When is a double central extension *universal*?

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joint work with George Peschke

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Introduction

- The concept of *universality* of central extensions extends from the context of (co)homology of groups to semi-abelian categories under mild additional assumptions (roughly, (SH) + (NH)).
- Universal central extensions occur when H_1 vanishes, so that in this case the interplay between H_2 (Hopf formula) and H^2 (central extensions) becomes particularly nice.
- For $n > 1$, an interpretation of H_{n+1} through higher Hopf formulae and H^{n+1} via n-fold central extensions has recently become available. Whence the question:

What does it mean for an *n*-fold central extension to be *universal*?

The aim of my talk is to explain

1. why the naive approach fails;
2. how to correct this, so that a general existence result is obtained;
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Universal central extensions of groups, I

An **extension** under A and over X is a short exact sequence

$$0 \longrightarrow A \rightarrowtail E \overset{f}{\twoheadrightarrow} X \longrightarrow 0.$$

It is **central** if and only of $[A, E] = 0$: all $aea^{-1}e^{-1}$ vanish, $a \in A$, $e \in E$. Then, in particular, A is an abelian group.

Theorem

Considering A as a trivial X-module, we get $\mathrm{H}^2(X, A) \cong \mathrm{Centr}^1(X, A)$, the group of equivalence classes of central extensions under A and over X.

- By the Short Five Lemma, equivalence class = isomorphism class.

The theorem remains true [Gran & VdL, 2008] in any semi-abelian category [Janelidze, Márki & Tholen, 2002] with enough projectives; centrality is now defined via categorical Galois theory [Janelidze & Kelly, 1994].

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Universal central extensions of groups, II

X is **perfect** when $[X,X]=X$, that is, $X/[X,X]=\mathrm{ab}(X)=\mathrm{H}_1(X)=0$.

A central extension $u\colon U\to X$ is **universal** iff it is initial in $\mathsf{CExt}_X(\mathsf{Gp})$:

$$\begin{tikzcd}
U \arrow[rr, dotted, "\exists!"] \arrow[dr, "u"'] & & E \arrow[dl, "\forall f \text{ central}"] \\
& X &
\end{tikzcd}$$

For perfect groups, computing H_2 is easy

- If $u\colon U\to X$ is universal then X and U are perfect, while $\mathrm{Ker}(u)\cong \mathrm{H}_2(X)$; conversely,
- any perfect X admits a universal central extension, unique up to iso.

Construction for X perfect

1. Take a projective presentation $R\to F\to X$;
2. centralise to a weakly universal central extension $\frac{R}{[R,F]}\to\frac{F}{[R,F]}\to X$;
3. take commutators: $\frac{R\wedge[F,F]}{[R,F]}\to\left[\frac{F}{[R,F]},\frac{F}{[R,F]}\right]\to[X,X]$.

$[X,X]=X$ since X is perfect, and $\frac{R\wedge[F,F]}{[R,F]}\cong\mathrm{H}_2(X)$ by the Hopf formula.

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Universal central extensions of groups, III

Recognition theorem

For any perfect group U, the following are equivalent:

1. every central extension $u\colon U \to X$ is universal;
2. all universal central extensions of U split;
3. U is *superperfect*: $\mathrm{H}_1(U) = \mathrm{H}_2(U) = 0$.

By [Casas & VdL, 2014] combined with [Gray & VdL, 2014], this remains true in semi-abelian categories with enough projectives which are *peri-abelian* [Bourn, 2010].

- Algebraic coherence ⇒ (SH) + (NH) ⇒ peri-abelianness by [Cigoli, Gray & VdL, 2015].
- Hence we find as examples: associative algebras; Lie, Leibniz, Poisson algebras; n-nilpotent or n-solvable groups, rings, algebras; torsion-free groups, reduced rings; crossed modules of such; n-cat-groups; compact Hausdorff models of such.

Universal *double* central extensions for *super*perfect objects?

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Universal *double* central extensions for *super*perfect objects?

Higher central extensions and cohomology

Theorem [Rodelo & VdL, 2011, 2012]

In a semi-abelian category $\mathbb{X}$ with enough projectives satisfying (SH), for any $X \in \mathbb{X}$, $A \in \mathsf{Ab}(\mathbb{X})$ and $n \geqslant 1$,

$$\mathsf{H}^{n+1}(X, A) \cong \mathsf{Centr}^n(X, A)$$

where the group on the right consists of equivalence classes of *n-fold central extensions* in the sense of [Everaert, Gran & VdL, 2008] and [Janelidze, 1991] over X and under A.

- We focus on the case $n = 2$: double central extensions
- Extension = regular pushout $\Leftrightarrow 3 \times 3$ diagram
  Classes determined by zigzags
- $A = K \wedge L$
  Central $\Leftrightarrow [K, L] = [A, E] = 0$

$$\begin{tikzcd}
A \arrow[r, dotted] \arrow[d, dotted] & L \arrow[r, dotted] \arrow[d, dotted, "\ker(d)"'] & \cdot \arrow[d, dotted] \\
K \arrow[r, dotted, "\ker(c)"] \arrow[d, dotted] & E \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d] \\
\cdot \arrow[r, dotted] & D \arrow[r] & X
\end{tikzcd}$$

Naive approach: when is a double central extension *initial* over X?

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A \arrow[r, dotted, tail] \arrow[d, dotted, tail] & L \arrow[r, dotted, two heads] \arrow[d, dotted, tail, "\ker(d)"'] & \cdot \arrow[d, dotted, tail] \\
K \arrow[r, dotted, tail, "\ker(c)"] \arrow[d, dotted, two heads] & E \arrow[r, two heads, "c"] \arrow[d, two heads, "d"'] & C \arrow[d, two heads] \\
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Naive approach: when is a double central extension *initial* over X?

If X admits an initial double central extension then $X = 0$

- Suppose the double central extension on the left is initial.

$$
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\end{tikzcd}
$$

- Consider $B \in \mathrm{Ab}(\mathbb{X})$, $X, Y \in \mathbb{X}$ and the 3×3 diagram on the right. The square is a double central extension, because $[B, Y \times B] = [B, X \times Y \times B] = 0$ since B is abelian.
- Hence there exists a unique arrow making the diagram commute. So for any chosen object $Y \in \mathbb{X}$, there is a unique $y\colon C \to Y$.
- Hence C is initial in $\mathbb{X}$, which implies $X = 0$.

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\end{tikzcd}
\qquad
\begin{tikzcd}
Y & Y \times B \arrow[l, dotted] \arrow[d, dotted] & B \arrow[l, dotted] \arrow[d, dotted, equal] \\
X \times Y \arrow[from=1-1, dotted] \arrow[d] & X \times Y \times B \arrow[l] \arrow[d] & B \arrow[l, dotted] \arrow[d, dotted] \\
X \arrow[r, equal] & X & 0 \arrow[l, dotted]
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- Consider $B \in \mathrm{Ab}(\mathbb{X})$, X, $Y \in \mathbb{X}$ and the 3×3 diagram on the right.
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K \arrow[r, dotted] \arrow[d, dotted] & E \arrow[r, "c"'] \arrow[d, "d"'] \arrow[rrr, bend left] & C \arrow[r, "{\langle f, y \rangle}"'] \arrow[d, "f"] & X \times Y \arrow[d] & X \times Y \times B \arrow[l] \arrow[d] & B \arrow[l, dotted] \arrow[d, dotted] \\
\cdot \arrow[r, dotted] & D \arrow[r, "g"] \arrow[rrr, bend right, "g"'] & X \arrow[r, equal] & X \arrow[r, equal] & X & 0 \arrow[l, dotted]
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\begin{tikzcd}
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K \arrow[r, dotted] \arrow[d, dotted] & E \arrow[r, "c"'] \arrow[d, "d"'] \arrow[rrr, bend left, blue] & C \arrow[r, blue, "{\langle f, y\rangle}"'] \arrow[d, "f"] & X \times Y \arrow[d] & X \times Y \times B \arrow[l] \arrow[d] & B \arrow[l, dotted] \arrow[d, dotted] \\
\cdot \arrow[r, dotted] & D \arrow[r, "g"] \arrow[rrr, bend right, blue, "g"'] & {\color{red}X} \arrow[r, equal, blue] & {\color{red}X} \arrow[r, equal] & X & 0 \arrow[l, dotted]
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Universal one-fold central extensions revisited

Proposition

For X perfect, a central extension $u \in \overline{u} \in \mathrm{Centr}^1(X, H)$ is universal iff

$$\mu^{\overline{u}} \colon \mathrm{Hom}(H, -) \to \mathrm{Centr}^1(X, -) \colon \mathrm{Ab}(\mathbb{X}) \to \mathrm{Ab}$$
$$(h \colon H \to A) \mapsto \mathrm{Centr}^1(X, h)(\overline{u})$$

determines a natural isomorphism.

Proof. $\Leftarrow$ The extension u is weakly initial by surjectivity:

$$\forall f \quad \exists h \colon \qquad \begin{tikzcd} 0 \arrow[r] & H \arrow[r, tail] \arrow[d, dotted, "\exists h"'] & U \arrow[r, "u"] \arrow[d, dotted, "\exists h'"'] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, tail] & E \arrow[r, "f"'] & X \arrow[r] & 0 \end{tikzcd}$$

The object U is perfect by injectivity:

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Universal one-fold central extensions revisited

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Theorem [Peschke & VdL, 2014]

In a semi-abelian variety satisfying (SH), for any X such that

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$$\begin{tikzcd}
{\color{red}A} \arrow[r, tail, dotted] \arrow[d, tail, dotted] & K_1 \arrow[r, dotted, "\triangleright"{marking}] \arrow[d, tail, dotted] & \cdot \arrow[d, tail, dotted] \\
K_0 \arrow[r, tail, dotted] \arrow[d, dotted] & F \arrow[r, "f_0"] \arrow[d, "f_1"'] & P_0 \arrow[d] \\
\cdot \arrow[r, tail, dotted] & P_1 \arrow[r] & {\color{red}X}
\end{tikzcd}$$

1. **Take a two-fold presentation of X: by covering with projectives, or as a truncation of a simplicial resolution;**
2. centralise: divide $[K_0, K_1] \vee [A, F]$ out of F;
3. take commutators $[-, -]$ of the induced square;
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$$
\begin{tikzcd}
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\frac{K_0}{[K_0,K_1]\vee[A,F]} \arrow[r, dotted] \arrow[d, dotted] & \frac{F}{[K_0,K_1]\vee[A,F]} \arrow[r] \arrow[d] & P_0 \arrow[d] \\
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\cdot \arrow[r, dotted, tail] \arrow[d, dotted] & \frac{[F,F]}{[K_0,K_1] \vee [A,F]} \arrow[r] \arrow[d] & {[P_0, P_0]} \arrow[d] \\
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Happy Birthday, Rudger!