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Various perspectives on the extension of graphic statics to the third dimension

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Abstract

This paper discusses the strengths, limitations and potential of three future computer-aided approaches for three-dimensional graphic statics. It concludes that a fully three-dimensional extension of graphics statics would create new possibilities for dealing with complex 3D structures during both the design process and structural analysis.

Key words: structural design, graphic statics, three-dimensional static equilibrium, form finding

1. Introduction

This paper presents the different perspectives for fully extending graphic statics to the third dimension. Its objective is to consider the different approaches they represent and point out their strengths and limitations.

1.1. Context

Considerations related to correctly managing resources, such as energy and materials, are crucial in the building context and require efficient structural conception. The development of computer-based and numerical methods has encouraged a disjunction between architects and engineers. The design process is mainly based on a cycle in which the architect first designs a shape (which with CAD programs allows them to design complex 3D shapes, regardless of their structural behavior), and then the engineer proposes a structural solution. In this process, formal and structural needs do not necessarily coincide. One reason for this is that there has been no common tool.

As far as structures are concerned, an integrated and synthetic approach provides an understanding of the structural behavior in the early stages of the design (which can therefore be taken into account during the definition of the shape). For example, Gaudi's hanging models were attempting to find a shape that is directly at a state of equilibrium, and not design a shape and then try and make it stand. This allows the design process to be mastered so that a kind of structural efficiency is achieved [1].

1.2. Pertinence of graphical methods

Graphical methods have proven their considerable potential for controlling the relationship between a structural shape and its inner stresses. That relationship is expressed in a geometrical dependency between two diagrams: the spatial diagram and the force diagram. Great structures have been designed using geometrically-constructed drawings. This makes it a highly relevant tool as far as structural morphogenesis is concerned. Built examples in the works of great masters such as Robert Maillart attest to the pertinence of these methods [1]. The Salginatobel Bridge is a benchmark of the structural efficiency discussed above [1][2].

1.3. Challenges associated with the extension of graphic statics to the third dimension

There is a shortage of design tools today for shaping the equilibrium of three-dimensional structures that are more complex than grid shells, for instance structures that combine tension and compression bars in a geometry that is out of plane and cannot be described by 3D curved surfaces, or structures that combine various elements

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and materials. Therefore there is considerable interest in extending graphic statics to the third dimension, which can be justified on different levels.

Figure 1: Eladio Dieste, Citricos caputto fruit packing plant – 1971-72,1986-87 (from [3])

Except for the special case of tensegrity structures, built examples (by the likes of Gaudi, Isler, Frei, Candela and Dieste) and contemporary structural tools (dynamic relaxation, force density and thrust network analysis) [4] only consider shells. The development of 3D graphic statics would allow this field of application to be enlarged and thus extended to typologies other than curved surfaces. It would enable significant variations in the thickness of such surfaces to be considered and controlled, and would also allow the definition of the stresses involved in every element (such as bars) to be mastered and controlled separately.

One of the useful applications of this involves structural analysis, for example for complex three-dimensional masonry structures such as historical buildings whose shapes have deformed. One chapter in the book "La coupole de Brunelleschi. Histoire et avenir d'une grande construction"[5] explains the difficulty of understanding the force flow in the duomo after the redistribution of its inner stresses as a result of cracks. Such a theory (3D graphic statics) would be very useful for developing efficient structures in 3D when used as a form-finding tool in the early stages of design, while offering an understanding of the structural behavior at the same time. 3D graphic statics would certainly allow the emergence of new typologies of three-dimensional structures. It would at least provide a tool enabling geometries to be corrected for problematic set of forces encountered.

Other benefits of this method are also worth mentioning. Firstly it deals with structures at equilibrium. Furthermore, the user can control forces starting from the geometry, as well as the converse (geometry starting from chosen internal stresses). All of this is done in a visual and contextualized way.

This paper discusses the software implementation of current approaches for three-dimensional graphic statics and presents new perspectives.

2. Background (for the examination of existing approaches)

2.1. Categorization

The full extension of graphic statics to the third dimension can be addressed in a variety of ways and the objective of this paper is to present an overview of the different approaches that could lead to this. Based on existing literature in this field, it is proposed that the methods be categorized into the following three main approaches: the projective approach, the composite approach and the full 3D approach.

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For each approach (and every method classified within them) the definition of the form and force diagrams and the constraints applied, as well as the relationship between the form and the force diagrams and the way these configurations are represented and handled, will be considered.

The strengths and limitations of each approach will then be discussed. Consequently, this paper presents an outline of the future research efforts required in this field.

2.2. Basic concepts of graphic statics

This section briefly describes the basic concepts of 2D graphics statics, a graphical method that was theorized in the second half of the 19th century (Rankine [6], Maxwell [7], Cullmann [8], Cremona [9]). As previously mentioned, the particular feature of this method is that it deals simultaneously with the geometry of a structure and its forces, i.e. inner stresses and external loads. The geometry of the structure is represented in the form diagram, while forces in static equilibrium are represented in a force diagram (Maxwell-Cremona diagram). The local and overall equilibrium can be verified using this diagram. The magnitude of the forces can be established by measuring their lengths in the force diagram. To satisfy the equilibrium of the structure, the force diagram should be closed.

The relationship between the two diagrams is reciprocal and this remains important as far as the extension to the third dimension is concerned. The set of conditions that secure this reciprocity are as follows [10]:

(1)"Corresponding lines, whether representing constituent parts of the frame, or external forces, which meet at a point in the frame, form a closed polygon in the [force] diagram
(2) Corresponding lines, which represent constituent parts of the frame and form a closed polygon in it, meet in a point in the [force] diagram
(3) The lines representing all the external forces acting on the frame should form a closed polygon in the [force] diagram
(4) Lines — some of which represent external forces — which meet in a point in the [force] diagram, have the corresponding lines contiguous in the frame; but these may form a partial boundary to an infinite area [...]."

This set of conditions does not prevent multiple representations of one configuration because a rod can be represented by more than one line in the force diagram. This means that opposite forces forming a rod in the form diagram will not necessarily form a rod in the force diagram because they are not adjacent in it. Another condition is needed for this to happen which adds some interesting properties, but is not required for static equilibrium and so is not always included by some authors (Zalewski & Allen [11] for example). This condition states:

(5) "*There should be only one line in the diagram of forces to represent any one force acting on the frame or in a part of it."*

As highlighted by Pirard [12], this condition is always fulfilled if the forces are read in both diagrams according to a unique reading cycle. In order for each force, strut-and-tie to be drawn only once in each diagram, the forces applied to any sub-structure must be read in the form diagram on a cycle that (a) is always read either clockwise or anti-clockwise, and (b) is identical to the order described by the corresponding closed polygon in the force diagram.

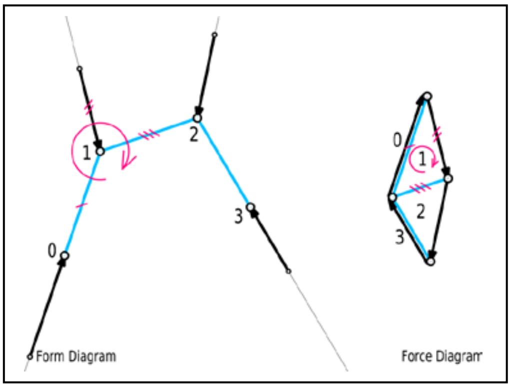

Figure 2: Graphic statics – Uniform reading cycle of forces. Forces applied on point 1 in the form diagram are read in the same order as in the force diagram (from [13])

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This fifth condition adds the following properties [14]:

(a) corresponding force polygons are identified more easily

(b) diagrams are more compact and confusion between lines is avoided

(c) the identification of rods in the force polygon is unambiguous since it is always equivalent to two opposite forces.

This property allows there to be different applications of force (push/pull) on nodes, which offers the possibility of dealing with more typologies of structures.

2.3. Geometrical constraints

In order to enjoy fully the benefits of the graphical representation of the form and the forces, applying geometrical constraints to the diagrams is of great interest [15]. This means that the points (that define rods) are constrained to a region of space (along a line, at a certain distance of a point, in a defined part of the space *etc.*). In the form diagram, this allows the geometry to be constrained to make it fit the architectural and formal needs. In the force diagram, this allows the forces to be constrained to specific needs (magnitude, orientation of the force...), because it is defined geometrically by the size of the segments, an angle *etc*.

Due to graphic statics rules, a condition in one diagram will impose a direct geometrical condition in the other. The visual representation of the geometrical constraints makes them very easy to handle and visualize.

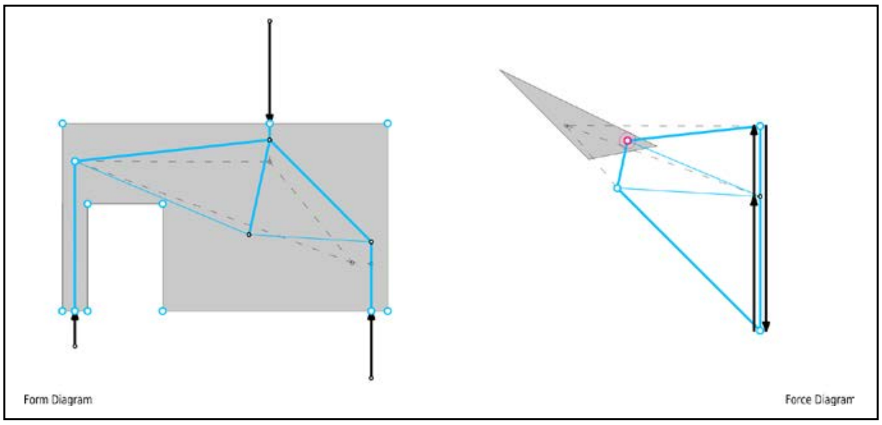

Figure 3: Geometrical constraints – The strut-and-tie network remains within the shaded area on the form diagram as long as the highlighted point in the force diagram remains within the shaded triangle limiting its position. The dashed lines represent another solution (from [13])

3. Discussion

3.1. The projective approach

The first approach is the main one that has been explored in literature so far. Projections are used to solve the issue of the representation and manipulation of a spatial structure. This starting point makes it very close to two-dimensional graphic statics.

With this approach, the form diagram is defined in a three-dimensional space but mastered by means of parallel projections. Each of these projections can be linked to a corresponding force diagram. These force diagrams are consequently planar. In this case, the geometric constraints must be planar too.

This method is based on the following property: "*If forces in space are in equilibrium, their projections on any plane are also in equilibrium. […] For a system of forces in space to be in equilibrium, it is both necessary and sufficient for the orthogonal projections on three rectangular planes to be in equilibrium*" [16]. It allows the use of 2D-graphic statics theory and thus the relationship securing the reciprocity between the form and force diagram remains valid. There is not really any need for a new theoretical background to be produced for this method.

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It has already been developed in literature (see for example the constructions of Föppl [17]). In 1903 Mayor published "Application de la Statique Graphique aux systèmes de l'espace" [18] in which he developed a method using orthogonal projections on a plane based on the theory of complex linear. He already pointed out that the main difficulty came in the representation method of spatial systems. His work was then expanded on by Foulon [19]. At that time, drawing projections by hand was quite difficult and took away the essential characteristic of graphic statics, i.e. the rapid sketch. In 2006, Huerta highlighted that "*Graphic statics, [...], can be used comfortably in two dimensions (on the drawing surface). To fix the position of a line in space, three projections are needed, thus making space problems very laborious to solve*" [20]. This is probably the reason why this particular method has never been used in practice. It was relevant when working with drawings by hand, but the issue is now overcome by the possibility of doing this by computer.

It can be established that software implementation of this kind would work with two (or more) projective planes. Some of the strengths of this method are that:

- it can easily link the projected situation with the 2D theory
- the displacement of points is unambiguous and precise
- the representation of the systems corresponds to the medium used for working on it (2D screen).

Some of the difficulties identified are:

- the difficulty of the representation of superimposed elements on a projected plane (problem of representing a sphere in two dimensions). As a result structures cannot fold back onto themselves. This could perhaps be resolved if the projected plane were able to move and work like a "section".
- The difficulty of describing 3D geometrical constraints (the shape would never be seen, just a section/projection of it). For example, the slope of a half space will only be represented well if one of the projection planes is perpendicular to one of the two main directions of the plane limiting the half space. The composite approach (see 3.2) is a proposal that can overcome this problem.

These limitations cannot be overcome for a full 3D extension to be achieved. The projective approach is interesting because it helps resolve some new 3D cases, but it cannot be considered as a full extension of spatial systems because its nature will not allow the treatment of all typologies of spatial structures. However, at the moment it manages to do some things that approaches such as the full 3D method have not yet achieved.

3.2. The composite approach

The main concept of this approach is that the form diagram, force diagram and geometric constraints are all defined in a three-dimensional space, but represented and handled on axonometric projections, allowing planar force diagrams to be the projections of different spatial force diagrams. This approach is halfway between the two other approaches presented in this paper (see points 3.1 and 3.3). The main strength of this approach is that it would take advantage of the existing 2D theory and so would secure the reciprocity – i.e. parallelism and reading cycles – of the 2D diagrams displayed. These remarks make it a very interesting area for development in future. The crucial point in order for this method to work is to establish the relationship between the 3D diagrams and their projections and to define these projections clearly.

3.3. The full 3D approach

In the third approach, the form diagram, force diagram and geometric constraints are all defined and handled in a 3D space.

3.3.1. The full 3D approach with polyhedrons

First the method proposed by Akbarzadeh, Van Mele and Block in "Equilibrium of spatial structures using 3D reciprocal diagrams" [21] will be discussed. The basis of this work is the paper by Rankine in which the magnitude of the forces is represented by polyhedral frames [22]. As a result, the equilibrium of the structure is vectorially represented by polyhedral frames. It states that forces acting on a point perpendicular and proportional to the areas of the faces of a polyhedron are in equilibrium. Maxwell [7] considered this problem in a purely geometrical manner and theorized the existence of the two polyhedrons and their reciprocal relationship.

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The method proposed by Akbarzadeh, Van Mele and Block is an extension of Maxwell's proposal that allows more general structural systems to be modeled because it allows external forces. The system is based on two polyhedrons, one for the form and one for the forces. Their paper defines these polyhedrons and their reciprocity.

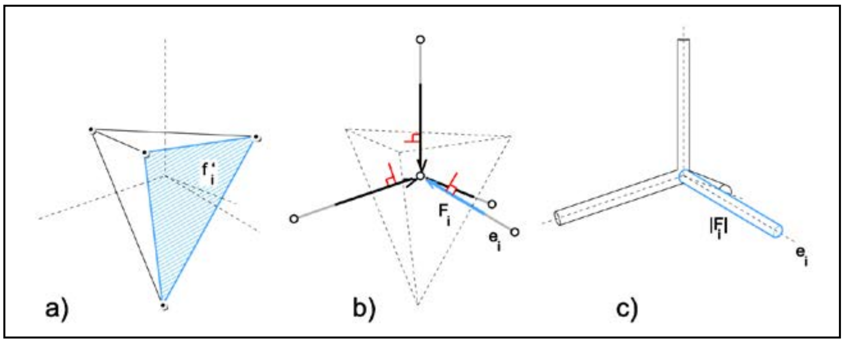

Figure 4: *Equilibrium of a single node in space: a) a single tetrahedron as force polyhedron; b) its reciprocal form polyhedron consisting of four lines and the applied force; c) a piped representation of the magnitude of the applied forces proportional to the areas of the face of the tetrahedron.* Figures from [21]

What is particular in the method proposed is its characterization through polyhedrons. Indeed what secures the equilibrium of a form configuration is the existence of the closed polyhedron (force). So a closed polygon of forces (that would be 3D) is not built classically as in graphic statics.

The reciprocal polyhedron can be found through iterative geometric operations. Both the reciprocity and unicity of the polyhedrons are guaranteed if the system involves compression or tension only. In this case, each surface representing a rod in the force diagram can be seen as equivalent to the superposition (in terms of area, position and orientation) of two surfaces, each belonging to a different closed polyhedron (one convex and one concave) at the place of the considered surface. So the existence of a unique reading cycle is displaced here for the polyhedrons. If a reading cycle exists, then the rods that are acting in compression or tension could be identified.

Nevertheless, if this method did overcome the problem of dealing with compression and tension forces in the structure, the polyhedrons do not yet have the qualities of a force diagram in terms of visual representation of a force flow, nor as far as handling the forces is concerned either. Some graphical creativity would be necessary to establish a representation of the polyhedrons and overcome this point.

3.3.2. The full 3D approach with spatial skew polygons

Another interesting method is the one developed by Schrems and Kotnik in "Statically motivated form-finding based on extended graphical statics" [23].

Their work is based on the principle that a finite number of forces in space can be reduced to a force pair. This force pair corresponds in intensity and direction to the sum of all forces. With an iterative process, three forces can be combined into two force pairs, and then linked to another force. This means a finite number of forces can be reduced to a force pair (one of the many possible).

With this method, one configuration (of the polygon of forces) will be made out of the many configurations possible and therefore the unicity of the force diagram cannot be guaranteed. As a result, this highlights one main limitation: there cannot be different applications of force (push/pull) on a node and therefore traction/compression cannot be combined.

The perspective of this research is about showing that planar graphic statics is a special case of 3D. The building of a 3D force diagram related to a 3D space configuration seems to be the most appropriate way of dealing with this. The approach taken is interesting, but the resolution method is not exactly the general case in 2D theory and has some limitations, as mentioned previously.

3.3.3. The full 3D approach based on a geometrical axiomatization of Graphic statics

Another approach is currently being developed by the authors. Its point of view is purely geometrical. The main concept is also to extend 2D theory so that it becomes a particular case of 3D theory. However the basis of this

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2D theory is the geometrical axiomatization of graphic statics in constraints-based graphic statics [15]. Indeed, the fundamental new axioms defined geometrically characterize graphic statics diagrams. The reciprocity of the force and form diagrams is secured by purely geometrical rules that are only applied on points. The uniform reading cycle order (i.e. here, the rule chosen is the clockwise order) has been theorized with simple geometrical relationships. In this approach, a bar of the form diagram corresponds to a bar in the force diagram (unlike the approach based on polyhedrons developed in [21]).

The concept here is therefore to generalize these axioms. The extension of some of these axioms is almost direct, but others require further research. The crux of the matter is located in the extension of the uniform reading cycle. This means that an equivalence of the clockwise order has to be defined in 3D, i.e. a relationship established concerning the order of reading forces applied on a point in a three-dimensional space. This is equivalent to defining a notion of order of reading of points located on the surface of a sphere.

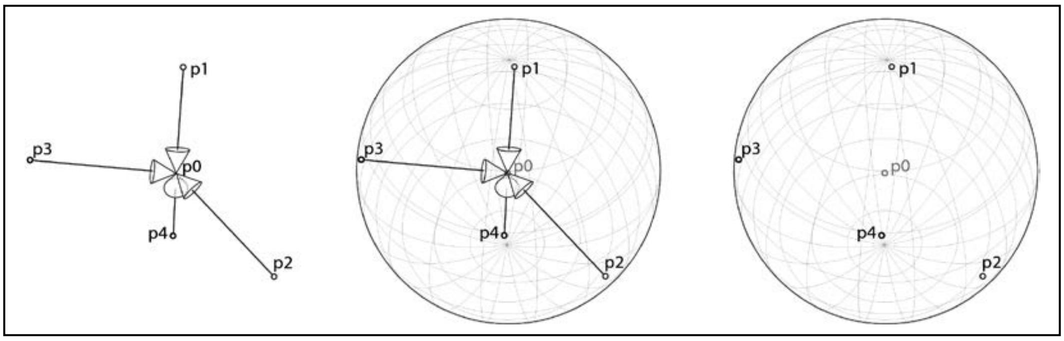

Figure 5: Extension of the uniform reading cycle in 3D – the problem can be considered as defining a notion of order of reading of points located on the surface of a sphere

The existence of constraints-based static graphics in three dimensions is directly linked to the existence of this uniform reading cycle in three dimensions. At present, nothing has proved that it is not possible. If this axiom is demonstrated, all that is required will be to establish the definition of the 3D geometrical constraints.

The main benefit of this axiom is that it means it is feasible to cope with systems combining compression and tension bars in complex 3D geometry (i.e. not shells, for instance).

The strength of a fully 3D approach is that it would certainly lead to a full extension of all the 2D properties, and maybe more. However it requires new theoretical research on the properties of tridimensional graphic statics, based on premises that are different from those that have already been developed in literature up to now. It could be said that the issue of representation has been reduced thanks to 3D CAD tools, because they provide a dynamic live projection of a 3D geometry dealing with geometrical properties (lengths, angles *etc.*). Owing to the experimental nature of this approach, there are different possible ways of embarking on this. Furthermore, it is still uncertain that all the methods can prove the fundamental relationship that establishes the unicity of the force diagram with regard to a spatial configuration. However, some ideas for this have been presented and a principal approach has been identified (i.e. a unique reading cycle in 3D) that combines the various other approaches.

4. Conclusions

This paper has described different approaches that could lead to the full extension of graphic statics in 3D.

Section 1 clarified the pertinence of structural morphogenesis in the design of structures related to their durability and savings on material and energy. It also highlighted the pertinence of the graphical methods for structural morphogenesis. Finally, it explained why it would be very useful to extend graphic statics to the third dimension and highlighted the great potential and interest of this method.

Section 2 briefly explained the methodology used for reviewing the different methods already developed. It also presented a brief reminder of the basis of 2D graphic statics theory and presented the geometrical constraints.

Section 3 then distinguished three main approaches: (1) the projective approach that has a quite limited purpose; (2) the composite approach that partially overcomes some of these limitations; (3) the full 3D approach that

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arouses most research interests at the moment. This last approach is currently being developed in three different directions. This paper has pointed out that a common prerequisite for further development is the demonstration of a unique reading cycle of forces in 3D, a necessary condition to theoretically handle structures with tensile and compressive forces combined.

As a result, this paper gives a global perspective on three-dimensional graphic statics, its benefits and its present stumbling blocks.

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