Pilots allocation for sparse channel estimation in multicarrier systems

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Abstract

Wireless channels experience multipath fading which can be modeled by a sparse discrete multi-tap impulse response. Estimating this channel is of crucial importance to allow the receiver to properly recover the transmitted signal. This paper investigates the issue of allocating the pilots for sparse channel estimation applied to multicarrier systems. When the number of pilots is larger than or equal to the channel maximal length, this issue is well-known and the optimal allocation is equispaced. However for long channels, this would require a very large number of pilots decreasing the throughput of the system. Therefore, compressed sensing (CS) techniques are considered to estimate the sparse channel from a limited number of pilots. In that case, the problem of placing the pilots remains an open issue. This paper proposes a two-step hybrid allocation of the pilots that takes the maximal channel length into account to restrict the frequency candidates. The performance of this allocation is demonstrated through simulations and comparisons with other classical allocations.

1 Introduction

In wireless telecommunication systems, multiple reflections induce distortion on the transmitted signal. The receiver has to estimate the channel impulse response to properly compensate this effect commonly known as multipath fading. The wireless channel is often characterized by a sparse discrete multi-tap impulse response of maximal delay L. In a multicarrier system with M subcarriers, the receiver probes the channel frequency response thanks to L_p known pilot symbols transmitted at well-chosen subcarriers. Using this partial information ($L_p \ll M$) the receiver has to estimate or interpolate the entire channel frequency response. In this paper, we investigate the optimal choice for the pilot subcarrier positions as a function of L_p and leveraging sparsity of the channel impulse response.

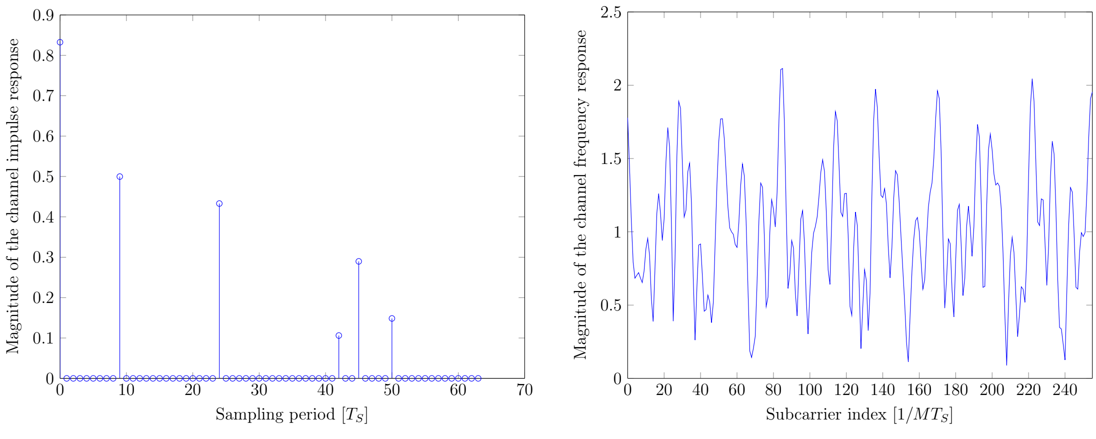
In practice, the maximal channel length L (number of time samples before the last tap), is much smaller than the number of subcarriers M, *i.e.*, $L \ll M$. For $L_p \geq L$, traditional approaches based on the least squares (LS) criterion have shown that the optimal pilot subcarrier positions are equispaced [1]. However, for very long channels, the required number of pilots L_p can be prohibitive, wasting subcarriers that are therefore unavailable for data transmission.

Compressed sensing (CS) theory allows to robustly estimate a sparse signal from a limited number of incoherent linear measurements [2]. In particular, a signal of length M with only K nonzero elements is perfectly recovered with high probability from about $K \log(M)$ randomly selected Fourier samples [3]. Some works have applied CS recovery methods to channel estimation showing that significant gains can be obtained [4] compared with classical methods. The problem of optimally placing the pilots has been investigated only very recently. Some works proposed sub-optimal research algorithms to select the pilot locations based on different criteria, i.e. the average mean squared error (MSE) using known channel models in [5], the coherence of the measurement matrix in [6] or its mutual and modified mutual coherence in [7]. However, none of those approaches effectively take the maximal channel length L into account to restrict the pilot possible positions.

Instead of randomly selecting the pilots among the M subcarriers, this paper proposes to restrict the frequency candidates to a subset of L equispaced subcarriers and randomly select L_p positions among these L candidates. This hybrid allocation strategy between equispaced and purely random gives a clear gain in performance and reduces the required number of pilots with respect to L.

The rest of this paper is structured as follows. Section 2 describes the system under study by introducing the linear forward model. It also presents three options for pilots allocation and gives the main idea behind CS principles and the recovery algorithm. Section 3 compares the different pilots allocations through numerical simulations and shows that the proposed hybrid allocation allows one to significantly reduce the number of pilots.

2 System model



(a) Channel impulse response

(b) Channel frequency response

Figure 1: Sparse channel model, $M = 256$, $K = 6$, $L = M/4 = 64$

The channel impulse response (CIR) $h[l]$ $(0 \geq l \geq L-1)$ is modeled as a discrete channel as depicted in Figure 1a for $K = 6$ non zero taps, $M = 256$ subcarriers, maximal channel length $L = 64$ and sampling period T_S. This is a typical length for very long channels, e.g. ITU Vehicular B channel model. Moreover an exponentially decaying power delay profile (PDP) is assumed for the channel. Figure 1b depicts the channel frequency response (CFR), obtained by performing a discrete Fourier transform (DFT) on $h[l]$ evaluated at M equispaced frequency points.

The system model for channel estimation in a multicarrier system based on transmitted pilots can be formulated as follows

$$\boldsymbol{H}_{L_p} = \underbrace{\boldsymbol{S}\boldsymbol{F}_{M\times M}\boldsymbol{\Sigma}}_{\boldsymbol{\Phi}}\,\boldsymbol{h} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{H}_{L_p}$ is the $L_p \times 1$ stacked vector of observations at pilot subcarriers, $\boldsymbol{S}$ is a $L_p \times M$ matrix which select the L_p rows of the full $M \times M$ DFT matrix $\boldsymbol{F}_{M\times M}$ corresponding to the pilot subcarrier indexes, $\boldsymbol{\Sigma}$ is a $M \times L$ matrix which selects the L first columns of $\boldsymbol{F}_{M\times M}$, $\boldsymbol{h}$ is a $L \times 1$ vector corresponding to the stacked CIR $h[0], ..., h[L-1]$ and $\boldsymbol{n}$ is an additive white Gaussian noise of covariance $\boldsymbol{C}_n = \mathbb{E}(\boldsymbol{n}\boldsymbol{n}^H) = \frac{2N_0}{E_{\text{Pilot}}}\boldsymbol{I}_{L_p}$. A constraint on the total energy E_T used by the pilots is assumed such that $E_{\text{Pilot}} = \frac{E_T}{L_p} = \frac{ME_S}{L_p}$. The $L_p \times L$ matrix $\boldsymbol{\Phi}$ is commonly known as the measurement matrix.

If the number of transmitted pilots is larger than or equal to the maximal channel length, $L_p \geq L$, a classical approach in the literature [1] is to use a least square (LS) criterion to estimate $\boldsymbol{h}$. The estimator of the CIR is then given by

$$\begin{aligned}\hat{\boldsymbol{h}}_{\text{LS}} &= \text{argmin}_{\tilde{\boldsymbol{h}}}\, \|\boldsymbol{H}_{L_p} - \boldsymbol{\Phi}\tilde{\boldsymbol{h}}\|^2 \\ &= (\boldsymbol{\Phi}^H\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^H\boldsymbol{H}_{L_p} \\ &= \boldsymbol{h} + (\boldsymbol{\Phi}^H\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^H\boldsymbol{n}.\end{aligned} \tag{2}$$

This estimator converges asymptotically towards the true CIR $\boldsymbol{h}$ at high SNR. The work in [1] has shown that to minimize the noise amplification due to the inverse in last expression, one should place the pilot subcarriers in an equispaced way. This could actually be predicted by Shannon sampling theory. Since $h[l]$ has a length limited to L samples, there will not be any aliasing if its DFT is sampled uniformly at maximal sampling period $1/L$ which is obtained as soon as $L_p \geq L$.

However, if $L_p < L$, the least squares problem is underdetermined. Inspired by sparse approximation theory, most CS reconstruction techniques leverage the sparsity of $\boldsymbol{h}$ to regularize the least squares problem, *i.e.*,

$$\hat{\boldsymbol{h}} = \text{argmin}_{\tilde{\boldsymbol{h}}}\, \|\boldsymbol{H}_{L_p} - \boldsymbol{\Phi}\tilde{\boldsymbol{h}}\|^2 \quad \text{s.t.} \quad \|\tilde{\boldsymbol{h}}\|_0 \leq K, \tag{3}$$

where $\|.\|_0$ is the number of non-zero coefficients in a vector. The estimator $\hat{\boldsymbol{h}}$ of (3) is NP-hard to compute due to the combinatorial number of possible supports for $\tilde{\boldsymbol{h}}$. However, there exist several ways to find a good approximation to $\hat{\boldsymbol{h}}$, *e.g.*, by ℓ_1-norm relaxation [2]. In this work, the Iterative Hard Thresholding (IHT) algorithm is for estimating $\hat{\boldsymbol{h}}$. This procedure is simple, fast and it provides guarantees of good reconstruction quality [8].

In this particular setup, the optimal pilot positions are not known. The problem addressed in this paper is the choice of the pilot subcarrier positions in order to recover $\boldsymbol{h}$ based on the channel observations $\boldsymbol{H}_{L_p}$.

Equispaced allocation One could think at first that placing the subcarriers in an equispaced manner over the band is still a good idea. Nevertheless, regarding Shannon sampling theory, this allocation would result in strong aliasing since the frequency band is not sampled at a sufficient rate. Sparse approximation theory also supports this fact. Indeed, it is known that one can find a unique sparse representation (here $\boldsymbol{h}$), of a signal (here $\boldsymbol{H}_{L_p}$) in a redundant dictionary $\boldsymbol{\Phi}$ only when the coherence of that dictionary is small [9]. The coherence μ of $\boldsymbol{\Phi}$ is defined as the maximum normalized absolute correlation between two columns,

$$\mu = \max_{0 \leq m < n \leq L-1} \frac{\boldsymbol{\phi}_m^H \boldsymbol{\phi}_n}{\|\boldsymbol{\phi}_m\|_2 \|\boldsymbol{\phi}_n\|_2}, \tag{4}$$

that is, in this case,

$$\begin{aligned} \mu &= \max_{0 \leq m < n \leq L-1} \frac{1}{L_p} \sum_{l=0}^{L_p-1} e^{-j\frac{2\pi}{M}k_l(n-m)} \\ &= \max_{1 \leq c \leq L-1} \frac{1}{L_p} \sum_{l=0}^{L_p-1} \omega_M^{k_l c}, \end{aligned} \tag{5}$$

where $\omega_M^k = e^{-j\frac{2\pi}{M}k}$ and $c = n-m$. Let's assume that L_p divides M and the subcarriers are placed equispaced, that is, $k_l = l\frac{M}{L_p}$. While the condition $L_p \geq L$ gives a zero coherence due to the orthogonality property of the root of unity, the underdetermined case $L_p < L$ allows $c = L_p$, that is,

$$\mu = \max_{1 \leq c \leq L-1} \frac{1}{L_p} \sum_{l=0}^{L_p-1} e^{-j\frac{2\pi}{M}l\frac{M}{L_p}c} = \frac{1}{L_p} \sum_{l=0}^{L_p-1} \omega_{L_p}^{lL_p} = 1. \tag{6}$$

This shows that if $L_p < L$, placing the subcarrier uniformly is the worst choice in the sense of the minimal coherence criterion.

Fully random allocation Following CS theory, when $L = M$, *i.e.*, the time range of $\boldsymbol{h}$ is equal to the number of subcarriers, a good choice for the subcarrier assignment scheme (SAS) is simply to choose fully randomly L_p subcarriers and one could expect much better performance than with equispaced placement. In particular, it has been shown that in this case, the condition

$$L_p \geq CK\log(M) \tag{7}$$

with C a reasonably small fixed constant, guarantees perfect recovery in the noiseless case and robust recovery in the presence of noise [3]. This SAS will be further referred to as the fully random SAS.

The limitation of the fully random SAS is that it does not take into account the maximal channel length L that is smaller than M in practice. Fully randomly selecting the subcarriers would wastefully allow to have a time range up to M taps in time domain (TD). A second way to interpret this limitation is that the CFR should not be sampled too fast. Indeed, a small distance between pilot subcarriers would give useless information on the taps which are far from the origin and known to be zero. Yet another way to see this is by acknowledging the fact that two frequency samples that are next to each other are strongly correlated and their mutual information is very low.

Proposed hybrid allocation Considering the limitation of the fully random SAS, a constraint can be added to restrict the distance between pilot subcarriers to be bigger than $\frac{M}{L}$. Instead of fully randomly selecting among the M subcarriers this paper proposes to perform a two-step hybrid SAS:

- The frequency candidates are first restricted to a subset of L equispaced subcarriers.
- The L_p positions are randomly selected among these L candidates.

This allocation is referred to as "hybrid" between equispaced and purely random since it randomly selects L_p subcarriers among the frequency candidates of the $L_p = L$ equispaced SAS which corresponds to an optimal Shannon sampling. Doing so, we try to ensure minimum correlation between pilots.

Furthermore, if L is assumed[1] to divide M, this hybrid SAS leads to a simplified system model as

$$\begin{aligned} \boldsymbol{H}_{L_p} &= \boldsymbol{S}\boldsymbol{F}_{M\times M}\boldsymbol{\Sigma}\boldsymbol{h} + \boldsymbol{n} \\ &= \boldsymbol{S}'\boldsymbol{F}_{L\times L}\boldsymbol{h} + \boldsymbol{n}. \end{aligned} \tag{8}$$

where $\boldsymbol{S}'$ is the new selection matrix with indices belonging to $[0, L-1]$. The previous DFT simplification comes from the fact that the L first columns and the L equispaced rows of $\boldsymbol{F}_{M\times M}$ are selected.

This simplification has two advantages. On the one hand, the complexity of the reconstruction algorithm is reduced since the DFT size decreases by a factor $\frac{M}{L_p}$. On the other hand, this allows to use the results of [3] but with L instead of M, namely,

$$L_p \geq CK\log(L), \tag{9}$$

which implies a reduction in the required number of pilots to guarantee perfect reconstruction in the noiseless case and robust recovery in the presence of noise.

3 Simulation results

In the simulations, $M = 256$ subcarriers are assumed, the maximal length of the channel is set to $L = M/4 = 64$ and $K = 6$ taps are non negligible. The first tap is placed in 0 and the five remaining tap delays follow a uniform distribution in $\{1, \cdots, L-1\}$. A uniform PDP of the channel is considered such that each non zero tap has a zero mean and a variance exponentially decaying with the delay with a 20dB attenuation of the last tap with respect to the first. 100 channel realizations and 100 SAS realized for each of the hybrid and fully random SAS while only one realization for the equispaced SAS (since it is deterministic). The metric used to evaluate each method is the normalized mean squared error (NMSE) defined as

$$\text{NMSE} = \mathbb{E}\left\{\frac{\|\boldsymbol{h} - \hat{\boldsymbol{h}}\|_2^2}{\|\boldsymbol{h}\|_2^2}\right\}, \tag{10}$$

which is averaged over all channel and SAS realizations. The method used to reconstruct the channel based on the observations is Iterative Hard Thresholding (IHT) [8].

[1] If it is not true, one can increase L up to the point where it becomes true.

We fix the number of iterations to 50 and the gradient step size parameter of the line search is computed at each iteration to minimize the LS criterion. This method assumes the channel maximal length and the number of non zero taps are known by the receiver. However, one could as well use another CS technique which does not know exactly the sparsity a priori, *e.g.* basis pursuit denoising [2].

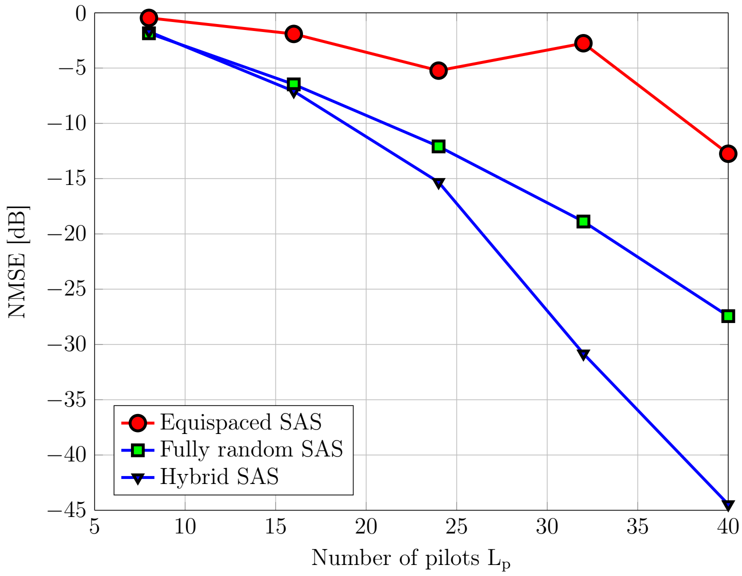


Figure 2: NMSE as a function of the number of pilots for different SAS and $E_S/N_0 = 30$dB.

Figure 2 shows the evolution of the NMSE as a function of the number of pilots for $E_S/N_0 = 30$dB. As expected, the equispaced SAS performs very badly[2]. Moreover, the hybrid SAS clearly outperforms the fully random SAS. There is a 12dB gap and a 17dB gap between the two methods for $L_p = 32$ and $L_p = 40$ respectively.

The results in Figure 2 are based on the NMSE which gives no information on the distribution of the normalized squared error (NSE) of reconstruction. Figure 3 depicts the cumulative density function (CDF) of the NSE for the different SAS and $L_p = 32$. As explained before, the reconstruction fails for almost all realizations using the equispaced SAS. We also see that the probability of good reconstruction for the hybrid SAS is higher than for the random SAS, e.g. about 99% of the NSE realizations are below -35dB for the hybrid allocation compared to about 70% of the NSE realization using the fully random SAS.

Figure 4 shows the NMSE as a function of the E_S/N_0 ratio for different SAS. As explained, if $L_p < L$, the equispaced SAS performs very badly. For $L_p = L$, a LS estimator can be computed for which the equispaced SAS is optimal. Moreover, the LS estimator thresholded to the 6 more significant taps is also shown[3] and is the best performance obtained. The fully random SAS performs well at low SNR while at high

[2]Note that the equispaced is only averaged over one SAS realization since it is deterministic versus 100 for the two other SAS.

[3]Since the IHT method is assumed to know the number of non zero taps, it is also more fair to threshold the result of the LS estimator.

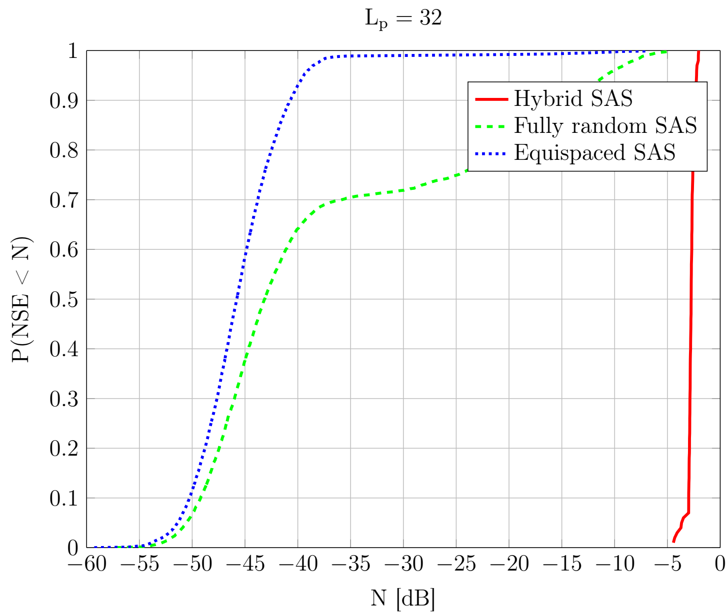


Figure 3: Cumulative density function (CDF) of the NSE for different SAS and $E_S/N_0 =$ 30dB, $L_p = 32$.

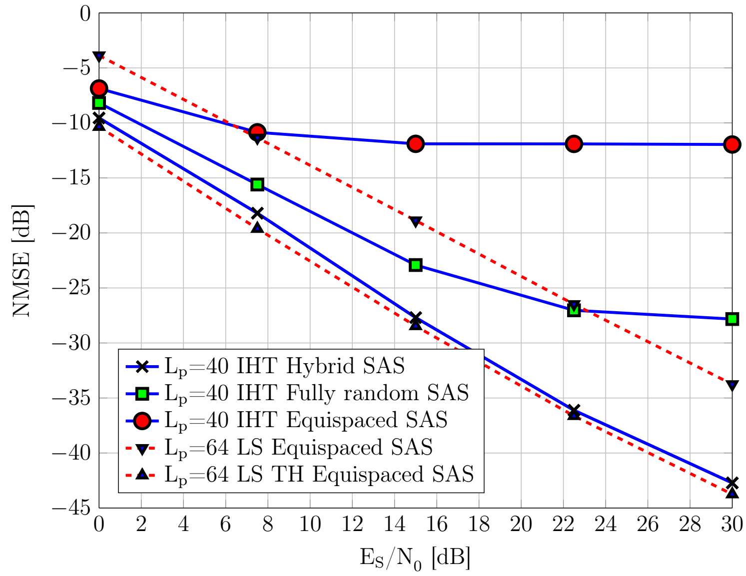


Figure 4: NMSE as a function of E_S/N_0 ratio ($L = 64$). The presented SAS are tested with $L_p = 40 < L$. A classical LS estimator is computed with $L_p = 64 = L$, for which the equispaced SAS is optimal. The LS estimator thresholded to its K highest coefficients is also shown for fair comparison with an ideal case.

SNR, the reconstruction error imposes a NMSE floor. Then, the two LS methods perform better at the cost of more pilots. However, the hybrid technique performs almost as well as the LS thresholded estimator still at high SNR and with only $L = 40$ pilots.

4 Conclusion

This paper investigated the issue of allocating the pilots for sparse channel estimation. In situations where the number of pilots is larger than or equal to the channel maximal length, this issue is well-known and the optimal allocation is equispaced. However, if the number of pilots is strictly smaller than the channel length, the problem remains open. This paper showed that still placing the subcarriers in an equispaced way is the worst choice in the sense of the minimal coherence criterion. Rather than selecting the pilots at random among the subcarriers, this paper proposed to use a two-step hybrid allocation. The first step restricts the frequency candidates to a subset of equispaced subcarriers and the second step randomly selects positions among these candidates. This allocation allows to significantly reduce the complexity of the reconstruction by decreasing the DFT matrix size. The performance of the method was demonstrated through simulations and compared with fully random allocation approach and other classical approaches based on the LS criterion. The hybrid allocation clearly outperforms the fully random allocation while reaching almost the same performance as a LS thresholded estimator requiring much more pilots.

Acknowledgment

The research reported herein was partly funded by F.N.R.S and F.R.I.A.

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