An improved finiteness test and a systematic procedure to compute the strong $\mathcal{H}_2$ norm of differential algebraic systems with multiple delays

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Abstract

We study the strong $\mathcal{H}_2$ norm of systems modeled by semi-explicit Delay Differential Algebraic Equations (DDAEs). We recall that the finiteness of the strong $\mathcal{H}_2$ norm is linked to an algebraic decision problem that can be solved by checking a finite numbers of equalities. We first improve the verification of the finiteness condition. In particular, the complexity of our new condition removes a dependency on the number of delays. We also show that, without imposing further conditions on the system, the number of checks cannot be further reduced. The methodology relies on interpreting the verification of the finiteness conditions in terms of a Polynomial Identity Testing problem. Second we show, in a constructive way, that if the strong $\mathcal{H}_2$ norm is finite, the system can always be transformed into a regular neutral-type system with the same $\mathcal{H}_2$ norm, without derivatives in the input or in the output equations. This result closes a gap in the literature as such a transformation was known to exist only under additional assumptions on the system. The transformation enables the computation of the strong $\mathcal{H}_2$ norm using delay Lyapunov matrices. Illustrative examples are provided throughout the paper.

1 Introduction

In this paper we consider systems described by semi explicit DDAEs (Delay Differential Algebraic Equations) of the form

$$\begin{aligned}\dot{x}_1(t) &= \sum_{i=0}^{m} \mathrm{A}_i^{(11)} x_1(t-\tau_i) + \sum_{i=0}^{m} \mathrm{A}_i^{(12)} x_2(t-\tau_i) + \mathrm{B}_1 u(t)\\ x_2(t) &= \sum_{i=0}^{m} \mathrm{A}_i^{(21)} x_1(t-\tau_i) + \sum_{i=1}^{m} \mathrm{A}_i x_2(t-\tau_i) + \mathrm{B}u(t)\\ y(t) &= \mathrm{C}_1 x_1(t) + \mathrm{C}x_2(t)\end{aligned} \tag{1}$$

where $x_1(t) \in \mathbb{R}^r$, $x_2(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ denote the state, input and output variables at time t, respectively. The delays satisfy $0 = \tau_0 < \tau_1 \leq \cdots \leq \tau_m$, with no loss of generality. All matrices are assumed real valued and of compatible dimensions.

Let $\varphi(\theta) := [\varphi_1^{\mathrm{T}}(\theta) \;\; \varphi_2^{\mathrm{T}}(\theta)]^{\mathrm{T}}$, $\varphi_1(\theta) \in \mathbb{R}^r$, $\varphi_2(\theta) \in \mathbb{R}^n$, $\theta \in [-\tau_m, 0]$, be the initial function of (1) taken from the set of $\mathbb{R}^{r+n}$-valued absolutely continuous functions $\mathrm{AC}\left([-\tau_m,\, 0], \mathbb{R}^{r+n}\right)$. We call the initial function *consistent* if the corresponding initial value problem of (1) at $t = 0$ has at least one solution [2]. A function $x(t,\varphi) = [x_1^{\mathrm{T}}(t) \;\; x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is called a *(classical) solution* of (1) if it is absolutely continuous and satisfies (1) almost everywhere on $[0,\infty)$, and $x(\theta,\varphi) = \varphi(\theta)$ for $\theta \in [-\tau_m,\, 0]$, where φ is a consistent initial function. For continuously differentiable input functions, the set of consistent initial functions of (1) is (see, e.g., [4]):

$$\mathrm{X} := \{\varphi \in \mathrm{AC}\left([-\tau_m, 0], \mathbb{R}^{r+n}\right) : \varphi_2(0) - \sum_{i=0}^{m} \mathrm{A}_i^{(21)} \varphi_1(-\tau_i) - \sum_{i=1}^{m} \mathrm{A}_i \varphi_2(-\tau_i) - \mathrm{B}u(0) = 0\}.$$

For every consistent initial function, a forward solution is uniquely defined [7,2,10].

Equations of the form (1) naturally appear in a systematic and automated modeling of interconnected systems.

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Equations describing (sub)systems, controllers and their interconnections can simply be augmented into one DDAE model, without need to eliminate inputs and outputs governing the internal connections, which is generally not possible in the presence of delays. DDAEs can be found in applications of lossless propagation models in electrical and fluid engineering [15,14,5,1], as well as many other fields; see [18] for a non-exhaustive list.

The model class allows one to handle multiple input and output delays and nontrivial feedthrough terms, and to describe neutral-type systems by introducing slack variables; see [6, Section 2]. Suppose that, for example, the original model formulation has multiple delays in state x, multiple delays in input u and output y, and possibly explicit feedthrough terms. Then, a system of the form (1) can be obtained by defining additional variables (z_1, z_2), substituting $(x, u, y) \leftarrow (x, z_1, z_2)$, adding the equation $z_1(t) = u(t)$ and defining the output as $y(t) = z_2(t)$.

System (1), with zero input, is *exponentially stable* if there exist $\gamma_1, \gamma_2 > 0$ such that for every consistent initial condition, the solution $x(t, \varphi)$ with $u = 0$ satisfies

$$\|x(t,\varphi)\| \le \gamma_1 e^{-\gamma_2 t} \sup_{\theta\in[-\tau_m,0]} \|\varphi(\theta)\|, \quad t > 0$$

or, equivalently (see, e.g., [12, Proposition 1] [2, Theorem 3.1]), if its spectral abscissa

$$\alpha(\vec{\tau}) := \sup_{s\in\mathbb{C}} \{\mathrm{Re}(s) : \ s \in \Lambda\}$$

is strictly negative, where $\vec{\tau} = [\tau_1, \ldots, \tau_m]$ denotes the vector of delay values, $\Lambda := \{s \in \mathbb{C} : \det(\mathrm{H}(s)) = 0\}$ denotes the spectrum of (1), with

$$\mathrm{H}(s) := \begin{bmatrix} \mathrm{A}_{11}(s) & \mathrm{A}_{12}(s) \\ \mathrm{A}_{21}(s) & \mathrm{A}_{22}(s) \end{bmatrix},$$

and

$$\mathrm{A}_{11}(s) := s\mathrm{I} - \sum_{i=0}^{m} \mathrm{A}_i^{(11)} e^{-s\tau_i}, \mathrm{A}_{12}(s) := -\sum_{i=0}^{m} \mathrm{A}_i^{(12)} e^{-s\tau_i}$$
$$\mathrm{A}_{21}(s) := -\sum_{i=0}^{m} \mathrm{A}_i^{(21)} e^{-s\tau_i}, \mathrm{A}_{22}(s) := \mathrm{I} - \sum_{i=1}^{m} \mathrm{A}_i e^{-s\tau_i}.$$

Motivated by the fact that the exponential stability of DDAEs may not be robust against small delay perturbations, the notion of strong stability is introduced in [12], along with necessary and sufficient conditions.

Definition 1 *[12] System* (1)*, with zero input, is strongly stable if there exists a number* $\epsilon > 0$ *such that* $\alpha(\vec{\tau}_\epsilon) < 0$ *for all* $\vec{\tau}_\epsilon \in \mathbf{B}(\vec{\tau}, \epsilon)$*, with*

$$\mathbf{B}(\vec{\tau}, \epsilon) := \{\vec{\theta} \in \mathbb{R}^m_{\ge 0} : \|\vec{\theta} - \vec{\tau}\| < \epsilon\}. \tag{2}$$

Under assumption of exponential stability, the $\mathcal{H}_2$ norm of (1) is defined as

$$\|\mathrm{G}\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Tr}(\mathrm{G}^*(j\omega)\mathrm{G}(j\omega))\,d\omega}, \tag{3}$$

where G is the transfer matrix of system (1), i.e.,

$$\mathrm{G}(s) = \begin{bmatrix} \mathrm{C}_1 & \mathrm{C} \end{bmatrix} \mathrm{H}^{-1}(s) \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B} \end{bmatrix}, s \in \mathbb{C} \setminus \Lambda \tag{4}$$

and $\cdot^*$ is the Hermitian conjugate. The $\mathcal{H}_2$ norm of (1) has non-intuitive features that complicate its study, as illustrated by the following example.

Example 2 *Let* $m = 3$ *and matrices*

$$\mathrm{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}, \mathrm{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} p \\ 0 \\ p \end{bmatrix},$$

$$\mathrm{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{A}_0^{(11)} = -1, \quad \mathrm{B}_1 = \mathrm{C}_1 = 1,$$

with p *a parameter, and all other matrices equal to zero. One can show that the system is strongly exponentially stable and its transfer function equals*

$$\mathrm{G}(s) = \frac{1}{s+1} - p\frac{1}{4}\frac{\left(e^{-s(\tau_1+\tau_2)} - e^{-s\tau_3}\right)}{8 - e^{-s\tau_2}}. \tag{5}$$

The $\mathcal{H}_2$ *norm is finite if and only if the second term vanishes. Hence, for* $p \neq 0$*, the* $\mathcal{H}_2$ *norm of (5) is finite whenever* $\tau_3 = \tau_1 + \tau_2$ *and infinite otherwise.*

Thus, despite the system given in Example 2 being exponentially stable and also strongly exponentially stable, its $\mathcal{H}_2$ norm may be infinite due to "hidden" feedthrough terms between input and output, and can also be sensitive to infinitesimal perturbations of the delays. In Example 2, this hidden feedthrough term gives rise to the second term in (5). We call it "hidden" because the output y in (1) does not depend directly on input u. To handle this sensitivity problem, the notion of strong $\mathcal{H}_2$ norm is introduced in [4],

$$|||\mathrm{G}(\cdot;\vec{\tau})|||_{\mathcal{H}_2} := \lim_{\epsilon\to 0+} \sup\{\|\mathrm{G}(\cdot;\vec{\tau}_\epsilon)\|_{\mathcal{H}_2} : \vec{\tau}_\epsilon \in \mathbf{B}(\vec{\tau}, \epsilon)\}.$$

Note that the strong $\mathcal{H}_2$ norm of Example 2 is finite if and only if $p = 0$.

The above properties induce two major questions in the $\mathcal{H}_2$ norm analysis: 1) determining whether the strong $\mathcal{H}_2$ norm is finite, and, if this is the case, 2) its actual computation. Given the implicit description in (1), an appropriate approach to address the second question is a "regularization" in terms of explicit delay differential equations

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of neutral type (see Definition 7 below), enabling the approach of [9] to compute the $\mathcal{H}_2$ norm. Such an approach is also amenable for other problems [2,7]. A main challenge, however, is that derivatives of inputs and outputs in the model must be avoided.

1.1 Finiteness conditions of the strong $\mathcal{H}_2$ norm

Remarkably, one can decide on the finiteness of the strong $\mathcal{H}_2$ norm from the solution of an algebraic problem on the semigroup of matrices generated by $\mathscr{A} = (A_1, \dots, A_m)$. To make this more precise, we need matrix polynomials $P_{k_1,\dots,k_m}(\mathscr{A})$ with $k_i \in \mathbb{Z}, i = 1,\dots,m$, recursively defined as follows: $P_{0,\dots,0}(\mathscr{A}) := I$,

$$\begin{aligned} P_{k_1,\dots,k_m}(\mathscr{A}) := A_1 P_{k_1-1,k_2,\dots,k_m}(\mathscr{A}) \\ + A_2 P_{k_1,k_2-1,\dots,k_m}(\mathscr{A}) + \dots \\ + A_m P_{k_1,k_2,\dots,k_m-1}(\mathscr{A}) \end{aligned} \tag{6}$$

if $k_i \in \mathbb{Z}_{\geq 0}, i = 1,\dots,m$ and $P_{k_1,\dots,k_m}(\mathscr{A}) := 0$ if any $k_i \in \mathbb{Z}_{<0}, i = 1,\dots,m$.

Example 3 *Let $\mathscr{A} = \{M, N, Q\}$, then we have*

$$P_{1,0,0} = M, \quad P_{0,1,0} = N, \quad P_{0,0,1} = Q.$$

We give a few more examples,

$$P_{1,1,0} = MN+NM, \quad P_{0,0,3} = Q^3, \quad P_{1,0,2} = MQ^2+QMQ+Q^2M.$$

In general, $P_{i,j,k}$ is the sum of all possible ways to multiply i times matrix M, j times matrix N and k times matrix Q. A different interpretation of $P_{\mathbf{k}}$ will be given in Lemma 12: $P_{\mathbf{k}}$ correspond to the coefficient of the polynomial expansion $(\sum x_i A_i)^{\|\mathbf{k}\|_1}$. In what follows, we exploit this interpretation to obtain Theorem 13.

Proposition 4 *[4] Let system (1) be strongly exponentially stable. Then the following assertions are equivalent:*

(1) The strong $\mathcal{H}_2$ norm of (4) is finite.
(2) The following condition holds:

$$C P_{k_1,\dots,k_m}(\mathscr{A}) B = 0, \quad \forall (k_1,\dots,k_m) \in \mathbb{N}^m. \tag{7}$$

Furthermore, if the strong $\mathcal{H}_2$ norm of (1) is finite, it equals its $\mathcal{H}_2$ norm.

In [3, Theorem 5], it is shown that checking condition (7) is equivalent to checking

$$\begin{aligned} &C P_{k_1,\dots,k_m}(\mathscr{A}) B = 0, \\ &\qquad \forall (k_1,\dots,k_m) \in \mathbb{N}^m : \sum_{i=1}^{m} k_i < mn, \end{aligned} \tag{8}$$

which induces a *finite* test for the finiteness of the strong $\mathcal{H}_2$ norm. However, the number of equalities to be verified, $\sum_{i=0}^{mn-1} \frac{(m+i-1)!}{(m-1)!i!}$, scales badly with the number of delays, m, and dimension n.

In [4], the following relaxation of (7) is proposed:

$$\begin{aligned} &CB = 0, \quad CA_{\sigma_1} \dots A_{\sigma_k} B = 0, \ \forall k \in \mathbb{N}, \\ &\qquad \forall \sigma_i \in \{1,\dots,m\}, i = 1,\dots,k. \end{aligned} \tag{9}$$

Example 5 *Let $\mathscr{A} = \{X, Y\}$. Condition (7) required checking that $C(XY+YX)B = 0$, and this relaxation amount to asking that $C(XY)B = 0 = C(YX)B$. If (9) holds, then we can deduce (7) by linearity.*

On the one hand, checking (9) is computationally more tractable than (8), as demonstrated in [4, Section 3]. On the other hand, condition (9) is sufficient but not necessary for a finite strong $\mathcal{H}_2$ norm of an exponentially stable DDAE.

Example 6 *Consider matrices*

$$A_1 = \frac{1}{8}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$B^T = \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix},$$

which can be complemented such that (1) is strongly exponentially stable. It can be verified by direct calculation that (8) and thus (7) holds, but (9) is not satisfied as $CA_1A_2B = -\frac{1}{8}$.

1.2 Transformation to delay equations of neutral type

Knowing that the $\mathcal{H}_2$ norm is finite, one may wish to compute its value. Here we show how to transform the DDAE (1) into a neutral type system with the same $\mathcal{H}_2$ norm, for which techniques based on delay Lyapunov matrices can be adopted.

Definition 7 *A system of the form*

$$\begin{aligned} \mathscr{D}_0 \dot{x}(t) - \sum_{i=1}^{m} \mathscr{D}_i \dot{x}(t-\tau_i) &= \mathscr{F}_0 x(t) + \sum_{i=1}^{m} \mathscr{F}_i x(t-\tau_i) + \mathscr{B} u(t), \\ y(t) &= \mathscr{C} x(t), \end{aligned} \tag{10}$$

with vectors $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^{n_u}, y(t) \in \mathbb{R}^{n_y}$ and matrices $\mathscr{D}_0, \mathscr{D}_i, \mathscr{F}_0, \mathscr{F}_i \in \mathbb{R}^{n\times n}, \mathscr{B} \in \mathbb{R}^{n\times n_u}, \mathscr{C} \in \mathbb{R}^{n_y\times n}$, where $\mathscr{D}_0$ is

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non-singular, is called a regular neutral-type system. Furthermore, the transfer matrix of system (10) *is*

$$\mathscr{G}(s) := \mathscr{C}\mathscr{H}^{-1}(s)\mathscr{B}, \quad s \in \mathbb{C} \setminus \Lambda_n$$

with

$$\mathscr{H}(s) = s\mathscr{D}_0 - s\sum_{i=1}^{m} \mathscr{D}_i e^{-s\tau_i} - \mathscr{F}_0 - \sum_{i=1}^{m} \mathscr{F}_i e^{-s\tau_i}$$

and $\Lambda_n := \{s \in \mathbb{C} : \det(\mathscr{H}(s)) = 0\}$.

Similarly to (1), neutral-type system (10), with zero input, is exponentially stable if there exist positive numbers η_1 and η_2 such that

$$\|x(t,\varphi_n)\| \le \eta_1 e^{-\eta_2 t} \sup_{\theta \in [-\tau_m, 0]} \|\varphi_n(\theta)\|, \quad t > 0$$

for any initial function of the system, $\varphi_n(\theta)$, $\theta \in [-\tau_m, 0]$, taken from the set of $\mathbb{R}^n$-valued continuously differentiable functions $C^{(1)}([-\tau_m, 0], \mathbb{R}^n)$. Moreover, system (10) is *strongly exponentially stable* if it is exponentially stable when subjected to small perturbations in the delay [8].

Let the delay-difference equation associated to (10)

$$\mathscr{D}_0 x(t) - \sum_{i=1}^{m} \mathscr{D}_i x(t - \tau_i) = 0$$

be strongly exponentially stable, which can be characterised by

$$\max_{\vec{\theta} \in [0, 2\pi]^m} \rho\left(\mathscr{D}_0^{-1} \sum_{i=1}^{m} \mathscr{D}_i e^{j\theta_i}\right) < 1, \qquad (11)$$

where ρ denotes the spectral radius [13, Proposition 1.30]. Then, if the spectrum Λ_n of (10) lies in the left-half complex plane, the system is strongly exponentially stable [13, Proposition 1.43].

The $\mathscr{H}_2$ norm of an exponentially stable neutral system is defined as in (3) with the corresponding transfer function. As remarked above, the interest of transforming the transfer function of (1) into the transfer function of a neutral equation is that the computational problem of the $\mathscr{H}_2$ norm for this class of system has already been addressed in [9].

Two approaches exist to perform the transformation of (1) to a neutral system, which we will refer to as *regularization mechanisms* (RMs):

RM1 If $B = 0$, then one can apply the operator $\left(\frac{d}{dt} + I\right)$ to the second set of equations, leading to

$$\dot{x}_2(t) - \sum_{i=1}^{m} A_i \dot{x}_2(t-\tau_i) - \sum_{i=0}^{m} A_i^{(21)} \dot{x}_1(t-\tau_i) = -x_2(t) + \sum_{i=1}^{m} A_i x_2(t-\tau_i) + \sum_{i=0}^{m} A_i^{(21)} x_1(t-\tau_i),$$

which corresponds to a multiplication by $(s+1)$ in the frequency domain. The reason for combining differentiation with addition to the original equation lies in the preservation of internal stability.

RM2 If $C = 0$, one can define a new variable $\hat{x}_2$ via the stable differential equation $\dot{\hat{x}}_2 + \hat{x}_2 = x_2$ and subsequently substituting x_2 in (1), leading to

$$\begin{aligned}
\dot{x}_1(t) - \sum_{i=0}^{m} A_i^{(12)} \dot{\hat{x}}_2(t-\tau_i) &= \sum_{i=0}^{m} A_i^{(11)} x_1(t-\tau_i) \\
&\quad + \sum_{i=0}^{m} A_i^{(12)} \hat{x}_2(t-\tau_i) + B_1 u(t) \\
\dot{\hat{x}}_2(t) - \sum_{i=1}^{m} A_i \dot{\hat{x}}_2(t-\tau_i) &= -\hat{x}_2(t) + \sum_{i=1}^{m} A_i \hat{x}_2(t-\tau_i) \\
&\quad + \sum_{i=0}^{m} A_i^{(21)} x_1(t-\tau_i) + B u(t), \\
y(t) &= C_1 x_1(t).
\end{aligned}$$

Note that RM2 is equivalent to applying RM1 to the dual (transposed) system of (1), followed by taking the transposed system again, as done in [5,4]. Note also that if $B \neq 0$, then RM1 would lead to differentiation of the input, whereas if $C \neq 0$, the change of variable of RM2 would imply the appearance of derivatives terms in the output equation.

It was recently shown in [4, Theorem 2] that if the sufficient condition (9) for a finite strong $\mathscr{H}_2$ norm is satisfied, there always exists a change of variables enabling a combination of RM1 and RM2 to the equations and variables of the delay-difference part *individually*, resulting in a standard neutral delay differential equation of dimension $r+n$, the same dimension as the vector $[x_1^T \; x_2^T]^T$, without derivatives of input and output signals. The situation is however more complicated if (7) is satisfied but (9) not.

Example 8 *We continue on Example 6. Both RM1 and RM2 are useful for the first and last rows, but they are not applicable in the second and the third one. At the same time, the proposed change of variables in [4, Lemma 2] is not applicable since it requires (9) to be satisfied.*

1.3 Contribution and outline

Two open problems naturally arise from the above discussion on the state-of-the-art.

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Problem 9 *Can we more efficiently verify the necessary and sufficient condition (7), in particular by lowering the upper bound on $k_1+\ldots+k_m$ in (8)?*

Problem 10 *Is there a procedure to transform (1) into a neutral-type system without derivatives in the input and the output equation, that is also applicable if* (7) *is satisfied but* (9) *is not?*

In this paper, we provide affirmative answers to these questions. In Section 2, we present a solution to Problem 9. We also discuss the computational aspects of the new condition, and show that the improved bound is tight. In Section 3, we solve the second problem by introducing an augmented system of DDAEs, amenable for RM1 and RM2. This solution enables the computation of the strong $\mathscr{H}_2$ norm via a standard Lyapunov matrix based formula [9, Theorem 1], which we will also discuss and illustrate. Some concluding remarks end the paper.

2 Improving the bound of Condition (8)

We provide a solution to Problem 9 in Subsection 2.1, and explore some consequences of the derived results in Subsection 2.2.

For ease of use, we introduce the following notation: $\mathbf{k}=(k_1,\ldots,k_m)\in\mathbb{N}^m$ is a choice of integer coefficients, whose *weight* is denoted by $\|k\|_1=\sum_{k=1}^m k_i$. We note $\mathrm{P}_{\mathbf{k}}(\mathscr{A})$ for $\mathrm{P}_{k_1,\ldots,k_m}(\mathrm{A}_1,\ldots,\mathrm{A}_m)$, with $\mathrm{A}_i\in\mathbb{R}^{n\times n}$, and for variables $x_1,\ldots,x_m$ we denote by $x^{\mathbf{k}}$ the product $x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}$.

2.1 Solution to Problem 9

In this subsection, we will use the following lemma, which is a well known consequence of the Cayley-Hamilton theorem and a special case of Theorem 4 for $m=1$.

Lemma 11 *Consider a matrix $\mathrm{A}\in\mathbb{R}^{n\times n}$ and $h\in\mathbb{N}$. Then:*

$$\mathrm{CA}^h\mathrm{B}\neq 0 \implies \exists l<n:\mathrm{CA}^l\mathrm{B}\neq 0. \tag{12}$$

Proof. By the Cayley-Hamilton theorem, A^n can be rewritten as a linear combination of its lower powers. That is, for a general $h\in\mathbb{N}$, there exist some $\alpha_i\in\mathbb{R}$ such that

$$\mathrm{A}^h=\sum_{i=0}^{n-1}\alpha_i\mathrm{A}^i. \tag{13}$$

Therefore if $\mathrm{CA}^h\mathrm{B}=\sum_{i=0}^{n-1}\alpha_i\mathrm{CA}^i\mathrm{B}$ is nonzero, then one element of the right side sum must be nonzero, concluding the proof. □

In the next lemma, we show that the quantities P_k, which are the object of condition (7), have an interpretation in terms of the coefficients of some matrix polynomial. This is the key property underlying the proof of Theorem 13.

Lemma 12 *Let $\mathscr{A}=\{\mathrm{A}_1,\ldots,\mathrm{A}_m\}$ be a set of matrices, with $\mathrm{A}_i\in\mathbb{R}^{n\times n}$. Then, for any $h\in\mathbb{N}$, it holds that:*

$$(x_1\mathrm{A}_1+\cdots+x_m\mathrm{A}_m)^h=\sum_{\|k\|_1=h}x^{\mathbf{k}}\mathrm{P}_{\mathbf{k}}(\mathscr{A}) \tag{14}$$

where $\mathrm{P}_{\mathbf{k}}$ are defined in (6). *This polynomial will be denoted $\mathscr{P}_h(x_1,\ldots,x_m)$.*

Proof. The proof is obtained by induction. First, the property is verified for $h=0$.
Suppose now that the property holds true for all values smaller than h, let us prove it for $h+1$:

$$\begin{aligned}(x_1\mathrm{A}_1&+\cdots+x_m\mathrm{A}_m)^{h+1}=\\&=(x_1\mathrm{A}_1+\cdots+x_m\mathrm{A}_m)\sum_{\|\mathbf{k}\|_1=h}x^{\mathbf{k}}\mathrm{P}_{\mathbf{k}}(\mathscr{A})\\&=\sum_{\|\mathbf{k}\|_1=h}x_1x^{\mathbf{k}}\mathrm{A}_1\mathrm{P}_{\mathbf{k}}(\mathscr{A})+\cdots+x_mx^{\mathbf{k}}\mathrm{A}_m\mathrm{P}_k(\mathscr{A}).\end{aligned} \tag{15}$$

We can change the order of summation to obtain

$$\begin{aligned}(x_1\mathrm{A}_1&+\cdots+x_m\mathrm{A}_m)^{h+1}=\\&=\sum_{\|\mathbf{j}\|_1=h+1}x^{\mathbf{j}}(\mathrm{A}_1\mathrm{P}_{\mathbf{j}-1_1}(\mathscr{A})+\cdots+\mathrm{A}_m\mathrm{P}_{\mathbf{j}-1_m}(\mathscr{A}))\\&=\sum_{\|\mathbf{j}\|_1=h+1}x^{\mathbf{j}}\mathrm{P}_{\mathbf{j}}(\mathscr{A})\end{aligned} \tag{16}$$

where $\mathbf{k}-1_i=(k_1,\ldots,k_i-1,\ldots,k_m)$ and we used (6) for the last line. □

The above lemma allows us to prove our first main result, which shows that condition (7) can be checked more efficiently than relying on (8). The proof will make use of the well known fact that a polynomial is identically zero if and only if its coefficients are all equal to zero.

Theorem 13 *Let $\mathscr{A}=\{\mathrm{A}_1,\ldots,\mathrm{A}_m\}$ be a set of matrices, with $\mathrm{A}_i\in\mathbb{R}^{n\times n}$. Let $\mathrm{B}\in\mathbb{R}^{n\times n_2}$ and $\mathrm{C}\in\mathbb{R}^{n_3\times n}$. The following conditions are equivalent*

(1) $\mathrm{CP}_{\mathbf{k}}(\mathscr{A})\mathrm{B}=0$ for all $\mathbf{k}\in\mathbb{N}^m$;
(2) The polynomials $\mathscr{P}_h(x_1,\ldots,x_m)=\mathrm{C}(x_1\mathrm{A}_1+\cdots+x_m\mathrm{A}_m)^h\mathrm{B}$ are zero for all values of $h\in\mathbb{N}$;
(3) The polynomials $\mathscr{P}_h(x_1,\ldots,x_m)=\mathrm{C}(x_1\mathrm{A}_1+\cdots+x_m\mathrm{A}_m)^h\mathrm{B}$ are zero for all values of $h<n$;
(4) $\mathrm{CP}_{\mathbf{k}}(\mathscr{A})\mathrm{B}=0$ for all k with $\|\mathbf{k}\|_1=\sum_{i=1}^m k_i<n$.

Proof. Scalar case Let us first consider the case of single-input single-output systems, for which B and C are vectors, meaning that $\mathrm{CP}_{\mathbf{k}}(\mathscr{A})\mathrm{B}$ are real numbers, and the polynomials of conditions 2) and 3) are real valued.

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Received March 10, 2022 06:36:54 Pacific Time

$1 \Leftrightarrow 2$ and $3 \Leftrightarrow 4$: Using the fact that a polynomial is identically zero if and only if its coefficients are all equal to zero, one has

$$\begin{aligned} \mathrm{C}(x_1\mathrm{A}_1 + \cdots + x_m\mathrm{A}_m)^h\mathrm{B} &= \mathrm{C}\left(\sum_{\|k\|_1=h} x^{\mathbf{k}}\mathrm{P}_{\mathbf{k}}\right)\mathrm{B} \\ &= \sum_{\|\mathbf{k}\|_1=h} x^{\mathbf{k}}\mathrm{C}\mathrm{P}_{\mathbf{k}}\mathrm{B}. \end{aligned} \tag{17}$$

$2 \implies 3$: Trivial since the latter is a particular case of the former.

$3 \implies 2$: The proof is obtained by contraposition. Suppose there exists a value h such that the polynomial $\mathrm{C}(\mathrm{A}_1x_1 + \cdots + \mathrm{A}_mx_m)^h\mathrm{B}$ is nonzero. Then, there exist $\mathrm{X}_1, \ldots, \mathrm{X}_n \in \mathbb{R}$ such that $\mathrm{C}(\mathrm{A}_1\mathrm{X}_1 + \cdots + \mathrm{A}_m\mathrm{X}_m)^h\mathrm{B} \neq 0$.
Let us consider the matrix $(\mathrm{X}_1\mathrm{A}_1 + \cdots + \mathrm{X}_m\mathrm{A}_m)$, from Lemma 11 we can conclude that there must be $h' < n$ such that $\mathrm{C}(\mathrm{X}_1\mathrm{A}_1 + \cdots + \mathrm{X}_m\mathrm{A}_m)^{h'}\mathrm{B} \neq 0$. As a consequence, the polynomial $\mathrm{C}(x_1\mathrm{A}_1 + \cdots + x_m\mathrm{A}_m)^{h'}\mathrm{B}$ is nonzero, concluding the proof for the scalar case.

General case Replacing B, C with matrices of arbitrary size, the argument in each proof remains unchanged, noting the following: $\mathrm{C}\mathrm{P}_{\mathbf{k}}(\mathscr{A})\mathrm{B}$ is a matrix with entries $(\mathrm{C}\mathrm{P}_{\mathbf{k}}(\mathscr{A})\mathrm{B})_{i,j} = \mathrm{C}_i\mathrm{P}_{\mathbf{k}}(\mathscr{A})\mathrm{B}_j$ with C_i the i-th row of C and B_j the j-th column of B. Therefore the matrix $\mathrm{C}\mathrm{P}_{\mathbf{k}}(\mathscr{A})\mathrm{B}$ is nonzero if one of its entries is nonzero, which happens if and only if the same entry is nonzero for a $\mathbf{k}$ of weight smaller than n, as demonstrated above. □

Hereinafter, a collection of matrices $\mathscr{A}$, B, C that respect properties 1-4 of Theorem 13 will be called an *$\mathscr{H}_2$-finite system.*

2.2 Corollaries

It is clear that Theorem 13 improves the upper bound in (8), previously proposed in [4,3], as it becomes independent of the number m of matrices: checking condition (7) can be restricted to considering the finite number of choices of $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ such that $\|\mathbf{k}\|_1 < n$. We now state a formula for the computational effort needed to check the new condition.

Corollary 14 *Condition (7) can be verified by checking the equalities for no more than $\frac{(n+m-1)!}{(n-1)!m!}$ different values of k.*

Moreover, since $\mathbf{k}$ has at most $\|\mathbf{k}\|_1$ nonzero entries, we obtain the following result:

Corollary 15 *Let $\mathscr{A} \subset \mathbb{R}^{n\times n}$, $\mathrm{B} \in \mathbb{R}^{n\times n_2}$, $\mathrm{C} \in \mathbb{R}^{n_3\times n}$. Then $\mathscr{A}$, B, C form a $\mathscr{H}_2$-finite system if and only if any subset $\{\mathrm{A}_1, \ldots, \mathrm{A}_m\} \subset \mathscr{A}$, B, C, with $m < n$ also forms a $\mathscr{H}_2$-finite system.*

Proof. Let us note $\mathbf{k}(\mathrm{A}_i)$ for the component $k_i \in \mathbb{N}$ associated to the matrix A_i when computing $\mathrm{P}_{\mathbf{k}}(\mathscr{A})$. It may help the reader to think of $\mathbf{k}$ as a function from $\mathscr{A}$ to $\mathbb{N}$. With this in mind, the proof consists of restricting or extending its domain.
$\Leftarrow$ Suppose the system is not $\mathscr{H}_2$-finite. That means that there is a $\mathbf{k}$ of weight $\|\mathbf{k}\|_1 < n$ for which property 4) of Theorem 13 does not hold. Restricting ourselves to the matrices used in $\mathrm{P}_{\mathbf{k}}$ creates a system $\mathscr{A}_m = \{\mathrm{A}_i \mid \mathbf{k}(\mathrm{A}_i) \neq 0\}$, B, C that is also not $\mathscr{H}_2$-finite. One can check that $|\mathscr{A}_m| \leq \|k\|_1 < n$.
$\Rightarrow$ Suppose we have a subsystem $\mathscr{A}_m = \{\mathrm{A}_1, \ldots, \mathrm{A}_m\}$, B, C that is not $\mathscr{H}_2$-finite. That means there is some $\mathbf{k} \in \mathbb{N}^m$ such that $\mathrm{C}\mathrm{P}_{\mathbf{k}}(\mathscr{A}_m)\mathrm{B} \neq 0$. We can create a suitable vector $\mathbf{k}' \in \mathbb{N}^{|\mathscr{A}|}$ by putting

$$\mathbf{k}'(\mathrm{A}_i) = \begin{cases} \mathbf{k}(\mathrm{A}_i), & \text{if } \mathrm{A}_i \in \mathscr{A}_m \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

It is easy to check that $0 \neq \mathrm{C}\mathrm{P}_{\mathbf{k}}(\mathscr{A}_m)\mathrm{B} = \mathrm{C}\mathrm{P}_{\mathbf{k}'}(\mathscr{A})\mathrm{B}$. □

Finally, we show that the novel bound $\|k\|_1 < n$ introduced by Theorem 13 is tight and cannot be improved.

Corollary 16 *The bound of Theorem 13 is tight. More precisely, for every $n \in \mathbb{N}$, there exists a system of matrices $\mathscr{A}$, B, C that satisfies the equality in (7) for all values of $\mathbf{k}$ with $\|\mathbf{k}\|_1 < n - 1$ and which cannot be $\mathscr{H}_2$-finite.*

Proof. We will construct a system that satisfies condition 4) of Theorem 13 for all $\mathbf{k}$ with $\|\mathbf{k}\|_1 < n - 1$, and which, however, is not $\mathscr{H}_2$-finite. Pick an orthogonal basis of $\mathbb{R}^n$, let us denote it $\{e_1, \ldots, e_n\}$. For $i < j$, define $\mathrm{A}_{i\to j}$ as the operator mapping $e_i \mapsto e_{i+1}$ (i.e., $e_{i+1} = \mathrm{A}_{i\to j}e_i$), $e_{i+1} \mapsto e_{i+2}, \ldots, e_{j-1} \mapsto e_j, e_j \mapsto e_i$, and acting as the identity on the rest of the basis (see Example 17 below). We fix some $i \in 1, \ldots n$ and consider the system given by $\mathrm{B} = e_1, \mathrm{C} = e_n^{\mathrm{T}}, \mathscr{A} = \{\mathrm{A}_{1\to i}, \mathrm{A}_{i\to n}\}$. Observe that $\mathrm{C}\mathrm{P}_{\mathbf{k}}(\mathscr{A})\mathrm{B} = 0$ for all k of weight $\|\mathbf{k}\|_1 < n - 1$ but that $\mathrm{C}\mathrm{P}_{\mathbf{k}}(\mathscr{A})\mathrm{B} = 1$ for $\mathbf{k} = (i-1, n-i)$. □

Note that the construction used in the proof of Corollary 16 not only reaches the bound, but it also works for any partition of the interval $[1, n] \subset \mathbb{N}$ into $m < n$ interlocking sub-intervals $[1, i_1], [i_1, i_2], \ldots, [i_{m-1}, i_m]$.

The next example illustrates the previous result and its proof.

Example 17 *Let $n = 5$ and consider the standard basis of $\mathbb{R}^5$, i.e., $\{e_1, \ldots, e_5\}$ where e_i is the vector whose i-th entry is one, and zero otherwise. Now consider $\mathrm{B} = e_1 = [1\ 0\ 0\ 0\ 0]^{\mathrm{T}}$ and $\mathrm{C} = e_5^{\mathrm{T}} = [0\ 0\ 0\ 0\ 1]$. For $i = 4$, we have*

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Received March 10, 2022 06:36:54 Pacific Time

$\mathscr{A} = \{A_{1\to4}, A_{4\to5}\}$ *with:*

$$A_{1\to4} = \begin{bmatrix} 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{bmatrix},\ A_{4\to5} = \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{bmatrix}. \quad (19)$$

One can easily check that $CP_{3,1}B = 1$ *and* $CP_{\mathbf{k}}B = 0$ *for all values of* $\mathbf{k}$ *with* $\|\mathbf{k}\|_1 < 4$.

Remark 18 *Finally, let us note that Lemma 12 reduces checking Condition* (7) *to the* Polynomial Identity Testing *problem (PIT). Particular cases such as integer-valued matrices could potentially be evaluated faster by computing directly* $C(A_1x_1 + \cdots + A_mx_m)^hB$ *for a suitably chosen value of* x. *This view also allows for a probabilistic approach, especially in light of the Schwartz-Zippel Lemma [17]. For a survey of the tools already developed to treat the PIT, see [16].*

3 Transformation of DDAEs into neutral-type system

In this section, we provide an affirmative answer to the question posed in Problem 10. The result is stated in Theorem 19, whose proof relies on the following core ideas. First, the system (1) is augmented with additional dynamics described by a new variable x_3, such that these additional dynamics are stable, unobservable and such that the output can be explicitly expressed as a function of x_1 and x_3. Second, the absence of x_2 in the alternative output equation enables both RM1 and RM2.

Theorem 19 *Let system* (1) *with transfer matrix* $G(s)$ *and spectrum* Λ *be strongly stable and condition* (7) *be satisfied. There exists a regular neutral-type system* (10) *with transfer matrix* $\mathscr{G}(s)$ *and spectrum* Λ_n *such that*

$$G(s) = \mathscr{G}(s),\ s \in \mathbb{C} \setminus \Lambda_n. \quad (20)$$

Moreover, the spectrum of this neutral system satisfies

$$\Lambda_n = \Lambda \cup \Lambda_d \cup \{-1\}, \quad (21)$$

where $\Lambda_d := \{s \in \mathbb{C} : \det(A_{22}(s)) = 0\}$.

Proof. We directly give the system. It is of dimension $2n+r$, and $x(t) \in \mathbb{R}^{2n+r}$ is an extended state vector composed by x_1 and x_2 defined as in (1), and a new state $x_3 \in \mathbb{R}^n$. The matrices are $\mathscr{B}^T := \begin{bmatrix} B_1^T & B^T & 0 \end{bmatrix}$, $\mathscr{C} := \begin{bmatrix} C_1 & 0 & C \end{bmatrix}$,

$$\mathscr{D}_0 = \begin{bmatrix} I & -A_0^{(12)} & 0\\ 0 & I & 0\\ -A_0^{(21)} & 0 & I \end{bmatrix},\ \mathscr{D}_i = \begin{bmatrix} 0 & A_i^{(12)} & 0\\ 0 & A_i & 0\\ A_i^{(21)} & 0 & A_i \end{bmatrix} \quad (22)$$

and

$$\mathscr{F}_0 = \begin{bmatrix} A_0^{(11)} & A_0^{(12)} & 0\\ A_0^{(21)} & -I & 0\\ A_0^{(21)} & 0 & -I \end{bmatrix},\ \mathscr{F}_i = \begin{bmatrix} A_i^{(11)} & A_i^{(12)} & 0\\ A_i^{(21)} & A_i & 0\\ A_i^{(21)} & 0 & A_i \end{bmatrix}, \quad (23)$$

for $i = 1, \ldots, m$. Now, by direct calculations, we obtain

$$\mathscr{H}(s) = \begin{bmatrix} A_{11}(s) & (s+1)A_{12}(s) & 0\\ A_{21}(s) & (s+1)A_{22}(s) & 0\\ (s+1)A_{21} & 0 & (s+1)A_{22}(s) \end{bmatrix},$$

and we can compute

$$\det(\mathscr{H}(s)) = (s+1)^{2n}\det(A_{22}(s))\det(H(s)),$$

which proves (21).

Augmenting (1) Notice that $\Lambda_n = \Lambda \cup \Lambda_d \cup \{-1\}$ implies that $\det(A_{22}(s)) \neq 0$ and $\det(H(s)) \neq 0$ for all $s \in \mathbb{C} \setminus \Lambda_n$, which enables us to apply the formula for inversion of a block matrix to obtain

$$H^{-1}(s) = \begin{bmatrix} F^{-1} & -F^{-1}A_{12}A_{22}^{-1}\\ -A_{22}^{-1}A_{21}F^{-1} & A_{22}^{-1}\left(A_{21}F^{-1}A_{12}A_{22}^{-1} + I\right) \end{bmatrix},$$

where $F(s) := A_{11} + A_{12}A_{22}^{-1}A_{21}$ and, for simplicity of presentation, we omitted the arguments in the matrices $A_{ij}(s)$, $i, j \in \{1,2\}$. Then, it follows from the above equality and

$$\begin{bmatrix} X_1(s)\\ X_2(s) \end{bmatrix} = H^{-1}(s)\begin{bmatrix} B_1\\ B \end{bmatrix}U(s),$$

that

$$\begin{aligned} X_1(s) &= (F^{-1}B_1 - F^{-1}A_{12}A_{22}^{-1}B)U(s) = F^{-1}\left(B_1 - A_{12}A_{22}^{-1}B\right)U(s)\\ X_2(s) &= -A_{22}^{-1}A_{21}F^{-1}B_1U(s) + A_{22}^{-1}\left(A_{21}F^{-1}A_{12}A_{22}^{-1} + I\right)BU(s)\\ &= A_{22}^{-1}A_{21}F^{-1}\left(-B_1 + A_{12}A_{22}^{-1}B\right)U(s) + A_{22}^{-1}BU(s)\\ &= -A_{22}^{-1}A_{21}X_1(s) + A_{22}^{-1}BU(s). \end{aligned}$$

Now, by [3, Proposition 1], if (7) is satisfied then $CA_{22}^{-1}(s)B \equiv 0$ and we have

$$\begin{aligned} Y(s) &= C_1X_1(s) + CX_2(s)\\ &= C_1X_1(s) - CA_{22}^{-1}A_{21}X_1(s) + CA_{22}^{-1}BU(s) \quad (24)\\ &= C_1X_1(s) - CA_{22}^{-1}A_{21}X_1(s). \end{aligned}$$

By defining the variable

$$X_3(s) := -A_{22}^{-1}A_{21}X_1(s),$$

we have that

$$Y(s) = C_1X_1(s) + CX_3(s) \quad (25)$$

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Received March 10, 2022 06:36:54 Pacific Time

and

$$\mathrm{H}_0(s)\begin{bmatrix}\mathrm{X}_1(s)\\ \mathrm{X}_2(s)\\ \mathrm{X}_3(s)\end{bmatrix} = \begin{bmatrix}\mathrm{B}_1\\ \mathrm{B}\\ 0\end{bmatrix}\mathrm{U}(s), \tag{26}$$

where

$$\mathrm{H}_0(s) := \left[\begin{array}{c|c} \mathrm{H}(s) & \begin{matrix}0\\0\end{matrix} \\ \hline \mathrm{A}_{21}(s)\ \ 0 & \mathrm{A}_{22}(s)\end{array}\right].$$

Thus,

$$\mathrm{G}(s) = \begin{bmatrix}\mathrm{C}_1 & 0 & \mathrm{C}\end{bmatrix}\mathrm{H}_0^{-1}(s)\begin{bmatrix}\mathrm{B}_1\\ \mathrm{B}\\ 0\end{bmatrix}, \; s \in \mathbb{C}\setminus\Lambda_n.$$

<u>*Concluding the proof*</u> Now, direct calculations allow to see that

$$\mathrm{T}_1(s)\mathrm{H}_0(s)\mathrm{T}_2(s) = \mathscr{H}(s), \tag{27}$$

where

$$\mathrm{T}_1(s) := \begin{bmatrix}\mathrm{I} & 0 & 0\\ 0 & \mathrm{I} & 0\\ 0 & 0 & (s+1)\mathrm{I}\end{bmatrix}, \; \mathrm{T}_2(s) := \begin{bmatrix}\mathrm{I} & 0 & 0\\ 0 & (s+1)\mathrm{I} & 0\\ 0 & 0 & \mathrm{I}\end{bmatrix}.$$

Since $\mathscr{B} = \mathrm{T}_1(s)\begin{bmatrix}\mathrm{B}_1^{\mathrm{T}} & \mathrm{B}^{\mathrm{T}} & 0\end{bmatrix}$ and $\mathscr{C} = \begin{bmatrix}\mathrm{C}_1 & 0 & \mathrm{C}\end{bmatrix}\mathrm{T}_2(s)$, we get

$$\mathrm{G}(s) = \mathscr{C}\mathrm{H}_0^{-1}(s)\mathscr{B} = \mathscr{C}\mathrm{T}_2^{-1}\mathrm{H}_0^{-1}(s)\mathrm{T}_1^{-1}\mathscr{B} = \mathscr{C}\mathscr{H}^{-1}(s)\mathscr{B},$$

$s \in \mathbb{C}\setminus\Lambda_n$, proving (20). □

For the sake of clarity, let us show how one may obtain neutral-type system (10) with (22),(23) from system (1).

Equations (25),(26) corresponds to an augmented DDAE in variables x_1, x_2 and x_3 that can be realized as (1) with the additional delay-difference equation

$$x_3(t) - \sum_{i=1}^{m}\mathrm{A}_i x_3(t-\tau_i) - \mathrm{A}_0^{(21)}x_1(t) - \sum_{i=1}^{m}\mathrm{A}_i^{(21)}x_1(t-\tau_i) = 0 \tag{28}$$

and output

$$y(t) = \mathrm{C}_1x_1(t) + \mathrm{C}x_3(t).$$

From an input-output perspective, this augmentation is highly redundant. However, introducing the alternative equation for the output $y(t)$ and the additional equation in the new variable $x_3(t)$ makes the application of RM1 and RM2 possible.

<u>*Transformation into neutral-type*</u>

a) Application of RM1

Since the equation in $x_3(t)$ does not depend on the input, we can apply the operator $\left(\frac{d}{dt}+\mathrm{I}\right)$, which results in

$$\begin{aligned}\dot{x}_3(t) - \sum_{i=1}^{m}\mathrm{A}_i\dot{x}_3(t-\tau_i) - \sum_{i=0}^{m}\mathrm{A}_i^{(21)}\dot{x}_1(t-\tau_i) = \\ -x_3(t) + \sum_{i=1}^{m}\mathrm{A}_i x_3(t-\tau_i) + \sum_{i=0}^{m}\mathrm{A}_i^{(21)}x_1(t-\tau_i).\end{aligned} \tag{29}$$

b) Application of RM2

Change of variable $x_2(t) = \left(\frac{d}{dt}+\mathrm{I}\right)\hat{x}_2(t)$ in (1) leads to

$$\begin{aligned}&\dot{x}_1(t) - \sum_{i=0}^{m}\mathrm{A}_i^{(12)}\dot{\hat{x}}_2(t-\tau_i) = \sum_{i=0}^{m}\mathrm{A}_i^{(11)}x_1(t-\tau_i)\\ &\quad+\sum_{i=0}^{m}\mathrm{A}_i^{(12)}\hat{x}_2(t-\tau_i) + \mathrm{B}_1u(t)\\ &\dot{\hat{x}}_2(t) - \sum_{i=1}^{m}\mathrm{A}_i\dot{\hat{x}}_2(t-\tau_i) = -\hat{x}_2(t) + \sum_{i=1}^{m}\mathrm{A}_i\hat{x}_2(t-\tau_i)\\ &\quad+\sum_{i=0}^{m}\mathrm{A}_i^{(21)}x_1(t-\tau_i) + \mathrm{B}u(t),\\ &y(t) = \mathrm{C}_1x_1(t) + \mathrm{C}x_3(t).\end{aligned} \tag{30}$$

System (29), (30) can be written as the neutral-type system (10) with matrices (22), (23), whose regularity is implied by the fact that $\det(\mathscr{D}_0) \neq 0$.

Remark 20 *Note that, in general, the solution of a DDAE as in* (1) *does not need to be differentiable, so the solution may not be preserved from system* (1) *to system* (10) *through the regularization mechanism RM1 and RM2. In fact, the set of allowable initial functions may not even be the same. Nevertheless, this is irrelevant in our analysis, since the $\mathscr{H}_2$ norm defined by* (3) *does not depend on any particular solution of* (1)*, but on its transfer matrix.*

Some comments on Theorem 19 are in order. The neutral-type system (10) has dimension $2n+r$, which is higher than system (1). Hence, if condition (9) is satisfied, the approach proposed in [4] is still advisable. It is also important to note that the regularization mechanism used within the proof of the theorem preserves the stability properties of system (1), as formalized in the following result.

Corollary 21 *Let system* (1) *be strongly stable and let condition* (7) *hold. Then the neutral system defined by* (22),(23) *is strongly exponentially stable.*

Preprint submitted to Automatica
Received March 10, 2022 06:36:54 Pacific Time

Proof. In order to prove that the neutral system defined by (22),(23) is strongly exponentially stable, one needs to show two facts: i) that the associated difference equation is strongly stable, and ii) that the spectrum Λ_n lies in the left half complex plane. As a preliminary step, let us examine the difference equation associated to the DDAE (1), given by:

$$x_2(t) = \sum_{i=1}^{n} A_i x_2(t - \tau_i). \tag{31}$$

One can now compute

$$\det\left(\mathscr{D}_0 - \sum_{i=1}^{m} \mathscr{D}_i e^{-s\tau_i}\right) = (\det(A_{22}(s)))^2,$$

with matrices $\mathscr{D}_i$, $i = 0, 1, \ldots, m$, given by (22), that is, the spectrum of the delay-difference equation of (1) coincides with the spectrum of the delay-difference equation of neutral-type system (22),(23). Moreover, the associated delay-difference equation of a strongly stable DDAE (1) is exponentially stable and satisfies [12, Theorem 1]

$$\max_{\vec{\theta}\in[0,2\pi]^m} \rho\left(\sum_{i=1}^{m} A_i e^{j\theta_i}\right) < 1. \tag{32}$$

This implies (11), so that the first part is done. Second, by (21) we have $\Lambda_n = \Lambda \cup \Lambda_d \cup \{-1\}$. Since the DDAE is strongly stable we know that Λ is strictly contained in the left half of the complex plane. The same argument holds for (31), giving us that Λ_d is in the left half of the complex plane as well. This concludes the proof. □

Theorem 19 allows us to recast the transfer matrix of a DDAE into an equivalent transfer matrix of a neutral-type system and use all the tools developed for regular neutral systems. As mentioned in the introductory part, this is relevant since the computation problem of the $\mathscr{H}_2$ norm of (10) has already been addressed in [9], where a delay Lyapunov matrix based formula is provided (see [11] for a study of Lyapunov matrices for time-delay systems). We summarize this result in the following corollary.

Corollary 22 *Assume that system* (1) *is strongly stable and that condition* (7) *is satisfied. Then,*

$$|||G|||_{\mathscr{H}_2} = \|G\|_{\mathscr{H}_2} = \|\mathscr{G}\|_{\mathscr{H}_2} = \sqrt{tr\left(\mathscr{B}^T U(0)\mathscr{B}\right)},$$

where $U : [-\tau_m, \tau_m] \to \mathbb{R}^{2n+r\times 2n+r}$ *is the delay Lyapunov matrix of system* (10) *associated with* $\mathscr{C}^T\mathscr{C}$.

Proof. Finiteness of the strong $\mathscr{H}_2$ norm and that $|||G|||_{\mathscr{H}_2} = \|G\|_{\mathscr{H}_2} = \|\mathscr{G}\|_{\mathscr{H}_2}$ follow from Theorem 4 and Theorem 19, whereas the equality $\|\mathscr{G}\|_{\mathscr{H}_2} = \sqrt{tr\left(\mathscr{B}^T U(0)\mathscr{B}\right)}$ follows from [9, Theorem 1]. □

With the above corollary at hand, we now possess a formula to compute the finite strong $\mathscr{H}_2$ norm of any system of the form (1), in contrast with [4, Corollary 1], where sufficient condition (9) was required to be satisfied. We close the section by illustrating the computation of $|||G|||_{\mathscr{H}_2}$ for a system constructed from Example 6. The formula provided in [4] cannot be used in this case since matrices (A_1, A_2, B, C) do not fulfill condition (9), as already explained in the introduction.

Example 23 *We consider a strongly stable system* (1) *with matrices* $A_0^{(11)} = -2$,

$$A_1^{(12)} = \begin{bmatrix} -1 & -2 & 1 & 0 \end{bmatrix}, \quad A_1^{(21)} = \begin{bmatrix} 0 & 0 & 0 & 0.75 \end{bmatrix}^T,$$

$$A_2^{(21)} = \begin{bmatrix} -0.5 & -0.25 & 0 & -0.75 \end{bmatrix}^T$$

$A_0^{(12)} = A_2^{(12)} = 0$, $A_0^{(21)} = 0$, $A_1^{(11)} = A_2^{(11)} = 0$, $C_1 = B_1 = 1$ *and matrices* A_1, A_2, B *and* C, *corresponding to the delay-difference part, as in Example 6. The nominal delays are* $\tau = (\tau_1, \tau_2) = (1, 2)$. *By Theorem 4, the strong* $\mathscr{H}_2$ *norm of the system is finite. We construct the neutral-type system* (10) *and, from Corollary 22, compute* $|||G|||_{\mathscr{H}_2}$ *for the nominal delays and perturbed delays* $\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2) = (1, 2 + 0.005\pi)$, *which gives*

$$|||G(\cdot;\tau)|||_{\mathscr{H}_2} \approx 1.9654$$

and

$$|||G(\cdot;\hat{\tau})|||_{\mathscr{H}_2} \approx 1.9869$$

respectively. Note the two values are close since the strong $\mathscr{H}_2$ *norm changes continuously with respect to the delays.*

4 Concluding remarks

In this paper we have closed a gap between paper [3] and [4]. The former reference gives a necessary and sufficient, but hard to check condition for the finiteness of the strong $\mathscr{H}_2$ norm of a DDAE, while the latter gives a more conservative sufficient condition, which is easier to check and is at the basis of a transformation to a neutral equation. Theorem 13 allows to significantly lower the computational cost of checking the necessary and sufficient finiteness condition. In addition, Theorem 19 allows for a transformation of (1) into an augmented neutral-type system if the strong $\mathscr{H}_2$ norm is finite, enabling to compute the value of the strong $\mathscr{H}_2$ norm based on the delay Lyapunov matrix.

Future research work includes investigating whether the PIT problem reformulation (see Remark 18) allows for a faster algorithm for checking (7) and exploring other applications of the regularization procedure to a neutral system.

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