

When is a double central extension *universal*?

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joint work with George Peschke

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Introduction

- ▶ The concept of *universality* of central extensions extends from the context of (co)homology of groups to semi-abelian categories under mild additional assumptions (roughly, (SH) + (NH)).
- ▶ Universal central extensions occur when H_1 vanishes, so that in this case the interplay between H_2 (Hopf formula) and H^2 (central extensions) becomes particularly nice.
- ▶ For $n > 1$, an interpretation of H_{n+1} through higher Hopf formulae and H^{n+1} via n -fold central extensions has recently become available. Whence the question:

What does it mean for an n -fold central extension to be *universal*?

The aim of my talk is to explain

- 1 why the naive approach fails;
- 2 how to correct this, so that a general existence result is obtained;
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Universal central extensions of groups, I

An **extension** under A and over X is a short exact sequence

$$0 \longrightarrow A \rightrightarrows E \xrightarrow{f} X \longrightarrow 0.$$

It is **central** if and only if $[A, E] = 0$: all $aea^{-1}e^{-1}$ vanish, $a \in A$, $e \in E$.
Then, in particular, A is an abelian group.

Theorem

Considering A as a trivial X -module, we get $H^2(X, A) \cong \text{Centr}^1(X, A)$, the group of equivalence classes of central extensions under A and over X .

- By the Short Five Lemma, equivalence class = isomorphism class.

The theorem remains true [Gran & VdL, 2008] in any semi-abelian category [Janelidze, Márki & Tholen, 2002] with enough projectives; centrality is now defined via categorical Galois theory [Janelidze & Kelly, 1994].

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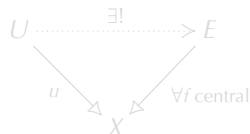
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Universal central extensions of groups, II

X is **perfect** when $[X, X] = X$, that is, $X/[X, X] = \text{ab}(X) = H_1(X) = 0$.

A central extension $u: U \rightarrow X$ is **universal** iff it is initial in $\text{CExt}_X(\text{Gp})$:



For perfect groups, computing H_2 is easy

- ▶ If $u: U \rightarrow X$ is universal then X and U are perfect, while $\text{Ker}(u) \cong H_2(X)$; conversely,
- ▶ any perfect X admits a universal central extension, unique up to iso.

Construction for X perfect

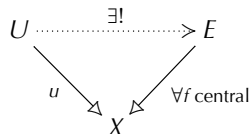
- 1 Take a projective presentation $R \rightarrow F \rightarrow X$;
- 2 centralise to a weakly universal central extension $\frac{R}{[R, F]} \rightarrow \frac{F}{[R, F]} \rightarrow X$;
- 3 take commutators: $\frac{R \wedge [F, F]}{[R, F]} \rightarrow \left[\frac{F}{[R, F]}, \frac{F}{[R, F]} \right] \rightarrow [X, X]$.

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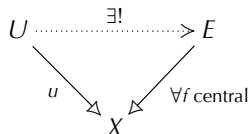
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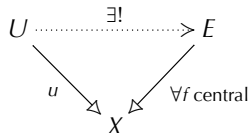
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Universal central extensions of groups, III

Recognition theorem

For any perfect group U , the following are equivalent:

- 1 every central extension $u: U \rightarrow X$ is universal;
- 2 all universal central extensions of U split;
- 3 U is *superperfect*: $H_1(U) = H_2(U) = 0$.

By [Casas & VdL, 2014] combined with [Gray & VdL, 2014], this remains true in semi-abelian categories with enough projectives which are *peri-abelian* [Bourn, 2010].

- ▶ Algebraic coherence $\Rightarrow$ (SH) + (NH) $\Rightarrow$ peri-abelianness by [Cigoli, Gray & VdL, 2015].
- ▶ Hence we find as examples: associative algebras; Lie, Leibniz, Poisson algebras; n -nilpotent or n -solvable groups, rings, algebras; torsion-free groups, reduced rings; crossed modules of such; n -cat-groups; compact Hausdorff models of such.

Universal *double* central extensions for *superperfect* objects?

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Higher central extensions and cohomology

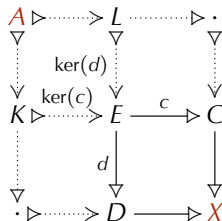
Theorem [Rodelo & VdL, 2011, 2012]

In a semi-abelian category $\mathbb{X}$ with enough projectives satisfying (SH), for any $X \in \mathbb{X}$, $A \in \text{Ab}(\mathbb{X})$ and $n \geq 1$,

$$H^{n+1}(X, A) \cong \text{Centr}^n(X, A)$$

where the group on the right consists of equivalence classes of *n-fold central extensions* in the sense of [Everaert, Gran & VdL, 2008] and [Janelidze, 1991] over X and under A .

- ▶ We focus on the case $n = 2$:
double central extensions
- ▶ Extension = regular pushout
 $\Leftrightarrow 3 \times 3$ diagram
Classes determined by zigzags
- ▶ $A = K \wedge L$
Central $\Leftrightarrow [K, L] = [A, E] = 0$



Naive approach: when is a double central extension *initial* over X ?

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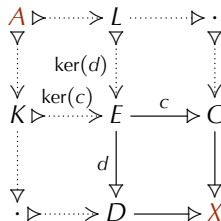
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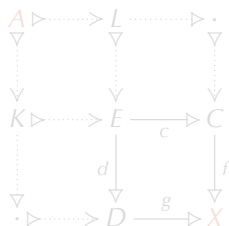
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If X admits an initial double central extension then $X = 0$

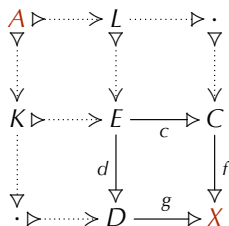
- Suppose the double central extension on the left is initial.



- Consider $B \in \text{Ab}(\mathbb{X})$, $X, Y \in \mathbb{X}$ and the 3×3 diagram on the right. The square is a double central extension, because $[B, Y \times B] = [B, X \times Y \times B] = 0$ since B is abelian.
- Hence there exists a unique arrow making the diagram commute. So for any chosen object $Y \in \mathbb{X}$, there is a unique $y: C \rightarrow Y$.
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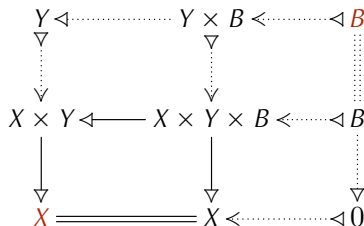
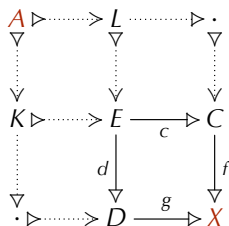
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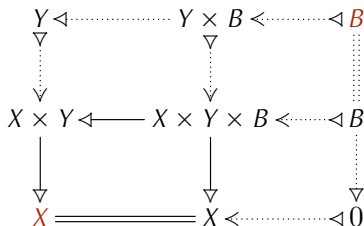
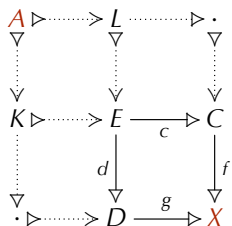
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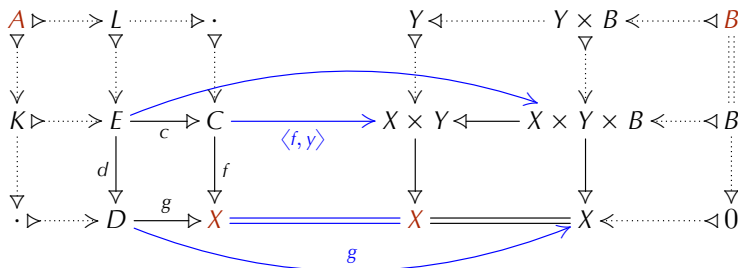
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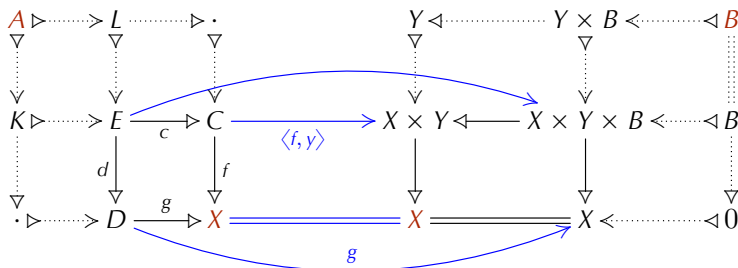
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Universal one-fold central extensions revisited

Proposition

For X perfect, a central extension $u \in \bar{u} \in \text{Centr}^1(X, H)$ is universal iff

$$\mu^{\bar{u}}: \text{Hom}(H, -) \rightarrow \text{Centr}^1(X, -): \text{Ab}(\mathbb{X}) \rightarrow \text{Ab}$$

$$(h: H \rightarrow A) \mapsto \text{Centr}^1(X, h)(\bar{u})$$

determines a natural isomorphism.

Proof. $\Leftarrow$ The extension u is weakly initial by surjectivity:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \rightrightarrows & U & \xrightarrow{u} & X & \longrightarrow & 0 \\ & & \downarrow \exists h & & \downarrow \exists h' & & \parallel & & \\ \forall f \quad \exists h: & 0 & \longrightarrow & A & \rightrightarrows & E & \xrightarrow{f} & X & \longrightarrow 0 \end{array}$$

The object U is perfect by injectivity:

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Universal one-fold central extensions revisited

Proposition

For X **perfect**, a central extension $u \in \bar{u} \in \text{Centr}^1(X, H)$ is universal iff

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In a semi-abelian variety satisfying (SH), for any X such that

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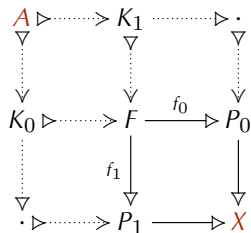
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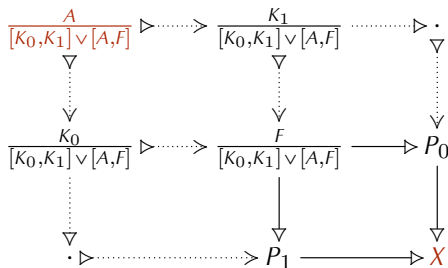
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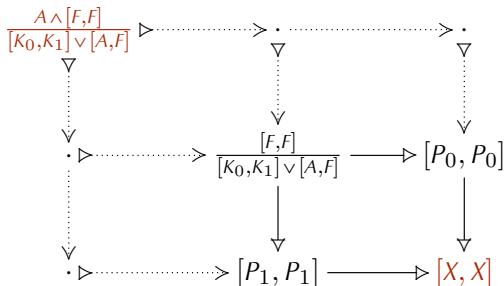
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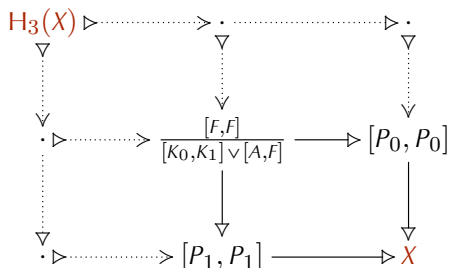
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 H_3(X) & \rhd & \cdots & \rhd & \cdots \\
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 \cdot & \rhd & \frac{[F,F]}{[K_0,K_1] \vee [A,F]} & \longrightarrow & [P_0, P_0] \\
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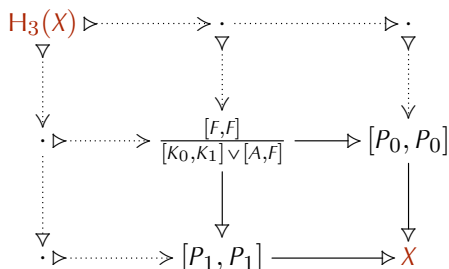
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- ▶ Note that the interpretation of cohomology in terms of higher central extensions, and the existence of universal higher central extensions, are new results even for groups and Lie algebras.

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Happy Birthday, Rudger!

