$\Psi(y) := \int_0^1 t^{-1}\left((q(y)+p(y)t)^I - q(y)^I\right) dt\,.$$

The proof is given in Appendix D.

The above result is general in the sense that it applies to any extension of the ASRF model, hence, holds in *Approach 1* and *Approach 2*. Only the specific expressions of $\lambda(y)$ and $v(y)$ depend on the chosen model. In addition, it is important to note that this formula does not rely on the LP approximation; it is applicable to small credit portfolios. The homogeneity restriction imposed in Theorem 1 is needed for the sake of getting a semi-analytical expression for the empirical correlation but does not entail the applicability of our approach. A possible way to handle heterogeneous pools is briefly discussed in the conclusion.[10]

The analytical expressions of c_I, C_I and r_I can be checked using Monte Carlo simulations (MCS). This is done in Fig. 2 for the homogeneous credit portfolio studied in Section 4.3 and setting $I = 1500$.

In line with the LP approximation, it is interesting to compute the asymptotic expressions of the empirical correlation and empirical covariance, defined as the expressions of r_I and c_I as $I \to \infty$. This is the purpose of the next corollary. For further reference, we also compute the *empirical normalised covariance*, defined as the empirical covariance normalised by the conditional standard deviation of the default rate:

$$C_I := \frac{c_I}{\sqrt{\mathbb{V}\left[\hat{N}|N>0\right]}}\,. \tag{19}$$

The next corollary gives the expressions of c_∞, r_∞ and C_∞ together with some of their properties. We obtain the intuitive result that, compared to the finite-size expressions given above, it suffices to drop the $\{N > 0\}$ condition and replace $\hat{N}$ by $p(Y)$ and $\hat{L}$ by $\lambda(Y)$.

Corollary 1. *Let $0 < p < 1$ and $\sigma > 0$. The asymptotic empirical covariance is*

$$c_\infty = \mathbb{C}\mathrm{ov}\left(\lambda(Y), p(Y)\right) = \left(\lambda - \mathbb{E}[\lambda(Y)]\right) \times p\,.$$

If ρ, ρ^{LGD} are strictly positive, the asymptotic empirical correlation is

$$r_\infty = \frac{c_\infty}{\sqrt{\mathbb{V}[p(Y)]\mathbb{V}[\lambda(Y)]}}\,.$$

If $\rho > 0$, the asymptotic empirical normalised covariance is

$$C_\infty = \frac{c_\infty}{\sqrt{\mathbb{V}[p(Y)]}}\,.$$

In these expressions,

$$\mathbb{V}[p(Y)] = \lim_{I\to\infty} \mathbb{V}\left[\hat{N}|N>0\right] = \Phi_2(\Phi^{-1}(p), \Phi^{-1}(p), \rho) - p^2 \quad \text{and}$$
$$\mathbb{V}[\lambda(Y)] = \lim_{I\to\infty} \mathbb{V}\left[\hat{L}|N>0\right]\,.$$

Moreover, both c_∞ and C_∞ are monotonic increasing functions of ρ^{LGD} on $(0,1)$ whenever $\rho > 0$ and $p > 0$, while counter-examples exist for r_∞.

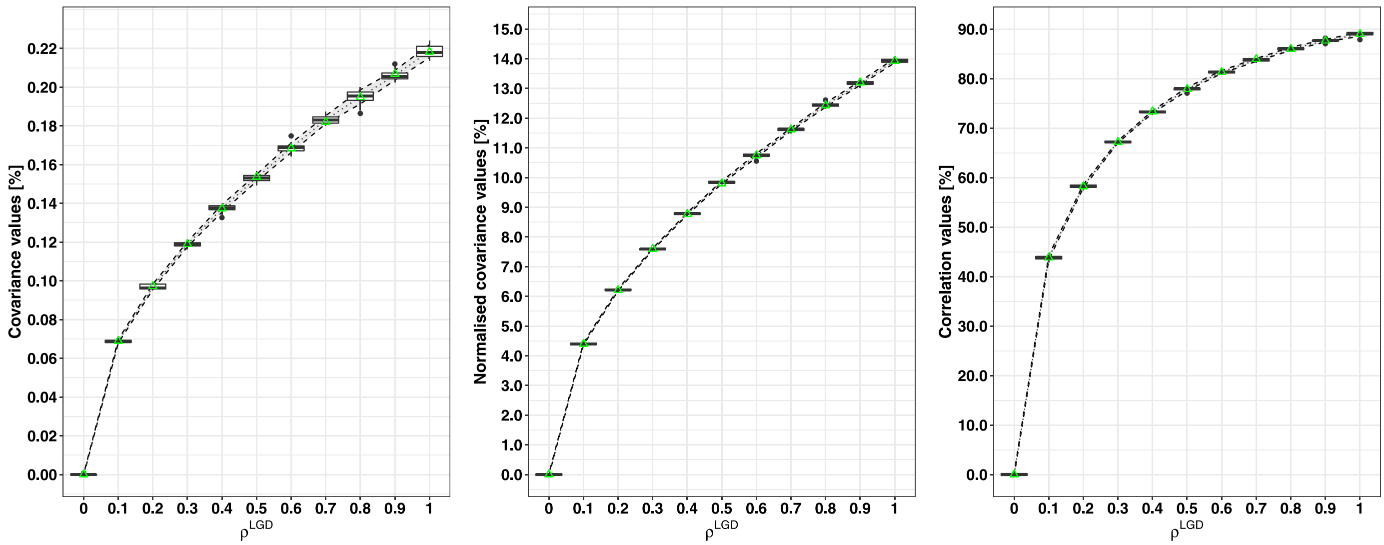
The proof is given in Appendix E.

6. Calibration criterion

The calibration of most of the parameters featured in the ASRF extensions is relatively intuitive. For instance, p_i and $\rho_i = \rho(p_i)$ can be computed following the Basel guidelines. Similarly, the parameters of the LGD marginals F_i can be calibrated from empirical distributions of recovery rates using maximum likelihood or moment-matching techniques. Therefore, this section focuses on the calibration of the ρ^{LGD} parameter controlling the PD-LGD relationship.

The dependence metrics presented in Theorem 1 and their asymptotic versions in Corollary 1 all capture the strength of the PD-LGD link, hence, are a natural calibration criteria for ρ^{LGD}. The asymptotic versions are particularly interesting because the knowledge of the pool size is not needed. Moreover, their mathematical expressions are simpler and using $I \to \infty$ is consistent with the LP approximation underlying the Basel ASRF framework. Among those statistics, the most intuitive criterion to calibrate ρ^{LGD} is the empirical correlation. Unfortunately,

[10] We can easily extend this analysis to a heterogeneous pool setting. The only required homogeneity is at the level of ρ^{LGD}.



(a) c_I vs. ρ^{LGD} (b) C_I vs. ρ^{LGD} (c) r_I vs. ρ^{LGD}

Fig. 2. Empirical covariance (a), normalised covariance (b) and correlation (c) as a function of the PD-LGD dependency parameter for Approach 2. The empty triangles (△) depict the values obtained from the analytical formula in Theorem 1. The boxplots are formed by using 20 repetitions of Monte Carlo estimates, each of which being computed from $M = 50{,}000$ simulations of $(\hat{N}, \hat{L})$. The grey area delimited by the dashed lines, represents ± one standard deviation from the estimated mean value. We consider a homogeneous credit portfolio made of $I = 1500$ obligors with $p = 1\%$, $\rho = \rho(1\%) = 19.28\%$ and marginal LGD distributions $Beta(2,3)$. The corresponding figures for Approach 1 can be found in Fig. G.8 of Appendix A.

r_I and r_∞ suffer from important weaknesses when considering a one-factor models. The empirical normalised covariance C_∞ offers a better alternative, as we now explain.

An important problem of the empirical correlation as a criterion to estimate ρ^{LGD} is that the impact of ρ^{LGD} on r_I is limited when dealing with large pools. To understand this, let us plot in panels (a) and (b) of Fig. 3 the graph of the function $\rho^{LGD} \mapsto r_I(\rho^{LGD})$ using a logarithmic scale for two sets of parameters (λ, σ, p) and increasing pool sizes. As expected, these curves converge to the asymptotic empirical correlation $r_\infty(\rho^{LGD})$ as I increases. However, r_∞ takes its values in a narrow interval (inside [75%-90%] and [80%-90%] in panel (a) and (b), respectively). This shows that for large pools, the impact of ρ^{LGD} on r_I is limited, i.e., r_I conveys little information about ρ^{LGD} which is annoying because ρ^{LGD} has a significant impact on the value-at-risk of credit portfolios. In addition, this also means that generating moderate empirical correlations can only be achieved when inserting small values for I and ρ^{LGD} in the model, leading to vanishing PD-LGD dependency. In the case of panels (a) and (b), for instance, empirical correlations below 75% and 80% cannot be matched when using the asymptotic version.

To understand this phenomenon, let us consider random LGDs ($\sigma > 0$) and assume further that $0 < p < 1$ and $\rho > 0$, which is always verified in a Basel setup. It is clear from Corollary 1 that r_∞ measures the correlation between $U := p(Y)$ and $V := \lambda(Y)$. These two random variables depend on Y in a deterministic way via the decreasing continuous functions $p(.)$ and $\lambda(.)$. Inverting the map $Y \mapsto U = p(Y)$ shows that (U, V) are linked by a deterministic increasing function:

$$V = \lambda(p^{-1}(U)) = \lambda\left(\frac{\Phi^{-1}(p) - \sqrt{1-\rho}\,\Phi^{-1}(U)}{\sqrt{\rho}}\right).$$

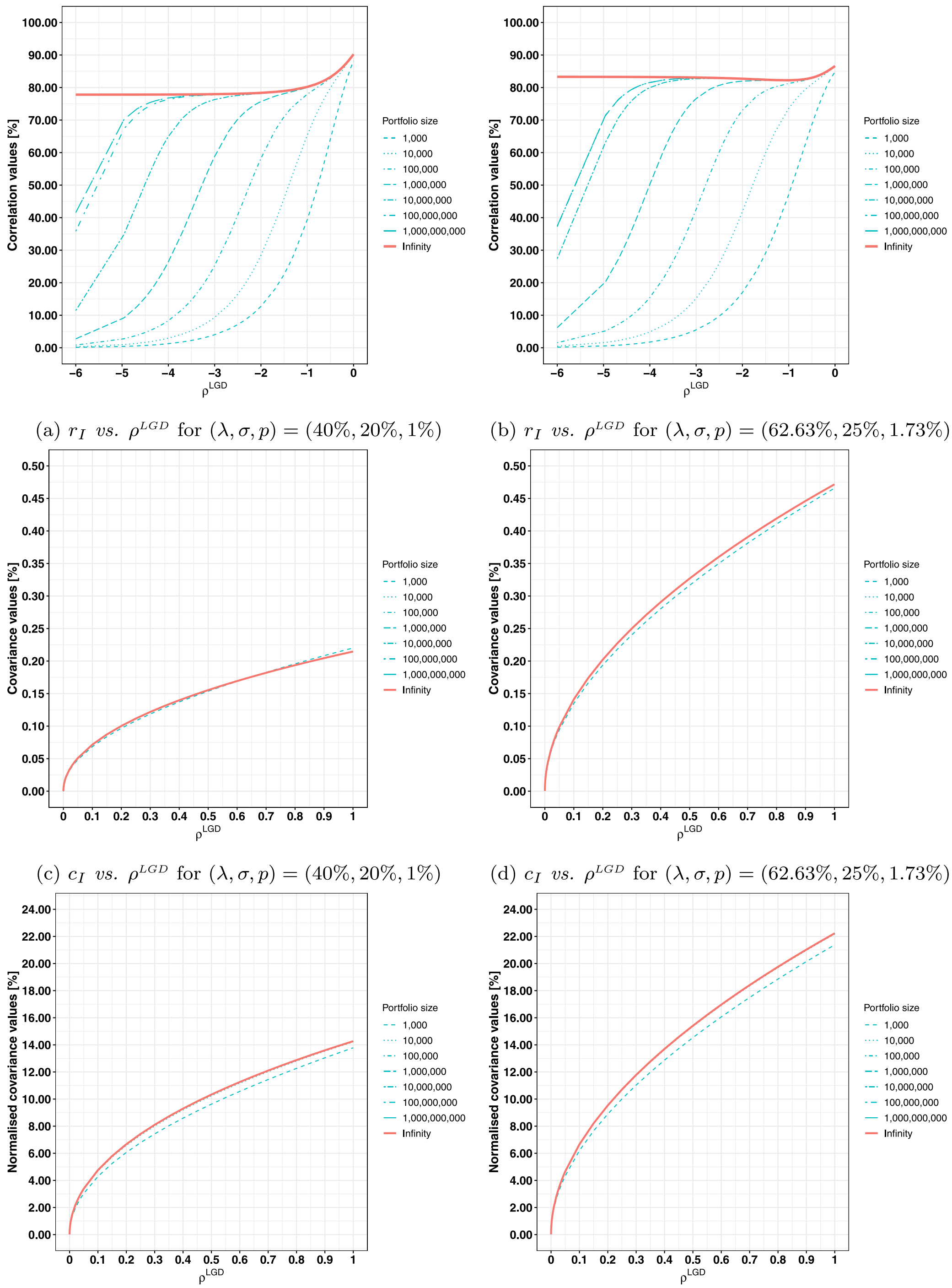
In other words, $p(Y)$ and $\lambda(Y)$ are *comonotonic* variables and their rank correlation is one for every positive value of ρ^{LGD}. Consequently, the Pearson correlation $r_\infty(\rho^{LGD})$ merely quantifies 'how linear' is the deterministic function connecting $p(Y)$ to $\lambda(Y)$. This function is flat at 0 when $\rho^{LGD} = 0$, but is increasing from 0 to 1 whenever $\rho^{LGD} > 0$. This explains that r_∞ is relatively constant and close to one for every $\rho^{LGD} > 0$. The same phenomenon applies to r_I provided that I is large.

It is important to point out that this phenomenon is not specific to *Approach 2*, but is expected to happen in any single-factor framework. Indeed, as $I \to \infty$, the risk that is not fully diversified away exclusively results from Y, showing that $\hat{N} \to \mathbb{E}[\hat{N}|Y]$ and $\hat{L} \to \mathbb{E}[\hat{L}|Y]$ will be deterministic functions of Y. Consequently, $\hat{N}$ and $\hat{L}$ will either be constant (if $\rho^{LGD} = 0$) or comonotonic (if $\rho^{LGD} > 0$). For instance, the very same problem happens in *Approach 1*, too (see Fig. G.10 in the Appendix A).

By contrast, it can be seen from panels (c) and (d) and panels (e) and (f) of Fig. 3 that this phenomenon does not materialise for the covariance and the normalised covariance: c_I and C_I strongly depend on ρ^{LGD}, even asymptotically as $I \to \infty$, because they both depend on the dispersion of $\hat{L}$, hence, on σ and ρ^{LGD}. Another advantage of c and C over r is that the pool size has a limited impact: c_I and C_I are virtually indistinguishable from their asymptotic versions c_∞ and C_∞ as from a few thousands of loans. This is handy because there is no need to know I precisely. In addition, it indicates that the asymptotic (simpler) expressions can be used in most practical cases. Covariance-based criteria benefit from other advantages compared to the empirical correlation. First, $r_\infty(\rho^{LGD})$ is not monotonic, meaning that several values of ρ^{LGD} may lead to a same empirical correlation value r_∞ causing identification issues.[11] This is not the case for the (normalised) covariance which, from Corollary 1, are increasing functions. Next, the expressions of r are substantially more difficult than that of c and C because the most complicated quantities in the finite and asymptotic cases are $\mathbb{V}[\hat{L}|N > 0]$ and $\mathbb{V}[\lambda(Y)]$, respectively. In particular, only $\mathbb{E}[\lambda(Y)]$ needs to be evaluated numerically when computing c_∞ or C_∞. Finally, in contrast with r_∞, calibrating ρ^{LGD} from c_∞ or C_∞ does not lead to a vanishing PD-LGD link.

Among the asymptotic covariance c_∞ and normalised covariance C_∞, we believe that the latter is more meaningful than the former for the sake of calibrating ρ^{LGD}. This is because the empirical covariance $\mathbb{C}\mathrm{ov}(\hat{N}, \hat{L}|N > 0)$ seen in the data is impacted both by the variance of

[11] For example, it can be checked that the curve $r_\infty(\rho^{LGD})$ is U-shaped for small ρ^{LGD} when considering a homogeneous pool with parameters $(\lambda, \sigma, p) = (58\%, 20\%, 1\%)$.



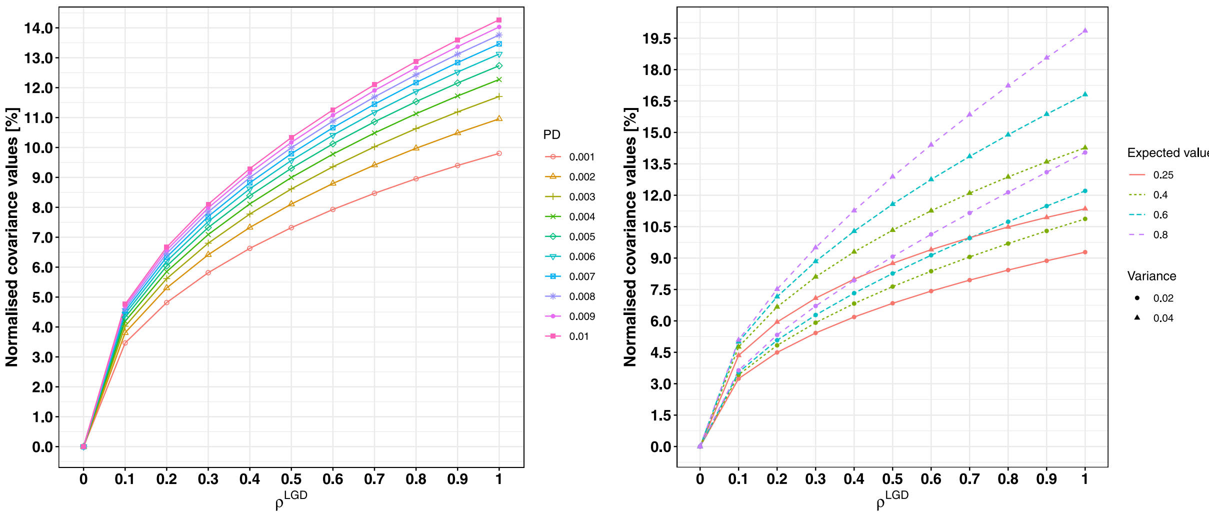
(a) r_I *vs.* ρ^{LGD} for $(\lambda, \sigma, p) = (40\%, 20\%, 1\%)$ (b) r_I *vs.* ρ^{LGD} for $(\lambda, \sigma, p) = (62.63\%, 25\%, 1.73\%)$

(c) c_I *vs.* ρ^{LGD} for $(\lambda, \sigma, p) = (40\%, 20\%, 1\%)$ (d) c_I *vs.* ρ^{LGD} for $(\lambda, \sigma, p) = (62.63\%, 25\%, 1.73\%)$

(e) C_I *vs.* ρ^{LGD} for $(\lambda, \sigma, p) = (40\%, 20\%, 1\%)$ (f) C_I *vs.* ρ^{LGD} for $(\lambda, \sigma, p) = (62.63\%, 25\%, 1.73\%)$

Fig. 3. Empirical correlation $r_I(\rho^{LGD})$ (top), empirical covariance $c_I(\rho^{LGD})$ (middle), empirical normalised covariance $C_I(\rho^{LGD})$ (bottom) *vs.* ρ^{LGD} for increasing pool sizes I and two sets of parameters (λ, σ, p) for Approach 2. The curves associated with $I = \infty$ correspond to the asymptotic expressions. Panels (a) and (b) use a logarithmic scale for the x-axis to emphasise the behaviour for small values of ρ^{LGD}. In panels (c) to (f), all the curves collapse to the asymptotic version except $I = 1000$. The corresponding figures for Approach 1 can be found in Fig. G.10 of Appendix A.

$\hat{L}$ and $\hat{N}$, where the latter strongly depends on the joint distribution of the default events. In the Basel ASRF model, however, the underlying copula is controlled by the regulatory formula in (9). Therefore, there is no chance one can fit the variance of $\hat{N}$ seen in the data with that computed in the model. The asymptotic normalised covariance, on the other hand, features the scaling factor $\sqrt{\mathbb{V}[p(Y)]}$ which eliminates the impact of the asymptotic dispersion of $\hat{N}$. By capturing the effect of ρ^{LGD} both on the PD-LGD correlation and in the variance of the averaged realised losses, C_∞ makes full use of the LGD data while being unaffected by the mismatch impacting the variance of the DR in the data and the regulatory model. The results obtained in the next section confirm that calibrating the model by matching the asymptotic normalised covariance provides realistic values of empirical correlation, implied DLGD and UL, and delivers a pretty good fit to the data. Fig. 4, confirms the nice behaviour of the asymptotic normalised covariance across a wide range of PD and LGD parameters.



(a) C_∞ *vs.* ρ^{LGD} for various values of p (b) C_∞ *vs.* ρ^{LGD} for various values of (λ, σ^2)

Fig. 4. Sensitivity analysis of the asymptotic empirical normalised covariance C_∞ for Approach 2. Panel (a) shows the impact of p when the distribution of the LGDs is a $Beta(2,3)$, setting $\rho = \rho(p)$. Panel (b) shows the impact of the moments (λ, σ^2) of the LGD distribution when setting $p = 1\%$ and $\rho = \rho(1\%) = 19.28\%$. In panel (b), we start from $\lambda = 25\%$ to acknowledge the existence of the 25% floor on the LGD under the A-IRB, for unsecured corporate exposures (BCBS, 2023b, §32.16). The corresponding figures for Approach 1 can be found in Fig. G.9 of Appendix A.

7. Application on Moody's database

In this section, we perform an empirical analysis of our proposed model on VaR and UL figures using the Moody's Analytics Default and Recovery Database. The pool of credit exposures we consider here is a pool of corporate bonds,[12] hence formulas (9) and (10) apply under the Basel framework. We start by presenting the summary statistics of the dataset and calibrate the *Approach 2* model in order for C_∞ to match the empirical normalised covariance seen in the data. Evidences confirm that the model fits the data well. Next, we compare the theoretical VaR figures computed under the LP approximation with those obtained in a finite size Monte Carlo setup. This comparison illustrates that the LP approximation featured in our theoretical setup is satisfactory as from a few hundreds of exposures. Finally, we present a simple procedure for recovering the *implied DLGD*, defined as the value of λ^* one should insert in the RWA formula (9) to obtain an UL equal to the one found in our ASRF model (8) when calibrated to the observed empirical normalised covariance.

7.1. Data

In this empirical study, we seek to consider a prototypical credit portfolio of corporate bonds held by an internationally active bank eligible to use the IRB approach. In absence of data about a specific pool, we follow García-Céspedes and Moreno (2020) and consider the corporate world bond portfolio (Moody's, 2015) as a suitable proxy. The full yearly time series of DR and ARLGD are provided in Table F.5 of Appendix A. The summary statistics are reported in Table 1.

The sample mean of the DR is 1.73%, suggesting taking $p = 1.73\%$ as our TTC estimate of the PD. The average of the ARLGD series is 57.93%, but this should not be interpreted as the 'average LGD' because one must account for the fact that there are more losses in the years where the DR is high. Instead, the average LGD is given by the average of the ARLGDs *weighted by the relative DRs* (defined as DR_i/p), leading to take $\lambda = 62.63\%$ as the TTC estimate for the LGD. The mean and variance of DR can be interpreted as estimates of $\mathbb{E}[\hat{N}]$ and $\mathbb{V}[\hat{N}]$ and, similarly, the mean and variance of the ARLGD can be interpreted as estimates of $\mathbb{E}[\hat{L}]$ and $\mathbb{V}[\hat{L}]$.

Table 1
Summary statistics of the yearly default rates and average realised LGD observed during the period 1982–2014 extracted from the corporate world bond portfolio (Moody's, 2015).

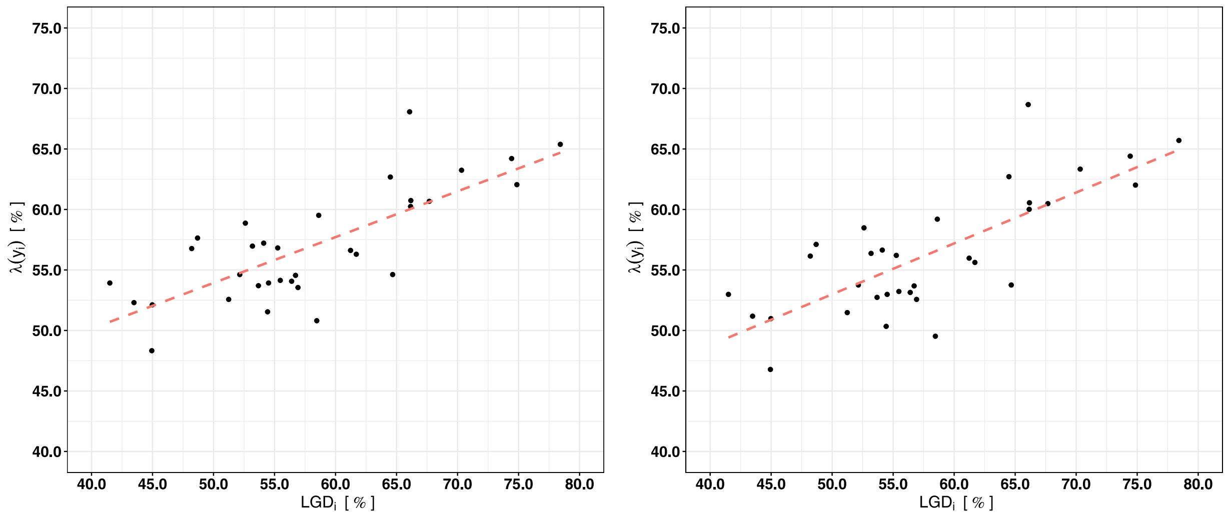
Statistic	Mean	St. Dev.	Min	Median	Max
Default Rate (DR)	1.73%	1.24%	0.40%	1.38%	6.02%
Average realised LGD (ARLGD)	57.93%	9.26%	41.50%	56.40%	78.42%

Note that 9.26% is the standard deviation of yearly averaged LGDs, which is different from the standard deviation of the individual LGDs. To estimate the latter, we refer to Gambetti et al. (2019) who extract a sample of 1831 observed RR of defaulted American corporate bonds from the same Moody's database, on a very similar time frame (from 1990–2013). The authors find a standard deviation of 25.60%; hence, we choose to set $\sigma = 25\%$. The correlation and covariance between DR and ARLGD are 72.82% and 0.08% respectively, while the covariance normalised by the standard deviation of the DR is 6.74%. The correlation, covariance and normalised covariance computed on this dataset can be compared to the expressions r_I, c_I and C_I, respectively.

This dataset deals with the market of American corporate bonds, whose size is large, but not fixed. Therefore, a reasonable approach consists in adopting the asymptotic version of these expressions (as $I \to \infty$) given in Corollary 1. The expectation and variance of $\hat{L}$ both depend on ρ^{LGD}. By contrast, the expectation and variance of the DR are given by $\mathbb{E}[\hat{N}] = \mathbb{E}[p(Y)] = p$ and $\mathbb{V}[\hat{N}] = \mathbb{V}[p(Y)] = \Phi_2(k, k, \rho(p)) - p^2$ where $p = 1.73\%$, $k = \Phi^{-1}(p)$ and $\rho = \rho(p) = 17.05\%$ is given in the Basel setup by (10). This leads to an asymptotic standard deviation of the DR equal to 2.12%, which is close to (but different from) the 1.24% value found in Table 1. This difference should not come as a surprise because even in the unlikely case where the default events associated with the data would be driven by a one-factor Gaussian copula, there is no reason for the value of the underlying default correlation ρ to perfectly match $\rho(p)$, a specific number imposed by a regulatory formula.[13]

[12] This is just one out of many possible examples of credit exposures classified as corporate, sovereign and bank exposures. The latter does not impact the feasibility of this analysis, since, as explained in Section 3.3, the IRB formula (9) applies across all regulatory asset classes.

[13] The default correlation ρ one needs to insert in the asymptotic model to match the value of 1.24% found in the data is 6.95%. A larger value is needed in the finite size case.



(a) Calibration using normalised covariance ($\rho^{LGD} = 10.20\%$) (b) Calibration using Min. RMSE ($\rho^{LGD} = 12.40\%$)

Fig. 5. Scatter plot of model-implied LGDs $\lambda(y_i)$ vs. realised LGDs LGD_i given in Table 1. The value y_i is found by looking for the value of y such that the conditional PD agrees with the DR observed on year i: $p(y_i) = DR_i$. The parameter ρ^{LGD} is found by matching the empirical normalised covariance (a) or by minimising the RMSE between $\lambda(y_i)$ and LGD_i (b). The trend lines are computed using linear regression.

The calibration strategy consists in finding the value of ρ^{LGD} to insert in C_∞ leading to match the observed normalised covariance of 6.74%. The latter is performed for the given set of pool parameters fitting the Moody's data: $(p, \rho, \lambda, \sigma) = (1.73\%, 17.05\%, 62.63\%, 25\%)$ and marginal $Beta$-distributed LGDs, i.e., $\Lambda_i \sim Beta(1.72, 1.03)$. This yields $\rho^{LGD} = 10.20\%$ and leads to $C_\infty = 0.14\%$ (vs. $\mathbb{C}\mathrm{ov}(\hat{N}, \hat{L}) = 0.08\%$ in the data), $r_\infty = 82.23\%$ (vs. $\mathbb{C}\mathrm{orr}(\hat{N}, \hat{L}) = 72.82\%$ in the data) and an asymptotic standard deviation of $\hat{L}$ equal to $\sqrt{\mathbb{V}[\lambda(Y)]} = 8.18\%$ (vs. $\sqrt{V[\hat{L}]} = 9.26\%$ in the data). The mismatch in the correlation can be explained from Fig. 3(b), which shows that the smallest attainable correlation value when $I \to \infty$ is about 82%. The ratio $0.08/0.14 = 0.57$ in the covariances can be largely explained from the ratio $1.24/2.12 = 0.58$ in the standard deviations of the DRs, which stems from the fact that ρ is not calibrated on the data but is set to its regulatory value. This is one of the reasons why the normalised covariance has been preferred over the plain covariance for calibration purposes.

Inspired by Giese (2005), we perform the following exercise to provide evidence of a satisfactory goodness-of-fit. For each year i in the time series of Table F.5, we compare the ARLGD observed that year, LGD_i, to the value given by the function $\lambda(y_i)$ implied by the model where y_i is the implied value of the systematic factor Y found by inverting the relation $DR_i = p(y_i)$.[14] Repeating this exercise for each year between 1982 and 2014 yields 33 pairs $(LGD_i, \lambda(y_i))$, shown in Fig. 5(a). The root-mean-squared error (RMSE) between these pairs can be computed using

$$\mathrm{RMSE} = \sqrt{\frac{1}{33}\sum_{1982}^{2014}(LGD_i - \lambda(y_i))^2}\,. \tag{20}$$

This suggests a second calibration method, which consists in looking for the value of ρ^{LGD} minimising (20). This leads $\rho^{LGD} = 12.40\%$ which is close to the 10.20% found by matching the empirical normalised covariance. The corresponding scatter plot is shown in Fig. 5(b).

7.2. Value-at-risk

The simplest approach to compute VaR consists in estimating our risk figure using Monte Carlo simulation, by sampling the risk factors $Y, Z_1, \ldots, Z_I, \hat{Z}_1, \ldots, \hat{Z}_i, \tilde{Z}_1, \ldots, \tilde{Z}_i$. Under the conditional independence framework detailed in Sections 4.1 and 4.2, the $\mathrm{VaR}_\alpha(L)$ is estimated by the corresponding empirical quantile out of M simulations. Denoting by $l_{(1)} \le l_{(2)} \le \cdots \le l_{(M)}$ the order statistics of the M simulated relative portfolio losses $l_1, \ldots, l_M$, the value-at-risk is approximated by:

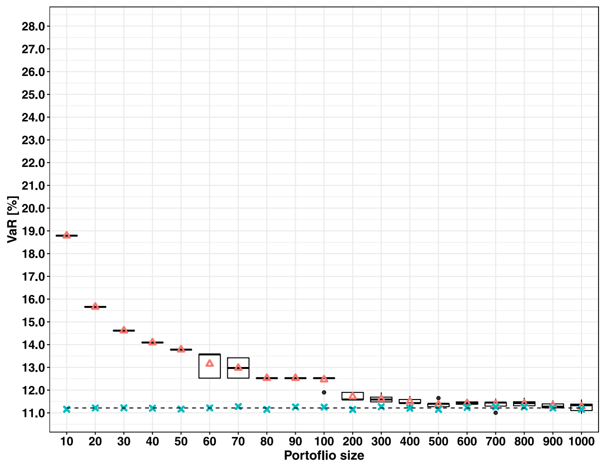
$$\mathrm{VaR}_\alpha^{(M)}(L) := l_{(i_\alpha)} \qquad \text{for } \; i_\alpha := min\{i \in \{1, \ldots, M\} : i/M \ge \alpha\}\,. \tag{21}$$

This is known to be an asymptotically unbiased and consistent estimator of $\mathrm{VaR}_\alpha(L)$ as discussed in Pitera and Schmidt (2018). More advanced methods, featuring interpolation between empirical quantiles, can be considered as well (see Hyndman and Fan (1996)). As for the EL, we simply use the sample estimator.

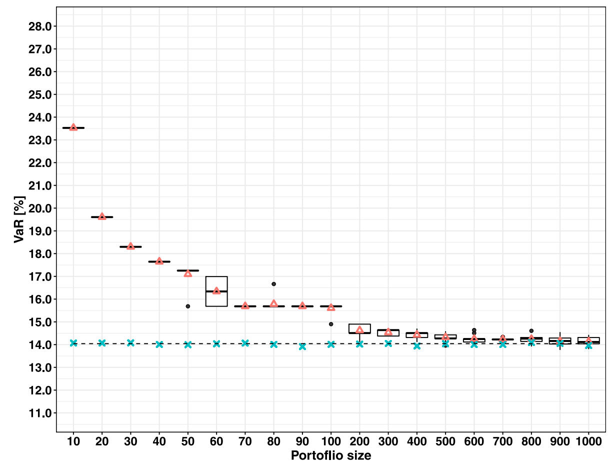
In Fig. 6, we compare the Monte Carlo to the finite-size and asymptotic VaR figures under four different LGD setups featuring comparable homogeneous credit portfolios with $p = 1.73\%$ calibrated to the data and $\rho = 17.05\%$ given by (10). In panels (a) and (b) of Fig. 6, we use the VaR section of the Basel formula (9) and insert a deterministic DLGD equal to the long-run average LGD estimate $\lambda^* = 62.63\%$ (a) or to the worst yearly average LGD for our pool $\lambda^* = 78.42\%$ (b). This leads to an analytical VaR (dotted line) equal to 11.22% and 14.04%, respectively. The finite-size and asymptotic Monte Carlo estimates are depicted by the triangular and cross markers in all sub-figures. In panels (c) and (d), the VaR is computed using (8) in the case of *Approach 2* with Beta-distributed LGDs with parameters complying with $(\lambda, \sigma) = (62.63\%, 25\%)$, i.e., $\Lambda_i \sim Beta(1.72, 1.03)$. In panel (c), we set $\rho^{LGD} = 0$ which is equivalent to using deterministic LGDs equal to λ when $I \to \infty$. This can be checked by comparing panels (a) and (c), where the analytical VaR values (dotted line) are identical on both graphs. Note that the Monte Carlo estimates in the finite pool size I (triangles) differ in panels (a) and (c) but converge to the same value (analytical VaR, dotted line) as the size of the pool increases. Panel (d) shows the effect of the PD-LGD dependency when $\rho^{LGD} = 10.20\%$, leading to $C_\infty = 6.74\%$ and an analytical VaR of 13.94%.

In all graphs of Fig. 6, we observe first that the Monte-Carlo estimates of the VaR with finite pool sizes converge to the corresponding LP approximation as the number of exposures I increases. This suggests that the LP approximation under the Basel IRB approach (panels (a)

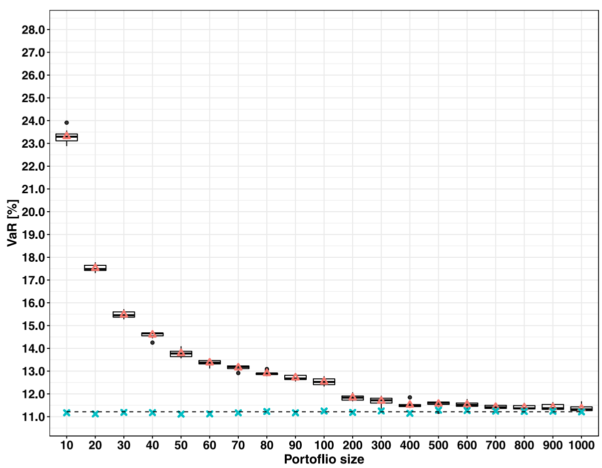
[14] For example, the average LGD in 2014 was $LGD_{2014} = 52.15\%$. The default rate that year was $DR_{2014} = 1.04\%$, leading to $y_{2014} = (\Phi^{-1}(1.73\%) - \sqrt{1-\rho(1.73\%)}\Phi^{-1}(1.04\%))/\sqrt{\rho(1.73\%)} = -0.0189$ while the value implied by the calibrated model is $\lambda(-0.0189) = 54.62\%$.



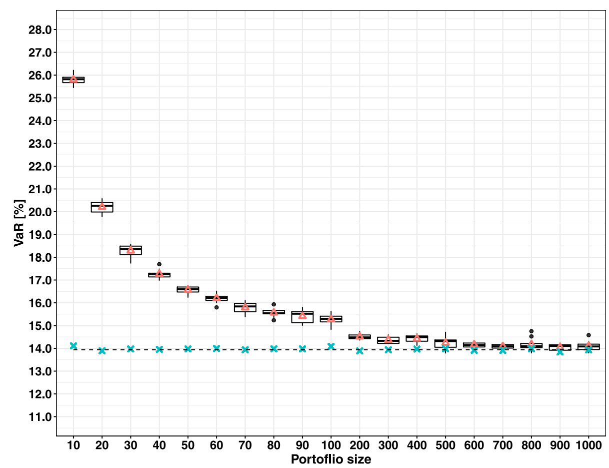
(a) Deterministic downturn LGD ($\lambda^* = 62.63\%$)



(b) Deterministic downturn LGD ($\lambda^* = 78.42\%$)



(c) Independent random LGDs



(d) Approach 2 with $\rho^{LGD} = 10.20\%$

Fig. 6. VaR at confidence level 99.9% for the prototypical corporate bond world portfolio as a function of the portfolio size I in different LGD frameworks. The dotted line shows the analytical expression obtained under the LP approximation specific to each model. The symbol (x) refers to the corresponding LP Monte Carlo estimator, found by estimating the 99.9% quantile of $\mathbb{E}[L|Y]$ via a sampling of the common factor Y, while (△) shows the quantile of L estimated in Monte Carlo but without relying on the LP approximation, i.e., when sampling all the random variables involved in the model. The boxplots are built from 1000 (in (a) and (b)) and 10 (in (c) and (d)) simulated VaR, each of which being estimated from $M = 250,000$ simulated relative portfolio losses. We choose the calibrated parameters $(p, \rho) = (1.73\%, 17.05\%)$ and, for stochastic LGDs in panels (c) and (d), we assume the marginal distribution $Beta(1.72, 1.03)$. The corresponding figure for Approach 1 can be found in Fig. G.11 of Appendix A.

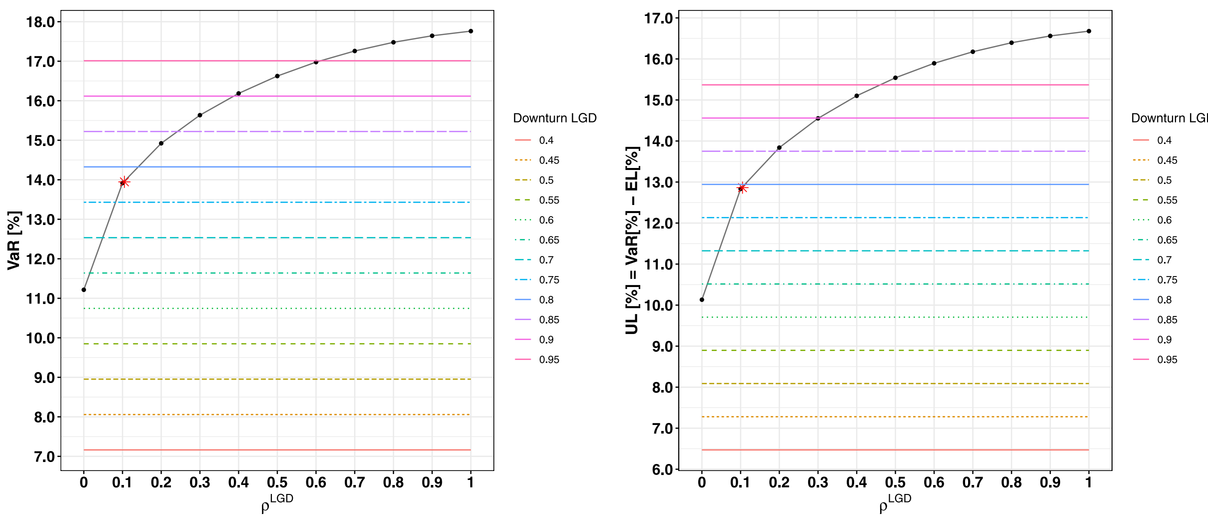
and (b)), the independent stochastic LGD setup (panel (c)), and our *Approach 2* (panel (d)) are correct. The convergence speed of the VaR to the LP result as the size of the pool increases does not seem to much depend on the considered model.[15] In particular, the addition of a new PD-LGD dependency parameter does not entail the quality of the estimation provided by the corresponding LP formula compared to the Basel IRB framework.

7.3. Implied Basel downturn LGD

This section aims to evaluate the impact of adopting *Approach 2* when computing capital requirements. To that end, we compare in Fig. 7 the VaR (left) and UL (right) generated by the Basel IRB approach featuring various values of the DLGD λ^* to those obtained from *Approach 2* with calibrated parameters for various values of ρ^{LGD}. For the UL, we compare $\text{UL}_{\text{IRB}}(\lambda^*) = \lambda^* \times (p(z_{-0.999}) - p)$ to $\text{UL}_{\text{Model}}(\rho^{LGD}) = \lambda(z_{-0.999})p(z_{-0.999}) - \lambda p$. The VaRs are computed similarly, but adding $\lambda^* p$ and λp, respectively. The asterisk points out the VaR and UL values associated with *Approach 2* when using $\rho^{LGD} = 10.20\%$ calibrated on the data.

From panels (a) and (b), one can extract *implied DLGDs* defined as the value of λ^* such that the Basel VaR (or UL) matches the VaR (or UL) computed using the calibrated model. For instance, the implied DLGD to match the VaR is $\lambda^* = 77.85\%$ (panel (a)) and the implied DLGD to match the UL is $\lambda^* = 79.48\%$ (panel (b)). The concept of *implied DLGDs* is particularly interesting for policymakers and practitioners: it is the value of λ^* one should consider in the Basel IRB formula in order to get the UL found in *Approach 2* calibrated on a value of the *empirical normalised covariance*. To put it differently, it is the DLGD needed to get the capital requirements implied by a model capturing the PD-LGD link whose strength is calibrated to relevant data. Hence, we can map banks internally estimated downturn LGDs into a fully observable risk-metric, granting to a value λ^* a clear and transparent economic interpretation. *Approach 2* can serve two regulatory purposes: (i) as a methodology to harmonise the estimation procedure for DLGDs under

[15] The variability in the width of the boxes in Fig. 6(a) and (b) results from the fact that when the LGDs are all set to a same constant, the loss is actually just a multiple of the number of defaults. Hence, the cdf of the loss is discrete and displays jumps at specific values. In this case, the variance of the quantile estimator depends on a complex way of the pool size.



(a) Value-at-risk (VaR) *vs.* ρ^{LGD} (b) Unexpected loss (UL) *vs.* ρ^{LGD}

Fig. 7. Sensitivity of the VaR (a) and UL (b) at a 99.9% confidence level for a homogeneous credit portfolio with parameters $(p, \rho, \lambda, \sigma) = (1.73\%, 17.05\%, 62.63\%, 25\%)$ chosen consistently with the Moody's Analytics Default and Recovery Database and Basel regulations. The solid curves show the values obtained in Approach 2 as a function of the ρ^{LGD} parameter. The asterisk ($*$) corresponds to the case where ρ^{LGD} is calibrated using the empirical normalised covariance observed in the Moody's data. The horizontal lines show the corresponding terms computed in the Basel formula for various values of DLGD λ^*. The corresponding figures for Approach 1 can be found in Fig. G.12 of Appendix A.

the IRB and (ii) as a back-testing tool by national supervisory entities to map banks supplied DLGDs into a value of the observed empirical normalised covariance at the level of their credit pool.

In Table 2, we report the DLGD λ^* associated with six 'schemes' compatible with the Basel IRB approaches. For two F-IRB schemes, we consider $\lambda^* = 40\%$ (imposed for senior unsecured corporate bonds) and $\lambda^* = 75\%$ (imposed for subordinated corporate bonds). We then consider four different estimates for the A-IRB approach. As explained in Section 1, in any Basel compliant jurisdiction, the final DLGD under the A-IRB cannot be less than the long-run default-weighted average LGD observed for the given pool. Therefore, no matter the LGD model used by a bank, the value 62.63% is a floor for the corresponding DLGD. Choosing $\lambda^* = 62.63\%$ is a valid but 'naive' choice, which implicitly discards the PD-LGD link. Focusing on the EU implementation of the Accords, we compute three DLGD values compatible with the EBA (2019) guidelines, as in Hartmann-Wendels and Imanto (2021). Type 1 (based on observed impact) amounts to choose the worst ARLGD: $\lambda^* = \max(LGD_i) = 78.42\%$. In Type 3 (free modelling with minimum fixed add-on), the final DLGD cannot be less than the long-run ARLGD for the considered pool plus 15 percentage points, leading to $\lambda^* = 77.63\%$. The non-binding reference value (R.V.) on the overall LGD estimation procedure is defined as the 'simple average of the ARLGDs from the two individual years with the highest observed economic losses' (EBA, 2019, p. 50). Given our dataset, this yields $\lambda^* = 76.64\%$. For each of these capital requirement schemes, we compute $\mathrm{UL}_{\mathrm{IRB}}(\lambda^*)$ and, if it exists, the value of ρ^{LGD} one must insert in $\mathrm{UL}_{\mathrm{Model}}(\rho^{LGD})$ to match this level of UL. Next, we consider two schemes related to *Approach 2*. We set ρ^{LGD} to the value found using the empirical normalised covariance or the min. RMSE calibration criterion. For each of them, we compute $\mathrm{UL}_{\mathrm{Model}}(\rho^{LGD})$ and the implied DLGD. For each scheme, we compute the conversion ratio, defined as the scaling factor λ^*/λ between the DLGD and the average LGD.

One concludes that for the considered pool, the capital requirements suggested by the Basel IRB approaches would underestimate the UL found using our calibrated model for any DLGD below 79.48%. This is a real concern because, as shown in Table 2, all the DLGDs associated with the considered Basel IRB approaches are below this level. What is most striking is to observe that the conditional PD implied by the regulatory mapping function, $\mathrm{PD}(-z_{0.999}) = 17.90\%$ is far larger than the worst-case PD observed in the data 6.02%, most notably to the DR (of 4.35%) in the same year of our selected 'worst-case' ARLGD. This suggests two things: (i) if one would apply the same downturn procedure to the PD, as depicted here for the LGD, one would considerably underestimate risk (6.02% *vs.* 17.90%), (ii) given the positive PD-LGD link, one should select an ARLGD in a year where the observed DR should be equal to 17.90%, hence the Basel model suggests that ARLGD should be much larger than the one observed in the data.

Finally, observe that the Basel DLGD estimated using the current approaches are all smaller than (or equal to) the worst ARLGD of 78.42%. However, there is no reason to believe that these values are close to the highly conservative approach suggested by the interpretation of minimum capital requirements in terms of capacity to absorb losses at confidence level 99.9%, which requires using $\lambda(-z_{0.999})$. To see this, suppose that one would treat the PD using a similar 'downturn procedure' instead of computing the value $p(-z_{0.999}) = 17.90\%$ suggested by the mapping function $p(Y)$. Taking the worst observed DR would lead to

Table 2
In the Basel 'schemes', one imposes the value of λ^* and computes the value of $\mathrm{UL}_{\mathrm{IRB}}$ using (9). The value of ρ^{LGD} is the value of the parameter one must insert in Approach 2 in order to get the same UL when using (8), when it exists. In the last two approaches, one sets ρ^{LGD} to its value calibrated using the empirical normalised covariance or the min. RMSE criterion, respectively, and compute the UL using (8). For these two approaches, λ^* is the implied DLGD. The conversion ratio λ^*/λ shows the scaling factor between the DLGD and the average LGD. All cases consider the same homogeneous pool with Beta-distributed LGDs and parameters (λ, p, σ) calibrated on the data.

Approach	λ^*	λ^*/λ	ρ^{LGD}	$\mathrm{UL}_{\mathrm{IRB}}$	$\mathrm{UL}_{\mathrm{Model}}$
Basel F-IRB (Senior unsec.)	40%	0.64	–	6.47%	–
Basel F-IRB (Subordinated)	75%	1.20	*5.30%*	12.13%	12.13%
Basel A-IRB (Naive scheme)	62.63%	1.00	*0%*	10.13	10.13%
Basel A-IRB (EBA Type 1)	78.42%	1.25	*8.88%*	12.69%	12.69%
Basel A-IRB (EBA Type 3)	77.63%	1.24	*7.96%*	12.56%	12.56%
Basel A-IRB (EBA R.V. scheme)	76.64%	1.22	*6.89%*	12.40%	12.40%
Calibrated model (C_∞)	*79.48%*	1.27	*10.20%*	12.86%	12.86%
Calibrated model (RMSE)	*81.08%*	1.29	*12.40%*	13.12%	13.12%

Table 3
Impact of σ on λ^* and the ratio λ^*/λ without (left) and with (right) recalibration of the parameter ρ^{LGD} for a homogeneous pool with parameters $(p, \rho, \lambda) = (1.73\%, 17.05\%, 62.63\%)$. The value of $\rho^{LGD} = 10.20\%$ corresponds to the calibrated value when choosing $\sigma = 25\%$.

σ	ρ^{LGD}	λ^*	λ^*/λ	ρ^{LGD}	λ^*	λ^*/λ
10%	10.20%	69.83%	1.115	57.18%	79.63%	1.271
15%	10.20%	73.25%	1.170	26.84%	79.60%	1.271
20%	10.20%	76.49%	1.221	15.54%	79.56%	1.270
25%	10.20%	79.49%	1.269	10.24%	79.51%	1.270
30%	10.20%	82.13%	1.311	7.41%	79.47%	1.269
35%	10.20%	84.36%	1.347	5.79%	79.45%	1.269

insert a 'downturn PD' of $p^* = 6.02\%$. Considering $\lambda^* = 78.42\%$ as in EBA Type 1 would yield an UL of $(6.02\%–1.73\%) \times 78.42\% = 3.36\%$ instead of $(17.90\%–1.73\%)\times 78.42\% = 12.69\%$ found in Table 2. This suggests that, if applied to the PD, a 'downturn procedure' would underestimate the UL by a factor close to 3.5 compared to the value suggested by the mapping function approach in the ASRF. By transposition, it is likely that the same problem applies to the LGD, although the impact is expected to be less severe because the DLGD is closer to the upper bound of 100%. This is exactly what we learn from our model: the values obtained using our mapping function $\lambda(-z_{-0.999})$ are slightly more conservative compared to choosing as DLGD the largest ARLGD.

7.4. Robustness check

Although compatible with similar data, the standard deviation of the marginal LGDs is the only parameter that is estimated from a slightly different set of defaulted bonds. Therefore, we conclude the empirical section with a robustness check, analysing the impact of σ on the results of Table 2. To that end, we start with our calibrated model, using $\rho^{LGD} = 10.20\%$. We then let σ vary between 10% and 35% by steps of 5% while leaving the other parameters unchanged.[16] The corresponding values of λ^* and λ^*/λ are given in left part of Table 3 and are proven to be quite dependent upon σ. Of course, these numbers make little sense because playing with σ while keeping ρ^{LGD} fixed impacts the asymptotic normalised covariance, hence, breaks the calibration of the model. A better analysis consists in recalibrating the parameter ρ^{LGD} for each value of σ. The corresponding values for the parameter ρ^{LGD}, the implied DLGD λ^* and for the ratio λ^*/λ are given in the right part of the table. We note that ρ^{LGD} tends to compensate for the variation of σ in a way that makes the implied DLGD and the conversion ratio pretty much constant: around 79.5% and 1.27, respectively. This is a desirable result, for two reasons. First, because it shows that the calibration method exploits the information embedded in the dispersion of the LGDs. Second, because it suggests that one does not need to have a precise estimation of σ: the value of λ^* is pretty insensitive to σ provided that the parameter ρ^{LGD} is computed so as to match the asymptotic normalised covariance. This is good news because it shows that a rough estimate of σ leads to meaningful results.

8. Conclusion

Following the recent financial crises, the Basel Committee promoted the development of capital requirement models displaying better *risk-sensitivity, simplicity* and *comparability*. Our research contributes to this effort by extending the asymptotic single risk factor (ASRF) model underlying the internal ratings-based approach governing regulatory capital requirements in order to take *PD-LGD dependency* into account.

[16] Note that $\sigma = 35\%$ is actually a pretty high value given that the maximum variance for a variable bounded in $[0, 1]$ with mean μ is that of the Bernoulli of same mean, $\sigma^2 = \mu(1-\mu)$. Setting $\mu = 62.63\%$, the upper bound for σ is thus 48.38%.

This phenomenon refers to the financial risk resulting from the well-known co-movements between the probability of default (PD) and the loss given default (LGD). Failing to account for this reality can lead to severe underestimation of realised losses, hence, of capital requirements. Therefore, properly capturing the PD-LGD dependency is a major concern when it comes to computing reliable credit risk metrics.

In the current Basel framework, PD-LGD dependency is addressed via the concept of *downturn LGD* (DLGD), which represents the average LGD during stressed periods. The guidelines for computing DLGDs are currently designed by the regional regulators and need to be very precise to ensure transparency and comparability. The DLGD setup relies on the simple constant LGD ASRF machinery, which has the advantage of being simple from a modelling perspective. However, it amounts to transferring the PD-LGD dependency to the notion of DLGD, hence, leads to several drawbacks. First, PD-LGD dependency is 'externalised' from the model and is thus treated very differently from the default dependency. Second, the concept of DLGD becomes somewhat disconnected from the underlying loss model. For instance, there is no reason for the estimated DLGD to comply with the 99.9% quantile of the LGD assuming an infinitely granular pool, which is, however, the natural interpretation suggested by the ASRF model. This raises questions about whether capital requirements found in this way can indeed be interpreted as a 99.9% VaR, even under the ASRF assumptions. Moreover, by construction, the DLGD is most often smaller than the worst observed value which, given the interpretation of capital requirements in terms of VaR, may not be conservative enough.

In this paper, we follow another route: we propose a simple extension of the Basel ASRF model where the link between PD and LGD is handled similarly as the dependency between the default events. Just as the mapping function $p(Y)$ converts the average PD into a PD conditional upon the systematic factor Y, we derive the mapping function $\lambda(Y)$ converting the average LGD into a conditional LGD, leading to a clear expression for the loss quantile in a large pool setup. The strength of either types of link is controlled using a specific parameter: ρ and ρ^{LGD}, respectively. Moreover, the model offers full flexibility in terms of marginal LGD distributions while leading to portfolio-invariant semi-analytical formula for the computation of VaR. By doing so, *risk-sensitivity* is enhanced while *simplicity* is preserved. A simple and robust procedure is proposed to calibrate the new parameter from the default rate and the average LGD observed over a period of time. Modelling the PD-LGD dependency through an explicit factor model whose parameters can be connected to observable metrics further enhances both transparency and explainability, and contributes to improve models' *comparability*. Finally, we consider an empirical study exploiting the dynamics of defaulted corporate bonds. The model is calibrated on the data and is proven to deliver a good fit. Moreover, it delivers intuitive values for value-at-risk and capital requirements. Interestingly, our model is simple to implement and can be designed in standard tools such as Excel, which is an appealing practical feature. Although the highlight is put on homogeneous loans, heterogeneous pools can be handled using a similar approach as in the current Basel framework for the defaults, i.e., by assigning a same PD-LGD parameter to loans having similar characteristics, and treating the pool as a set of homogeneous sub-pools.

Our model serves multiple purposes, such as calculating economic capital, assessing model risk, or pricing and risk-management of mortgage backed securities, to name a few. Focusing on banking supervision, our model provides insights on how to adjust the current Basel regulatory framework to capture the PD-LGD link. Alternatively, our model can be used to compute the *implied downturn LGD*, defined as the DLGD one needs to insert into the current risk-weighted assets formula to obtain the unexpected loss found using a model that explicitly accounts for the PD-LGD link and is calibrated to observable statistics, in the same vein as implied volatility in option pricing. Implied downturn

LGDs provide a simple procedure to compute DLGDs, serve as benchmark DLGDs in back testing exercises, or as stressed values in stress testing tools. Within an ASRF setup, such tools constitute a key element of supervisory projects conducted by regulatory institutions worldwide, such the targeted review of internal models within the European Union. Finally, our results could also help policymakers to design a simple regulatory mapping function $\lambda(Y)$ and PD-LGD parameter, in the same vein as what is done for $p(Y)$ and the default correlation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data is available in Table F.5.

Acknowledgements

Previous versions of this paper benefited from discussions when presented at several research events, including: the 7th Workshop on Recent Developments in Dependence Modelling with Applications in Finance and Insurance (September 2022, Agistri), the 38th International Conference of the French Finance Association (May 2022, Saint-Malo), the PRISM research seminars (November 2021, Université Paris 1 Panthéon-Sorbonne), the 14th edition of the International Risk Management Conference (October 2021, Cagliari) and the International Finance and Banking Society 2021 Oxford Conference (September 2021, Oxford). The authors are grateful to Jean-Paul Laurent, Yannick Malevergne, Carole Bernard, Olivier Scaillet, Edward I. Altman and the anonymous referees for providing valuable feedback, insightful comments and suggestions that contributed to strengthening the quality of this work. All errors remain ours. The work of Frédéric Vrins is supported by the Fonds de la Recherche Scientifique - (F.S.R.-FNRS) de Belgique, Belgium grant CDR J.0037.18 and by the grant ARC 18-23/089 of the Belgian Federal Science Policy Office, Belgium. Computational resources have been provided by the supercomputing facilities of the Université catholique de Louvain (CISM/UCL) and the Consortium des équipements de Calcul Intensif en Fédération Wallonie Bruxelles (CÉCI) funded by the Fond de la Recherche Scientifique de Belgique (F.R.S.-FNRS) under convention 2.5020.11 and by the Walloon Region.

Appendix A. Proof of results

A.1. Proof that the CDF of Λ_i in (11) is F

Because Λ_i is only defined conditional upon default, it is enough to check that $U_i := 1 - \frac{\Phi(X_i)}{p_i}$ is uniformly distributed conditional upon $\{B_i = 1\}$. To see this, observe first that

$$B_i = 1 \quad \Leftrightarrow \quad X_i \le k_i \quad \Leftrightarrow \quad \Phi(X_i) \le p_i$$

which, in turns, implies that

$$\mathbb{P}(U_i \le 0|B_i = 1) = 0 \quad \text{and} \quad \mathbb{P}(U_i \le 1|B_i = 1) = 1\,.$$

Now, for any $x \in [0, 1]$,

$$\begin{aligned}\mathbb{P}(U_i \le x|B_i = 1) &= \mathbb{P}(U_i \le x|X_i \le k_i)\\ &= \frac{\mathbb{P}\left(\frac{\Phi(X_i)}{p_i} \ge 1 - x \cap X_i \le k_i\right)}{\Phi(k_i)}\\ &= \frac{\mathbb{P}\left(\Phi^{-1}(p_i \times (1-x)) \le X_i \le k_i\right)}{p_i}\\ &= \frac{\mathbb{P}\left(\Phi^{-1}(p_i \times (1-x)) \le X_i \le \Phi^{-1}(p_i)\right)}{p_i}\\ &= \frac{p_i - p_i \times (1-x)}{p_i} = x\,.\end{aligned}$$

A.2. Proof of Lemma 1

Notice that the distribution of Λ_i remains equal to F_i for all $\rho_i^{LGD} \in [0, 1]$. This is clear for $\rho^{LGD} = 1$, and it is obvious for $\rho^{LGD} = 0$ since then

$$\mathbb{P}(L_i \le l|B_i = 1) = \mathbb{P}(F_i^{-1}(\Phi(\hat{Z}_i)) \le l) = \mathbb{P}(\hat{Z}_i \le \Phi^{-1}(F_i(l))) = F_i(l). \quad \text{(A.1)}$$

For the other values of ρ_i^{LGD} we have

$$\begin{aligned}\mathbb{P}(L_i \le l|B_i = 1) &= \mathbb{P}(\Lambda_i \le l|B_i = 1)\\ &= \mathbb{P}\left(\Lambda_i \le l \cap X_i \le k_i\right)/p_i\\ &= \mathbb{P}\left(F_i^{-1}\left(\Phi\left(\sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1 - \frac{\Phi(X_i)}{p_i}\right)\right.\right.\right.\\ &\qquad \left.\left.\left.+\sqrt{1-\rho_i^{LGD}}\hat{Z}_i\right)\right) \le l \cap X_i \le k_i\right)/p_i\\ &= \mathbb{P}\left(\sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1 - \frac{\Phi(X_i)}{p_i}\right)\right.\\ &\qquad \left.+\sqrt{1-\rho_i^{LGD}}\hat{Z}_i \le \Phi^{-1}(F_i(l)) \cap X_i \le k_i\right)/p_i\\ &= \mathbb{P}\left(\hat{Z}_i \le \frac{\Phi^{-1}(F_i(l)) - \sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1 - \frac{\Phi(X_i)}{p_i}\right)}{\sqrt{1-\rho_i^{LGD}}}\right.\\ &\qquad \left.\cap X_i \le k_i\right)/p_i\\ &= \frac{1}{p_i}\int_{-\infty}^{k_i}\mathbb{P}\left(\hat{Z}_i \le \frac{\Phi^{-1}(F_i(l)) - \sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1 - \frac{\Phi(x)}{p_i}\right)}{\sqrt{1-\rho_i^{LGD}}}\right)\\ &\qquad \times \phi(x)dx\\ &= \frac{1}{p_i}\int_{-\infty}^{\Phi^{-1}(p_i)}\Phi\left(\frac{\Phi^{-1}(F_i(l)) - \sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1 - \frac{\Phi(x)}{p_i}\right)}{\sqrt{1-\rho_i^{LGD}}}\right)\\ &\qquad \times \phi(x)dx\,.\end{aligned}$$

Now, let us apply two changes of variables: $u = \Phi(x)$ ($du = \phi(x)dx$) and $v = \Phi^{-1}(1 - u/p_i)$ ($dv = \frac{-1}{p_i\phi(v)}du$), leading to write

$$\begin{aligned}\mathbb{P}(L_i \le l|B_i = 1) &= \frac{1}{p_i}\int_0^{p_i}\Phi\left(\frac{\Phi^{-1}(F_i(l)) - \sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1 - \frac{u}{p_i}\right)}{\sqrt{1-\rho_i^{LGD}}}\right)du\\ &= -\frac{1}{p_i}\int_{-\infty}^{\infty}\Phi\left(\frac{\Phi^{-1}(F_i(l)) - \sqrt{\rho_i^{LGD}}v}{\sqrt{1-\rho_i^{LGD}}}\right)(-p_i\phi(v))dv\end{aligned}$$

Defining

$$f(Y, p, \rho) := \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right) \quad \text{with} \quad \rho \in [0, 1)$$

and cancelling the -p_i's, the above integral takes the form $\mathbb{E}[f(Y, F_i(l), \rho_i^{LGD})]$. The proof is concluded by noting that $\mathbb{E}[p_i(Y)] = \mathbb{E}[f(Y, p_i, \rho_i)] = p_i$:

$$\mathbb{P}(L_i \le l|B_i = 1) = \mathbb{E}[f(Y, F_i(l), \rho^{LGD})] = F_i(l)\,.$$

A.3. Proof of Lemma 2

The expectation can be computed by fixing Y in (12) (which is hidden behind X_i) and imposing further $\{B_i = 1\}$. For Y given, the default event becomes

$$\{B_i = 1\} = \{X_i \le \Phi^{-1}(p_i)\}$$

$$= \left\{ Z_i \le (\Phi^{-1}(p_i) - \sqrt{\rho_i} Y)/\sqrt{1-\rho_i} \right\}$$
$$= \left\{ Z_i \le k_i(Y) \right\},$$

where we have used $k_i(y) := \Phi^{-1}(p_i(y))$. Hence,

$$\mathbb{E}[\Lambda_i^m | B_i = 1, Y] = \mathbb{E}\left[\Lambda_i^m | Z_i \le k_i(Y), Y\right].$$

The proof is concluded by computing the expectation of $\Lambda_i = F_i^{-1}(\Phi(\hat{X}_i))$ over the set $\{Z_i \le k_i(Y)\}$ assuming Y is fixed, where $\hat{X}_i$ is expressed as a function of the i.i.d. variables $Y, Z_i, \hat{Z}_i$ given in (13) and rescaling by the probability of the conditioning event given Y, $\mathbb{P}(Z_i \le k_i(Y)) = p_i(Y)$.

A.4. Proof of Lemma 3

The proof consists in showing that H_i is the cumulative distribution function of $\tilde{X}_i$ conditional upon $\{B_i = 1\}$, and then applying the probability integral transform.

Let us first compute the cdf of $\tilde{X}_i$ conditional upon $\{B_i = 1\}$ evaluated at x:

$$\begin{aligned}
F_{\tilde{X}_i|B_i=1}(x) &= \mathbb{P}(\tilde{X}_i \le x | B_i = 1) \\
&= \mathbb{P}\left(\tilde{X}_i \le x | X_i \le k_i\right) \\
&= \mathbb{P}\left(\tilde{X}_i \le x \cap X_i \le k_i\right)/p_i, \\
&= \mathbb{P}\Big(-\sqrt{\rho_i^{LGD}} Y + \sqrt{1-\rho_i^{LGD}} \tilde{Z}_i \le x \cap \sqrt{\rho_i} Y \\
&\quad + \sqrt{1-\rho_i} Z_i \le k_i\Big)/p_i \\
&= \mathbb{E}\Big[\mathbb{P}\Big(-\sqrt{\rho_i^{LGD}} Y + \sqrt{1-\rho_i^{LGD}} \tilde{Z}_i \le x \cap \sqrt{\rho_i} Y \\
&\quad + \sqrt{1-\rho_i} Z_i \le k_i | Y\Big)\Big]/p_i \\
&= \mathbb{E}\left[\mathbb{P}\left(\tilde{Z}_i \le \frac{x + \sqrt{\rho_i^{LGD}} Y}{\sqrt{1-\rho_i^{LGD}}} \middle| Y\right) \times \mathbb{P}\left(Z_i \le \frac{k_i - \sqrt{\rho_i} Y}{\sqrt{1-\rho_i}} \middle| Y\right)\right]/p_i \\
&= \mathbb{E}\left[\Phi\left(\frac{x + \sqrt{\rho_i^{LGD}} Y}{\sqrt{1-\rho_i^{LGD}}}\right) \times \Phi\left(\frac{k_i - \sqrt{\rho_i} Y}{\sqrt{1-\rho_i}}\right)\right]/p_i,
\end{aligned}$$

where the numerator takes the form $\mathbb{E}\left[\Phi\left(a_1 + b_1 Y\right) \times \Phi\left(a_2 + b_2 Y\right)\right]$ with

$$a_1 := x/\sqrt{1-\rho_i^{LGD}}, \quad b_1 := \sqrt{\rho_i^{LGD}}/\sqrt{1-\rho_i^{LGD}},$$

$$a_2 := k_i/\sqrt{1-\rho_i}, \quad b_2 := -\sqrt{\rho_i}/\sqrt{1-\rho_i}.$$

This shows that H_i is the cdf of $\tilde{X}_i$ conditional upon $\{B_i = 1\}$ because, for $X \sim \mathcal{N}(0,1)$ and any real values a_1, a_2, b_1, b_2, it holds that

$$\mathbb{E}\left[\Phi\left(a_1 + b_1 X\right) \Phi\left(a_2 + b_2 X\right)\right] = \Phi_2\left(\frac{a_1}{\sqrt{1+b_1^2}}, \frac{a_2}{\sqrt{1+b_2^2}}, \frac{b_1 b_2}{\sqrt{1+b_1^2}\sqrt{1+b_2^2}}\right).$$

To prove this relationship, define $Y_1 = a_1 + b_1 X$ and $Y_2 = a_2 + b_2 X$ where $X \sim \mathcal{N}(0,1)$. Noting $Z_1, Z_2 \sim \mathcal{N}(0,1)$ i.i.d., we have

$$\begin{aligned}
\mathbb{E}[\Phi(Y_1)\Phi(Y_2)] &= \mathbb{E}[\mathbb{P}(Z_1 \le a_1 + b_1 X | X)\mathbb{P}(Z_2 \le a_2 + b_2 X | X)] \\
&= \mathbb{E}[\mathbb{P}(Z_1 \le a_1 - b_1 X | X)\mathbb{P}(Z_2 \le a_2 - b_2 X | X)] \\
&= \mathbb{P}(Z_1 + b_1 X \le a_1, Z_2 + b_2 X \le a_2) \\
&= \mathbb{P}(X_1 \le a_1, X_2 \le a_2)
\end{aligned}$$

where we have used $X \sim -X$ and $X_1 := Z_1 + b_1 X$ and $X_2 = Z_2 + b_2 X$.

Clearly, the pair (X_1, X_2) defined as forms a bivariate normal random vector with mean vector $(\mu_{X_1}, \mu_{X_2}) = (0,0)$, standard deviation vector $(\sigma_{X_1}, \sigma_{X_2}) = \left(\sqrt{1+b_1^2}, \sqrt{1+b_2^2}\right)$ and correlation $r = (b_1 b_2)/(\sigma_{X_1}\sigma_{X_2})$. Hence,

$$(X_1, X_2) \sim (\sigma_{X_1}\tilde{X}, \sigma_{X_2}(r\tilde{X} + \sqrt{1-r^2}\tilde{Y})) \quad \text{where} \quad \tilde{X}, \tilde{Y} \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

The last probability above can thus be written as

$$\mathbb{P}\left(\tilde{X} \le \frac{a_1}{\sigma_{X_1}}, r\tilde{X} + \sqrt{1-r^2}\tilde{Y} \le \frac{a_2}{\sigma_{X_2}}\right) = \Phi_2\left(\frac{a_1}{1+b_1^2}, \frac{a_2}{1+b_2^2}, r\right)$$

since $(\tilde{X}, r\tilde{X} + \sqrt{1-r^2}\tilde{Y})$ forms a standard bivariate normal pair with correlation r.

This shows that the conditional cdf of $\tilde{X}_i$ defined in Eq. (14) is the function H_i given in the Lemma.

The last step of the proof consists in noting that, from the probability integral transform, the distribution of $H_i(\tilde{X}_i)$ conditional upon $\{B_i = 1\}$ is a uniform random variable, hence:

$$\mathbb{P}(H_i(\tilde{X}_i) \le u | B_i = 1) = \max(0, \min(u, 1)).$$

Because F_i is increasing and invertible we have, using the definition of Λ_i given in (14),

$$\mathbb{P}(\Lambda_i \le l | B_i = 1) = \mathbb{P}(F_i^{-1}(H_i(\tilde{X}_i)) \le l | B_i = 1) = \mathbb{P}(H_i(\tilde{X}_i) \le F_i(l) | B_i = 1).$$

The proof is concluded by noting that $u := F_i(l)$ is included in the unit interval, leading to

$$\mathbb{P}(\Lambda_i \le l | B_i = 1) = \mathbb{P}(H_i(\tilde{X}_i) \le F_i(l) | B_i = 1) = F_i(l).$$

Table B.4
Nodes and weights associated with the 8-point Gaussian quadrature.

i	$z_{i,8}$	$\kappa_{i,8}$
1	−4.1445472	0.000112615
2	−2.8024859	0.00963522
3	−1.636519	0.117239908
4	−0.5390798	0.373012258
5	0.5390798	0.373012258
6	1.636519	0.117239908
7	2.8024859	0.00963522
8	4.1445472	0.000112615

Appendix B. Numerical schemes

In the paper, the integrals have been computed using the function `integrate` in R. Note, however, that these integrals take the form of the expectation of some function f with respect to a standard Normal variable Z. Moreover, in *Approach 2* which is most relevant to us, these functions are smooth and bounded, and the integrals can be computed in a very simple and reliable way using the n-point Gaussian quadrature

$$\mathbb{E}[f(Z)] = \int_{-\infty}^{\infty} f(z)\phi(z)dz \approx \sum_{i=1}^{n} \kappa_{i,n} f(z_{i,n})$$

where the $\kappa_{i,n}$'s and the $z_{i,n}$'s are the weights and nodes that depend on n but not on f. They can be found by calling `gauss.quad.prob(n, dist = ''norm'')` in R (`statmod` package). We find that considering $n = 8$ already provides a very accurate estimation of the integrals related to *Approach 2*. The 8-point Gaussian quadrature is extremely simple to implement since any expectation of a function of a standard normal variable takes the form of a weighted sum of 8 terms. The weights and nodes are given in Table B.4 for $n = 8$.

With regards to Φ_2, the approximation considered in Drezner and Wesolovski (1990) is very easy to implement. A variant of this approach is discussed in Hull (2021), and an Excel version can be downloaded https://www-2.rotman.utoronto.ca/~hull/software/bivar.xls.

Appendix C. Proof that $\lambda_i(y)$ in Approach 1 is decreasing

Setting $m=1$ in (13), we have $\lambda_i(Y)=I(Y)/p_i(Y)$ where

$$I(y) := \int_{-\infty}^{k_i(y)} f(y,\hat{z})\phi(\hat{z})d\hat{z} \tag{C.1}$$

and

$$f(y,\hat{z}) = \int_{-\infty}^{\infty} F_i^{-1}\left(\Phi\left(\sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1-\frac{\Phi(\sqrt{\rho_i}y+\sqrt{1-\rho_i}z)}{p_i}\right) + \sqrt{1-\rho_i^{LGD}}\hat{z}\right)\right)\phi(z)dz.$$

Hence,

$$\lambda_i'(y) = \frac{d}{dy}\lambda_i(y) = \frac{I'(y)p_i(y)-I(y)p_i'(y)}{(p_i(y))^2}.$$

We compute the derivative of I using Leibniz rule,

$$I'(y) = \underbrace{f(y,k_i(y))\times\phi(k_i(y))\times k_i{}'(y)}_{:=I_1(y)} + \underbrace{\int_{-\infty}^{k_i(y)}\phi(\hat{z})\frac{\partial}{\partial y}f(y,\hat{z})d\hat{z}}_{:=I_2(y)}, \tag{C.2}$$

where

$$k_i(y) = (\Phi^{-1}(p_i)-\sqrt{\rho_i}y)/\sqrt{1-\rho_i}, \quad \text{and} \quad k_i{}'(y) := -\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}.$$

To show that λ_i is decreasing, we must prove that

$$I_1(y)p_i(y)+I_2(y)p_i(y)-I(y)p_i'(y)\le 0\,.$$

Let us first show that $I_2(y)p_i(y)$ is negative, which amounts to prove that $I_2(y)\le 0$ for all y. To that end, we rewrite f using $G_i := F_i^{-1}$, $\Psi := \Phi^{-1}$ and $g(y,z) := \sqrt{\rho_i}y+\sqrt{1-\rho_i}z$ and interchange integration with respect to z and differentiation with respect to y,

$$\frac{\partial}{\partial y}f(y,\hat{z}) = \int_{-\infty}^{\infty}\frac{\partial}{\partial y}G_i\left(\Phi\left(\sqrt{\rho_i^{LGD}}\Psi\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\rho_i^{LGD}}\hat{z}\right)\right) \times \phi(z)dz\,. \tag{C.3}$$

Using the chain rule for differentiation, we find

$$\begin{aligned}\frac{\partial}{\partial y}G_i(\cdot) = {} & G_i'\left(\Phi\left(\sqrt{\rho_i^{LGD}}\Psi\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\rho_i^{LGD}}\hat{z}\right)\right)\\ & \times\ \Phi'\left(\sqrt{\rho_i^{LGD}}\Psi\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\rho_i^{LGD}}\hat{z}\right)\\ & \times\ \sqrt{\rho_i^{LGD}}\Psi'\left(1-\frac{\Phi(g(y,z))}{p_i}\right)\times\frac{-\Phi'(g(y,z))}{p_i}\times g'(y,z)\,.\end{aligned}$$

Moreover, $\Phi'(x)=\phi(x)$, $g'(x,z)=\sqrt{\rho_i}x$ and, from the expression of the derivative of an inverse, $G_i'(x)=1/F_i'(F_i^{-1}(x))$ and $\Psi'(x)=1/\phi(\Phi^{-1}(x))$. Rearranging the terms, one gets

$$\begin{aligned}\frac{\partial}{\partial y}G_i(\cdot) = {} & -\frac{\phi(g(y,z))}{p_i}\times\frac{\sqrt{\rho_i\rho_i^{LGD}}}{\phi\left(\Phi^{-1}\left(1-\frac{\Phi(g(y,z))}{p_i}\right)\right)}\\ & \times\ \phi\left(\sqrt{\rho_i^{LGD}}\Psi\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\rho_i^{LGD}}\hat{z}\right)\\ & \times\ \frac{1}{F_i'\left(F_i^{-1}\left(\Phi\left(\sqrt{\rho_i^{LGD}}\Phi^{-1}\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\rho_i^{LGD}}\hat{z}\right)\right)\right)}.\end{aligned}$$

Because F_i' and ϕ are probability densities and $p_i>0$, we conclude that the derivative in (C.3) is negative, and so is the function appearing inside the integral I_2 in (C.2). Consequently, the integral $I_2(y)$ itself is negative for all y.

It remains to show that

$$I_1(y)p_i(y)-I(y)p_i'(y)\le 0\,.$$

Plugging $I_1(y)$ in this expression and noting that $p_i(y)=\Phi(k_i(y))$, the above inequality becomes

$$I(y)\phi(k_i(y))k_i{}'(y)\ge f(y,k_i(y))\phi(k_i(y))k_i{}'(y)p_i(y)$$

or, dividing both sides by the negative quantity $k_i{}'(y)\phi(k_i(y))$,

$$I(y)\le f(y,k_i(y))p_i(y)\,.$$

To see that this equality indeed holds, it suffices to notice that $f(y,\hat{z})$ is increasing in $\hat{z}$. This implies that $f(y,\hat{z})\le f(y,k_i(y))$ for all $\hat{z}\le k_i(y)$, leading to

$$I(y) = \int_{-\infty}^{k_i(y)} f(y,\hat{z})\phi(\hat{z})d\hat{z} \le f(y,k_i(y))\int_{-\infty}^{k_i(y)}\phi(\hat{z})d\hat{z} = f(y,k_i(y))p_i(y)\,.$$

Appendix D. Proof of Theorem 1

Density of Y given $N>0$

To start with, let us compute the conditional density of Y given $\{N>0\}$ for further reference. Observe first that

$$\mathbb{P}(Y\le y|N>0) = \frac{\mathbb{P}(Y\le y\cap N>0)}{\mathbb{P}(N>0)}\,. \tag{D.1}$$

Using the fact that $N|Y\sim Bin(I,p(Y))$, $\mathbb{P}(N>0|Y)=1-\mathbb{P}(N=0|Y)=1-\prod_{i=1}^I q_i(Y)$, and the numerator and the denominator in the right-hand side above read

$$\int_{x=-\infty}^{y}(1-q(x)^I)\phi(x)dx = \Phi(y)-\int_{x=-\infty}^{y}q(x)^I\phi(x)dx$$

and

$$\mathbb{P}(N>0) = \mathbb{E}[1-\mathbb{P}(N=0|Y)] = 1-\int_{-\infty}^{\infty}q(x)^I\phi(x)dx\,. \tag{D.2}$$

Plugging those expressions in Eq. (D.1) and differentiating with respect to y yields the conditional density $f_{Y|N>0}$ given in the lemma.

Covariance

We compute the analytical expression of each of the three terms in the conditional covariance expression

$$\mathbb{C}\mathrm{ov}(\hat{N},\hat{L}|N>0) = \mathbb{E}[\hat{N}\hat{L}|N>0]-\mathbb{E}[\hat{N}|N>0]\mathbb{E}[\hat{L}|N>0]\,. \tag{D.3}$$

Using the tower law, the first expectation in (D.3) can be written as

$$\begin{aligned}\mathbb{E}[\hat{N}\hat{L}|N>0] &= \frac{1}{I}\mathbb{E}\left[\sum_{i=1}^{I}L_i\middle|N>0\right]\\ &= \frac{1}{I}\sum_{i=1}^{I}\mathbb{E}\left[\mathbb{E}\left[L_i|N>0,Y\right]|N>0\right]\,.\end{aligned}$$

The inner expectation is

$$\begin{aligned}\mathbb{E}\left[L_i|N>0,Y\right] &= \mathbb{E}[L_i|B_i=1,N>0,Y]\mathbb{P}(B_i=1|N>0,Y)\\ &\quad+\underbrace{\mathbb{E}[L_i|B_i=0,N>0,Y]}_{=0}\mathbb{P}(B_i=0|N>0,Y)\\ &= \mathbb{E}[\Lambda_i|B_i=1,N>0,Y]\times\mathbb{P}(B_i=1|N>0,Y)\,.\end{aligned}$$

Because $\{B_i=1\cap N>0\}=\{B_i=1\cap N\ge 0\}=\{B_i=1\}$, we have

$$\mathbb{E}\left[\Lambda_i|B_i=1,N>0,Y\right] = \mathbb{E}\left[\Lambda_i|B_i=1,Y\right] = \lambda_i(Y)$$

for all $i\in\{1,\dots,I\}$ where the last equality results from the homogeneity of the pool. On the other hand, Bayes rule yields

$$\begin{aligned}\mathbb{P}(B_i=1|N>0,Y) &= \frac{\mathbb{P}(B_i=1\cap N>0|Y)}{\mathbb{P}(N>0|Y)} = \frac{\mathbb{P}(B_i=1|Y)}{\mathbb{P}(N>0|Y)}\\ &= \frac{p_i(Y)}{1-\prod_{j=1}^I q_j(Y)}\,,\end{aligned} \tag{D.4}$$

where the last equality comes from the conditional independence of the default indicators:

$$\mathbb{P}(N>0|Y)=1-\mathbb{P}(N=0|Y)=1-\prod_{j=1}^{I}\mathbb{P}(B_j=0|Y)=1-\prod_{j=1}^{I}q_j(Y)\,.$$

Using the pool homogeneity ($\lambda_i=\lambda$, $p_i=p$ and $q_j=q$), one gets

$$\mathbb{E}[\hat{N}\hat{L}|N>0]=\mathbb{E}\left[\frac{\lambda(Y)p(Y)}{1-q(Y)^I}\middle|N>0\right]=\frac{\mathbb{E}[\lambda(Y)p(Y)]}{\mathbb{P}(N>0)} \tag{D.5}$$

because

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\lambda(y)p(y)}{1-q(y)^I}f_{Y|N>0}(y)dy&=\int_{-\infty}^{+\infty}\frac{\lambda(y)p(y)}{1-q(y)^I}\frac{(1-q(y)^I)\phi(y)}{\mathbb{P}(N>0)}dy\\&=\frac{\int_{-\infty}^{+\infty}\lambda(y)p(y)\phi(y)dy}{\mathbb{P}(N>0)}\,.\end{aligned}$$

Using the definitions of $\lambda(Y)=\lambda_i(Y)$ and $p(Y)=p_i(Y)$ and because $\Lambda_i=L_i$ on $\{B_i=1\}$,

$$\begin{aligned}\mathbb{E}[\lambda(Y)p(Y)]&=\mathbb{E}[\mathbb{E}[L_i|B_i=1,Y]\mathbb{E}[1_{\{B_i=1\}}|Y]]\\&=\mathbb{E}\left[\frac{\mathbb{E}[L_i1_{\{B_i=1\}}|Y]}{\mathbb{E}[1_{\{B_i=1\}}|Y]}\mathbb{E}[1_{\{B_i=1\}}|Y]\right]\,.\end{aligned}$$

From the tower law and using that $L_i=0$ on the set $\{B_i=0\}$, the last expectation becomes

$$\mathbb{E}[L_i1_{\{B_i=1\}}]=\mathbb{E}[L_i|B_i=1]\mathbb{P}(B_i=1)=\mathbb{E}[\Lambda_i|B_i=1]p_i=\lambda p$$

showing that $\mathbb{E}[\hat{N}\hat{L}|N>0]=\lambda p/\mathbb{P}(N>0)$ for a homogeneous pool.

The second expectation in (D.3) can be decomposed using the same trick as before,

$$\mathbb{E}[\hat{N}|N>0]=\frac{1}{I}\mathbb{E}[N|N>0]=\frac{1}{I}\mathbb{E}[\mathbb{E}[N|N>0,Y]|N>0]\,.$$

Using $\mathbb{E}\left[B_i\,|N>0,Y\right]=\mathbb{P}(B_i=1|N>0,Y)$ given in (D.4), the inner expectation can be computed as follows

$$\begin{aligned}\mathbb{E}[N|N>0,Y]&=\mathbb{E}\left[\sum_{i=1}^{I}B_i\middle|N>0,Y\right]\\&=\sum_{i=1}^{I}\mathbb{P}(B_i=1\,|N>0,Y)\\&=\frac{\sum_{i=1}^{I}p_i(Y)}{1-\prod_{j=1}^{I}q_j(Y)}\\&=\frac{I\times p(Y)}{1-q(Y)^I}\,,\end{aligned}$$

where the last equality results from the homogeneity of the pool. Hence,

$$\mathbb{E}[\hat{N}|N>0]=\mathbb{E}\left[\frac{p(Y)}{1-q(Y)^I}\middle|N>0\right]\,. \tag{D.6}$$

Using $f_{Y|N>0}$, the conditional expectation of the default rate collapses to

$$\mathbb{E}[\hat{N}|N>0]=\int_{-\infty}^{\infty}\frac{p(y)}{1-q(y)^I}\times\frac{(1-q(y)^I)\phi(y)}{\mathbb{P}(N>0)}dy=\frac{p}{\mathbb{P}(N>0)}\,. \tag{D.7}$$

The last expectation in (D.3) is

$$\mathbb{E}[\hat{L}|N>0]=\mathbb{E}\left[\frac{\sum_{i=1}^{I}L_i}{N}\middle|N>0\right]=\sum_{i=1}^{I}\mathbb{E}\left[\mathbb{E}\left[\frac{L_i}{N}\middle|N>0,Y\right]\middle|N>0\right].$$

To evaluate the inner expectation, we first partition the event $\{N>0\}$ as

$$\{N>0\}=\left\{B_i=0\cap\sum_{j\neq i}B_j>0\right\}\quad\cup\quad\left\{B_i=1\cap\sum_{j\neq i}B_j\geq 0\right\}.$$

Note that $L_i=0$ on the first event, and that the condition $\{\sum_{j\neq i}B_j\geq 0\}$ showing up in the second event is trivial, this leads to

$$\begin{aligned}\mathbb{E}\left[\frac{L_i}{N}\middle|N>0,Y\right]&=\mathbb{E}\left[\frac{\Lambda_i}{1+\sum_{j\neq i}B_j}\middle|B_i=1,Y\right]\times\mathbb{P}(B_i=1|N>0,Y)\\&=\mathbb{E}\left[\Lambda_i\middle|B_i=1,Y\right]\times\mathbb{E}\left[\frac{1}{1+\sum_{j\neq i}B_j}\middle|Y\right]\\&\quad\times\mathbb{P}(B_i=1|N>0,Y)\\&=\lambda_i(Y)\times\mathbb{E}\left[\frac{1}{1+W(Y)}\right]\times\frac{p_i(Y)}{1-\prod_{j=1}^{I}q_j(Y)}\,.\end{aligned}$$

The second equality above results from $\Lambda_i\perp B_{j\neq i}$ conditionally on Y, which is valid under both *Approach 1* and *Approach 2*. Under the homogeneous pool assumption, we further have that $W(Y):=\sum_{j\neq i}B_j|Y$ is distributed according to a Binomial distribution with parameters $(I-1,p(Y))$. From Chao and Strawderman (1972), it is known that if $W\sim Bin(n,p)$, then

$$\mathbb{E}\left[\frac{1}{1+W}\right]=\frac{1-q^{n+1}}{(n+1)p}\,,\quad q:=1-p,$$

Hence,

$$\mathbb{E}\left[\frac{L_i}{N}\middle|N>0,Y\right]=\lambda(Y)\times\frac{1-q(Y)^I}{Ip(Y)}\times\frac{p(Y)}{1-q(Y)^I}=\lambda(Y)/I$$

so that one simply gets

$$\begin{aligned}\mathbb{E}[\hat{L}|N>0]=\mathbb{E}[\lambda(Y)|N>0]&=\int_{-\infty}^{+\infty}\lambda(y)\frac{1-q(y)^I}{\mathbb{P}(N>0)}\phi(y)dy\\&=\frac{\mathbb{E}[\lambda(Y)(1-q(Y)^I)]}{\mathbb{P}(N>0)}\,.\end{aligned} \tag{D.8}$$

Combining the three expectations yields the claim.

Variances

We compute the analytical expression of the conditional variances of $\hat{L}$ and $\hat{N}$ which, combined to (D.3) lead to their correlation:

$$\mathbb{V}[\hat{N}|N>0]=\mathbb{E}[\hat{N}^2|N>0]-\left(\mathbb{E}[\hat{N}|N>0]\right)^2$$

$$\mathbb{V}[\hat{L}|N>0]=\mathbb{E}[\hat{L}^2|N>0]-\left(\mathbb{E}[\hat{L}|N>0]\right)^2.$$

Regarding the variance of $\hat{N}$, the second expectation is known from (D.6). The first one can be recovered from the tower law, noting that

$$\mathbb{E}[\hat{N}^2|N>0]=\frac{1}{I^2}\mathbb{E}[\mathbb{E}[N^2|N>0,Y]|N>0]$$

The inner expectation is

$$\begin{aligned}\mathbb{E}[N^2|N>0,Y]&=\mathbb{E}\left[\left(\sum_i B_i\right)^2\middle|N>0,Y\right]\\&=\mathbb{E}\left[\sum_i B_i^2+\sum_{i,j\neq i}B_iB_j\middle|N>0,Y\right]\\&=\sum_i\mathbb{E}\left[B_i^2\middle|N>0,Y\right]+\sum_i\sum_{j\neq i}\mathbb{E}\left[B_iB_j\middle|N>0,Y\right]\,.\end{aligned}$$

The expectation in the first sum is just $\mathbb{P}(B_i=1|N>0,Y)$ because $B_i^2=B_i$ for a Bernoulli random variable. The expectation in the second sum is

$$\begin{aligned}\mathbb{P}(B_i=1\cap B_j=1|N>0,Y)&=\frac{\mathbb{P}(B_i=1|Y)\times\mathbb{P}(B_j=1|Y)}{\mathbb{P}(N>0|Y)}\\&=\mathbb{P}(B_i=1|N>0,Y)p_j(Y)\end{aligned}$$

Using (D.4) for a homogeneous pool,

$$\begin{aligned}\mathbb{E}[N^2|N>0,Y]&=\sum_i\mathbb{P}(B_i=1|N>0,Y)\left(1+\sum_{j\neq i}p_j(Y)\right)\\&=\frac{Ip(Y)(1+(I-1)p(Y))}{1-q(Y)^I}\end{aligned}$$

One thus gets

$$\mathbb{E}[\hat{N}^2|N>0]=\frac{1}{I}\mathbb{E}\left[\frac{p(Y)}{1-q(Y)^I}\middle|N>0\right]+\frac{I-1}{I}\mathbb{E}\left[\frac{p(Y)^2}{1-q(Y)^I}\middle|N>0\right]$$

Combining (D.6) and (D.7), the first expectation is $p/\mathbb{P}(N>0)$. The second is

$$\int \frac{p(y)^2}{1-q(y)^I} f_{Y|N>0}(y)dy = \int_{-\infty}^{\infty} \frac{p(y)^2}{1-q(y)^I} \times \frac{(1-q(y)^I)\phi(y)}{1-\int_{-\infty}^{\infty} q(x)^I \phi(x)dx} dy = \frac{\mathbb{E}[p_i(Y)^2]}{\mathbb{P}(N>0)}$$

On the other hand, we have that for $X \sim \mathcal{N}(0,1)$ and any real values a and b,

$$\mathbb{E}[\Phi^2(a+bX)] = \Phi_2\left(\frac{a}{\sqrt{1+b^2}}, \frac{a}{\sqrt{1+b^2}}, \frac{b^2}{1+b^2}\right) .$$

Applying this result to $a = \frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}$ and $b = \frac{-\sqrt{\rho}}{\sqrt{1-\rho}}$ leads to

$$\mathbb{E}[p(Y)^2] = \Phi_2(\Phi^{-1}(p), \Phi^{-1}(p), \rho)$$

so that the considered variance then becomes

$$\mathbb{V}[\hat{N}|N>0] = \frac{p}{I\mathbb{P}(N>0)}\left(1 - \frac{Ip}{\mathbb{P}(N>0)} + (I-1)\frac{\Phi_2(\Phi^{-1}(p), \Phi^{-1}(p), \rho)}{p}\right) \tag{D.9}$$

Let us now proceed to the computation of the variance of $\hat{L}$. The second expectation is known from (D.8). The first one can be decomposed using the same trick as before,

$$\mathbb{E}[\hat{L}^2|N>0] = \mathbb{E}[\mathbb{E}[\hat{L}^2|N>0,Y]|N>0] .$$

The inner expectation is

$$\mathbb{E}[\hat{L}^2|N>0,Y] = \mathbb{E}\left[\frac{\sum_i L_i^2 + \sum_{i,j\neq i} L_i L_j}{N^2}\middle| N>0,Y\right] = \sum_i \mathbb{E}\left[\frac{L_i^2}{N^2}\middle| N>0,Y\right] + \sum_i \sum_{j\neq i} \mathbb{E}\left[\frac{L_i L_j}{N^2}\middle| N>0,Y\right] . \tag{D.10}$$

Let us evaluate the expectation in the first sum of (D.10) by partitioning the event $\{N>0\}$ as

$$\{N>0\} = \left\{B_i = 0 \cap \sum_{k\neq i} B_k > 0\right\} \cup \left\{B_i = 1 \cap \sum_{k\neq i} B_k \geq 0\right\}.$$

note that $L_i = 0$ on the first event, and that the condition $\{\sum_{k\neq i} B_k \geq 0\}$ showing up in the second event is trivial. Hence,

$$\begin{aligned}\mathbb{E}\left[\frac{L_i^2}{N^2}\middle| N>0,Y\right] &= \mathbb{E}\left[\frac{\Lambda_i^2}{N^2}\middle| B_i=1,Y\right] \times \mathbb{P}(B_i=1|N>0,Y) \\ &= \mathbb{E}\left[\frac{\Lambda_i^2}{(1+\sum_{k\neq i} B_k)^2}\middle| B_i=1,Y\right] \\ &\quad \times \mathbb{P}(B_i=1|N>0,Y) \\ &= \underbrace{\mathbb{E}[\Lambda_i^2|B_i=1,Y]}_{\gamma_i(Y)} \times \mathbb{E}\left[\frac{1}{(1+\sum_{k\neq i} B_k)^2}\middle| Y\right] \\ &\quad \times \mathbb{P}(B_i=1|N>0,Y) .\end{aligned}$$

The above result is valid under both *Approach 1* and *Approach 2*, as conditionally on Y, we have that $\Lambda_i \perp B_{j\neq i}$. Moreover, if $W \sim Bin(n,p)$ with probability mass function $P(i,n,p) := \mathbb{P}(W=i)$, we have

$$g(n,p,k) := \mathbb{E}\left[\frac{1}{(k+W)^2}\right] = \sum_{i=0}^{n} \frac{P(i,n,p)}{(k+i)^2} ,$$

which admits the integral representation (see, e.g., Chao and Strawderman (1972))

$$g(n,p,k) = \int_0^1 t^{-1} \int_0^t u^{k-1}((1-p)+pu)^n du dt .$$

Under the homogeneous pool assumption, $\sum_{k\neq i} B_k|Y \sim Bin(I-1, p(Y))$. Hence,

$$G_1(Y) := \mathbb{E}\left[\frac{1}{(1+\sum_{k\neq i} B_k)^2}\middle| Y\right] = g(I-1, p(Y), 1) .$$

Applying (D.4) to the homogeneous pool case, we finally have that

$$\sum_i \mathbb{E}\left[\frac{L_i^2}{N^2}\middle| N>0,Y\right] = \gamma(Y)G_1(Y) \times \frac{Ip(Y)}{1-q(Y)^I} ,$$

where $\gamma_i(Y) = \gamma(Y), \forall i \in \{1,\dots,I\}$.

To evaluate the expectation in the second sum of (D.10), we further partition the event $\{N>0\}$ as

$$\begin{aligned}\{N>0\} = &\left\{B_i = 0 \cap B_j = 0 \cap \sum_{s\neq i,j} B_s > 0\right\} \\ &\cup \left\{B_i = 1 \cap B_j = 0 \cap \sum_{s\neq i,j} B_s \geq 0\right\} \\ &\cup \left\{B_i = 0 \cap B_j = 1 \cap \sum_{s\neq i,j} B_s \geq 0\right\} \\ &\cup \left\{B_i = 1 \cap B_j = 1 \cap \sum_{s\neq i,j} B_s \geq 0\right\} .\end{aligned}$$

Note that $L_i L_j = 0$ on all the events except the last one, and that the condition $\{\sum_{s\neq i,j} B_s \geq 0\}$ is trivial. For the second term, imposing $j \neq i$ leads to

$$\begin{aligned}\mathbb{E}\left[\frac{L_i L_j}{N^2}\middle| N>0,Y\right] &= \mathbb{E}\left[\frac{\Lambda_i \Lambda_j}{(2+\sum_{s\neq i,j} B_s)^2}\middle| B_i=1, B_j=1, Y\right] \\ &\quad \times \mathbb{P}(B_i=1, B_j=1|N>0,Y) \\ &= \mathbb{E}\left[\Lambda_i|B_i=1,Y\right] \times \mathbb{E}\left[\Lambda_j|B_j=1,Y\right] \\ &\quad \times \mathbb{E}\left[\frac{1}{(2+\sum_{s\neq i,j} B_s)^2}\middle| Y\right] \\ &\quad \times \mathbb{P}(B_i=1, B_j=1|N>0,Y) \\ &= \lambda_i(Y)\lambda_j(Y) \times \mathbb{E}\left[\frac{1}{(2+\sum_{s\neq i,j} B_s)^2}\middle| Y\right] \\ &\quad \times \frac{p_i(Y)\times p_j(Y)}{1-\prod_{s=1}^I q_s(Y)}\end{aligned}$$

The above result is valid under both *Approach 1* and *Approach 2*, as conditionally on Y, we have that $\Lambda_i \perp B_{j\neq i}$. Under the homogeneous pool assumption, we further have that $\sum_{s\neq k,j} B_s|Y \sim Bin(I-2, p(Y))$. Hence,

$$G_2(Y) := \mathbb{E}\left[\frac{1}{(2+\sum_{s\neq i,j} B_s)^2}\middle| Y\right] = g(I-2, p(Y), 2) .$$

Due to the homogeneity of the pool we have that

$$\sum_i \sum_{j\neq i} \mathbb{E}\left[\frac{L_i L_j}{N^2}\middle| N>0,Y\right] = \frac{Ip(Y)}{1-q(Y)^I} \times (I-1)\lambda(Y)^2 G_2(Y)p(Y)$$

One finally gets,

$$\mathbb{E}[\hat{L}^2|N>0,Y] = \frac{Ip(Y)}{1-q(Y)^I}\left(\gamma(Y)G_1(Y) + (I-1)\lambda(Y)^2 G_2(Y)p(Y)\right)$$

Using $f_{Y|N>0}$, the conditional expectation of the default rate leads to

$$\mathbb{E}[\hat{L}^2|N>0] = \frac{I}{\mathbb{P}(N>0)}\left(\mathbb{E}\left[\gamma(Y)G_1(Y)p(Y)\right] + (I-1)\mathbb{E}\left[\lambda(Y)^2 G_2(Y)p(Y)^2\right]\right) . \tag{D.11}$$

It is easy to see that the integral form of g satisfies the recursion

$$g(n,p,2) = \frac{1-(1-p)^{n+2} - p(n+2)g(n+1,p,1)}{p^2(n+2)(n+1)} .$$

Hence, we conclude that

$$G_2(Y) = g(I-2, p(Y), 2) = \frac{1 - q(Y)^I - I p(Y) G_1(Y)}{p(Y)^2 I(I-1)} .$$

Plugging this expression in the second expectation of the RHS of (D.11) yields

$$\begin{aligned}(I-1)\mathbb{E}\left[\lambda(Y)^2 G_2(Y) p(Y)^2\right] &= \mathbb{E}\left[\lambda(Y)^2 \frac{1-q(Y)^I - I p(Y) G_1(Y)}{I}\right] \\ &= \frac{1}{I}\left(\mathbb{E}\left[\lambda(Y)^2(1-q(Y)^I)\right]\right) \\ &\quad -\mathbb{E}\left[\lambda(Y)^2 p(Y) G_1(Y)\right]\end{aligned}$$

Hence, noting that

$$\begin{aligned}\gamma_i(Y) - \lambda_i(Y)^2 &= \mathbb{E}[\Lambda_i^2 | B_i = 1, Y] - (\mathbb{E}[\Lambda_i | B_i = 1, Y])^2 \\ &= \mathbb{V}[\Lambda_i | B_i = 1, Y] =: v(Y) ,\end{aligned}$$

$$\begin{aligned}\mathbb{V}[\hat{L}|N>0] &= \frac{I}{\mathbb{P}(N>0)} \times \mathbb{E}\left[(\gamma(Y) - \lambda(Y)^2) G_1(Y) p(Y)\right] \\ &\quad + \frac{\mathbb{E}\left[\lambda(Y)^2(1-q(Y)^I)\right]}{\mathbb{P}(N>0)} \\ &\quad - (\mathbb{E}[\lambda(Y)|N>0])^2 \\ &= \frac{I}{\mathbb{P}(N>0)} \times \mathbb{E}\left[v(Y) G_1(Y) p(Y)\right] + \mathbb{E}\left[\lambda(Y)^2 | N>0\right] \\ &\quad - (\mathbb{E}[\lambda(Y)|N>0])^2 \\ &= \frac{I}{\mathbb{P}(N>0)} \times \mathbb{E}\left[v(Y) G_1(Y) p(Y)\right] + \mathbb{V}\left[\lambda(Y)|N>0\right]\end{aligned}$$

The proof is concluded by noting that $I G_1(Y) p(Y) = \Psi(Y)$, which can be obtained by inserting the integral representation of g in $G_1(y) = g(I-1, p(y), 1)$:

$$\begin{aligned}G_1(y) &= \int_0^1 t^{-1} \int_0^t (q(y) + p(y)u)^{I-1} du dt \\ &= \frac{1}{p(y) I} \int_0^1 t^{-1} \left((q(y) + p(y)t)^I - q(y)^I\right) dt .\end{aligned}$$

Appendix E. Proof of Corollary 1

We assume $0 < p < 1$, which guarantees that $0 < p(y) < 1$ for any $y \in \mathbb{R}$ and similarly for $q(y) = 1 - p(y)$. This last assumption implies that the probability to have no default collapses to zero as the size of the pool increases:

$$\lim_{I\to\infty} \mathbb{P}(N>0) = 1 - \lim_{I\to\infty} \mathbb{P}(N=0) = 1 - \mathbb{E}\left[\lim_{I\to\infty} q(Y)^I\right] = 1 .$$

This shows that as $I \to \infty$, one can drop the $\{N > 0\}$ condition and replace $\mathbb{P}(N > 0)$ by 1. This also means that the conditional density $f_{Y|N>0}(y)$ converges to the unconditional one $f_Y(y) = \phi(y)$. Let us now compute the limit of three expectations and the two variances from the expressions given in Theorem 1.

The expectation and variance of $\hat{N}$ collapse to that of $p(Y)$:

$$\lim_{I\to\infty} \mathbb{E}[\hat{N}|N>0] = \lim_{I\to\infty} \mathbb{E}\left[\left.\frac{p(Y)}{1-q(Y)^I}\right| N>0\right] = \mathbb{E}[p(Y)] = p ,$$

The asymptotic conditional variance of $\hat{N}$ is

$$\begin{aligned}\lim_{I\to\infty} \mathbb{V}[\hat{N}|N>0] &= \lim_{I\to\infty} \frac{p}{I\mathbb{P}(N>0)} \\ &\quad \times \left(1 - \frac{Ip}{\mathbb{P}(N>0)} + (I-1)\frac{\Phi_2(k,k,\rho)}{p}\right) \\ &= \Phi_2(k,k,\rho) - p^2 \\ &= \mathbb{V}[p(Y)] .\end{aligned}$$

Similarly, the expectation and variance of $\hat{L}$ collapse to that of $\lambda(Y)$:

$$\lim_{I\to\infty} \mathbb{E}[\hat{L}|N>0] = \lim_{I\to\infty} \frac{\mathbb{E}[\lambda(Y)(1-q(Y)^I)]}{\mathbb{P}(N>0)} = \mathbb{E}[\lambda(Y)] ,$$

and

$$\lim_{I\to\infty} \mathbb{V}[\hat{L}|N>0] = \lim_{I\to\infty} \frac{1}{\mathbb{P}(N>0)} \times \mathbb{E}\left[v(Y)\Psi(Y)\right] + \mathbb{V}\left[\lambda(Y)|N>0\right]$$

where Ψ can be written as

$$\Psi(y) = \int_0^1 t^{-1}((1 - p(y)(1-t))^I - q(y)^I) dt .$$

Clearly, for any t between the integration bounds, both $(1 - p(y)(1-t))$ and $q(y)$ are non-negative numbers smaller than one, hence, converge to 0 when set at the power $I \to \infty$. This shows that $\Psi(y) \to 0$ for all y in this case. The same applies to the product $v(y)\Psi(y)$ because $v(Y)$ is the conditional variance of Λ, hence, $v(y)$ is a bounded function. This shows that the asymptotic variance of $\hat{L}$ is

$$\lim_{I\to\infty} \mathbb{V}[\hat{L}|N>0] = \mathbb{V}\left[\lambda(Y)\right] .$$

Finally, the asymptotic expectation of the product $\hat{N}\hat{L}$ is

$$\lim_{I\to\infty} \mathbb{E}[\hat{N}\hat{L}|N>0] = \lim_{I\to\infty} \frac{\mathbb{E}[\lambda(Y)p(Y)]}{\mathbb{P}(N>0)} = \mathbb{E}[\lambda(Y)p(Y)] = \lambda p .$$

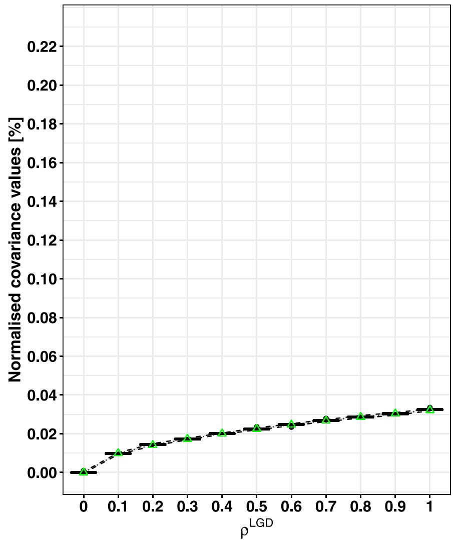
This shows that

$$c_\infty = \mathrm{Cov}(p(Y), \lambda(Y)) = \mathbb{E}[\lambda(Y)p(Y)] - \mathbb{E}[\lambda(Y)]\mathbb{E}[p(Y)] = (\lambda - \mathbb{E}[\lambda(Y)]) \times p .$$

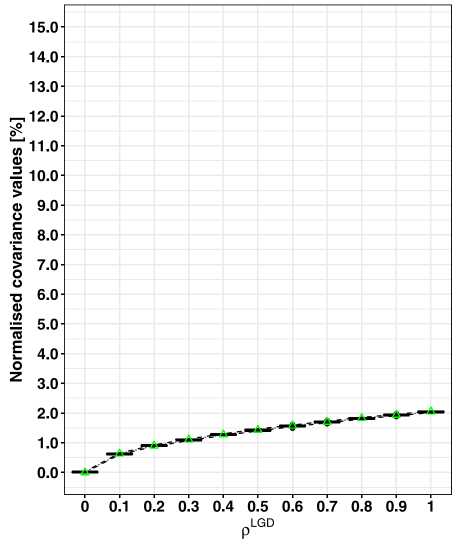
Note that under the considered assumptions, the variance of $p(Y)$ (resp. $\lambda(Y)$) is zero if and only if $\rho = 0$ (resp. $\rho^{LGD} = 0$) leading to an indeterminate 0/0 value for the empirical correlation and normalised

Table F.5
Yearly observed default rate (DR) and average realised loss given default (ARLGD) for the corporate world bond portfolio rated by Moody's, going from 1982–2014. The ARLGD presented here, are simply computed as one minus the observed recovery rate (RR) on all bonds extracted from the Moody's Analytics Default and Recovery Database. The implied values y_i of the systematic risk factor Y are found by inverting the relation $DR_i = p(y_i)$, leading to $y_i := (\Phi^{-1}(p) - \sqrt{1-\rho(p)}\Phi^{-1}(DR_i))/\sqrt{\rho(p)}$, where $p = 1.73\%$ and $\rho(p) = 17.05\%$. A similar dataset is provided in García-Céspedes and Moreno (2020, p. 1870, Table 1). The summary statistics of these two times series are provided in Table 1.

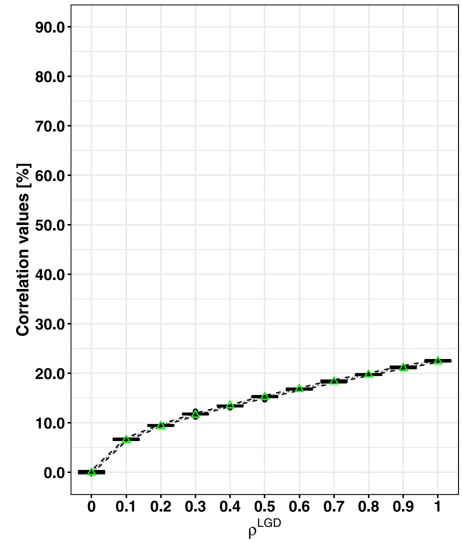
Year	Default rate	Average LGD all bonds	y_i
1982	1.04%	64.68%	−0.0189
1983	0.97%	55.47%	0.0388
1984	0.94%	54.51%	0.0647
1985	0.96%	56.4%	0.0474
1986	1.88%	52.61%	−0.5314
1987	1.59%	48.69%	−0.3821
1988	1.38%	61.23%	−0.2585
1989	2.39%	67.68%	−0.7521
1990	3.76%	74.43%	−1.1929
1991	3.1%	64.49%	−1.0010
1992	1.5%	54.11%	−0.3310
1993	0.89%	56.92%	0.1094
1994	0.66%	54.43%	0.3488
1995	1.03%	56.72%	−0.0108
1996	0.59%	58.46%	0.4365
1997	0.77%	51.24%	0.2265
1998	1.32%	61.7%	−0.2203
1999	2.41%	66.17%	−0.7599
2000	2.86%	74.86%	−0.9227
2001	4.35%	78.42%	−1.3425
2002	3.33%	70.33%	−1.0714
2003	2.05%	58.62%	−0.6101
2004	0.94%	41.5%	0.0647
2005	0.74%	43.48%	0.2582
2006	0.72%	44.98%	0.2800
2007	0.4%	44.94%	0.7320
2008	2.26%	66.15%	−0.6999
2009	6.02%	66.07%	−1.6917
2010	1.41%	48.21%	−0.2771
2011	0.91%	53.68%	0.0912
2012	1.42%	55.26%	−0.2833
2013	1.45%	53.19%	−0.3014
2014	1.04%	52.15%	−0.0189
Average	*1.73%*	*57.93%*	



(a) c_I *vs.* ρ^{LGD}

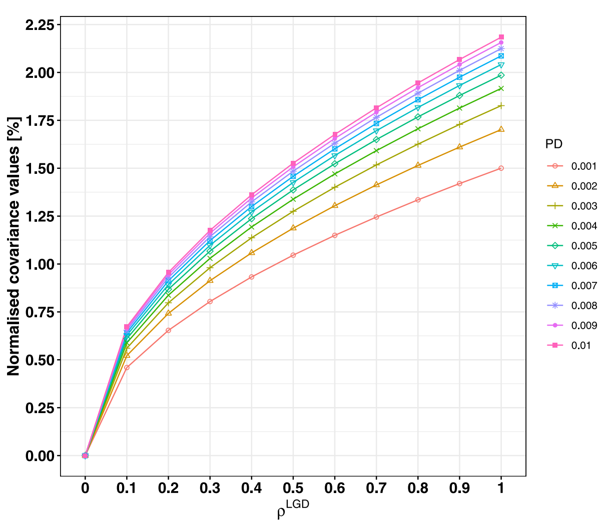


(b) C_I *vs.* ρ^{LGD}

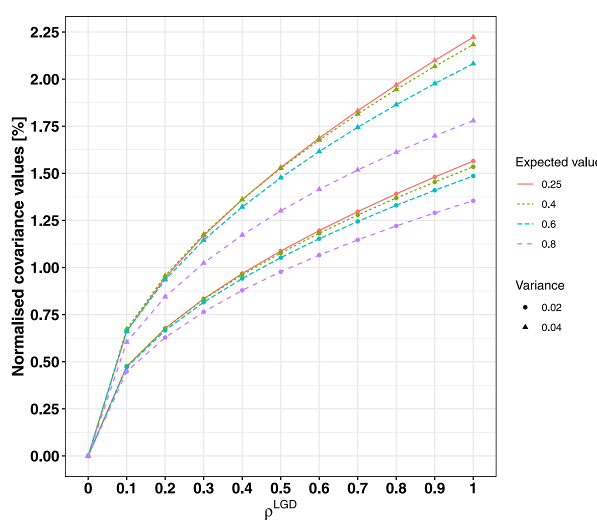


(c) r_I *vs.* ρ^{LGD}

Fig. G.8. Empirical covariance (a), normalised covariance (b) and correlation (c) as a function of the PD-LGD dependency parameter for Approach 1. The empty triangles (△) depict the values obtained from the analytical formula in Theorem 1. The boxplots are formed by using 20 repetitions of Monte Carlo estimates, each of which being computed from $M = 50{,}000$ simulations of $(\hat{N}, \hat{L})$. The grey area delimited by the dashed lines, represents ± one standard deviation from the estimated mean value. We consider a homogeneous credit portfolio made of $I = 1500$ obligors with $p = 1\%$, $\rho = \rho(1\%) = 19.28\%$ and marginal LGD distributions $Beta(2,3)$. The corresponding figures for Approach 2 can be found in Fig. 2.



(a) C_∞ *vs.* ρ^{LGD} for various values of p



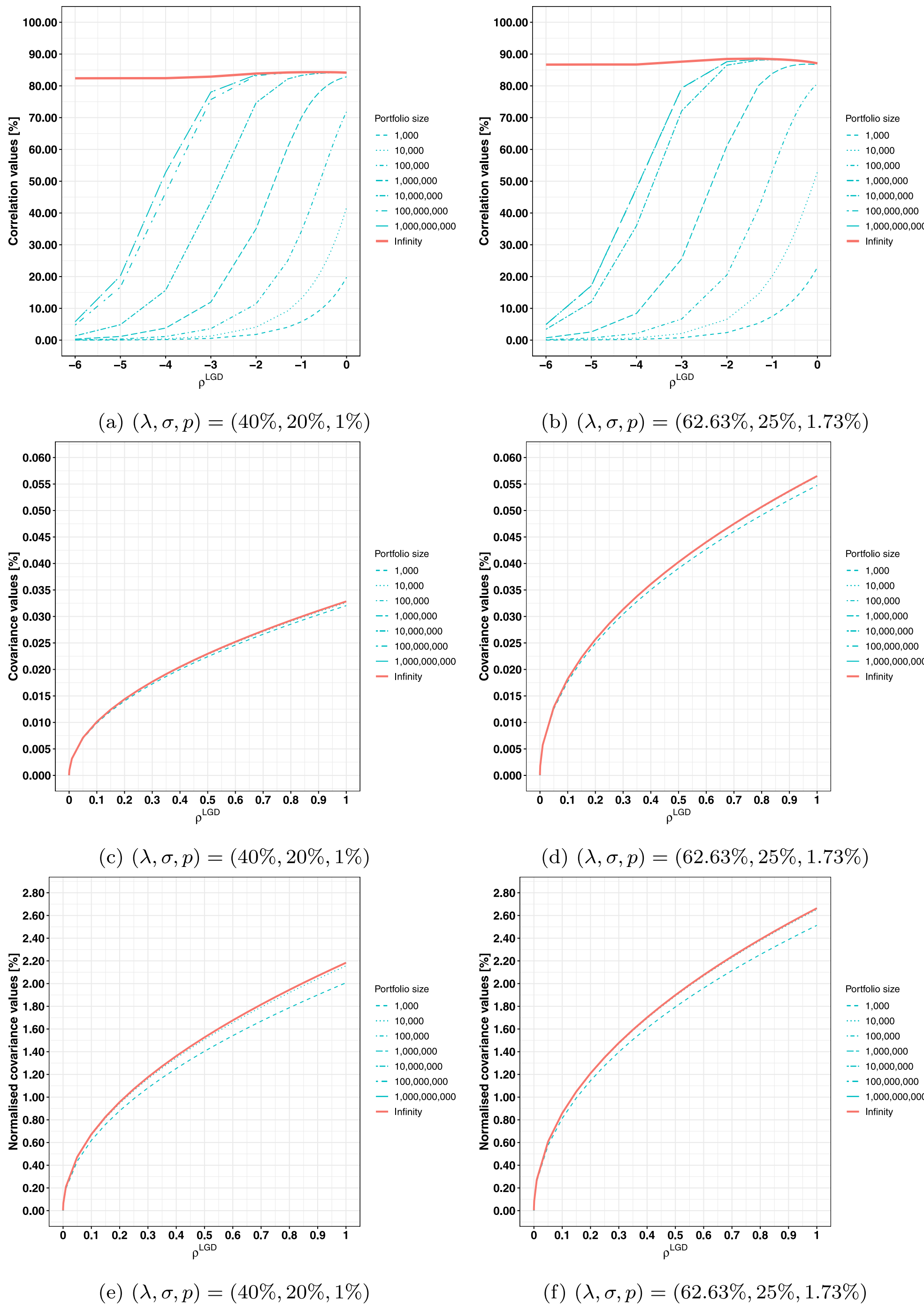
(b) C_∞ *vs.* ρ^{LGD} for various values of (λ, σ^2)

Fig. G.9. Sensitivity analysis of the asymptotic empirical normalised covariance C_∞ for Approach 1. Panel (a) shows the impact of p when the distribution of the LGDs is a $Beta(2,3)$, setting $\rho = \rho(p)$. Panel (b) shows the impact of the moments (λ, σ^2) of the LGD distribution when setting $p = 1\%$ and $\rho = \rho(1\%) = 19.28\%$. In panel (b), We start from $\lambda = 25\%$ to acknowledge the existence of the 25% floor on the LGD under the A-IRB, for unsecured corporate exposures (BCBS, 2023b, §32.16). The corresponding figures for Approach 2 can be found in Fig. 4.

covariance. Otherwise, all the above limits exist, the variances $\mathbb{V}[p(Y)]$ and $\mathbb{V}[\lambda(Y)]$ are strictly positive, and the expressions of r_∞ and C_∞ follow by replacing the conditional variances in the denominator by their limits.

To show that r_∞ is not necessarily monotonic, we provide a counter-example. It can be checked that the curve $r_\infty(\rho^{LGD})$ is U-shaped for $\rho^{LGD} \in [0, 0.2]$ when considering a pool with parameters $(\lambda, \sigma, p) = (58\%, 20\%, 1\%)$. It remains to prove that c_∞ and C_∞ are monotonic. Because those quantities are related by a scaling factor which does not depend on ρ^{LGD}, we restrict ourselves to prove that $c_\infty(\rho^{LGD})$ is increasing monotonously on $[0,1]$. Noting that

$$c_\infty(\rho^{LGD}) = (\lambda - \mathbb{E}[\lambda(Y)])p\,,$$



(a) $(\lambda, \sigma, p) = (40\%, 20\%, 1\%)$ (b) $(\lambda, \sigma, p) = (62.63\%, 25\%, 1.73\%)$

(c) $(\lambda, \sigma, p) = (40\%, 20\%, 1\%)$ (d) $(\lambda, \sigma, p) = (62.63\%, 25\%, 1.73\%)$

(e) $(\lambda, \sigma, p) = (40\%, 20\%, 1\%)$ (f) $(\lambda, \sigma, p) = (62.63\%, 25\%, 1.73\%)$

Fig. G.10. Empirical correlation $r_I(\rho^{LGD})$ (top), empirical covariance $c_I(\rho^{LGD})$ (middle), empirical normalised covariance $C_I(\rho^{LGD})$ (bottom) *vs.* ρ^{LGD} for increasing pool sizes I and two sets of parameters (λ, σ, p) for Approach 1. The curves associated with $I = \infty$ correspond to the asymptotic expressions. Panels (a) and (b) use a logarithmic scale for the x-axis to emphasise the behaviour for small values of ρ^{LGD}. In panels (c) to (f), all the curves collapse to the asymptotic version except $I = 1000$. The corresponding figures for Approach 2 can be found in Fig. 3.

it remains to prove that $\frac{\partial}{\partial \rho^{LGD}} \mathbb{E}[\lambda(Y)] < 0$. Observe first that setting $k = \Phi^{-1}(p)$, we have

$$\lambda(Y) = \mathbb{E}\left[F^{-1}\left(H(\sqrt{1-\rho^{LGD}}\tilde{Z} - \sqrt{\rho^{LGD}}Y, k, -\sqrt{\rho\rho^{LGD}}) \right) \middle| Y \right].$$

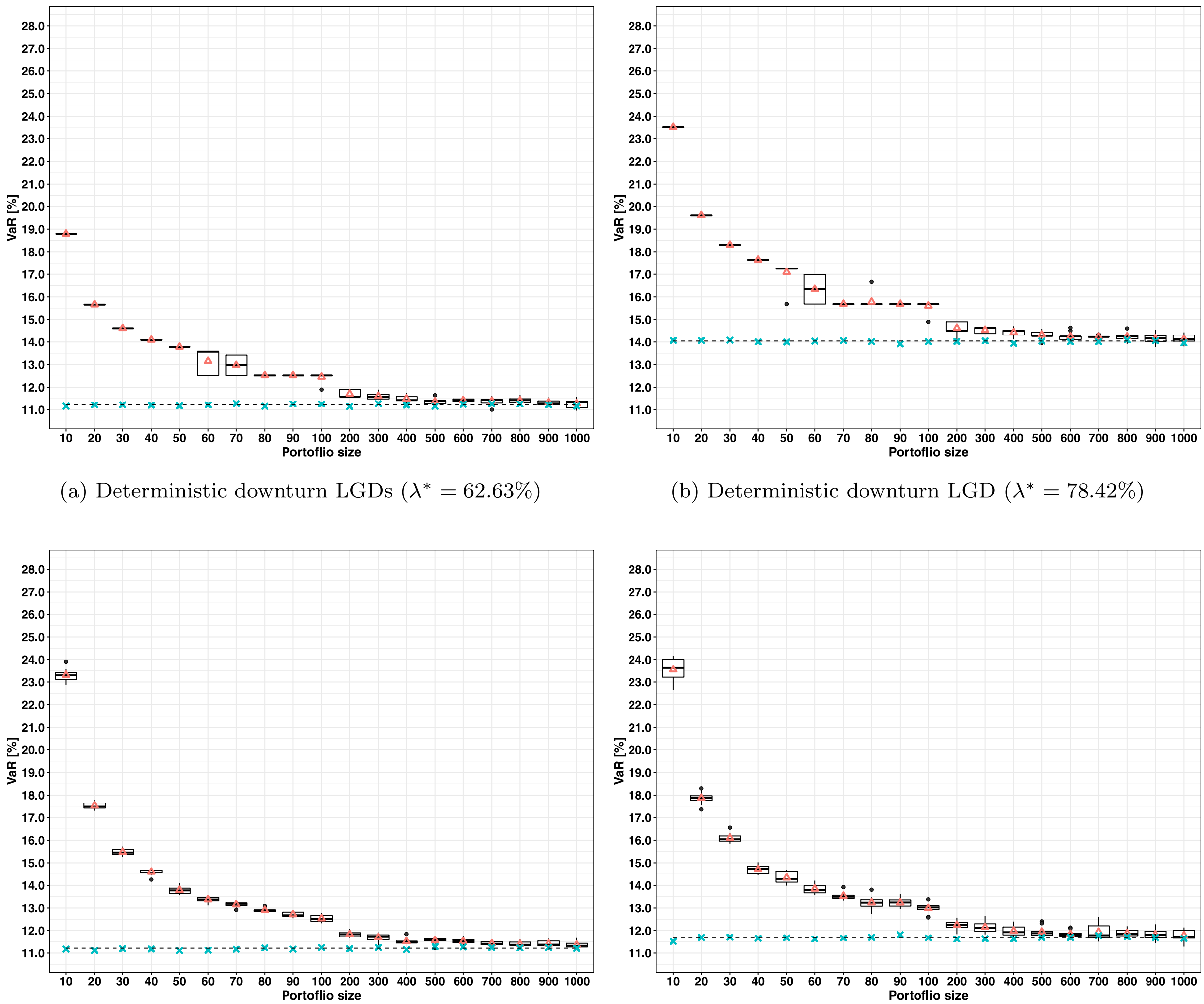
On the other hand, $Z := \sqrt{1-\rho^{LGD}}\tilde{Z} - \sqrt{\rho^{LGD}}Y \sim \mathcal{N}(0,1)$ for any ρ^{LGD}, hence

$$\begin{aligned}\mathbb{E}[\lambda(Y)] &= \mathbb{E}\left[F^{-1}\left(H(\sqrt{1-\rho^{LGD}}\tilde{Z} - \sqrt{\rho^{LGD}}Y, k, -\sqrt{\rho\rho^{LGD}}) \right) \right] \\ &= \mathbb{E}\left[F^{-1}\left(H(Z, k, -\sqrt{\rho\rho^{LGD}}) \right) \right] \\ &= \mathbb{E}\left[F^{-1}\left(\frac{\Phi_2(Z, k; -\sqrt{\rho\rho^{LGD}})}{p} \right) \right].\end{aligned}$$

Next, observe that $\Phi_2(x, y; r)$ is increasing with respect to the correlation parameter r for any (x, y, r), because

$$\Phi_{2,r}(x, y; r) := \frac{\partial}{\partial r}\Phi_2(x, y; r) = \phi_2(x, y; r) > 0$$

where $\phi_2(.,.;r)$ is the standard bivariate normal density with correlation coefficient r. Noting by $f = F'$ the density associated with the cumulative distribution F of the LGD, one finally gets



(a) Deterministic downturn LGDs ($\lambda^* = 62.63\%$)

(b) Deterministic downturn LGD ($\lambda^* = 78.42\%$)

(c) Independent random LGDs

(d) Approach 1 with $\rho^{LGD} = 10.20\%$

Fig. G.11. VaR at confidence level 99.9% for the prototypical corporate bond world portfolio as a function of the portfolio size I in different LGD frameworks. The dotted line shows the analytical expression obtained under the LP approximation specific to each model. The symbol (x) refers to the corresponding LP Monte Carlo estimator, found by estimating the 99.9% quantile of $\mathbb{E}[L|Y]$ via a sampling of the common factor Y, while (△) shows the quantile of L estimated in Monte Carlo but without relying on the LP approximation, i.e., when sampling all the random variables involved in the model. The boxplots are built from 1000 (in (a) and (b)) and 10 (in (c) and (d)) simulated VaR, each of which being estimated from $M = 250{,}000$ simulated relative portfolio losses. We choose the calibrated parameters $(p, \rho) = (1.73\%, 17.05\%)$ and, for stochastic LGDs in panels (c) and (d), we assume the marginal distribution $Beta(1.72, 1.03)$. The corresponding figure for Approach 2 can be found in Fig. 6.

$$
\begin{aligned}
\frac{\partial}{\partial \rho^{LGD}} \mathbb{E}[\lambda(Y)] &= \frac{\partial}{\partial \rho^{LGD}} \mathbb{E}\left[F^{-1}\left(\frac{\Phi_2(Z, k; -\sqrt{\rho \rho^{LGD}})}{p} \right) \right] \\
&= \mathbb{E}\left[\frac{\frac{\partial}{\partial \rho^{LGD}} \Phi_2(Z, k; -\sqrt{\rho \rho^{LGD}})}{p f(F^{-1}(H(Z)))} \right] \\
&= \mathbb{E}\left[\frac{\Phi_{2,r}(Z, k; -\sqrt{\rho \rho^{LGD}})}{p f(F^{-1}(H(Z)))} \times \frac{-\partial \sqrt{\rho \rho^{LGD}}}{\partial \rho^{LGD}} \right] \\
&= -\frac{\sqrt{\rho / \rho^{LGD}}}{2p} \mathbb{E}\left[\frac{\phi_2(Z, k; -\sqrt{\rho \rho^{LGD}})}{f(F^{-1}(H(Z)))} \right] .
\end{aligned}
$$

This is strictly negative if ρ, ρ^{LGD} and p are strictly positive because ϕ_2 and $f(F^{-1}(H(.)))$ are strictly positive, too. We conclude that under the considered assumptions, $c_\infty(\rho^{LGD})$ is increasing in ρ^{LGD} on $(0, 1)$.

Appendix F. Data

See Table F.5.

Appendix G. Analysis of Approach 1

This Appendix, contains the reciprocal of all results and analysis performed on *Approach 2*, for *Approach 1*.

G.1. Monte carlo validation

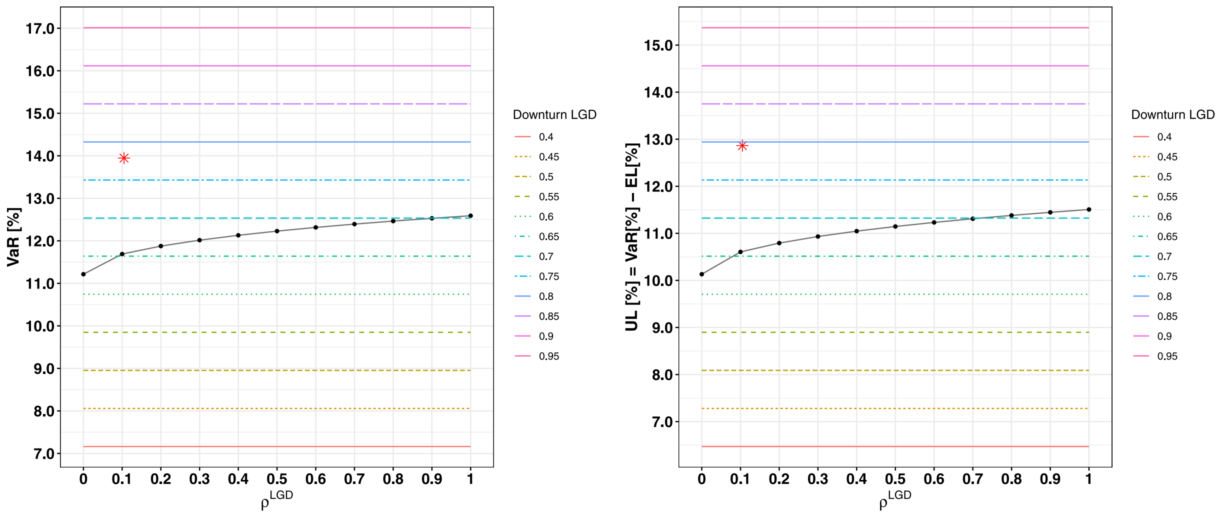
In this section, we compare for *Approach 1* the values obtained from our analytical formulas presented in Theorem 1 and Eq. (19), with the values found using Monte Carlo simulations for c_I in Fig. G.8(a), for C_I in Fig. G.8(b), and for r_I in Fig. G.8(c).

G.2. Sensitivity analysis

See Fig. G.9.

G.3. Calibration criterion

See Fig. G.10.



(a) Value-at-risk (VaR) *vs.* ρ^{LGD} (b) Unexpected loss (UL) *vs.* ρ^{LGD}

Fig. G.12. Sensitivity of the VaR (a) and UL (b) at a 99.9% confidence level for a homogeneous credit portfolio with parameters $(p, \rho, \lambda, \sigma) = (1.73\%, 17.05\%, 62.63\%, 25\%)$ chosen consistently with the Moody's Analytics Default and Recovery Database and Basel regulations. The solid curves show the values obtained in Approach 2 as a function of the ρ^{LGD} parameter. The asterisk ($*$) corresponds to the case where ρ^{LGD} is calibrated using the empirical normalised covariance observed in the Moody's data. The horizontal lines show the corresponding terms computed in the Basel formula for various values of DLGD λ^*. The corresponding figures for Approach 2 can be found in Fig. 7.

G.4. Empirical analysis

See Figs. G.11 and G.12.

References

Altman, E.I., 2011. Default Recovery Rates and LGD in Credit Risk Modeling and Practice. In: Lipton, A., Rennie, A. (Eds.), The Oxford Handbook of Credit Derivatives. Oxford University Press, Oxford, England, http://dx.doi.org/10.1093/oxfordhb/9780199546787.013.0003.

Altman, E.I., Brady, B., Resti, A., Sironi, A., 2005. The Link between Default and Recovery Rates: Theory, Empirical Evidence, and Implications. J. Bus. 78 (6), 2203–2228. http://dx.doi.org/10.1086/497044.

Bade, B., Rösch, D., Scheule, H., 2011. Default and Recovery Risk Dependencies in a Simple Credit Risk Model. Eur. Financial Manag. 17 (1), 120–144. http://dx.doi.org/10.1111/j.1468-036X.2010.00582.x.

BCBS, 2000. Principles for the Management of Credit Risk. Technical Report, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/publ/bcbs75.htm.

BCBS, 2005. An Explanatory Note on the Basel II IRB Risk Weight Functions. Technical Report, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/bcbs/irbriskweight.htm.

BCBS, 2006. International Convergence of Capital Measurement and Capital Standards. Technical Report June, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/publ/bcbs128.htm.

BCBS, 2013. Discussion Paper - The Regulatory Framework: Balancing Risk Sensitivity, Simplicity and Comparability. Technical Report July 2013, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/pub/bcbs258.htm.

BCBS, 2017. Basel III: Finalising Post-Crisis Reforms. Technical Report December, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/bcbs/publ/d424.htm.

BCBS, 2023a. Basel Framework - Calculation of RWA for Credit Risk - IRB Approach: Minimum Requirements to use IRB Approach. Technical Report December, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/basel_framework/chapter/CRE/36.htm.

BCBS, 2023b. Basel Framework - Calculation of RWA for Credit Risk - IRB Approach: Risk Components. Technical Report December, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/basel_framework/chapter/CRE/32.htm.

BCBS, 2023c. Basel Framework - Calculation of RWA for Credit Risk - IRB Approach: Risk Weight Functions. Technical Report December, Bank for International Settlements, Basel, Switzerland, URL https://www.bis.org/basel_framework/chapter/CRE/31.htm.

Betz, J., Kellner, R., Rösch, D., 2018. Systematic Effects among Loss Given Defaults and their Implications on Downturn Estimation. European J. Oper. Res. 271 (3), 1113–1144. http://dx.doi.org/10.1016/j.ejor.2018.05.059.

Calabrese, R., 2014. Downturn Loss Given Default: Mixture distribution estimation. European J. Oper. Res. 237 (1), 271–277. http://dx.doi.org/10.1016/j.ejor.2014.01.043.

Cantor, R., Emery, K., Keisman, D., Ou, S., 2007. Moody's Ultimate Recovery Database: Special Report. Technical Report, Moody's Investor Service, URL https://www.moodys.com/sites/products/defaultresearch/2006600000428092.pdf.

Caselli, S., Gatti, S., Querci, F., 2008. The Sensitivity of the Loss Given Default Rate to Systematic Risk: New Empirical Evidence on Bank Loans. J. Financ. Serv. Res. 34 (1), 1–34. http://dx.doi.org/10.1007/s10693-008-0033-8.

Chabaane, A., Laurent, J.-P., Salomon, J., 2004. Double Impact: Credit Risk Assessment and Collateral Value. Rev. Financ. 25 (1), 157–178, URL https://citeseerx.ist.psu.edu/pdf/cc3ee8646423c4c6b4f86c56541b0c5e5d2dec82.

Chao, M.-T., Strawderman, W., 1972. Negative Moments of Positive Random Variables. J. Amer. Statist. Assoc. 67 (338), 429–431. http://dx.doi.org/10.2307/2284399.

Drezner, Z., Wesolovski, G., 1990. On the computation of the bivariate normal integral. J. Stat. Comput. Simul. 35 (1–2), 101–107. http://dx.doi.org/10.1080/00949659008811236.

Düllmann, K., Gehde-Trapp, M., 2004. Systematic Risk in Recovery Rates: An Empirical Analysis of US Corporate Credit Exposures. Discussion Paper, Deutsche Bundesbank, No. 02/2004, pp. 1–53, URL https://www.bundesbank.de/resource/blob/704154/31bac58987486c54742e606a9f17abaf/mL/2004-08-01-dkp-02-data.pdf.

EBA, 2018. Final Draft Regulatory Technical Standards - On the Specification of the Nature, Severity and Duration of an Economic Downturn in Accordance with Articles 181(3)(a) and 182(4)(a) of Regulation (EU) No 575/2013. Technical Report November, European Banking Authority, Paris, France, URL https://www.eba.europa.eu/regulation-and-policy/model-validation/regulatory-technical-standards-on-the-specification-of-the-nature-severity-and-duration-of-an-economic-downturn.

EBA, 2019. Final Report - Guidelines for the Estimation of LGD Appropriate for an Economic Downturn ('Downturn LGD estimation'). Technical Report March, European Banking Authority, Paris, France, URL https://www.eba.europa.eu/regulation-and-policy/model-validation/regulatory-technical-standards-on-the-specification-of-the-nature-severity-and-duration-of-an-economic-downturn.

ECB, 2021. Targeted Review of Internal Models Project report. Technical Report April, European Central Bank, Frankfurt am Main, Germany, URL https://www.bankingsupervision.europa.eu/ecb/pub/pdf/ssm.trim_project_report~aa49bb624c.en.pdf.

Frontczak, R., Rostek, S., 2015. Modeling loss given default with stochastic collateral. Econ. Model. 44, 162–170. http://dx.doi.org/10.1016/j.econmod.2014.10.006.

Frye, J., 2000. Collateral Damage. Risk Mag. 13 (4), 91–94, URL https://www.bis.org/bcbs/events/oslo/frye.pdf.

Frye, J., 2014. The simple link from default to LGD. Risk Mag. 27 (3), 60–65, URL https://www.risk.net/risk-management/credit-risk/2328812/simple-link-default-lgd.

Gambetti, P., Gauthier, G., Vrins, F., 2019. Recovery rates: Uncertainty certainly matters. J. Bank. Financ. 106, 371–383. http://dx.doi.org/10.1016/j.jbankfin.2019.07.010.

García-Céspedes, R., Moreno, M., 2020. Random LGD adjustments in the Vasicek credit risk model. Eur. J. Finance 26 (18), 1856–1875. http://dx.doi.org/10.1080/1351847X.2020.1789685.

Genz, A., Brenton, K., Adelchi, A., 2015. pbivnorm: Vectorized Bivariate Normal CDF. Technical Report, URL https://CRAN.R-project.org/package=pbivnorm, R package version 0.6.0.

Giese, G., 2005. The impact of PD/LGD correlations on credit risk capital. Risk Mag. 18 (4), 79–84, URL https://www.risk.net/infrastructure/1530424/the-impact-of-pdlgd-correlations-on-credit-risk-capital.

Gordy, M.B., 2003. A risk-factor model foundation for ratings-based bank capital rules. J. Financ. Intermed. 12 (3), 199–232. http://dx.doi.org/10.1016/S1042-9573(03)00040-8.

Han, C., 2017. Modeling severity risk under PD–LGD correlation. Eur. J. Finance 23 (15), 1572–1588. http://dx.doi.org/10.1080/1351847X.2016.1212385.

Hartmann-Wendels, T., Imanto, C.P., 2021. Is the regulatory downturn LGD adequate? Performance analysis and alternative methods. J. Oper. Res. Soc. 74 (3), 736–747. http://dx.doi.org/10.1080/01605682.2021.1973349.

Hu, Y.-T., 2007. Extreme Correlation of Defaults and LGDs. February. Working Paper, Birkbeck University of London, URL https://eprints.bbk.ac.uk/id/eprint/26907/.

Hull, J., 2021. Options, Futures, and Other Derivatives, eleventh ed. Pearson Prentice Hall, Upper Saddle River, NJ [u.a.], URL https://www.pearson.com/en-us/subject-catalog/p/options-futures-and-other-derivatives/P200000005938/9780136939917.

Hyndman, R.J., Fan, Y., 1996. Sample Quantiles in Statistical Packages. Amer. Statist. 50 (4), 361–365. http://dx.doi.org/10.2307/2684934.

Jokivuolle, E., Virén, M., 2013. Cyclical default and recovery in stress testing loan losses. J. Financ. Stab. 9 (1), 139–149. http://dx.doi.org/10.1016/j.jfs.2011.10.001.

Juelsrud, R.E., Wold, E.G., 2020. Risk-weighted capital requirements and portfolio rebalancing. J. Financ. Intermed. 41, 100806. http://dx.doi.org/10.1016/j.jfi.2018.10.002.

Khieu, H.D., Mullineaux, D.J., Yi, H.-C., 2012. The determinants of bank loan recovery rates. J. Bank. Financ. 36 (4), 923–933. http://dx.doi.org/10.1016/j.jbankfin.2011.10.005.

Koopman, S.J., Lucas, A., Schwaab, B., 2011. Modeling frailty-correlated defaults using many macroeconomic covariates. J. Econometrics 162 (2), 312–325. http://dx.doi.org/10.1016/j.jeconom.2011.02.003.

Mariathasan, M., Merrouche, O., 2014. The manipulation of basel risk-weights. J. Financ. Intermed. 23 (3), 300–321. http://dx.doi.org/10.1016/j.jfi.2014.04.004.

Meng, Q., Levy, A., Kaplin, A., Wang, Y., Hu, Z., 2010. Implications of PD-LGD Correlation in a Portfolio Setting. Technical Report, Moody's Analytics, URL https://www.moodysanalytics.com/-/media/whitepaper/before-2011/10-05-02-Implications-of-PD-LGD-Correlation-in-a-Portfolio-Setting.pdf.

Merton, R.C., 1974. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. J. Finance 29 (2), 449–470. http://dx.doi.org/10.2307/2978814.

Miu, P., Ozdemir, B., 2006. Basel requirement of downturn LGD: Modeling and estimating PD & LGD correlations. J. Credit Risk 2 (2), 43–68. http://dx.doi.org/10.21314/JCR.2006.037.

Moody's, 2015. Corporate Default and Recovery Rates 1920–2014. Technical Report, Moody's Investor service, New York, URL http://www.alacrastore.com/moodys-credit-research/Annual-Default-Study-Corporate-Default-and-Recovery-Rates-1920-2014-PBC_179348.

Mora, N., 2012. What determines creditor recovery rates?. Fed. Reserv. Bank Kansas City Econ. Rev. 97 (2), 79–109, URL https://www.kansascityfed.org/documents/1381/2012-What%20Determines%20Creditor%20Recovery%20Rates%3F.pdf.

Perraudin, W.R.M., Hu, Y.-T., 2006. The dependence of recovery rates and defaults. Risk Control Res. Pap. 6 (1), 1–28, URL https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1961142.

Pitera, M., Schmidt, T., 2018. Unbiased estimation of risk. J. Bank. Financ. 91, 133–145. http://dx.doi.org/10.1016/j.jbankfin.2018.04.016.

Puzanova, N., Siddiqui, S., Trede, M., 2009. Approximate value-at-risk calculation for heterogeneous loan portfolios: Possible enhancements of the Basel II methodology. J. Financ. Stab. 5 (4), 374–392. http://dx.doi.org/10.1016/j.jfs.2009.02.003.

Pykhtin, M., 2003. Unexpected recovery risk. Risk Mag. 16 (8), 74–78, URL https://www.risk.net/risk-management/credit-risk/1530263/unexpected-recovery-risk.

Pykhtin, M., 2004. Multi-factor adjustment. Risk Mag. 17 (3), 85–90, URL https://www.risk.net/risk-management/credit-risk/1530333/multi-factor-adjustment.

R Core Team, 2022. R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria, URL https://www.R-project.org/.

Rossignolo, A.F., Fethi, M.D., Shaban, M., 2012. Value-at-Risk models and Basel capital charges: Evidence from Emerging and Frontier stock markets. J. Financ. Stab. 8 (4), 303–319. http://dx.doi.org/10.1016/j.jfs.2011.11.003.

Saurina, J., Trucharte, C., 2007. An Assessment of Basel II Procyclicality in Mortgage Portfolios. J. Financ. Serv. Res. 32, 81–101. http://dx.doi.org/10.1007/s10693-007-0008-1.

Schäfer, R., Koivusalo, A.F., 2013. Dependence of defaults and recoveries in structural credit risk models. Econ. Model. 30, 1–9. http://dx.doi.org/10.1016/j.econmod.2012.08.033.

Tarashev, N., Zhu, H., 2008. Specification and Calibration Errors in Measures of Portfolio Credit Risk: The Case of the ASRF Model. Int. J. Central Bank. 4 (2), 129–173, URL https://www.ijcb.org/journal/ijcb08q2a4.htm.

Tasche, D., 2004. The single risk factor approach to capital charges in case of correlated loss given default rates. URL https://arxiv.org/abs/cond-mat/0402390v2.

Thakor, A.V., 2021. Politics, credit allocation and bank capital requirements. J. Financ. Intermed. 45, 100820. http://dx.doi.org/10.1016/j.jfi.2019.03.005.

Tsai, M.-S., Chen, L.-C., 2011. The calculation of capital requirement using Extreme Value Theory. Econ. Model. 28 (1–2), 390–395. http://dx.doi.org/10.1016/j.econmod.2010.08.010.

Van Damme, G., 2011. A generic framework for stochastic Loss-Given-Default. J. Comput. Appl. Math. 235 (8), 2523–2550. http://dx.doi.org/10.1016/j.cam.2010.11.006.

Varma, P., Cantor, R., 2005. Determinants of Recovery Rates on Defaulted Bonds and Loans for North American Corporate Issuers: 1983–2003. J. Fixed Income 14 (4), 29–44. http://dx.doi.org/10.3905/jfi.2005.491110.

Vasicek, O.A., 1991. Limiting Loan Loss Probability Distribution. Technical Report, Moodys KMV Company Research Paper, pp. 1–4, URL https://doi.org/10.1002/9781119186229.ch18.

Vasicek, O.A., 2002. Loan portfolio value. Risk Mag. 15 (12), 160–162, URL https://www.risk.net/risk-management/credit-risk/1500333/loan-portfolio-value#:~:text=It%20means%20that%20the%20equity,corresponds%20to%20the%20desired%20probability.

Yao, X., Crook, J., Andreeva, G., 2017. Is it obligor or instrument that explains recovery rate: Evidence from US corporate bond. J. Financ. Stab. 28, 1–15. http://dx.doi.org/10.1016/j.jfs.2016.11.005.