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Income poverty measures
with relative poverty lines

Benoît Decerf

The word "CORE" in a bold, black, sans-serif font. A thin blue arc starts above the 'C', curves over the 'O' and 'R', and ends below the 'E'.

CORE

DISCUSSION PAPER

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**Income poverty measures
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Abstract

I derive poverty indices taking into account both the absolute and relative aspects of income well-being. The trade-off made by the social planner between those two aspects is captured at individual level by a well-being ordering. This ordering evaluates the well-being of an agent based on her income and a reference statistic on the income distribution, typically the mean. A family of poverty indices respecting the judgements held in the ordering is axiomatically characterized. Then, I study the consequences of requiring the poverty indices to grant a minimal precedence to the absolute over the relative aspect of income well-being. This compelling requirement has strong implications. In particular, the Poverty Gap Ratio is the only index in the popular Foster-Greer-Thorbecke family to satisfy it.

Keywords: relative poverty, absolute poverty, income poverty, poverty gap ratio.

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1 Introduction

In the last 15 years, several international institutions have identified the reduction of poverty as one of their major objectives. The United Nations set poverty reduction as one of its eight Millenium Developement Goals and the European Commission identified it as one of its five main headline targets for its EU 2020 strategy. Making policy recommendations or simply monitoring the progress in global poverty reduction requires an unambiguous definition of poverty. My objective is therefore to derive income poverty measures from a global perspective. Since the groundbreaking contribution of [Sen \(1976\)](#), the construction of an income poverty measure follows a three steps procedure. First, the space in which poverty is measured must be chosen. Then, a threshold allowing to identify the poor is selected. Finally, the poverty of all individuals in a population is aggregated into an overall poverty measure. Following [Atkinson and Bourguignon \(2001\)](#), I limit here the space of poverty to the inadequacy of command over economic resources. The command a person has over resources is measured by her income or the value of her consumption. This interpretation of poverty is obviously narrow as it does not account for deprivations in important dimensions such as health, education or safety. Income poverty is nevertheless an important aspect of poverty as income allows buying goods and services necessary for leading a decent life. In the identification step, a poverty line must be chosen. This line defines the income threshold below which an individual is considered to be poor. As noted by [Atkinson and Bourguignon \(2001\)](#), taking a world perspective raises the challenge of developing a globally inclusive measure, giving equal treatment to all citizens of the world. The difficulty is that different countries use different kinds of poverty lines (see [Datt et al. \(1991\)](#) or more recently [Ravallion \(2012\)](#)). Most developing countries use *absolute poverty lines*. These lines are usually anchored to the satisfaction of a minimal level of nutrition and their corresponding income threshold only evolve with inflation. On the other hand, many developed countries use *relative poverty lines*, for which the corresponding income threshold evolves as a constant fraction of a statistic on the income distribution, typically the mean or the median. National poverty measures using different poverty lines can hence not be compared from a global perspective, as individuals in different countries would not be treated equally. Furthermore, if a global measure is to be relevant for all countries in the world, it must combine both the absolute and relative aspects of income poverty.

In the literature, there are two main proposals for combining both absolute and relative aspects of income poverty. The first road, opened by [Atkinson and Bourguignon \(2001\)](#), consists in using two poverty lines, one absolute and one relative. Poor individuals can be considered as absolutely poor, relatively poor or both, depending on the position of their income with respect to both lines. They explored two ways of combining absolute and relative poverty. First, the social planner could consider there exists a hierarchy that ranks absolute poverty above relative poverty. There would hence be two measures of poverty to be considered in lexicographic order. Second, the social planner could construct the poverty measure as the result of a multidimensional exercise. The income shortfalls with respect to each poverty line are then aggregated to obtain the global poverty measure. Their proposal is based on several parameters which together control for the precedence given to absolute poverty, the ex-

tent of double-counting and poverty aversion.¹ This second approach has the merit of yielding a single measure taking both absolute and relative aspects into account. Nevertheless, the properties of their proposed measure are unclear as they do not provide an axiomatic characterization for it. Furthermore, the exact values of the three parameters are left unspecified, which makes its application challenging. The second road, opened by [Foster \(1998\)](#) and followed by [Ravallion and Chen \(2011\)](#) and [Ravallion \(2012\)](#), consists in defining an hybrid poverty lines taking into account both absolute and relative aspects. Hybrid lines have their income threshold evolve with mean income, but not necessarily as a constant fraction. Their income elasticity with respect to mean income is typically between 0 and 1, respectively the elasticity of absolute and purely relative lines. The poverty measure for a country taking a global perspective is then taken to be the fraction of individuals in that country whose income falls below the hybrid line. The poverty measure is hence obtained by using an aggregated index called the Head-Count Ratio. With absolute poverty line, [Foster and Shorrocks \(1991\)](#) have characterized the family to which this index belongs and therefore the properties it satisfies are well-known. Nevertheless, with relative or hybrid poverty lines, no such characterization exists. There is hence no solid foundations for using the Head-Count Ratio. As a result, global poverty measures based on this index might behave in counter-intuitive ways. Avoiding bad behavior requires then to derive axiomatically a suitable index. My proposal for solving these issues is to introduce, at the identification step, an *income well-being ordering*. This ordering is a social preference balancing, at the individual level, the absolute and relative aspects of income well-being. For simplicity, relative income is assumed to be the fraction of own income over society's standard of living. Following [Atkinson and Bourguignon \(2001\)](#) and [Ravallion \(2012\)](#), mean income is the reference statistic capturing standard of living. The income well-being of an agent will therefore depend on both her own income and the mean of the income vector. The hybrid poverty line is selected to be one of the indifference curves of this ordering. As a result, this line corresponds to the minimal level of income well-being below which an individual is poor. Implicitly, the single measure proposed by [Atkinson and Bourguignon \(2001\)](#) defines such an ordering. My proposal is to define this ordering explicitly before aggregating the income shortfalls.

At the aggregation step, income poverty indices are required to respect the judgements held in the well-being ordering. The endogeneous link existing between mean income and the income threshold is then automatically respected by the indices. On this basis, an extension of the family of additive poverty indices of [Foster and Shorrocks \(1991\)](#) is characterized. The indices obtained are thus in essence the average lack of income well-being in the population. Then, the indices are required to give a minimal precedence to absolute over relative aspects of income poverty. This precedence is a normative view not only shared by scholars (such as [Atkinson and Bourguignon \(2001\)](#)) but it emerged from a questionnaire survey conducted in different countries around the globe by [Corazzini et al. \(2011\)](#). This compelling requirement strongly constrains the numerical representations of the well-being ordering defining the additive poverty indices. In particular, among the numerical representations belonging to the

¹Double counting arises when an agent's income is below the two poverty lines. The distance between her income and the minimal of these two lines enters in both gaps.

Foster-Greer-Thorbecke (FGT) family defined in [Foster et al. \(1984\)](#), only the Poverty Gap Ratio satisfies it. The interest of this last result comes from the wide use of the FGT family in empirical applications. Those results are then extended to the use of other reference statistics than the mean for capturing standard of living.

The paper is organised as follows. Section 2 discusses the foundations in the capability approach of the indices derived. Notations and well-being orderings are introduced in section 3. The poverty axioms and their implications are presented in section 4. Finally, the extension to a class of reference statistics is studied in section 5. Section 6 concludes.

2 Capability underpinning

This section discusses how the global poverty indices derived here belong to the capability approach developed in [Sen \(1985\)](#). This book summarizes the important point made by Sen according to which economic (dis)advantages of individuals should not be measured in the space of income or utilities but in the space of capabilities and functionings. *Functionings* are all the doings and the beings that matter to individuals. Simple examples of functionings are being healthy, being able to move to places one needs to go to, being able to communicate with others, etc. The functioning vector achieved by an individual do not only depend on her external resources such as income but on her personal characteristics as well. For example, an handicaped individual might need more resources for traveling from one point to another than a non-handicaped individual. *Capabilities* are the set of functionings vectors an individual can achieve. Capabilities capture hence the individual freedom, the size of the choice set of an individual given her characteristics and the resources she commands.

Following [Atkinson and Bourguignon \(2001\)](#), a global income poverty measure must consider two crucial functionings, physical survival and social participation. Both functionings are achieved by consuming goods and services measured here in a unique space: income. The cost of goods and services associated with physical survival, like food, is assumed not to evolve with the income distribution. On the other hand, participating in the social life of a particular society requires a bundle of goods and services that depends on the standard of living in that society. The higher the standard, the more expensive this bundle will be. For example, in Adam Smith's England, wearing a linen shirt was a pre-requisite for social participation. As shown by [Atkinson \(1995\)](#), not only the composition of this bundle but also the real cost of some of its goods and services evolves with standard of living. Therefore, the level of social participation an individual can achieve given a particular income depends on her "personal" characteristic: the standard of living in her society. For simplicity, all other individual characteristics are omitted.

[Atkinson and Bourguignon \(2001\)](#) associate one poverty line to each functioning: the absolute line is associated physical survival and the relative line to social participation. My proposal, based on an income well-being ordering, considers a unique poverty line. This poverty line coincide with one of the indifference curves of the ordering. Each indifference curve can be constructed such that all

its points are associated with the same vector of functionings.² Therefore, an agent is identified to be poor if her income is not sufficient to buy a bundle of goods and services allowing her to reach the minimal level of *both* functionings together. Such a bundle would not only contain goods and services necessary for physical survival but also some required for social participation. As the cost of social participation increases with mean income, so must the indifference curves and hence the poverty line. Notice that the total cost of reaching the minimal levels in each functioning together need not be the sum of reaching each separately, as some goods may serve both ends. This is for example the case of a linen shirt which serves both clothing and social participation purposes.

A given income in a particular society defines an individual's capability, i.e. the set of functioning vectors she can achieve. This set is defined by the functioning vectors attached, in her society, to the different bundles of goods and services she can afford with her income. Different points of one indifference curve are attached different capabilities. Points with low mean have vectors with higher levels of social participation and lower physical survival in their capability set than points with high mean. Nevertheless, being on the same indifference curve, all of them are judged equally valuable by the ordering. As a result, the income well-being ordering defines an ordering in the space of capabilities.

3 The model

3.1 Basic notations

Let $y = (y_1, y_2, \dots, y_n)$ be an income vector composed of non-negative incomes. Its elements are sorted in non-decreasing order ($y_1 \leq y_2 \leq \dots \leq y_n$). The population size corresponding to y is $n(y)$. Mean income in y is written $\bar{y} = \frac{\sum y_i}{n}$. I refer to y_i as the *absolute income* of agent i and $\frac{y_i}{\bar{y}}$ as her *relative income*. The results derived with the mean are extended to a class of reference statistics in section 5.

The social planner evaluates individual well-being based on an ethical preference $\succeq$ balancing their absolute and relative income. This ethical preference $\succeq \in \mathcal{R}$ is a continuous ordering over the set of bundles $X = \{(y_i, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_{++}\}$.³ See Figure 1 for examples of such well-being orderings. Let $(y_i, \bar{y}) \succeq (y'_i, \bar{y}')$ denote the judgment that an agent receiving income y_i when mean income is $\bar{y}$ has a weakly higher well-being than an agent receiving y'_i when mean income is $\bar{y}'$. The symmetric and asymmetric parts of $\succeq$ are represented by $\sim$ and $\succ$ respectively.

The identification of poverty is based on $\succeq$. Agent i is considered as poor if and only if her bundle is deemed strictly worse than a reference bundle $(z_1, z_2) \in X$. Let $X_p = \{(y_i, \bar{y}) \in X | (z_1, z_2) \succ (y_i, \bar{y})\}$ be the subset of bundles yielding a lower well-being than the well-being threshold associated to the reference bundle. Agent i is poor if and only if $(y_i, \bar{y}) \in X_p$. Let $z : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ be the *threshold function* yielding at $\bar{y}$ the income threshold corresponding to the well-being threshold. The income threshold $z(\bar{y})$ solves $(z_1, z_2) \sim (z(\bar{y}), \bar{y})$, as illustrated in Figure 1.a. Agent i is then poor if and only if $z(\bar{y}) > y_i$. The

²The set of these vectors characterizing the indifference curves form an increasing path in the space of functioning.

³An ordering is a reflexive, transitive and complete binary relation.

number of poor agents in y is noted $q(y)$.

The objective is to rank income distributions according to the level of poverty they contain. A small limitation of the results derived is that they only hold when ranking income distributions for which at least one agent is non-poor. The set of income distribution is hence $Y = \{y \in \mathbb{R}_+^n \text{ with } n \in \mathbb{N} | y_n \geq z(\bar{y})\}$. Given $\succeq$ and (z_1, z_2) , a poverty index is a function $P : Y \rightarrow \mathbb{R}$ such that $P(y) > P(y')$ means there is strictly more poverty in y than in y' . I now turn to the definition of the domain $\mathcal{R}$ of well-being orderings.

3.2 Domain of well-being orderings

The ethical preference over bundles of the social planner is a continuous ordering belonging to the domain $\mathcal{R}$. The domain $\mathcal{R}$ is limited by four restrictions. Those minimal restrictions exclude exotic trade-offs between absolute and relative income. First, $\mathbf{R}^1$ assumes that relative income is not a bad for the well-being of the poor. Then, restrictions $\mathbf{R}^2$ to $\mathbf{R}^4$ limit the importance of relative income for individual well-being. In essence, restriction $\mathbf{R}^2$ requires that, if the income vector y' is obtained from y by distributing equally an extra amount of income, all poor agents are weakly better-off. Restriction $\mathbf{R}^3$ states that however large the relative income of an agent may be, there always exists an absolute income that, if earned by all agents, strictly increases that agent's well-being. It is as-if agents were always ready to give up their good relative income provided their absolute income is sufficiently increased.⁴ Restriction $\mathbf{R}^4$ requires there always be a limit to the portion of an agent's absolute income that, if traded against lower mean income, improves her well-being. An agent with zero income is therefore strictly worse-off than another agent with non-zero income.

- $\mathbf{R}^1$ (Relative income is not a bad):
For all $(y_i, \bar{y}), (y'_i, \bar{y}') \in X_p$, if $y_i = y'_i$ and $\bar{y} < \bar{y}'$, then $(y_i, \bar{y}) \succeq (y'_i, \bar{y}')$.
- $\mathbf{R}^2$ (Translation Monotonicity):
For all $(y_i, \bar{y}) \in X_p$, $a > 0$, we have $(y_i + a, \bar{y} + a) \succeq (y_i, \bar{y})$.
- $\mathbf{R}^3$ (Well-off's limited relative concern):
For all $(y_i, \bar{y}) \in X$ with $y_i > \bar{y}$ there exists $a > y_i$ such that $(a, a) \succ (y_i, \bar{y})$.
- $\mathbf{R}^4$ (Minimum absolute concern):
For all $y_i > 0, \bar{y}, \bar{y}' \in \mathbb{R}_{++}$, we have $(y_i, \bar{y}) \succ (0, \bar{y}')$.

Graphically, the slopes of the indifference curves below the poverty line must be non-negative ($\mathbf{R}^1$) but weakly smaller than one ($\mathbf{R}^2$). Indifference curves passing through bundles above the bissectrice must cross the bissectrice ($\mathbf{R}^3$). Finally, there is a flat indifference curve for $y_i = 0$, on the horizontal axis ($\mathbf{R}^4$). The domain $\mathcal{R}$ defined by $\mathbf{R}^1$ to $\mathbf{R}^4$ is very rich, as illustrated by the examples in Figure 1.

Two remarks must be made about orderings in $\mathcal{R}$. First, if the parameter $\bar{s}$ of the family illustrated in 1.b is set to zero, the threshold function is an *absolute poverty line*. If instead, the parameter z^0 of that family is set to zero, the threshold function is a constant fraction of the mean. This corresponds then

⁴This interpretation is abusive since $\succeq$ is the ethical preference of the social planner rather than the agent's preferences.

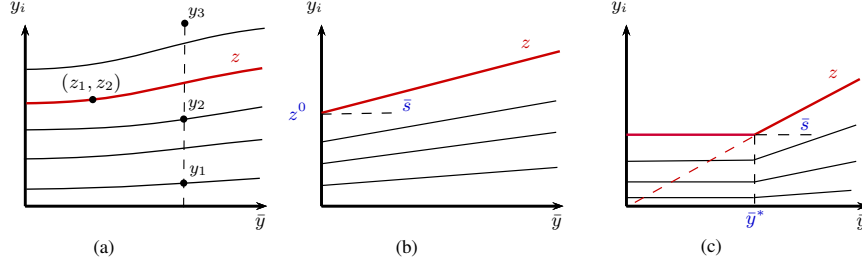


Figure 1: (a) Indifference curves of an ordering in the bundle space X , with a distribution $(y_1, y_2, y_3) \in Y$. The reference bundle (z_1, z_2) identifies the indifference curve from which the threshold function z is derived. (b) Family of linear orderings parametrized by intercept z^0 and slope $\bar{s}$ of the threshold function, with $z^0 \geq 0$ and $\bar{s} \in [0, 1)$. (c) Family of piecewise linear orderings with flat threshold function for $\bar{y} \leq \bar{y}^*$ and $z = \bar{s}\bar{y}$ for $\bar{y} > \bar{y}^*$, with $\bar{s} \in [0, 1)$.

to a *relative poverty line*. This does not conflict with $\mathbf{R}^4$, since the set X is defined for $\bar{y} > 0$. This framework can hence deal with both kinds of poverty lines. Nevertheless, this approach does not aim at separating purely absolute and purely relative income poverty. Rather, it aims at developing one notion of income poverty considering both absolute and relative income. Second, the poverty indices developed for absolute poverty lines in the literature surveyed in Zheng (1997) can be divided between those respecting *Translation invariance* and those respecting *Scale invariance*. Those axioms constrain the comparison of distributions facing different income thresholds.⁵ Although very convenient, there exists no convincing defense for these axioms, as noted in Zheng (1997). Using an income well-being ordering makes transparent how the individual well-being evolves when mean income and therefore the income threshold is changed. As a consequence, there is no need to resort to these axioms.

Two extra restrictions, $\mathbf{R}^5$ and $\mathbf{R}^6$, do not enter the definition of our basic domain $\mathcal{R}$, but will be referred to when they help simplify the exposition. Restriction $\mathbf{R}^5$ does not impose any additional constrain on the shape of the threshold function z . Rather, once z is chosen, all the lower indifference curves are fixed: they evolve as constant fractions of z . Restriction $\mathbf{R}^6$ is an extreme way of looking at relative income. It requires relative income to be irrelevant for the well-being of poor agents.

- $\mathbf{R}^5$ (Fraction of threshold function):

For all $(y_i, \bar{y}), (y'_i, \bar{y}') \in X_p$, if $\frac{y_i}{z(\bar{y})} = \frac{y'_i}{z(\bar{y}')}$, then $(y_i, \bar{y}) \sim (y'_i, \bar{y}')$.

- $\mathbf{R}^6$ (Flat indifference curves):

For all $(y_i, \bar{y}), (y'_i, \bar{y}') \in X_p$, if $y_i = y'_i$, then $(y_i, \bar{y}) \sim (y'_i, \bar{y}')$.

The selection of suitable deprivation ordering $\succeq \in \mathcal{R}$ and reference bundle (z_1, z_2) concludes the identification step. In the next section, I present the axioms defining the properties that the poverty indices should satisfy.

⁵Abusing slightly notations to let z become an argument of P , *Translation invariance* requires that $P(y + a1_n, z + a) = P(y, z)$ and *Scale invariance* that $P(ay, az) = P(y, z)$. The symbol 1_n refers to a n -dimensional vector of ones.

4 Axiomatic characterization

4.1 Additive poverty indices

This section aims at characterizing a family of income poverty indices. Many authors such as [Kakwani \(1980\)](#) or [Chakravarty \(1983\)](#) have characterized such families for absolute poverty lines. In my framework, the families derived for absolute poverty line correspond to a case where the underlying ordering satisfies the additional restriction $\mathbf{R}^6$. What follows is then the first characterization valid for any ordering in $\mathcal{R}$. Otherly stated, the first characterization of poverty indices for distribution-dependent poverty lines. Calling them *relative poverty indices* would nevertheless be reductive as *both* absolute and relative income are taken into account.

I present here six poverty axioms. Five of them are required in order to derive a generalization of the family characterized by [Foster and Shorrocks \(1991\)](#). The first axiom makes sure that poverty is only concerned with the well-being of poor agents. This was traditionally ensured by using *focus*, an axiom requiring the income of non-poor agents to be irrelevant for poverty measurement. However, when relative income matters, their incomes can no longer be discarded, as they affect mean income. *Weak Focus* imposes therefore that the distribution of income among non-poor agents is irrelevant, as long as it does not affect the mean, our reference statistic.

Poverty axiom 1 (*Weak Focus*).

For all $y, y' \in Y$, if $n(y') = n(y)$, $q(y') = q(y)$, $y'_i = y_i$ for all $i \leq q(y)$ and $\bar{y} = \bar{y}'$, then $P(y) = P(y')$.

The well-being ordering captures the trade-offs made by the social planner between absolute and relative income. *Domination among the Poor* makes sure the poverty index respects these trade-offs by imposing a monotonicity requirement in the space of well-being vectors, limited to poor agents.⁶ If the well-being of one poor agent increases, while all other poor agents are deemed at least as well-off, then the poverty index must decrease. This axiom implies *Weak Focus*.

Poverty axiom 2 (*Domination among the Poor*).

For all $y, y' \in Y$ such that $n(y) = n(y')$, if $(y'_i, \bar{y}') \succeq (y_i, \bar{y})$ for all $i \leq q(y')$, then $P(y) \geq P(y')$.

If, in addition, there is $j \leq q(y)$ such that $(y'_j, \bar{y}') \succ (y_j, \bar{y})$, then $P(y) > P(y')$.

Subgroup consistency is a classical axiom which requires that, if poverty decreases in a subgroup while it remains constant in the rest of the distribution, overall poverty must decrease. [Sen \(1992\)](#) has questioned the desirability of such axiom by arguing that what happens in a subgroup may affect poverty in another subgroup. [Foster and Sen \(1997\)](#) recommend not to use this axiom when the index aims at capturing relative aspects of income poverty. I subscribe to this point of view. The underlying issue becomes transparent once the channel through which one subgroup affects the other is modeled. This channel is modeled here by the well-being ordering. If the ordering satisfies restriction $\mathbf{R}^6$, relative income does not matter and subgroup consistency seems reasonable. If

⁶ *Domination among the Poor* is familiar in essence with the axiom of Strong Pareto among the Poor introduced in [Decancq et al. \(2012\)](#).

relative income does matter, then it is not always meaningful to extrapolate the judgments made on subgroups to the whole distribution. *Weak Subgroup Consistency* restricts such extrapolations to cases for which a subgroup does not influence the well-being of agents in the rest of the distribution. This happens when the two subgroups have the same mean income. In such cases, poverty judgments made on one subgroup are still valid for the total distribution.

Poverty axiom 3 (*Weak Subgroup Consistency*).

For all $y^1, y^2, y^3, y^4 \in Y$ such that $n(y^1) = n(y^3)$, $n(y^2) = n(y^4)$, $\bar{y}^1 = \bar{y}^2$, $\bar{y}^3 = \bar{y}^4$, if $P(y^1) > P(y^3)$ and $P(y^2) = P(y^4)$, then $P(y^1, y^2) > P(y^3, y^4)$.

The next three axioms are classical auxiliary axioms. *Symmetry* states that the identity of the agents does not matter. We can therefore work with sorted distributions. *Continuity* requires the poverty index P to be continuous in y . This is particularly important for empirical applications in order to avoid measurement errors to have excessive impacts. *Replication Invariance* allows to compare poverty in distributions of different sizes. If a distribution is obtained by replicating another several times, then their poverty are equal.

Poverty axiom 4 (*Symmetry*).

For all $y, y' \in Y$, if $y' = y \cdot \pi_{n(y) \times n(y)}$ for some permutation matrix $\pi_{n(y) \times n(y)}$, then $P(y) = P(y')$.

Poverty axiom 5 (*Continuity*).

For all $y \in Y$, P is continuous in y .

Poverty axiom 6 (*Replication Invariance*).

For all $y, y' \in Y$, if $n(y') = kn(y)$ for some positive integer k and $y' = (y, y, \dots, y)$, then $P(y) = P(y')$.

Those axioms allow to derive an extension of the additive separability result of Foster and Shorrocks (1991). Its formal statement is simplified by the introduction of the following definition. A *numerical representation* d is a continuous function representing the well-being ordering on the set of bundles deemed worse than the reference bundle (z_1, z_2) .

Definition 1 (Numerical Representations d). The function d is called a numerical representation of $\succeq \in \mathcal{R}$, if and only if it is a continuous real-valued functions $d : X \rightarrow [0, 1]$ such that

- $(y_i, \bar{y}) \succeq (y'_i, \bar{y}') \Leftrightarrow d(y_i, \bar{y}) \leq d(y'_i, \bar{y}')$ for all $(y_i, \bar{y}), (y'_i, \bar{y}') \in X_p$,
- $d(y_i, \bar{y}) = 0$ for all $(y_i, \bar{y}) \in X \setminus X_p$.

A numerical representation differ from a utility representation of the well-being ordering in two ways. First, it is constant for all well-being levels above the poverty threshold. Second, for poor agents, its value decreases when well-being increases. The value obtained by this function is in a sense the opposite concept of utility, as it measures the distance to the threshold. This value measures therefore the lack of income well-being. Equipped with this definition, the characterization of an additively separable family of poverty indices which takes both absolute and relative income into account is stated in the following Theorem.

Theorem 1 (Family of additive poverty indices). *$P : Y \rightarrow \mathbb{R}$ is an income poverty index satisfying *Domination among the Poor*, *Weak Subgroup Consistency*, *Symmetry*, *Continuity* and *Replication Invariance*, if and only if there exist a numerical representation d of the underlying well-being ordering $\succeq \in \mathcal{R}$ and a continuous and strictly increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $y \in Y$:*

$$P(y) = F\left(\frac{1}{n(y)} \sum_{i=1}^{n(y)} d(y_i, \bar{y})\right). \quad (1)$$

Proof. The proof is lengthy and rather technical and is therefore relegated in the Appendix. Basically, it shows that the result on additive separability of Gorman (1968) can be applied here. The crucial step for this is to show that our poverty index satisfies a separability property. After having applied Theorem 1 in Gorman (1968), the remaining part of the proof is an adaptation of Foster and Shorrocks (1991). ■

I will refer to poverty indices satisfying those five axioms as *additive* poverty indices. An additive poverty index is thus - an increasing function of - the average lack of well-being in the distribution. Two remarks can be made about this Theorem. To start with, this Theorem applies for a set of well-being ordering larger than $\mathcal{R}$. Restriction $\mathbf{R}^1$, stating that relative income is not a bad for the poor, is not necessary for it. Then, not all members of this family are compelling income poverty indices. So far, no restriction is imposed on the numerical representation which, together with the well-being ordering, completely characterize a poverty ordering over Y . Such restrictions will emerge from the imposition of other compelling properties of poverty indices, such as *Weak Transfer*. Most poverty indices derived after Sen (1976) satisfy this axiom. It encapsulates the Pigou-Dalton transfer principle which requires in this context that poverty does not to increase after a progressive transfer is made between two poor agents. This axiom is still compelling in this context since only the well-being of the two concerned agents is impacted, as the balanced transfer does not affect mean income.

Poverty axiom 7 (*Weak Transfer*).

For all $y, y' \in Y$, $a > 0$, if $y_j - a = y'_j > y'_k = y_k + a$, $z(\bar{y}) > y_j$ and $y'_i = y_i$ for all $i \neq j, k$, then $P(y) \geq P(y')$.

On top of it, in line with a view shared by specialists (see Atkinson and Bourguignon (2001)) and non-specialists (see Corazzini et al. (2011)), some precedence should be given to physical survival over social functionings.

4.2 Giving some precedence to absolute income

The specificity of the poverty indices derived in this paper is that they account for both absolute and relative income. Both dimensions of well-being are traded-off at individual level via the well-being ordering. *Domination among the Poor* makes sure poverty indices take this well-being ordering into account. When several agents are affected by a change in the distribution, the index must balance the absolute and relative gains and losses made by those agents. This balance should not give excessive attention to relative impacts. Let us consider

the following example. Imagine one poor agent receives an increment (i.e. an income increase) not sufficient to lift her out of poverty, while the income of all other agents remain constant. This improves her absolute income, but since mean income increases, the relative income of other poor agents decreases. I argue here that, when balancing the absolute gain versus the relative losses, the poverty index should never record this as a poverty increase. The fundamental reason for this is that no agent has less resources she can devote to her physical survival while one has strictly more. The relative income of some might be affected, but the higher the number of agents whose relative income is affected, the lower is this impact. Indeed, more agents in the distributions means a lower impact of the increment on mean income, the reference statistic. This minimal limitation of the attention given to relative income is encapsulated in *Monotonicity in Income*. In other words, destroying part of the income of some poor agent can never unambiguously decrease poverty.

Poverty axiom 8 (*Monotonicity in Income*).

For all $y, y' \in Y$, if $z(\bar{y}') > y'_i > y_i$ and $y'_j = y_j$ for all $j \neq i$, then $P(y) \geq P(y')$.

I have argued above that only additive poverty indices satisfying *Monotonicity in Income* comply with the idea that some precedence should be given to absolute over relative concerns. I derive next the constraints on numerical representations imposed by this axiom.

4.2.1 Constraints on numerical representation

Presenting the constraints inherent to *Monotonicity in Income* requires the definition of the *degree of priority* $DP_{ij}(y)$ given by the poverty index to an agent with income y_i over another agent with income y_j in the distribution y . This concept captures the relative impact on the poverty index of giving an increase in income to agent i instead of giving it to agent j .

Definition 2 (Degree of Priority of y_i over y_j in y). *The degree of priority given to y_i over y_j in the distribution y by an additive poverty index P is defined as:*⁷

$$DP_{ij}(y) = \frac{\frac{\partial d}{\partial y_i}(y_i, \bar{y})}{\frac{\partial d}{\partial y_j}(y_j, \bar{y})}$$

From this definition, if $DP_{ij}(y) = 2$, as in the example of Figure 2, transferring an arbitrarily small ϵ from a non-poor agent to i increases her well-being by two times the amount j 's well-being would increase if j was the one to receive that transfer.

The poverty orderings over Y represented by additive poverty indices are completely characterized by a pair $(\succeq, d)$. Such a pair is not a unique characterization, as another pair $(\succeq, d')$ such that $d' = a + bd$ with $b > 0$, i.e. a positive

⁷The function d is differentiable almost everywhere (d is continuous). The value of its partial derivatives at the points $(y_i, \bar{y})$ where d is not differentiable is defined as:

$$\frac{\partial d}{\partial y_i}(y_i, \bar{y}) = \lim_{\epsilon \rightarrow 0} \frac{\partial d}{\partial y_i}(y_i + \epsilon, \bar{y}) \quad \text{and} \quad \frac{\partial d}{\partial y_j}(y_i, \bar{y}) = \lim_{\epsilon \rightarrow 0} \frac{\partial d}{\partial y_j}(y_i, \bar{y} + \epsilon)$$

As usual, the partial derivative of a function $f(x_1, x_2)$ in the direction x_1 at point (a_1, a_2) is written $\frac{\partial f}{\partial x_1}(a_1, a_2)$.

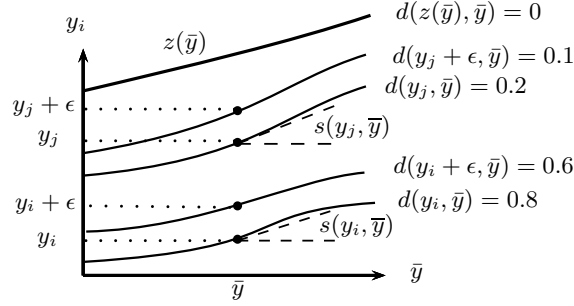


Figure 2: Illustration of the notions of degree of priority $DP_{ij}(y)$ and slope $s(y_i, \bar{y})$. In this example, $DP_{ij}(y) = \frac{0.8-0.6}{0.2-0.1} = 2$.

linear transformation of d , characterizes the same poverty ordering. On top of $\succeq$, the minimal information necessary to characterize the poverty ordering is the function DP_{ij} yielding the degrees of priority.

Monotonicity in Income sets a lower and an upper bound on the values $DP_{ij}(y)$ can take. These bounds, illustrated in Figure 4.b, depend on the slopes of the indifference curves of the concerned agents. These slopes are defined as follows:⁸

Definition 3 (Slope at $(y_i, \bar{y})$). $s(y_i, \bar{y}) = -\frac{\frac{\partial d}{\partial \bar{y}}(y_i, \bar{y})}{\frac{\partial d}{\partial y_i}(y_i, \bar{y})}$

We are now equipped to state the central result of this paper.

Theorem 2 (Bounds on degree of priority). *An additive poverty index satisfies Monotonicity in Income:*

1. (sufficient condition) if for all $y \in Y$, $y_i, y_j < z(\bar{y})$, we have:

$$s(y_j, \bar{y}) \leq DP_{ij}(y) \leq \frac{1}{s(y_i, \bar{y})} \quad (2)$$

2. (necessary condition) only if for all $y \in Y$ such that $z(\bar{y}) \leq \bar{y}$ and all $y_i, y_j < z(\bar{y})$, inequality 2 holds.

Proof. *Monotonicity in Income* requires that for all $y \in Y$ and $y_i \in [0, z(\bar{y})]$ we have $\frac{dP}{dy_i}(y) \leq 0$. By the additively separable form of P , we obtain by chain derivation:

$$\frac{\partial d}{\partial y_i}(y_i, \bar{y}) + \sum_{j=1}^{n(y)} \frac{\partial d}{\partial y_j}(y_j, \bar{y}) \frac{\partial \bar{y}}{\partial y_i} \leq 0 \quad (3)$$

From the definition of the mean, we have $\frac{\partial \bar{y}}{\partial y_i} = \frac{1}{n}$. From the definition of $s(y_j, \bar{y})$, we get $\frac{\partial d}{\partial y_j}(y_j, \bar{y}) = -\frac{\partial d}{\partial y_j}(y_j, \bar{y}) s(y_j, \bar{y})$ for all $(y_j, \bar{y}) \in X$. Inequality 3

⁸If the iso-deprivation curve has a kink in $(y_i, \bar{y})$,

$$s(y_i, \bar{y}) = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} s(y_i, \bar{y} + \epsilon)$$

becomes:

$$\underbrace{\frac{\partial d}{\partial y_i}(y_i, \bar{y}) - \frac{1}{n(y)} \sum_{j=1}^{n(y)} \frac{\partial d}{\partial y_j}(y_j, \bar{y}) s(y_j, \bar{y})}_{L_4} \leq 0 \quad (4)$$

In the rest of the proof, inequality 4 is shown to imply the necessary and sufficient conditions linked to inequality 2. As $DP_{ij}(y) = \frac{1}{DP_{ji}(y)}$, inequality 2 is equivalent to:

$$\underbrace{\frac{\partial d}{\partial y_i}(y_i, \bar{y}) - \frac{\partial d}{\partial y_j}(y_j, \bar{y}) s(y_j, \bar{y})}_{L_5} \leq 0 \quad (5)$$

Necessity is proved by contradiction. Suppose inequality 5 does not hold for some $y \in Y$ with $\bar{y} = \bar{g}$, $z(\bar{g}) \leq \bar{g}$, $y_i = a$, $y_j = b$ with $0 \leq a < b < z(\bar{g})$. Therefore, at $(a, \bar{g})$, $(b, \bar{g}) \in X_p$, we have for some $l > 0$ that $L_5 = l$. I prove that for all $\epsilon > 0$, there exists $y' \in Y$ with $\bar{y}' = \bar{g}$ such that $l - L_4 < \epsilon$ and hence there exists an y' such that $L_4 > 0$. Take $y'_1 = a$, $y'_k = b$ for all $2 \leq k \leq n-1$ and $y'_n = n\bar{g} - \sum_{h=1}^{n-1} y'_h$. Notice $y'_n \geq z(\bar{g})$ since $\bar{g} \geq z(\bar{g})$. For this y' , remembering that $\frac{\partial d}{\partial y_i}(y'_n, \bar{g}) = 0$, we have:

$$l - L_4 = L_5 - L_4 = -\frac{1}{n} \left(2 \frac{\partial d}{\partial y_i}(b, \bar{g}) s(b, \bar{g}) - \frac{\partial d}{\partial y_i}(a, \bar{g}) s(a, \bar{g}) \right).$$

Taking n sufficiently large, we can make $l - L_4 < \epsilon$, implying $L_4 > 0$, which violates inequality 4. The condition is therefore necessary.

Sufficiency follows from the fact that if there exists an $y \in Y$ violating inequality 4, inequality 5 is violated as well for a particular value of y_j . For all $y \in Y$ there exists $b \in [0, z(\bar{y}))$ such that, taking $y_j = b$ in L_5 , we have $L_4 < L_5$:

$$-\frac{1}{n(y)} \sum_{j=1}^{n(y)} \frac{\partial d}{\partial y_j}(y_j, \bar{y}) s(y_j, \bar{y}) < -\frac{\partial d}{\partial y_j}(b, \bar{y}) s(b, \bar{y}),$$

where the strict inequality comes from the presence of the non-poor agent n . This b is the solution of the following problem:

$$b = \arg \max_{y_j} -\frac{\partial d}{\partial y_j}(y_j, \bar{y}) s(y_j, \bar{y}).$$

■

In essence, the more important is relative income in the well-being ordering, the narrower are the possibilities for the index to give more priority to some well-being levels over others. The intuitions for this result and its implications are discussed in the remaining part of this section, based on particular cases. Before moving to an example, I should emphasize the following point. The existence of a numerical representation satisfying *Monotonicity in Income* and

Weak Transfer is not guaranteed for all well-being orderings in $\mathcal{R}$.⁹ This potential issue should not be seen as problematic as such numerical representations exists for the most natural $\succeq \in \mathcal{R}$, in particular, for all those satisfying restriction $\mathbf{R}^5$, as illustrated below.

4.2.2 Constraints on numerical representation: an example

Assume that the social planner chooses the well-being ordering in the *linear subdomain* illustrated in Figure 1.b. This linear subdomain $\mathcal{R}^{Lin} \subset \mathcal{R}$ is defined from $\mathcal{R}$ by two additional restrictions: (i) all $\succeq \in \mathcal{R}^{Lin}$ satisfy restriction $\mathbf{R}^5$, (ii) for all $\succeq \in \mathcal{R}^{Lin}$, the threshold function z must be linear: $z(\bar{y}) = z^0 + \bar{s} \bar{y}$, with $z^0 \geq 0$ and $\bar{s} \in [0, 1]$. Clearly, all numerical representations of the deprivation orderings satisfying $\mathbf{R}^5$ are function of a unique variable: $\frac{y_i}{z(\bar{y})}$. Assume further that the numerical representation d must belong to the *quadratic family* D^q illustrated in Figure 3.a, i.e. for all $\frac{y_i}{z(\bar{y})} \in [0, 1]$:

$$d(y_i, \bar{y}) = \left(1 - \frac{y_i}{z(\bar{y})}\right) + \alpha \left(\left(\frac{y_i}{z(\bar{y})}\right)^2 - \frac{y_i}{z(\bar{y})} \right) \quad \text{with } \alpha \in [-1, 1]$$

This family has no particular ethical appeal but simplifies drastically the exposition. The parameter α can be interpreted as an indicator of poverty aversion, controlling for the priority granted to agents at the bottom of the well-being distribution. If $\alpha = 0$, the poverty index will attach equal *priority* to all poor agents in the distribution, no matter their differences in well-being. The poverty index gives extra attention to agents with lower well-being if $\alpha > 0$ and to the poor agents whose income is closer to the income threshold $z(\bar{y})$ if $\alpha < 0$. The poverty index implied by the numerical representation does not satisfy *Domination among the Poor* when $\alpha \notin [-1, 1]$. The implications of *Monotonicity in Income* for this example are formally stated in Corollary 1 and illustrated in Figure 3.b. The coefficient of poverty aversion is bounded below and above and the mathematical expression of those bounds depends monotonically on the slope of the indifference curves of the well-being ordering.

Corollary 1 (Bounds on poverty aversion - example). *For all $\succeq \in \mathcal{R}^{Lin}$, an additive poverty index with $d \in D^q$ satisfies *Monotonicity in Income* if and only if:*

$$\frac{(\bar{s} - 1)}{(1 + \bar{s})} \leq \alpha \leq \frac{4 - \bar{s} + 4(1 - \bar{s})^{\frac{1}{2}}}{(\bar{s} + 8)}$$

Proof. See Appendix. ■

Granting a minimal precedence to absolute income generates hence a trade-off between the importance given to relative income in the well-being ordering and the poverty aversion implied by the numerical representation. The more important is relative income for well-being, the steeper is the slope of the threshold function, the narrower is the range of acceptable values of poverty aversion and the lower the priority that can be given to agents with very low well-being. Notice that if relative income does not matter for well-being ($\mathbf{R}^6$), then the previous result puts no additional constraints on the numerical representations. In that case, *Monotonicity in Income* is implied by *Domination among the Poor*.

⁹I am grateful to Martin Ravallion for pointing out this issue.

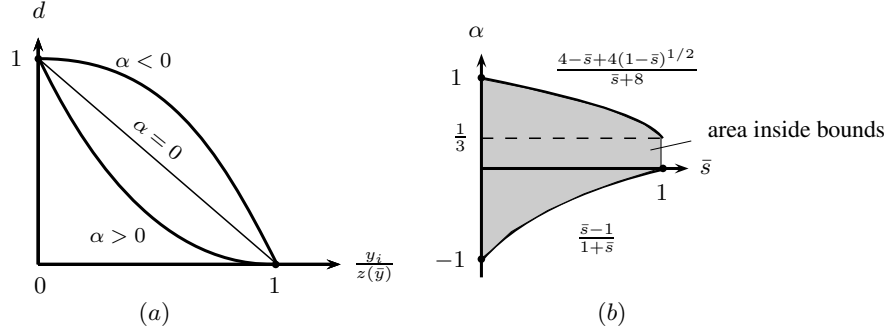


Figure 3: (a) Numerical representations of the quadratic family for several values of the poverty aversion parameter α . (b) The upper and lower bounds on the poverty aversion parameter α evolve monotonically as a function of the slope $\bar{s}$ of the threshold function.

The intuition for the bounds on poverty aversion is as follows: a positive increment to the income of a poor agent i will affect the relative income of another poor agent j . If the slope of the indifference curves is flat, agent j 's well-being is not affected, but the steeper the slope is, the larger the drop in j 's well-being. The additive poverty index will balance the gain made by agent i with the loss made by j . If the index give disproportionate attention to the well-being level of j compared to that of agent i , it could conclude that poverty has increased. This judgement would contradict the minimal precedence that the absolute component of poverty should have over its relative component.

This example was only meant to give intuition on the trade-off introduced by *Monotonicity in Income*. The next section studies the drastic implications of this axiom for FGT indices, the family most used in empirical applications.

4.2.3 Foster-Greer-Thorbecke indices

Assume that the well-being ordering chosen by the social planner satisfies restriction $\mathbf{R}^5$. Assume further that the threshold function is *non-flat*: there exists $\bar{g} \geq 0$ with $\bar{g} \geq z(\bar{g})$ such that $s(z(\bar{g}), \bar{g}) > 0$. This is typically the case of well-being orderings violating restriction $\mathbf{R}^6$. Assume finally that the social planner wishes to select a numerical representation d belonging to the Foster-Greer-Thorbecke family $\mathcal{D}^{FGT}$ proposed by Foster et al. (1984).

Definition 4 (FGT family of numerical representation). *The numerical representation d belongs to the Foster-Greer-Thorbecke family $\mathcal{D}^{FGT}$ if and only if for all $\frac{y_i}{z(\bar{y})} \in [0, 1]$:*

$$d(y_i, \bar{y}) = \left(1 - \frac{y_i}{z(\bar{y})}\right)^\alpha \quad \text{with } \alpha \geq 0$$

This family allows for a wide range of judgements by modifying the parameter α , the coefficient of poverty aversion. In particular, some among the most popular poverty indices are members of this family. The Head-Count Ratio is

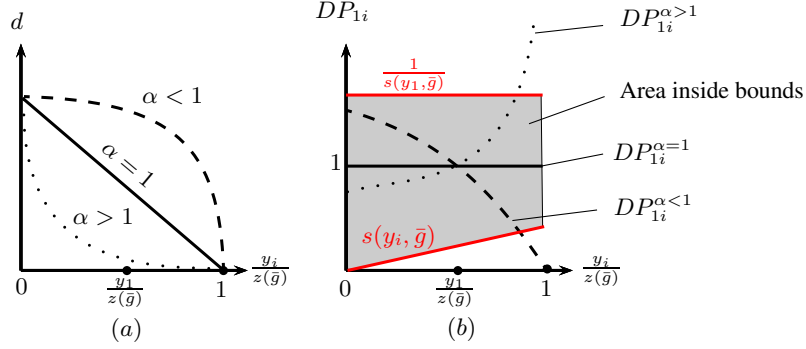


Figure 4: (a) Three Foster-Greer-Thorbecke numerical representations associated to different values of the coefficient of poverty aversion α . (b) Only when $\alpha = 1$ does the degree of priority always lie in the bounds imposed by *Monotonicity in Income*.

obtained when taking $\alpha = 0$ and the Poverty Gap Ratio for $\alpha = 1$.¹⁰ Let us examine the consequences of Theorem 2 for the coefficient α by means of an example illustrated in Figure 4. Figure 4.a shows three possible numerical representations attached to different values of the coefficient α . For which values of α does the additive poverty index respect *Monotonicity in Income*? Answering this question can be done by checking the necessary and sufficient conditions of Theorem 2 at mean income $\bar{y} = \bar{g}$ where the threshold curve is *non-flat*. The answer is illustrated in Figure 4.b. As soon as $\alpha > 1$, the degree of priority given to any poor income - say y_1 in Figure 4 - over y_i goes to infinity when y_i tends to $z(\bar{g})$. This arises due to the exponential form of the FGT numerical representations. Since by assumption $s(y_1, \bar{g}) > 0$, the upper-bound on the degree of priority $DP_{1i}(y)$ is therefore violated, as shown by the dotted curve. On the contrary, if $\alpha < 1$, the degree of priority $DP_{1i}(y)$ goes to zero when y_i tends to $z(\bar{g})$, for symmetric reason. As $s(z(\bar{g}), \bar{g}) > 0$, this violates the lower-bound, as shown by the dashed curve. Finally, when $\alpha = 1$, the degree of priority is always equal to one. This value lies automatically inside the bounds since the slopes of poor agent's indifference curves belong to $[0, 1]$ for all $R \in \mathcal{R}$. Therefore we have the following Corollary of Theorem 2:

Corollary 2 (Poverty Gap Ratio). *The Poverty Gap Ratio is an additive poverty index, obtained from FGT numerical representation with $\alpha = 1$ and defined as $PGR(y) = \frac{1}{n(y)} \sum_{i=1}^{q(y)} \left(1 - \frac{y_i}{z(\bar{y})}\right) = \frac{q(y)}{n(y)} \left(1 - \frac{1}{q(y)} \sum_{i=1}^{q(y)} \frac{y_i}{z(\bar{y})}\right)$. Let $\succeq$ be a well-being ordering satisfying $\mathbf{R}^5$ and equipped with a **non-flat** threshold function. The only additive poverty index of the Foster-Greer-Thorbecke family respecting *Monotonicity in Income* is the Poverty Gap Ratio.*

Proof. The argument of the proof is given above. A formal proof valid for a class of reference statistics is provided in section 5. $\blacksquare$

¹⁰The Head-Count Ratio is defined as $HCR(y) = \frac{q(y)}{n(y)}$ and Poverty Gap Ratio is defined in corollary 2.

In the particular family of FGT indices, the implications of *Monotonicity in Income* are particularly sharp. There is a collapse of the acceptable values of poverty aversion for non-flat threshold functions. As a result, the only member of the FGT family which gives a minimal precedence to absolute income is the Poverty Gap Ratio. Notice that the Poverty Gap Ratio also respects *Weak Transfer* since its numerical representation is convex.

5 Extension to a class of reference statistics

I have assumed so far that the reference statistic for relative income was mean income. I investigate in this section the robustness of the results when relative income is defined as $\frac{y_i}{f(y)}$, where f is a reference statistic on the distribution belonging to a certain class.

5.1 Class Definition

The question addressed here is the following: What is the class of reference statistics to which the results derived above can be extended? We will denote this class of functions by $\mathcal{F}$.

Definition 5 (Class of reference statistics). *The class $\mathcal{F}$ is the set of continuous and symmetric functions $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ respecting:*

- *Monotonicity:* For all $y \in \mathbb{R}_+^n$, $i \leq n$, we have $\frac{\partial f}{\partial y_i}(y) > 0$,
- *Separability:* For all $x, y \in \mathbb{R}_+^n$, if $f(x) = f(y)$, then $f(x, y) = f(x)$,
- *No upper-bound:* For all $y_{-i} \in \mathbb{R}_+^{n-1}$, $a \geq f(0, y_{-i})$, there exists $y_i \geq 0$ such that $f(y_i, y_{-i}) \geq a$.
- *Concavity:* For all $y_{-i} \in \mathbb{R}_+^{n-1}$, if $0 \leq a < b$, then $f(a, y_{-i}) + f(b, y_{-i}) \leq 2f(\frac{a+b}{2}, y_{-i})$.

A typical example of a reference statistic in $\mathcal{F}$ is a social welfare function $f(y) = \frac{1}{n} \sum_i u(y_i)$, where u is a concave and unbounded utility function. Three remarks must be made on this class of functions. First, concavity is not a necessary restriction but it simplifies greatly the exposition. Second, the median is not a member of $\mathcal{F}$. Theorem 1 does not hold when median income is the reference statistic because the median does not depend on the income of poor agents. In practice, median income is widely used for defining relative poverty lines. This is for example the case of the AROP income poverty measure used by the European Commission.¹¹ There are advantages in practice for using median income. The median is more robust than the mean in random samples, as established in Cowell and Victoria-Feser (1994). In Theory however, the median has no superior appeal which justifies its use as reference statistic. In particular, this statistic violates the monotonicity requirement, which seems compelling for my purpose. Furthermore, as it has been shown by de Mesnard (2007), poverty lines referring to the median lead to poverty measures that behave in a counter

¹¹The *At Risk Of Poverty* index is the Head-Count Ratio where the threshold function is defined as 60 % of median income.

intuitive way for income distributions experiencing an increase in inequality. An illustration is provided by [Easton \(2002\)](#) who comments on the decrease of the Head-Count Ratio in New-Zealand between 1981 and 1992. According to him, the implementation of policies inducing regressive transfers led to a decrease in the income of the bottom 80 % of households. Nevertheless, the median based Head-Count Ratio dropped due to the significant decline in median income. Third, the *no upper-bound* requirement is strong and not uncontroversial. In this paper, this requirement has been used in order to obtain additively separable poverty indices. Do the results carry over to statistics violating this condition? In essence, the answer is yes, but it comes at the cost of a reduction of the set Y^f of income distributions the poverty indices can rank. Typically, the reduced set $Y^{f'}$ will contain all income vectors with a minimal share of non-poor agents. I illustrate this point using an example. Let $Y^{f'}$ be the subset of income distributions with at least a percentage π of non-poor agents, such that $Y^{f'} = \{y \in \mathbb{R}_+^n \mid \frac{n-q}{n} \geq \pi\}$. Let the reference statistic be $f(y) = \frac{1}{n} \sum_i u(y_i)$ where $u(y_i)$ is a strictly increasing function infinitely close to $u'(y_i) = \min(y_i, t)$, with $t > 0$ defining the income threshold above which an agent becomes so rich that further increases in her income do not affect the reference statistic anymore. If $\pi > \frac{1}{t}$, then the results still hold in essence: Theorem 1 holds, the bounds derived for Theorem 2 are slightly less narrow, but the Poverty Gap Ratio is still the only FGT index respecting *Monotonicity in Income*.

5.2 Extension of results

Once a reference statistic in $\mathcal{F}$ is selected, the notations, restrictions and axioms must be adapted accordingly. Write $I = f(Y^f)$ the set of values that the reference statistic f can take, where $I \subseteq \mathbb{R}$ is a real interval. For all $a > 0$, let $f^a = f(a, \dots, a)$ be the value of the reference statistic for an equal income vector in which all agents earn a . The social planner picks a well-being ordering $\succeq$ on the set of bundles $X^f = \{(y_i, f(y)) \in \mathbb{R}_+ \times I\}$. Let the reference bundle defining the poverty threshold be (z_1^f, z_2^f) . The domain $\mathcal{R}^f$ of $\succeq$ is again defined by the corresponding restrictions, obtained by replacing mean income by the reference statistic in their definition.¹² Poverty axioms are adapted using the same convention. The additive separability result becomes:

Theorem 3 (Family of additive poverty indices). *$P : Y^f \rightarrow \mathbb{R}$ is an income poverty index satisfying *Domination among the Poor*, *Weak Subgroup Consistency*, *Symmetry*, *Continuity* and *Replication Invariance*, if and only if there exist a numerical representation d of the underlying well-being ordering $\succeq \in \mathcal{R}^f$ and a continuous and strictly increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $y \in Y^f$:*

$$P(y) = F\left(\frac{1}{n(y)} \sum_{i=1}^{n(y)} d(y_i, f(y))\right). \quad (6)$$

Proof. See Appendix. ■

¹²Restriction $\mathbf{R}^2$ becomes: For all $(y_i, f(y)) \in X_p^f$, $a > 0$, we have $(y_i + a, f(y + a1_n)) \succeq (y_i, f(y))$ and $\mathbf{R}^3$ becomes: For all $(y_i, f(y)) \in X^f$ with $f^{y_i} > f(y)$, there exists $a > 0$ such that $(a, f^a) \succ (y_i, f(y))$.

Straightforward adaptation of the notions of degree of priority and slope allows to derive the extended version of Theorem 2:

Theorem 4 (Bounds on degree of priority). *An additive poverty index satisfies **Monotonicity in Income**:*

1. (sufficient condition) if for all $y \in Y^f$, $y_i, y_j < z(f(y))$, we have:

$$s(y_j, f(y))n \frac{\partial f}{\partial y_i}(y) \leq DP_{ij}(y) \leq \left(s(y_i, f(y))n \frac{\partial f}{\partial y_j}(y) \right)^{-1} \quad (7)$$

2. (necessary condition) only if for all $y \in Y^f$ such that $f(y) = f^g$ where $g > 0$ and $z(f^g) < g$, and all $y_i, y_j < z(f(y))$, inequality 7 holds.

Proof. The difference between inequality 7 and inequality 2 is the factor $n \frac{\partial f}{\partial y_i}(y)$, which equals 1 when f is the mean. The *sufficient* and *necessary* conditions are proven using directly similar reasoning as in Theorem 2 and are hence omitted. ■

Theorem 4 shows the same trade-off appear between the importance of relative income and the range of acceptable values for poverty aversion. The implications of this extended Theorem on the poverty indices in the FGT family are as drastic as before: only the Poverty Gap Ratio satisfies **Monotonicity in Income**.

Corollary 3 (Poverty Gap Ratio extended). *Let $\succeq$ be a well-being ordering satisfying $\mathbf{R}^5$ and equipped with a **non-flat** threshold function. The only additive poverty index of the Foster-Greer-Thorbecke family respecting **Monotonicity in Income** is the Poverty Gap Ratio.*

Proof. When $d \in \mathcal{D}^{FGT}$, we have for all $y \in Y^f$, $y_i, y_j < z(f(y))$:

$$DP_{ij}(y) = \left(\frac{z(f(y)) - y_i}{z(f(y)) - y_j} \right)^{\alpha-1}. \quad (8)$$

Let $f(y) \geq f^g$ where $g \geq z(f^g)$ and $s(z(f(y)), f(y)) > 0$. By restriction $\mathbf{R}^3$, such g always exists and such $f(y)$ with strictly positive slope also exists since the threshold function is non-flat. Three cases can arise:

- Case 1: $\alpha < 1$. Take any y_i with $0 < y_i < z(f(y))$. From equation 8, for all $\epsilon > 0$ there exists $y_j < z(f(y))$ such that $DP_{ij}(y) < \epsilon$. From the necessary condition of Theorem 4, the lower bound on $DP_{ij}(y)$ is $s(y_j, f(y))n \frac{\partial f}{\partial y_i}(y)$. This lower bound takes a strictly positive value. We have in effect that $n > 0$, by monotonicity of f we have $\frac{\partial f}{\partial y_i}(y) > 0$ and finally by construction $s(y_j, f(y)) > 0$. This value does not tend to zero when $\epsilon \rightarrow 0$ because then $y_j \rightarrow z(f(y))$ and therefore $s(y_j, f(y)) \rightarrow s(z(f(y)), f(y)) > 0$.
- Case 2: $\alpha > 1$. Take any y_j with $0 < y_j < z(f(y))$. From equation 8, for all $\epsilon > 0$ there exists $y_i < z(f(y))$ such that $DP_{ij}(y) < \epsilon$. The lower bound on $DP_{ij}(y)$ is violated for the same reason.

- Case 3: $\alpha = 1$. This case defines the Poverty Gap Ratio. For all $y \in Y^f$, $y_i, y_j < z(f(y))$, we have $DP_{ij}(y) = 1$. From the sufficient condition of Theorem 4, we must show that for all $y \in Y^f$, $y_i, y_j < z(f(y))$, we have $s(y_j, f(y))n \frac{\partial f}{\partial y_i}(y) \leq 1$. If this holds, the upper-bound is automatically respected.

Lemma 1. *The well-being ordering $\succeq$ satisfies $\mathbf{R}^2$ (Translation Monotonicity) only if for all $f(y) \in I$, $y_i \in [0, z(f(y))]$ we have:*

$$s(y_i, f(y)) \leq \left(n \frac{\partial f}{\partial y_1}(y^*) \right)^{-1} \quad (9)$$

where y^* is such that $y_1^* = 0$ and $f(y^*) = f(y)$.

Proof. Suppose inequality 9 does not hold for some $(a, b_f) \in X^f$ for which $s(a, b_f) = \left(n \frac{\partial f}{\partial y_1}(y^*) \right)^{-1} + c$ with $c > 0$. Translation monotonicity requires that for all $y \in Y$ and all $i \leq q(y)$ we have $d(y'_i, f(y')) - d(y_i, f(y)) \leq 0$ when y' is obtained from y by a uniform increment. When considering an infinitely small increment, the previous requirement becomes

$$\begin{aligned} -\frac{\partial d}{\partial y_i}(y_i, f(y)) &\geq \frac{\partial d}{\partial f}(y_i, f(y)) \left(\frac{\partial f}{\partial y_1}(y) + \dots + \frac{\partial f}{\partial y_n}(y) \right), \\ 1 &\geq s(y_i, f(y)) \left(\frac{\partial f}{\partial y_1}(y) + \dots + \frac{\partial f}{\partial y_n}(y) \right). \end{aligned}$$

Let y be such that $y_j = 0$ for all $j \leq n-2$, $y_{n-1} = a$ and $y_n = l$ such that $f(y) = b_f$. Taking n sufficiently large, we have that $l \geq z(b_f)$ and hence $y \in Y^f$. For agent $n-1$, the last inequality can then be rewritten as:

$$1 \geq \left(\left(n \frac{\partial f}{\partial y_1}(y^*) \right)^{-1} + c \right) \sum_{j=1}^n \frac{\partial f}{\partial y_j}(y), \quad (10)$$

$$1 \geq \frac{\sum_{j=1}^n \frac{\partial f}{\partial y_j}(y)}{\left(n \frac{\partial f}{\partial y_1}(y^*) \right)} + c \sum_{j=1}^n \frac{\partial f}{\partial y_j}(y). \quad (11)$$

When n increases, y_n increases as well and by concavity of f , $\frac{\partial f}{\partial y_n}(y)$ decreases. As $n \rightarrow \infty$, the first term of the right-hand side of inequality 11 tends to 1. Since the second term is strictly positive and does not tend to zero, we have a violation of Translation Monotonicity. ■

There remains to show the sufficient condition in Theorem 4 is satisfied. Since $\succeq$ satisfies $\mathbf{R}^2$ (Translation Monotonicity), by the previous Lemma, we have for all $f(y) \in I$, $y_j \in [0, z(f(y))]$ that

$$s(y_j, f(y))n \frac{\partial f}{\partial y_i}(y) \leq \left(\frac{\partial f}{\partial y_1}(y^*) \right)^{-1} \frac{\partial f}{\partial y_i}(y) \leq 1.$$

By the concavity of f , the sufficient condition for *Monotonicity in Income* is always satisfied. The Poverty Gap Ratio satisfies hence this axiom. ■

This last result proves the existence of additive poverty indices satisfying *Monotonicity in Income* for all reference statistic $f \in \mathcal{F}$ and all well-being orderings in $\mathcal{R}^f$ satisfying $\mathbf{R}^5$. However, one should not deduce from the previous result that the Poverty Gap Ratio is the only additive index satisfying *Monotonicity in Income*. Depending on the choice of the reference statistic and well-being ordering, other additive poverty indices will respect that axiom. The FGT family has a simple mathematical expression but there is no ethical reason to stick to that particular family.

6 Conclusion

The objective of this paper is to derive income poverty measures from a world perspective. There is a need to develop a measure of income poverty that can be compared across time and space, between populations with different standards of living. The challenge associated to such measures is to combine the absolute and relative aspects of income poverty in a transparent and meaningful way. Taking these aspects separately, absolute poverty measures point to growth promotion policies whereas purely relative measures point to redistribution policies. A sensible measure combining both aspects would then point to a balanced mix of both kinds of policies, which is the middle position taken by many governments in practice.

I have argued that a global measure based on two poverty lines raises difficult aggregation issues. Furthermore, constructing a single poverty line combining both aspects is only part of the solution. There remains to provide an appropriate index aggregating poverty in a population when the poverty line is not absolute. This question has received little attention so far. My proposal is to capture the trade-offs to be made between absolute and relative aspects at the individual level, by introducing a well-being ordering. This ordering is a normative choice of the social planner that allows her to compare transparently the income well-being of agents in different societies. A family of additive poverty indices that respects the chosen ordering is axiomatically derived. The precedence of absolute over relative aspects of income poverty was encapsulated into the axiom of *Monotonicity in Income*. This axiom is shown to have strong consequences on the acceptable numerical representations of the well-being ordering. Granting such minimal precedence to absolute income introduces a trade-off between the importance of relative income in the well-being ordering and the poverty aversion of the numerical representation. The larger is the well-being decrease for a given increase of the reference statistic, the smaller is the priority that the numerical representation can give to agents at the bottom of the income distribution. This trade-off is particularly sharp for indices of the Foster-Greer-Thorbecke family. As soon as relative income matters for income well-being, only the Poverty Gap Ratio respects *Monotonicity in Income*. The popularity of this family in applications makes therefore the Poverty Gap Ratio a natural index for global poverty. Measuring poverty from a world perspective can therefore be done by combining this index with an appropriate global poverty line, like the one proposed in Ravallion (2012) or in Atkinson and Bourguignon (2001).

Atkinson and Bourguignon (2001) discuss the implications of using a FGT index in combination with a unique poverty line. They compute the marginal valua-

tion of income for two agents with equal absolute income but living in societies with different mean income, and therefore different income thresholds. They observe that, when using the Poverty Gap Ratio, the impact on poverty of an additional unit of income is larger in a society with lower mean. Nevertheless, this conclusion can be reversed for higher poverty aversion. They raise then the question why a marginal unit of income might be valued more in societies with higher mean income. Some poverty indices satisfying all the axioms presented in this paper (but outside the FGT family) might draw this conclusion. The reason would be that, having the same income, the agent in the society with larger mean income has lower relative income. The agent in the richer society has therefore a lower income well-being. If sufficient priority is granted to agents with lower well-being, the marginal unit of income has more impact if given to the agent living in the high income society.

7 Appendix

7.1 Proof of Theorem 1

Step 1: From an ordering on income vectors to an ordering on lack of income well-being (LIWB) vectors.

Take any d representing $\succeq$.¹³ Let $m : Y \rightarrow \times_{i=1}^{n-1} [0, 1]_i$ be a mapping returning the LIWB vector v corresponding to an income vector y : $v = (v_1, \dots, v_{n-1}) = m(y)$ with $v_i = d(y_i, \bar{y})$ for all $i \leq n-1$. The vector v has size $n-1$ because $y \in Y$, i.e. $d(y_n, \bar{y}) = 0$ and is hence omitted. Therefore $m(y) \in V_d = \times_{i=1}^{n-1} [0, 1]_i$. The mapping m is continuous since d is continuous in both its arguments and the mean is a continuous function of its arguments.

Domination among the Poor implies *Indifference among the Poor*:

Poverty axiom 9 (*Indifference among the Poor*). *For all $y, y' \in Y$ such that $n(y) = n(y')$ and $q(y') = q(y)$, if $(y'_i, \bar{y}') \sim (y_i, \bar{y})$ for all $i \leq q(y)$, then $P(y) = P(y')$.*

As proven below, we have that $m(Y) = V_d$, the image set of Y through the mapping m is V_d . Therefore, *Indifference among the Poor* has the following consequence: for all d representing $\succeq$, there exists an ordering $\succeq_{V_d}$ on V_d such that for all $y, y' \in Y$ we have $y \succeq y' \Leftrightarrow m(y) \succeq_{V_d} m(y')$. The ordering on V_d depends on the numerical representation d defining mapping m . By *Continuity* and the continuity of the mapping m , the ordering $\succeq_{V_d}$ is continuous. Therefore $\succeq_{V_d}$ can be represented by a continuous poverty index $P^v : V_d \rightarrow \mathbb{R}$. Without loss of generality, we can take $P^v(m(y)) = P(y)$ for all $y \in Y$.

Step 2: The ordering $\succeq_{V_d}$ on LIWB vectors is additively separable.

We verify that the assumptions of Theorem 1 in Gorman (1968) are all met. This allows us to derive the the following functional form for the index P^v , for a given $n \in \mathbb{N}$:

$$P^v(v) = \tilde{F}\left(\sum_{i=1}^{n-1} \tilde{\phi}(v_i)\right) \quad (12)$$

where $\tilde{F}$ and $\tilde{\phi}$ are strictly increasing functions. The assumptions required for this Theorem are the following:

Step 2.1: There exists a *complete* and *continuous* ordering $\succeq_{V_d}$ on a *product space* $V_d = \times_{i=1}^{n-1} [0, 1]_i$.

The properties of completeness and continuity of the ordering $\succeq_{V_d}$ follows from the completeness of the ordering $\succeq$, the continuity of the mapping m and *Continuity*. There remains to show that $m(Y) = \times_{i=1}^{n-1} [0, 1]_i$. This means both (i) $m(y) \in V_d$ for all $y \in Y$ and (ii) for all $v \in V_d$ there exists $y \in Y$ such that $v = m(y)$. If (i) follows directly from the definition of the mapping m , (ii) must be proven.

Take any $v \in V_d$. Let $\bar{g} > 0$ be such that $\bar{g} \geq z(\bar{g})$. Income $\bar{g}$ is therefore a level

¹³We can restrict ourselves to continuous numerical representations d without loss of generality.

of income such that, if consumed by all agents in the distribution, all agents are non-poor. Such $\bar{g}$ always exists as proven in the next Lemma.

Lemma 2. *For all $\succeq \in \mathcal{R}$, $(z_1, z_2) \in X$, there exists $\bar{g} > 0$ such that $\bar{g} \geq z(\bar{g})$.*

Proof. Suppose there exists $b > 0$ such that $b < z(b)$ and consider the bundle $(z(b), b)$. By $\mathbf{R}^3$, there exists $\bar{g} > 0$ such that $(\bar{g}, \bar{g}) \succ (z(b), b)$. This means $\bar{g} > z(\bar{g})$. ■

We need to construct y such that $v = m(y)$. The next Lemma shows it can always be done.

Lemma 3. *For all $\succeq \in \mathcal{R}$, $(z_1, z_2) \in X$, $\bar{g} \geq z(\bar{g})$, $v \in V_d$, there exists $y \in Y$ with $\bar{y} = \bar{g}$ such that $v = m(y)$.*

Proof. The existence of such $\bar{g}$ as been shown in Lemma 2. Let y be such that, for all $i \leq q$, $y_i = a_i$ defined implicitly by $v_i = d(a_i, \bar{g})$. By $\mathbf{R}^4$ and the continuity of d , we have that $a_i \in [0, z(\bar{g})]$ for all $i \leq q$. Let y' be such that $y'_i = y_i$ for all $i \leq q$ and $y'_j = \bar{g}$ for all $q+1 \leq j \leq n$. We have $\bar{y}' \leq \bar{g}$. By the properties of the mean, there exists $l \geq \bar{g}$ such that, if $y_j = l$ for all $q+1 \leq j \leq n$, then we have $\bar{y} = \bar{g}$. As $l \geq \bar{g} \geq z(\bar{g})$, all agents j with $q+1 \leq j \leq n$ are non-poor. ■

Step 2.2: Each sector $[0, 1]_i$ of V_d has a *countably dense subset*, is *arc-connected* and is *strictly essential*.

As all sectors are real intervals, they have a *countably dense subset* and are *arc-connected*. *Strict essentiality* means that given any $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n-1}) \in \times_{j=1}^{n-2} [0, 1]_j$, not all elements of $[0, 1]_i$ are indifferent for the ordering $\succeq_{V_d}$. This follows from *Domination among the Poor*.

Step 2.3: Let $N^* = \{1, \dots, n-1\}$ be the set of sectors in V_d and $A \subseteq N^*$ be any subset of sectors, we have that each A is *separable*. *Separability* means that for all $(u, w), (v, w), (u, t), (v, t) \in V_d$, we have $P^v(u, w) \geq P^v(v, w) \Leftrightarrow P^v(u, t) \geq P^v(v, t)$.

Step 2.3.1: Construct appropriate income vectors.

Construct $y^1, y^2, y^3, y^4 \in Y$ such that $m(y^1) = (u, w)$, $m(y^2) = (v, w)$, $m(y^3) = (u, t)$, $m(y^4) = (v, t)$ and $\bar{y}^1 = \bar{y}^2 = \bar{y}^3 = \bar{y}^4 = \bar{g}$ with $\bar{g} \geq z(\bar{g})$. Such vectors can be constructed as shown in Lemma 3.

Decompose in subgroups $y^1 = (y_A^1, y_B^1, y_n^1)$, such that subvectors y_A^1 and y_B^1 are associated to the LIWB subvectors u and w respectively. Typically, $\bar{y}_A^1 \neq \bar{y}_B^1 \neq \bar{g}$ but our next operations will aim at obtaining such equality.

TriPLICATE y^1 and re-organize the subgroups to obtain at least one non-poor agent per subgroup. Let $y^{1'} = (y^1, y^1, y^1) = (y_A^1, y_A^1, y_A^1, y_B^1, y_B^1, y_B^1, y_n^1, y_n^1, y_n^1)$. This triplication does not affect the mean: $\bar{y}^{1'} = \bar{y}^1$. Reorganize subgroups: $y^{1'} = (y_A^1, y_B^1, y_n^1)$ with $y_A^1 = (y_A^1, y_A^1, y_A^1, y_n^1)$ and $y_B^1 = (y_B^1, y_B^1, y_B^1, y_n^1)$. Letting $u' = (u, u, u)$ and $w' = (w, w, w)$, we have that $m(y^{1'}) = (u, u, u, 0, w, w, w, 0) = (u', 0, w', 0)$.

Construct y_A^{1*} such that $m(y_A^{1*}) = u'$ with $\bar{y}_A^{1*} = \bar{g}$ and y_B^{1*} such that $m(y_B^{1*}) = w'$ with $\bar{y}_B^{1*} = \bar{g}$. Those income vectors exist as proven in Lemma 3, as subgroups A' and B' hold each at least one non-poor agent. The income vector $y^{1*} = (y_A^{1*}, y_B^{1*}, \bar{g})$ is such that $m(y^{1*}) = (u', 0, w', 0)$. This vector is such that

$\bar{y}^{1*} = \bar{g}$ as its three subgroups have mean $\bar{g}$.

Using the same procedure (decomposition, triplication, reorganisation), construct $y^{2'}, y^{3'}, y^{4'}$ and y^{2*}, y^{3*}, y^{4*} such that:

$$\begin{aligned} y^{1*} &= (y_{A'}^{1*}, y_{B'}^{1*}, \bar{g}) & \text{with } m(y^{1*}) &= (u', 0, w', 0) = (u, u, u, 0, w, w, w, 0), \\ y^{2*} &= (y_{A'}^{2*}, y_{B'}^{2*}, \bar{g}) & \text{with } m(y^{2*}) &= (v', 0, w', 0) = (v, v, v, 0, w, w, w, 0), \\ y^{3*} &= (y_{A'}^{3*}, y_{B'}^{3*}, \bar{g}) & \text{with } m(y^{3*}) &= (u', 0, t', 0) = (u, u, u, 0, t, t, t, 0), \\ y^{4*} &= (y_{A'}^{4*}, y_{B'}^{4*}, \bar{g}) & \text{with } m(y^{4*}) &= (v', 0, t', 0) = (v, v, v, 0, t, t, t, 0). \end{aligned}$$

For all $m \in \{1, 2, 3, 4\}$, we have $P(y^{m'}) = P(y^m)$ by *Replication Invariance*. As $(y_i^{m'}, \bar{g}) \sim (y_i^{m*}, \bar{g})$ for all $i \leq q(y^{m*})$, we have $P(y^{m*}) = P(y^{m'})$ by *Indifference among the Poor*. Therefore, proving $P(y^{1*}) \geq P(y^{2*}) \Leftrightarrow P(y^{3*}) \geq P(y^{4*})$ is equivalent to proving $P^v(u, w) \geq P^v(v, w) \Leftrightarrow P^v(u, t) \geq P^v(v, t)$. For notational simplicity, drop the symbols $*$ and $'$ to name the new vectors and subgroups as the old ones.

Step 2.3.2: Prove $P(y_A^1, y_B^1, \bar{g}) \geq P(y_A^2, y_B^2, \bar{g}) \Leftrightarrow P(y_A^3, y_B^3, \bar{g}) \geq P(y_A^4, y_B^4, \bar{g})$. Our income vectors are constructed such that $P(y_A^1) = P(y_A^3)$, $P(y_A^2) = P(y_A^4)$, $P(y_B^1) = P(y_B^2)$ and $P(y_B^3) = P(y_B^4)$ by *Indifference among the Poor*. By assumption, we have $P(y^1) \geq P(y^2)$. As $P(y_B^1) = P(y_B^2)$, by *Weak Subgroup Consistency*, we have that $P(y_A^1, \bar{g}) \geq P(y_A^2, \bar{g})$ (remember all our subgroups have their mean equal to $\bar{g}$). By *Weak Subgroup Consistency* again, this implies $P(y_A^1) \geq P(y_A^2)$.

Then, $P(y_A^1) \geq P(y_A^2)$ together with $P(y_A^1) = P(y_A^3)$ and $P(y_A^2) = P(y_A^4)$ imply $P(y_A^3) \geq P(y_A^4)$. Two cases can arise.

- Case 1: $P(y_A^3) > P(y_A^4)$. As $P(y_B^3) = P(y_B^4)$, we have $P(y_B^3, \bar{g}) = P(y_B^4, \bar{g})$ by *Indifference among the Poor*. Together we obtain $P(y_A^3, y_B^3, \bar{g}) > P(y_A^4, y_B^4, \bar{g})$, by *Weak Subgroup Consistency*.
- Case 2: $P(y_A^3) = P(y_A^4)$. We can not have $P(y_A^3, y_B^3, \bar{g}) < P(y_A^4, y_B^4, \bar{g})$. Otherwise, as $P(y_A^3) = P(y_A^4)$, *Weak Subgroup Consistency* implies that $P(y_A^3, y_B^3, y_A^4, \bar{g}) < P(y_A^4, y_B^4, y_A^3, \bar{g})$. Again, as $P(y_B^3) = P(y_B^4)$, we obtain $P(y_A^3, y_B^3, y_A^4, y_B^4, \bar{g}) < P(y_A^4, y_B^4, y_A^3, y_B^3, \bar{g})$. This is a contradiction as the two vectors are identical.

We can therefore use Theorem 1 in Gorman (1968) and obtain, for all $v \in V_d$:

$$P^v(v) = \tilde{F}' \left(\sum_{i=1}^{n-1} \tilde{\phi}_i(v_i) \right)$$

where $\tilde{F}'$ and $\tilde{\phi}_i$ are strictly increasing functions. Functions $\tilde{\phi}_i$ might still depend on the rank i of the considered agent. Nevertheless, since $\succeq_{V_d}$ is separable, we must have $\tilde{\phi}_i = \tilde{\phi} + f(i)$. Defining $\tilde{F}(x) = \tilde{F}'(x + \sum f(i))$, a translation of $\tilde{F}'$, we can rewrite equation 12 with function $\tilde{\phi}$ independent of the rank i .

Step 3: Adapt the procedure of Foster and Shorrocks (1991) to show functions $\tilde{F}$ and $\tilde{\phi}$ do not depend on the number of agents n .

Theorem 1 in [Gorman \(1968\)](#) is valid for a fixed number of agents n . Therefore, when n is allowed to vary, equation 12 must be written:

$$P^v(v) = \tilde{F}_n \left(\sum_{i=1}^{n-1} \tilde{\phi}_n(v_i) \right)$$

Step 3.1: Define transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$ for normalization purposes. Let F_n and ϕ_n be the following transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$:

$$\begin{aligned} \phi_n(v_i) &= n[\tilde{\phi}_n(v_i) - \tilde{\phi}_n(0)], \\ F_n(x) &= \tilde{F}_n[x + (n-1)\tilde{\phi}_n(0)]. \end{aligned}$$

This yields: $P^v(v) = F_n\left(\frac{1}{n} \sum_{i=1}^{n-1} \phi_n(v_i)\right)$ with $\phi_n(0) = 0$. Since the last agent is always non-poor, we have $d(y_n, \bar{y}) = 0$. Therefore, we obtain that for all $n \geq 2$, by slightly abusing notations (by introducing the zero deprivation of agent n at the end of the vector v):

$$P^v(v) = F_n \left(\frac{1}{n} \left(\phi_n(0) + \sum_{i=1}^{n-1} \phi_n(v_i) \right) \right) = F_n \left(\frac{1}{n} \sum_{i=1}^n \phi_n(v_i) \right) \quad (13)$$

with F_n and ϕ_n continuous, strictly increasing and $\phi_n(0) = 0$.

Step 3.2: Use *Replication Invariance* to prove functions F_n and ϕ_n do not depend on n .

I do not reproduce here the full reasoning of [Foster and Shorrocks \(1991\)](#) that can be easily adapted to our framework. I rather provide the intuition behind it. Let $x, y \in Y$ be such that x is a k -replication of y . Letting $r = n(x)$ and $s = n(y)$, this means $r = ks$ and $x = (y, y, \dots, y)$ for some positive integer k . Let $u = m(x)$ and $v = m(y)$, using the same abuse of notations as before. Write $F = F_s$ and $\phi = \phi_s$. By equation 13, we have:

$$\begin{aligned} P^v(v) &= F \left(\frac{1}{s} \sum_{i=1}^s \phi(v_i) \right) = F \left(\frac{1}{s} (\phi(v_1) + \dots + \phi(v_s)) \right), \\ P^v(u) &= F_r \left(\frac{1}{r} \sum_{i=1}^r \phi_r(u_i) \right) = F_r \left(\frac{1}{ks} (k\phi_r(v_1) + \dots + k\phi_r(v_s)) \right) \\ &= F_r \left(\frac{1}{s} (\phi_r(v_1) + \dots + \phi_r(v_s)) \right) \end{aligned}$$

By *Replication Invariance*, we have that $P^v(v) = P^v(u)$ and therefore:

$$F \left(\frac{1}{s} \sum_{i=1}^s \phi(v_i) \right) = F_r \left(\frac{1}{s} \sum_{i=1}^s \phi_r(v_i) \right)$$

This expression holds for all k -replications. The reader interested by the extension of this domain of validity to all $y \in Y$ with $n(y) = r$ will find the full reasoning in [Foster and Shorrocks \(1991\)](#).

Even if r is not a multiple of s , the same expression can still be used, because the same reasoning can be applied between r and the least common multiple

between r and s . Use transformations d' and G of functions ϕ and F in order for the domain of image of d' to be $[0, 1]$. Letting $d'(y_i, \bar{y}) = \frac{\phi(d(y_i, \bar{y}))}{\phi(1)}$ and $G(x) = F(x\phi(1))$, we have for all $y \in Y$:

$$P(y) = G\left(\frac{1}{n} \sum_{i=1}^n d'(y_i, \bar{y})\right) \quad (14)$$

with G a continuous and strictly increasing function, d' is a numerical representation of $\succeq$.

7.2 Proof of Corollary 1

When $d \in \mathcal{D}^q$ and $\succeq \in \mathcal{R}^{Lin}$, we have $s(y_i, \bar{y}) = \bar{s} \frac{y_i}{z(\bar{y})}$ where $\bar{s} \in [0, 1]$ and:

$$\frac{\partial d}{\partial y_i}(y_i, \bar{y}) = -\frac{1}{z(\bar{y})} \left(1 + \alpha \left(1 - 2 \frac{y_i}{z(\bar{y})}\right)\right).$$

Inequality 4 becomes:

$$\frac{\frac{1}{n(y)} \sum_{j=1}^{n(y)} \left(1 + \alpha \left(1 - 2 \frac{y_j}{z(\bar{y})}\right)\right) \frac{y_j}{z(\bar{y})} \bar{s}}{\left(1 + \alpha \left(1 - 2 \frac{y_i}{z(\bar{y})}\right)\right)} \leq 1 \quad (15)$$

The following condition is necessary and sufficient for inequality 4 to hold for all $y \in Y$. Take any $\bar{y} > 0$, we must have that for all $y_i, y_j \in [0, z(\bar{y})]$:

$$\underbrace{\frac{y_j}{z(\bar{y})} \frac{1 + \alpha \left(1 - 2 \frac{y_i}{z(\bar{y})}\right)}{1 + \alpha \left(1 - 2 \frac{y_j}{z(\bar{y})}\right)}}_{L_{16}} \leq \frac{1}{\bar{s}}. \quad (16)$$

Inequality 16 is nothing else than inequality 5 adapted for $d \in \mathcal{D}^q$ and $\succeq \in \mathcal{R}^{Lin}$. The sufficiency of the condition is implied by the proof of Theorem 2. The particularity of this case is that inequality 16 is necessary for all $\bar{y} > 0$. Suppose this condition does not hold for some $\bar{y} = \bar{c} > 0$ and $y_i = a, y_j = b$ with $a, b \in [0, z(\bar{c})]$. Take any $\bar{g} > 0$ with $\bar{g} \geq z(\bar{c})$. Such $\bar{g}$ exists for all $\succeq \in \mathcal{R}$ and $(z_1, z_2) \in X$, as shown by Lemma 2 in the proof of Theorem 1. For $a', b' \in [0, z(\bar{g})]$ such that $\frac{a}{z(\bar{c})} = \frac{a'}{z(\bar{g})}$ and $\frac{b}{z(\bar{c})} = \frac{b'}{z(\bar{g})}$, inequality 16 is also violated. In effect, inequality 16 depends only on the fraction of own income over income threshold. This arises in this example because the slopes and the degrees of priority only depend on these fractions. The condition associated to inequality 16 is therefore also necessary.

Two cases can arise:

- Case 1: $\alpha < 0$, more priority is attached to poor agents with higher well-being. L_{16} is maximal for $y_i = 0$ and $\frac{y_j}{z(\bar{y})} \rightarrow 1$. Replacing those values yields the lower bound on α .
- Case 2: $\alpha \geq 0$, more priority is attached to poor agents with lower well-being. L_{16} is maximal for $y_i \rightarrow 1$ and $\frac{y_j}{z(\bar{y})} = \frac{(1+\alpha)}{4\alpha}$, when $\frac{1}{3} \leq \alpha \leq 1$. Replacing those values yields the upper bound on α .

7.3 Proof of Theorem 3

Few modifications to the proof of Theorem 1 are necessary in order to extend its validity for the class of reference statistics $\mathcal{F}$. Therefore, I do not reproduce the full proof but point to the necessary adaptations that go beyond the convention of replacing $\bar{y}$ by $f(y)$.

Step 1: The mapping $m^f : Y^f \rightarrow \times_{i=1}^{n-1} [0, 1]_i$ is still continuous as all reference statistic $f(y) \in \mathcal{F}$ must be *continuous* in y .

Step 2.1: The definition of income g whose existence is proved in Lemma 2 must be adapted. Let $g > 0$ be such that $g \geq z(f^g)$. As shown in Lemma 2, the existence of such g directly relies on restriction $\mathbf{R}^3$: for all $(y_i, f(y)) \in X^f$ with $f^{y_i} > f(y)$, there exists $a > 0$ such that $(a, f^a) \succ (y_i, f(y))$. Suppose there exists $b > 0$ such that $b < z(f^b)$. Consider the bundle $(z(f^b), f^b)$. By $\mathbf{R}^3$, there exists $g > 0$ such that $(g, f^g) \succ (z(f^b), f^b)$. Intuitively, g is a level of individual income which, if earned by all agents in the distribution, leaves all agents non-poor. Lemma 3 must be rewritten for the class $\mathcal{F}$:

Lemma 4. *For all $\succeq \in \mathcal{R}^f$, $(z_1^f, z_2^f) \in X^f$, $g \geq z(f(g))$, $v \in V_d$, there exists $y \in Y^f$ with $f(y) = f^g$ such that $v = m^f(y)$.*

Proof. Let y be such that, for all $i \leq q$, $y_i = a_i$ defined implicitly by $v_i = d(a_i, f(g))$. By $\mathbf{R}^4$ and the continuity of d , we have that $a_i \in [0, z(f^g)]$ for all $i \leq q$. Let y' be such that $y'_i = y_i$ for all $i \leq q$ and $y'_j = g$ for all $q+1 \leq j \leq n$. We have $f(y') \leq f^g$. As all reference statistics $f \in \mathcal{F}$ satisfy *continuity*, *monotonicity* and *No-upper Bound*, there exists $l \geq g$ such that, if $y_j = l$ for all $q+1 \leq j \leq n$, then we have $f(y) = f^g$. As $l \geq g \geq z(f^g)$, all agents j with $q+1 \leq j \leq n$ are non-poor. ■

Step 2.4.1: The triplication of income vectors leaves all agents unaffected since all reference statistics $f \in \mathcal{F}$ satisfy *separability* and therefore we have $f(y) = f(y, y, y)$. For the same reason, as $f(y_{A'}^{1*}) = f(y_{B'}^{1*}) = f(g)$, we have $f(y_{A'}^{1*}, y_{B'}^{1*}, g) = f^g$.

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