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On the Polyhedral Structure of a Multi–Item Production Planning Model with Setup Times

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Abstract

We present and study a mixed integer programming model that arises as a substructure in many industrial applications. This model provides a relaxation of various capacitated production planning problems, more general fixed charge network flow problems, and other structured mixed integer programs. After distinguishing the general case, which is $\mathcal{NP}$–hard, from a polynomially solvable case, we analyze the polyhedral structure of the convex hull of this model, as well as of a strengthened LP relaxation. Among other results, we present valid inequalities that induce facets of the convex hull in the general case. These inequalities suffice to solve this model by linear programming in the polynomially solvable case mentioned above, and they have proven computationally effective in solving capacitated lot–sizing problems (see also Miller et al. [2000a, 2000b]).

Keywords: Mixed integer programming, production planning, polyhedral combinatorics, capacitated lot–sizing, fixed charge network flow

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1 Introduction

One of the most successful techniques that has been used to solve integer programming (IP) and mixed integer programming (MIP) problems in recent years is the application of polyhedral results for structured relaxations of these problems. Often facet–defining families of valid inequalities can be defined for the relaxed models. Since these inequalities are also valid for the original problem, they can be applied in a branch–and–cut algorithm.

In this paper we introduce a model that occurs as a relaxation of many structured MIP problems. We derived this relaxation as a single–period relaxation of the multi–item capacitated lot–sizing problem with setup times (MCL), which is described in Section 2. In this relaxation we consider, among other decision variables, those that represent the amount of inventory carried over from the preceding period; therefore we call this relaxation PI, for *preceding inventory.* PI can be formulated as follows:

$$\min \quad \sum_{i=1}^{P} p^i x^i + \sum_{i=1}^{P} q^i y^i + \sum_{i=1}^{P} h^i s^i \tag{1}$$

subject to

$$x^i + s^i \geq d^i, i = 1, ..., P, \tag{2}$$

$$\sum_{i=1}^{P} x^i + \sum_{i=1}^{P} t^i y^i \leq c, \tag{3}$$

$$x^i \leq (c - t^i) y^i, i = 1, ..., P, \tag{4}$$

$$x^i, s^i \geq 0, i = 1, ..., P, \tag{5}$$

$$y^i \in \{0, 1\}, i = 1, ..., P. \tag{6}$$

Perhaps the simplest way to view PI is as a single–node, multi–item fixed charge flow model with two inbound arcs (see Figure 1.) The first of these arcs has capacity c, which is jointly imposed on P items. Moreover, a fixed capacity usage $t^i \geq 0$ must be incurred if any amount of item i is shipped along this arc. (If capacity is measured in time units, then t^i can be considered a setup time; we therefore refer to this quantity as the setup time for item i throughout this paper.) Thus x^i is the amount of flow shipped along this arc with unit cost p^i, and the binary variable y^i indicates whether the fixed capacity usage for item i is incurred; the cost for setting $y^i = 1$ is q^i. The second arc can also be used by each of these commodities. This arc is uncapacitated, and there are no fixed costs associated with flow on it. The variables s^i represent the amount of flow on this second arc for item i; the unit cost for shipping i along this arc is h^i. The total amount of flow shipped along these two arcs must be at least $d^i, i = 1, ..., P$.

We let $\mathcal{P} = \{1, ..., P\}$, and we denote the set of points defined by (2)–(6) as X^{PI}.

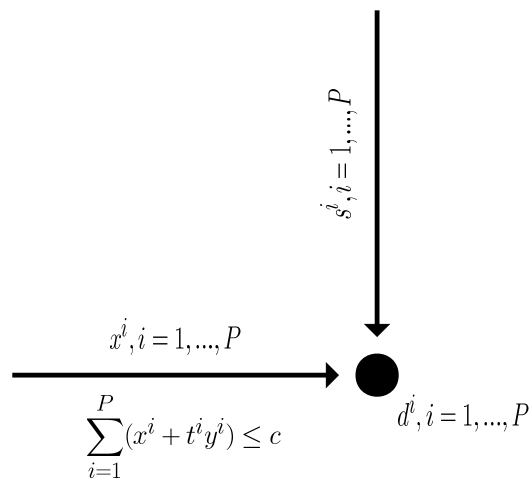

Figure 1: A Single–Node Multi–Commodity Model with Two Arcs

If $h^i < 0$ for any $i \in \mathcal{P}$, PI has unbounded optimum. We will therefore assume that $h^i \geq 0, i = 1, ..., P$. Also, we assume that $q^i > 0, i = 1, ..., P$, which ensures that in every optimal solution, $y^i = 1$ if and only if $x^i > 0$.

During the last two decades researchers have often used strong valid inequalities for structured MIP relaxations to solve more complicated models. Among those who pioneered such methods were Crowder, Johnson, and Padberg [1983], who demonstrated that the use of inequalities for knapsack relaxations defined by single constraints of 0–1 IP problems can be effective in solving IP's. Today the use of valid inequalities for the knapsack polytope is a common feature in IP solvers. Padberg, Van Roy, and Wolsey [1985] derived flow cover inequalities for a relaxation of MIP's defined by a single–node fixed charge (SNFC) network flow model, and the use of inequalities based on their work to solve MIP's has also become common.

When $t^i = 0, i = 1, ..., P$, PI resembles SNFC, but even in this case PI has a different, more complicated structure than SNFC. Both PI and SNFC have flow variables x^i, setup variables y^i, and a joint capacity constraint (3). Moreover, both have variable upper bounds similar to (4), although in SNFC these are generally tighter than in PI. PI, in addition, has constraints (2), which impose a demand on each of the flow variables, and variables s^i, which serve as slack variables for (2) and, for example, allow demand in excess of capacity to be satisfied.

All the results that we present for PI are valid for the case $t^i = 0, i = 1, ..., P$. The fact that our results also hold when setup times for some or all of the items are strictly positive only serves to make these results more applicable to industrial problems such as MCL (Miller et al. [2000b]).

An outline of the rest of the paper follows. In Section 2, we motivate our discussion of PI further by discussing applications for which it serves as a relaxation. These include not only MCL, but other lot–sizing problems and more general MIP's as well.

In Section 3, we analyze the polyhedral structure of PI. We first discuss briefly the complexity of PI. Although PI is $\mathcal{NP}$–hard, under certain conditions it is polynomially solvable. Thus, unless $\mathcal{NP} = \text{co–}\mathcal{NP}$, it is in general not possible to derive a good characterization of $\text{conv}(X^{PI})$ by families of linear inequalities (Karp and Papadimitrou [1982]). However, for the polynomially solvable case, we can hope to do so.

After identifying trivial facets, we analyze the extreme points and rays of $\text{conv}(X^{PI})$, and we show which of these extreme points are possible optimal solutions of PI, given the conditions on the objective function coefficients. This analysis not only allows us to characterize $\text{conv}(X^{PI})$ by extreme points and rays, it also allows us to describe which part of the polyhedron is the most interesting for our purposes. We perform a similar analysis for an LP relaxation of X^{PI} that is strengthened with a family of facet–inducing valid inequalities analogous to the *path inequalities* (Barany, Van Roy, and Wolsey [1984], Van Roy and Wolsey [1987]).

In Section 4, we derive new valid inequalities for PI. We give general conditions under which each of these classes of inequalities define facets of $\text{conv}(X^{PI})$, and we also show that these new inequalities cut off all possible fractional optimal solutions of the strengthened LP relaxation mentioned above. In defining the second of these classes, we use the concepts of superadditivity and lifting to expand the class into a more complete set of inequalities. After thus expanding this class, we can state that, under conditions that ensure that PI is polynomially solvable, the inequalities that we present allow it to be solved by linear programming (Miller et al. [2000a]).

2 Motivation

Our original motivation in studying PI was to derive strong valid inequalities for MCL by considering simplified models obtained from a single time period. A standard formulation for MCL is

$$\min \quad \sum_{i=1}^{P}\sum_{j=1}^{J} p_j^i x_j^i + \sum_{i=1}^{P}\sum_{j=1}^{J} q_j^i y_j^i + \sum_{i=1}^{P}\sum_{j=1}^{J} h_j^i s_j^i \tag{7}$$

subject to

$$x_j^i + s_{j-1}^i - s_j^i = d_j^i, i = 1, ..., P, j = 1, ..., J, \tag{8}$$

$$\sum_{i=1}^{P} x_j^i + \sum_{i=1}^{P} t_j^i y_j^i \leq c_j, j = 1, ..., J, \tag{9}$$

$$x_j^i \leq (c_j - t_j^i) y_j^i, i = 1, ..., P, j = 1, ..., J, \tag{10}$$

$$x_j^i, s_j^i \geq 0, i = 1, ..., P, j = 1, ..., J, \tag{11}$$

$$y_j^i \in \{0, 1\}, i = 1, ..., P, j = 1, ..., J. \tag{12}$$

The number of time periods in the problem is J. The production capacity, setup times, and demands in period j are given, respectively, by c_j, $t_j^i, i = 1, ..., P$, and $d_j^i, i = 1, ..., P$; p_j^i, q_j^i, and h_j^i represent production, setup, and holding costs, respectively. The variables x_j^i represent production, s_j^i represent ending inventory, and y_j^i are the binary setup variables, for item i in period j.

It is clear that an instance of PI can be obtained from each period j of MCL by considering x_j^i, y_j^i, s_{j-1}^i, and the constraints that link them. (Note also that constraints (8) are relaxed by not considering variables s_j^i.) It is also possible to consider tighter relaxations of MCL that are also instances of PI but that consider demand for multiple periods. This is discussed in detail in Miller et al. [2000b], where successful computational experience in applying and extending results for PI to solve instances of MCL is also given.

There are a number of other production planning problems for which PI serves as a relaxation. For example, Pochet and Wolsey [1991] showed how to model the multi–level capacitated lot–sizing problem using *echelon stocks*. The resulting model has a structure that is similar to (8)–(12), with additional constraints linking the echelon inventory variables. Using these results, it is possible to show that for any time period j, PI provides a single–period relaxation for this model as well.

In addition, PI provides a relaxation of more general MIP problems, such as fixed charge network flow problems. For example, PI can be used to model an instance of SNFC with side constraints. In such a relaxation constraints (3) and (4) correspond to the basic model SNFC, and (2) provide a potential relaxation for additional constraints involving the flow variables. For more on SNFC and related models, see, among others, Padberg, Van Roy, and Wolsey [1985], Gu, Nemhauser, and Savelsbergh [1999], and Marchand and Wolsey [1999].

3 Basic Polyhedral Results

In this section we present basic results concerning the polyhedral structure of PI. Where proofs are omitted, they can be found in Miller [1999], unless otherwise stated. The first basic result we state for PI concerns its complexity.

Proposition 1 *(Miller et al. [2000a]) If both setup times* and *demand are constant for all items (i.e., $t^i = t \geq 0$, $d^i = d \geq 0$, $i = 1, ..., P$), then PI is polynomially solvable. Otherwise, PI is $\mathcal{NP}$–hard.*

In Miller et al. [2000a], we present an $\mathcal{O}(P^2)$ algorithm for the case of constant setup times and demand. Because of Proposition 1, it is doubtful that we can find a complete description of conv(X^{PI}) given by facet–inducing inequalities in the general case.

Proposition 2 *If $c > t^i, i = 1, ..., P$, conv(X^{PI}) is full–dimensional. Moreover, for $i = 1, ..., P$, (2), (4), and the bound $x^i \geq 0$ induce facets of conv(X^{PI}).*

Proposition 3 *If, for every $(i, j) \in \mathcal{P} \times \mathcal{P}, i \neq j$, $c > t^i + t^j$, then (3) and the bounds $y^i \leq 1, i = 1, ..., P$ induce facets of conv(X^{PI}).*

Proposition 4 *If $c > t^i + d^i$, the valid inequalities*

$$s^i + d^i y^i \geq d^i \tag{13}$$

induce facets of conv(X^{PI}), $i = 1, ..., P$.

Note that (13) implies the bound $s^i \geq 0, i = 1, ..., P$; therefore this bound never induces a facet. Also note that these inequalities are similar to the (l, S) inequalities for the uncapacitated lot–sizing problem, introduced by Barany, Van Roy, and Wolsey [1984]. Van Roy and Wolsey [1987] later generalized these to the path inequalities for more general fixed charge network flow problems.

We will now discuss the extreme points and rays of conv(X^{PI}) and of an LP relaxation of X^{PI}. Given an extreme point $(\bar{x}, \bar{y}, \bar{s})$ of conv(X^{PI}), let $Q = \{i \in \mathcal{P} : \bar{y}^i = 1\}$. Also, let $Q_u = \{i \in Q : \bar{x}^i = d^i\}$, $Q_l = \{i \in Q : \bar{x}^i = 0\}$, and let $Q_r = \{i \in Q : \bar{x}^i > 0, \bar{x}^i \neq d^i\}$. Moreover, for a given i, we define types of (x^i, y^i, s^i) triples:

$$\begin{array}{ll} \text{Type 1:} & \bar{x}^i = d^i, \bar{y}^i = 1, \bar{s}^i = 0, \\ \text{Type 2:} & \bar{x}^i = 0, \bar{y}^i = 1, \bar{s}^i = d^i, \\ \text{Type 3:} & \bar{x}^i = 0, \bar{y}^i = 0, \bar{s}^i = d^i. \end{array}$$

Proposition 5 (Characterization of the extreme points of conv(X^{PI})) *In every extreme point $(\bar{x}, \bar{y}, \bar{s})$ of conv(X^{PI}), $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ is of Type 1, $i \in Q_u$; $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ is of Type 2, $i \in Q_l$; and $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ is of Type 3, $i \in \mathcal{P} \setminus Q$. Moreover, either $Q_r = \emptyset$, or $|Q_r| = 1$ and*

$$\bar{x}^r = c - \sum_{i \in Q_u} (t^i + d^i) - \sum_{i \in Q_l} t^i - t^r, \bar{y}^r = 1, \bar{s}^r = (d^r - \bar{x}^r)^+, \tag{14}$$

where $Q_r = \{r\}$.

The fact that $|Q_r| \leq 1$ in every extreme point of conv(X^{PI}) can be shown by contradiction. In effect, this says that there in only one $i \in \mathcal{P}$ for which the constraint (3) plays a part in determining the value $\bar{x}^i$. For the other $i \in \mathcal{P}$, $\bar{x}^i$ is determined by some interaction among (2), the variable upper bound (4), and the bound $x^i \geq 0$. If $|Q_r| > 1$ does not hold for some $(\bar{x}, \bar{y}, \bar{s}) \in X^{PI}$, then it is not difficult to construct two new points in X^{PI} such that $(\bar{x}, \bar{y}, \bar{s})$ is a convex combination of the two.

Now define X^{PI}_{LP} to be the set of feasible points to the LP relaxation of (2)–(6), (13); we can characterize its extreme points as well. Given an extreme point $(\bar{x}, \bar{y}, \bar{s})$ of X^{PI}_{LP}, let Q, Q_u, and Q_l be defined as for X^{PI}, and let $R = \{i \in \mathcal{P} : 0 < \bar{y}^i < 1\}$.

Proposition 6 (Characterization of the extreme points of X^{PI}_{LP}) *Let $(\bar{x}, \bar{y}, \bar{s})$ be an extreme point of X^{PI}_{LP}. Then $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ is of Type 1, $i \in Q_u$; $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ is of Type 2, $i \in Q_l$; and $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ is of Type 3, $i \in \mathcal{P} \setminus \{Q \cup R\}$. Moreover, if R is non–empty, it consists of a singleton $\{r\}$, and exactly one of the following statements is true:*

1. *$(\bar{x}^r, \bar{y}^r, \bar{s}^r)$ is defined as in (14), and thus $(\bar{x}, \bar{y}, \bar{s})$ is an extreme point of conv(X^{PI});*

2. $$\bar{y}^r = \frac{c - \sum_{i \in Q_u}(t^i + d^i) - \sum_{i \in Q_l} t^i}{t^r}, \bar{x}^r = 0, \bar{s}^r = d^r; \tag{15}$$

3. $$\bar{y}^r = \frac{c - \sum_{i \in Q_u}(t^i + d^i) - \sum_{i \in Q_l} t^i}{t^r + d^r}, \bar{x}^r = d^r \bar{y}^r, \bar{s}^r = d^r - \bar{x}^r; \tag{16}$$

4. $$\bar{y}^r = \frac{c - \sum_{i \in Q_u}(t^i + d^i) - \sum_{i \in Q_l} t^i}{c}, \bar{x}^r = (c - t^r)\bar{y}^r, \bar{s}^r = d^r(1 - \bar{y}^r). \tag{17}$$

The reasoning that $|R| \leq 1$ is similar to the reasoning that $|Q_r| \leq 1$ in Proposition 5. For fractional extreme points defined by (15), inequality (2) and the bound $x^r \geq 0$ are both tight for r, and inequalities (4) and (13) are not. For fractional extreme points defined by (16), inequalities (2) and (13) are both tight for r, inequality (4) is not, and $x^r > 0$. For fractional extreme points defined by (17), inequalities (4) and (13) are tight, inequality (2) is not, and $x^r > 0$.

Proposition 7 *All extreme rays of both conv(X^{PI}) and X^{PI}_{LP} have the form $s^i = 1, x^i = y^i = 0$, for some $i \in \mathcal{P}$, and $s^j = x^j = y^j = 0, j \neq i$.*

Thus PI has a bounded optimum if and only if $h^i \geq 0, i = 1, ..., P$, and PI has an unbounded optimal solution if and only if $h^i \leq 0$, for some $i \in \mathcal{P}$. If $h^i \geq 0, i = 1, ..., P$, and there is an element i for which $h^i = 0$, then PI has an infinite number of optimal solutions. So, given our assumption that $h^i \geq 0, i = 1, ..., P$, every optimal solution of PI is either bounded, or is the sum of another optimal solution that is bounded and of a linear combination of extreme rays.

These results provide a characterization of both conv(X^{PI}) and conv(X^{PI}_{LP}) by their extreme points and rays. Next, we discuss which extreme points are possible optimal solutions of X^{PI} and X^{PI}_{LP}. In discussing a set of points $X \subset \{(x, y, s) : x^i, s^i \geq 0, 0 \leq y^i \leq 1, i = 1, ..., P\}$, we will use

Definition 1 *The point* $(\bar{x}, \bar{y}, \bar{s})$ *is a* ***dominant solution*** *of* X *if there exists some cost vector* (p, q, h)*, such that* $q^i > 0, h^i \geq 0, i = 1, ..., P$*, for which optimizing (1) over* X *yields* $(\bar{x}, \bar{y}, \bar{s})$ *as the* ***unique*** *optimal solution.*

This definition allows us to obtain

Proposition 8 *The point* (x, y, s) *is a dominant solution of* X^{PI} *for PI if and only if (1)* (x, y, s) *is an extreme point of conv(*X^{PI}*); and (2)* $y^i = 1$ *if and only if* $x^i > 0, i = 1, ..., P$.

Thus, the condition $q^i > 0, i = 1, ..., P$, ensures that a setup for an item does not take place unless production of that item also occurs. This is a reasonable condition to enforce; for example, in both lot–sizing and fixed charge problems, a fixed charge to produce/transport an item is not paid unless some positive quantity of that item is produced/transported. A similar result holds for X^{PI}_{LP}.

Proposition 9 *Given an instance of PI, no dominant solution of* X^{PI}_{LP} *has the form (15). Moreover, every dominant solution of* X^{PI}_{LP} *is either a dominant solution of* X^{PI}*, or else it is of one of the forms (16) or (17) with* $Q_l = \emptyset$.

4 New Valid Inequalities for PI

Now we will derive two new families of valid inequalities for X^{PI} and present results concerning their strength. In order to define these inequalities, we first define a *cover* of PI to be a set S such that $\lambda = \sum_{i \in S}(t^i + d^i) - c \geq 0$. Similarly, define a *reverse cover* of PI to be a set $S \neq \emptyset$ such that $\mu = c - \sum_{i \in S}(t^i + d^i) > 0$. Note that a reverse cover S does not have to be maximal in any sense, merely non-empty. Because the development needed to define reverse cover inequalities is much simpler than that for cover inequalities, we will discuss the former class first.

4.1 Reverse Cover Inequalities

Before formally defining this class, we provide a brief motivation. Given $(x, y, s) \in X^{PI}$, let S be a reverse cover, and let i' be some element not in S. If $x^{i'} = (c - t^{i'})y^{i'}$, either

1. $y^{i'} = 0$; or
2. $y^{i'} = 1$ and no capacity is left for $i \in S$; therefore $y^i = 0, i \in S$, and $\sum_{i \in S} s^i \geq \sum_{i \in S} d^i$.

Such reasoning yields

$$\sum_{i \in S} s^i \geq (\sum_{i \in S}(t^i + d^i))y^{i'} - \sum_{i \in S} t^i(1 - y^i) - ((c - t^{i'})y^{i'} - x^{i'}),$$

which is generalized in the following proposition.

Proposition 10 (Reverse Cover Inequalities) *Let S be a reverse cover of PI, let $T = \mathcal{P} \setminus S$, and let (T', T'') be any partition of T. Then*

$$\sum_{i \in S} s^i \geq (\sum_{i \in S}(t^i + d^i)) \sum_{i \in T'} y^i - \sum_{i \in S} t^i(1 - y^i) - \sum_{i \in T'}((c - t^i)y^i - x^i) \tag{18}$$

is valid for X^{PI}.

Proof: Let $(\bar{x}, \bar{y}, \bar{s})$ be any point in X^{PI}. If $\bar{y}^i = 0$ (and thus $\bar{x}^i = 0$), $i \in T'$, then (18) is obviously valid. Let $T'^1 = \{j \in T' : \bar{y}_j = 1\}$; consider first the case in which $T'^1 = \{i'\}$ is a singleton. Then (18) becomes

$$\sum_{i \in S} \bar{s}^i \geq \sum_{i \in S} d^i + \sum_{i \in S} t^i \bar{y}^i - (c - t^{i'}) + \bar{x}^{i'}.$$

This inequality holds because

$$\begin{array}{rcll} (c - t^{i'}) & \geq & \sum_{i \in S}(\bar{x}^i + t^i \bar{y}^i) + \bar{x}^{i'} & \text{(by (3))} \\ & \geq & \sum_{i \in S}(d^i - \bar{s}^i + t^i \bar{y}^i) + \bar{x}^{i'}. & \text{(by (2))} \end{array}$$

When $|T'^1| > 1$, the argument is similar, except that in this case the inequality is weaker and cannot ever hold at equality. $\square$

Note that T' can be seen as a candidate set for the choice of the element $i' \notin S$ mentioned in the paragraph motivating Proposition 10.

Example 1

Consider an instance of PI with $P = 4$, with $c = 16$, and with demand and setup times given by

i	1	2	3	4
t^i	2	2	1	3
d^i	5	4	5	10

Choose $S = \{2, 3\}$ as a reverse cover; thus $\mu = 16 - (2 + 4) - (1 + 5) = 4$. Let $T' = \{1\}$; then (18) yields

$$\begin{aligned} s^2 + s^3 &\geq 12y^1 - 2(1 - y^2) - (1 - y^3) - (14y^1 - x^1) \\ &= -3 - 2y^1 + x^1 + 2y^2 + y^3, \end{aligned} \tag{19}$$

a valid inequality for X^{PI}. $\square$

One property of reverse cover inequalities concerns the fractional extreme points of X^{PI}_{LP} that they cut off.

Proposition 11 *For PI, every fractional dominant solution of X^{PI}_{LP} is cut off by an inequality of the form (18).*

We can prove this proposition by considering any fractional dominant solution (as characterized by Propositions 6 and 9) and taking $S = Q_u$ and $T' = \{r\}$ to define (18). It is interesting to recall here that X^{PI}_{LP} is defined by a formulation that has already been strengthened with a family of strong inequalities (13). We now state very general conditions under which reverse cover inequalities define facets of conv(X^{PI}).

Proposition 12 *If $S \neq \emptyset$, $T' \neq \emptyset$, and $t^i < \mu, i \in T$, (18) induces a facet of conv(X^{PI}).*

Proof: The conditions of the Proposition imply that $c > t^i, i = 1, ..., P$, and so from Proposition 2 conv(X^{PI}) is full–dimensional. It therefore suffices to show that the intersection of the hyperplane defined by (18) and of conv(X^{PI}) has dimension $3P - 1$. Note that the $|T|$ rays $s^i = 1, i \in T$, are extreme rays of conv(X^{PI}) and also lie in the hyperplane defined by (18). To obtain the results, therefore, it suffices to show $3P - |T|$ linearly independent points in X^{PI} that lie in this hyperplane, and that are also linearly independent of each of these extreme rays. Let i^* be some element in T'; consider the points

- for each $i' \in S$,

$$x^{i'} = 0, y^{i'} = 1, s^{i'} = d^{i'}$$
$$x^{i^*} = c - t^{i^*} - t^{i'}, y^{i^*} = 1, s^{i^*} = (d^{i^*} - x^{i^*})^+,$$
$$x^i = y^i = 0, s^i = d^i, i \neq i', i \neq i^*$$

($|S|$ points)

- for each $i' \in S$,

$$x^{i'} = d^{i'}, y^{i'} = 1, s^{i'} = 0$$
$$x^{i^*} = c - t^{i^*} - t^{i'} - d^{i'}, y^{i^*} = 1, s^{i^*} = (d^{i^*} - x^{i^*})^+$$
$$x^i = y^i = 0, s^i = d^i, i \neq i', i \neq i^*$$

($|S|$ points)

- for each $i' \in S$,

$$x^{i'} = c - t^{i'} - t^{i^*}, y^{i'} = 1, s^{i'} = 0$$
$$x^{i^*} = 0, y^{i^*} = 1, s^{i^*} = d^{i^*}$$
$$x^i = y^i = 0, s^i = d^i, i \neq i', i \neq i^*$$

($|S|$ points)

- for each $i' \in T' \setminus i^*$,

$$x^{i'} = \mu - t^{i'}, y^{i'} = 1, s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S$$
$$x^i = y^i = 0, s^i = d^i, i \in T \setminus \{i'\}$$

($|T'| - 1$ points)

- for each $i' \in T'$,

$$x^{i'} = c - t^{i'}, y^{i'} = 1, s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^i = y^i = 0, s^i = d^i, i \neq i'$$

($|T'|$ points)

- for each $i' \in T''$,

$$x^{i'} = 0, y^{i'} = 1, s^{i'} = d^{i'}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S$$
$$x^i = y^i = 0, s^i = d^i, i \in T \setminus \{i'\}$$

($|T''|$ points)

- for each $i' \in T''$,

$$x^{i'} = \mu - t^{i'}, y^{i'} = 1, s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S$$
$$x^i = y^i = 0, s^i = d^i, i \in T \setminus \{i'\}$$

  ($|T''|$ points)

- the point

$$x^i = d^i, y^i = 1, s^i = 0, i \in S$$
$$x^i = y^i = 0, s^i = d^i, i \in T.$$

It can be checked that the points listed are linearly independent of each other and of the $|T|$ rays $s^i = 1, i \in T$. □

Example 1 (continued)
The inequality (19) satisfies the conditions of Proposition 12 and is therefore a facet of $\text{conv}(X^{PI})$. □

4.2 Cover Inequalities

The form of the full set of cover inequalities is rather complex. Therefore, in presenting these inequalities, we first present the most basic form of the inequality. Next, we present a second form that, though more complex than the first, still does not encompass the entire class. We then lift additional variables into this second form, thus expanding this class into the full set of cover inequalities.

4.2.1 Basic Cover Inequalities

To motivate this class, consider a cover S of PI. Recall that

$$\lambda = \sum_{i \in S}(t^i + d^i) - c \geq 0.$$

So if $y^i = 1, i \in S$,

$$\sum_{i \in S} s^i \geq \sum_{i \in S} d^i - \sum_{i \in S} x^i \geq \sum_{i \in S} d^i - (c - \sum_{i \in S} t^i) = \sum_{i \in S}(t^i + d^i) - c \geq \lambda.$$

Moreover, if $y^{i'} = 0$ for exactly one $i' \in S$, then both

$$\sum_{i \in S} s^i \geq \lambda - t^i$$

and

$$\sum_{i \in S} s^i \geq s^{i'} \geq d^{i'}$$

must hold. From such reasoning we can derive that

$$\sum_{i \in S} s^i \geq \lambda + \sum_{i \in S} \max\{-t^i, d^i - \lambda\}(1 - y^i) \tag{20}$$

is valid for PI. A more formal statement and proof follows.

Proposition 13 (Cover Inequalities, basic form) *Given a cover S of PI, (20) is valid for X^{PI}.*

Proof: Let $(\bar{x}, \bar{y}, \bar{s})$ be any point in X^{PI}. Let $S^1 = \{i \in S : \bar{y}^i = 1\}$, and let $S^0 = \{i \in S : \bar{y}^i = 0\}$. If $\{i \in S^0 : t^i + d^i > \lambda\} = \emptyset$, then

$$\begin{aligned}
\sum_{i \in S} s^i &= \sum_{i \in S^0} s^i + \sum_{i \in S^1} s^i \\
&\geq \sum_{i \in S^0} d^i + (\sum_{i \in S^1} (t^i + d^i) - c)^+ \\
&= \sum_{i \in S^0} (t^i + d^i - t^i(1 - \bar{y}^i)) + (\sum_{i \in S^1} (t^i + d^i) - c)^+ \\
&\geq \sum_{i \in S} (t^i + d^i) - c - \sum_{i \in S^0} (t^i(1 - \bar{y}^i)) \\
&= \lambda + \sum_{i \in S} \max\{-t^i, d^i - \lambda\}(1 - \bar{y}^i). \qquad (21)
\end{aligned}$$

If $\{i \in S^0 : t^i + d^i > \lambda\} \neq \emptyset$, then

$$
\begin{aligned}
\sum_{i\in S} s^i &\geq \sum_{i\in S^0} s^i & (22)\\
&\geq \sum_{i\in S^0} d^i \\
&\geq \lambda + \sum_{i\in S^0: t^i+d^i>\lambda} (d^i-\lambda) + \sum_{i\in S^0: t^i+d^i\leq\lambda} d^i \\
&\geq \lambda + \sum_{i\in S} \max\{-t^i, d^i-\lambda\}(1-\bar{y}^i), & (23)
\end{aligned}
$$

and so the proposition holds. □

Example 1 (continued)
Choose as a cover $S = \{1,2,3\}$; then $\lambda = 7+6+6-16 = 3$, and (20) yields

$$s^1 + s^2 + s^3 \geq 3 + 2(1-y^1) + (1-y^2) + 2(1-y^3) = 8 - 2y^1 - y^2 - 2y^3, \qquad (24)$$

a valid inequality for X^{PI}. □

The family of inequalities defined by (20) cuts off many fractional extreme points. Specifically, we have

Proposition 14 *For PI, every dominant fractional solution of X^{PI}_{LP} of the form (16) is cut off by an inequality of the form (20).*

The proof is obtained by observing that $Q_l = \emptyset$ (from Proposition 9), and taking $S = Q_u \cup \{r\}$ in order to define (20).

Under general conditions, inequalities of the form (20) define facets of conv(X^{PI}).

Proposition 15 *Given an inequality of the form (20), order the $i \in S$ such that $t^{[1]} + d^{[1]} \geq \ldots \geq t^{[|S|]} + d^{[|S|]}$, let $\mu^1 = t^{[1]} + d^{[1]} - \lambda$, and define $T = \mathcal{P} \setminus S$. If $\lambda > 0$, $t^{[2]} + d^{[2]} > \lambda$, and $t^i < \mu^1, i \in T$, then (20) induces a facet of conv(X^{PI}).*

Proof: Similarly to the proof of Proposition 12, it suffices to show $3P - |T|$ points and $|T|$ rays in X^{PI} that are linearly independent and that lie in the hyperplane defined by (25). To simplify notation, we assume first that $d^i + d^{[1]} \geq \lambda, i \in S \setminus [1]$. Consider the $3|P| - |T|$ points

- for each $i' \in S \setminus \{[1]\}$,

$$x^{i'} = (d^{i'} - \lambda)^+, y^{i'} = 1, s^{i'} = d^{i'} - x^{i'}$$
$$x^{[1]} = d^{[1]} - (\lambda - d^{i'})^+, y^{[1]} = 1, s^{[1]} = d^{[1]} - x^{[1]}, (\text{ if } i' \neq [1])$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

($|S| - 1$ points)

- the point

$$x^{[1]} = (d^{[1]} - \lambda)^+, y^{[1]} = 1, s^{[1]} = d^{[1]} - x^{[1]}$$
$$x^{[2]} = d^{[2]} - (\lambda - d^{[1]})^+, y^{[2]} = 1, s^{[1]} = d^{[1]} - x^{[1]},$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

- for each $i' \in S \setminus \{[1]\}$,

$$x^{i'} = y^{i'} = 0, s^{i'} = d^{i'}$$
$$x^{[1]} = d^{[1]} - (\lambda - t^{i'} - d^{i'})^+, y^{[1]} = 1, s^{[1]} = d^{[1]} - x^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

($|S| - 1$ points)

- the point

$$x^{[1]} = 0, y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{[1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

- for each $i' \in S \setminus [1]$,

$$x^{i'} = d^{i'} + \mu^1, y^{i'} = 1, s^{i'} = 0$$
$$x^{[1]} = y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

($|S| - 1$ points)

- the point

$$x^{[1]} = d^{[1]} + t^{[2]} + d^{[2]} - \lambda, y^{[1]} = 1, s^{[1]} = 0$$
$$x^{[2]} = y^{[2]} = 0, s^{[2]} = d^{[2]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [2]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

- for each $i' \in T$,

$$x^{i'} = 0, y^{i'} = 1, s^{i'} = d^{i'}$$
$$x^{[1]} = y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus [1]$$
$$x^i = y^i = 0, s^i = d^i, i \notin S \cup i'$$

  ($|T|$ points)

- for each $i' \in T$,

$$x^{i'} = \mu^1 - t^{i'}, y^{i'} = 1, s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^{[1]} = y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus [1]$$
$$x^i = y^i = 0, s^i = d^i, i \notin S \cup i'$$

  ($|T|$ points).

It can be checked that these points and the $|T|$ rays $s^i = 1, i \in T$, are linearly independent. When the assumption $d^i + d^{[1]} \geq \lambda, i \in S \setminus [1]$, does not hold, $3P - |T|$ linearly independent points can be chosen similarly, but either a larger number of subsets or more complex notation is required to list them. □

Example 1 (continued)

The inequality (24) satisfies the conditions of Proposition 15, and is therefore a facet of $\text{conv}(X^{PI})$. □

4.2.2 Second Form of Cover Inequalities

We now expand the class of cover inequalities by incorporating another set into their definition. This allows us to cut off more (in many cases, all) fractional dominant solutions and to define more facets of conv(X^{PI}), and it is essential to the sufficiency result that is stated at the end of this section. This new set plays a role in cover inequalities similar to that played by the set T' in the reverse cover inequalities (18).

Proposition 16 (Cover Inequalities, second form) *Given a cover S of PI, order the $i \in S$ such that $t^{[1]} + d^{[1]} \geq ... \geq t^{[|S|]} + d^{[|S|]}$. Let $T = \mathcal{P} \setminus S$, and let (T', T'') be any partition of T. Define $\mu^1 = t^{[1]} + d^{[1]} - \lambda$; if $\mu^1 \geq 0$ and $|S| \geq 2$, then*

$$\sum_{i\in S} s^i \geq \lambda + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}(1 - y^i) + \frac{\sum_{i\in S} \min\{t^i + d^i, \lambda\} - \lambda}{(|S| - 1)(t^{[2]} + d^{[2]})} \sum_{i\in T'} (x^i - (\mu^1 - t^i)y^i) \tag{25}$$

is valid for X^{PI}.

Proof: Let $(\bar{x}, \bar{y}, \bar{s})$ be any point in X^{PI}. Let $S^1 = \{i \in S : \bar{y}^i = 1\}$, let $S^0 = \{i \in S : \bar{y}^i = 0\}$, and let $T'^1 = \{i \in T' : \bar{y}^i = 1\}$. If $T'^1 = \emptyset$, then the validity argument is exactly the case as for Proposition 13. So let $T'^1 \neq \emptyset$; we first assume that $T'^1 = \{i'\}$, a singleton. To show that $(\bar{x}, \bar{y}, \bar{s})$ satisfies (25), it suffices to show that

$$\begin{aligned} \sum_{i\in S} \bar{s}^i + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}\bar{y}^i \quad \geq \quad & \lambda + \sum_{i\in S} \max\{-t^i, d^i - \lambda\} \\ & + \frac{\sum_{i\in S} \min\{t^i + d^i, \lambda\} - \lambda}{(|S| - 1)(t^{[2]} + d^{[2]})} \sum_{i\in T'} (\bar{x}^i - (\mu^1 - t^i)\bar{y}^i). \end{aligned} \tag{26}$$

Now

$$\sum_{i\in S} \bar{s}^i + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}\bar{y}^i \geq \min_{(x,y,s)\in X^{PI}: x^{i'} = \bar{x}^{i'}, y^{i'} = 1} \left\{ \sum_{i\in S} s^i + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}y^i \right\}. \tag{27}$$

This is true because the right hand side is the minimum value that the left hand side can achieve, provided that $y^{i'} = 1$ and $x^{i'} = \bar{x}^{i'}$ are both still true. Therefore we will consider the minimization problem

$$\min_{(x,y,s)\in X^{PI}:x^{i'}=\bar{x}^{i'},y^{i'}=1} \left\{ \sum_{i\in S} s^i + \sum_{i\in S} \max\{-t^i, d^i - \lambda\} y^i \right\}, \tag{28}$$

and show that the optimal solution to this problem has value greater than or equal to that of the right hand side of (26).

To characterize the optimal value to the above minimization problem, we can consider the problem item by item. Note that if in a solution of (28) we set $y^{i^*} = 1, x^{i^*} = d^{i^*}, s^{i^*} = 0$, where $t^{i^*} + d^{i^*} \leq \lambda$, then we obtain a savings in the objective function of $t^{i^*} + d^{i^*}$ over the same solution in which $y^{i^*} = x^{i^*} = 0, s^{i^*} = d^{i^*}$. Moreover, setting $y^{i^*} = 1, x^{i^*} = d^{i^*}, s^{i^*} = 0$, requires the usage of $t^{i^*} + d^{i^*}$ units of capacity. However, if $t^{i^*} + d^{i^*} > \lambda$, then setting $y^{i^*} = 1, x^{i^*} = d^{i^*}, s^{i^*} = 0$, results in a savings in the objective function of only λ over the same solution in which $y^{i^*} = x^{i^*} = 0, s^{i^*} = d^{i^*}$, although doing so still uses $t^{i^*} + d^{i^*}$ units of capacity. Thus, as $t^{i^*} + d^{i^*}$ increases above λ, it becomes less economical, given the objective function of (28), to satisfy demand from production.

Therefore, it is possible to solve (28) greedily, by satisfying demand for $i \in S$ from production in *nondecreasing* order of $t^i + d^i$. Given this ordering, the demand for items in S is satisfied by production as long as there is sufficient capacity left over after the production of i' to do so. After capacity is used up, the remaining demand for items in S is satisfied from inventory. That is, for some integer j with $1 \leq j \leq P$, this solution has objective function value

$$\sum_{i=1}^{j-1} d^{[i]} + \sum_{i=j}^{|S|} \max\{-t^{[i]}, d^{[i]} - \lambda\} + (\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) + t^{i'} + \bar{x}^{i'} - c)^+. \tag{29}$$

The value of j is determined by the number of items S for which it is possible to satisfy demand by production, given the value of $\bar{x}^{i'}$ and the fact that we produce in nondecreasing order of $t^i + d^i$. So let j' be defined by

$$j' = \min\{i'' : \sum_{i=i''}^{|S|}(t^{[i]} + d^{[i]}) < c - t^{i'} - \bar{x}^{i'}\}.$$

Then the choice of j for which (29) attains the optimal value to (28) is

$$j = \begin{cases} j' - 1 & \text{if } \sum_{i=j'-1}^{|S|}(t^{[i]} + d^{[i]}) - (c - t^{i'} - \bar{x}^{i'}) \leq \lambda \\ j' & \text{otherwise.} \end{cases}$$

Depending on the value of $x^{i'}$, the capacity constraint (3) may or may not be tight at the optimal solution obtained in this way. If (3) is tight, then demand for item $[j]$ cannot be fully met. In this case $j = j' - 1$ and

$$\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) + t^{i'} + x^{i'} - c \tag{30}$$

will be positive. The sum (30) is the inventory necessary to satisfy the demand for $[j]$ that cannot be satisfied from production, given that all the demand for items $[j+1], ..., [S]$, is satisfied from production. If (3) is not tight, then demand for all of $[j], ..., |S|$, can be satisfied entirely from production. In this case $j = j'$, and (30) is nonpositive.

So we have that

$$\begin{gathered}\min_{(x,y,s)\in X^{PI}:x^{i'}=\bar{x}^{i'},y^{i'}=1}\left\{\textstyle\sum_{i\in S} s^i + \sum_{i\in S}\max\{-t^i, d^i - \lambda\}y^i\right\} = \\ \textstyle\sum_{i=1}^{j-1} d^{[i]} + \sum_{i=j}^{|S|}\max\{-t^{[i]}, d^{[i]} - \lambda\} + (\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) + t^{i'} + x^{i'} - c)^+ = \\ \textstyle\lambda + \sum_{i\in S}\max\{-t^i, d^i - \lambda\} + \sum_{i=1}^{j-1}\min\{t^{[i]} + d^{[i]}, \lambda\} - \lambda + (\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) + t^{i'} + x^{i'} - c)^+,\end{gathered}$$

where the last equality holds because

$$d^{[i]} = \max\{-t^{[i]}, d^{[i]} - \lambda\} + \min\{t^{[i]} + d^{[i]}, \lambda\}, i = 1, ..., j-1.$$

So now we need to show that

$$\begin{gathered}\textstyle\lambda + \sum_{i\in S}\max\{-t^i, d^i - \lambda\} + \sum_{i=1}^{j-1}\min\{t^{[i]} + d^{[i]}, \lambda\} - \lambda + (\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) + t^{i'} + x^{i'} - c)^+ \geq \\ \textstyle\lambda + \sum_{i\in S}\max\{-t^i, d^i - \lambda\} + \frac{\sum_{i=1}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}-\lambda}{(|S|-1)(t^{[2]}+d^{[2]})}(\bar{x}^{i'} - (\mu^1 - t^{i'})),\end{gathered}$$

or equivalently, that

$$\begin{gathered}\textstyle\sum_{i=1}^{j-1}\min\{t^{[i]} + d^{[i]}, \lambda\} - \lambda + (\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) + t^{i'} + x^{i'} - c)^+ \geq \\ \textstyle\frac{\sum_{i=1}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}-\lambda}{(|S|-1)(t^{[2]}+d^{[2]})}(\bar{x}^{i'} - (\mu^1 - t^{i'})).\end{gathered} \tag{31}$$

From here the proof proceeds by cases. First consider the case in which

$$\bar{x}^{i'} = \mu^1 + \sum_{i=2}^{j-1}(t^{[i]} + d^{[i]}) - t^{i'}.$$

In this case, by definition of μ^1, $\sum_{i=j}^{|S|}(t^{[i]}+d^{[i]})+t^{i'}+x^{i'}-c=0$. Also,

$$\begin{aligned}
\sum_{i=1}^{j-1}\min\{t^{[i]}+d^{[i]},\lambda\}-\lambda &= \sum_{i=2}^{j-1}\min\{t^{[i]}+d^{[i]},\lambda\} \\
&\geq \sum_{i=2}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}\frac{\sum_{i=2}^{j-1}(t^{[i]}+d^{[i]})}{(|S|-1)(t^{[2]}+d^{[2]})} \\
&= \frac{\sum_{i=1}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}-\lambda}{(|S|-1)(t^{[2]}+d^{[2]})}(\bar{x}^{i'}-(\mu^1-t^{i'})).
\end{aligned}$$

The above inequality follows from the fact that $t^{[1]}+d^{[1]}\geq\lambda$, and from the inequality

$$(|S|-1)(t^{[2]}+d^{[2]})\sum_{i=2}^{j-1}\min\{t^{[i]}+d^{[i]},\lambda\}\geq\sum_{i=2}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}\sum_{i=2}^{j-1}(t^{[i]}+d^{[i]}).$$

To see that this inequality holds, note that

$$(t^{[2]}+d^{[2]})\min\{t^{[j]}+d^{[j]},\lambda\}\geq\min\{t^{[k]}+d^{[k]},\lambda\}(t^{[2]}+d^{[2]})$$

when $2\leq j<k$; therefore

$$\begin{aligned}
(|S|-1)(t^{[2]}+d^{[2]})\sum_{i=2}^{j-1}\min\{t^{[i]}+d^{[i]},\lambda\} &= \sum_{i=2}^{j-1}(t^{[2]}+d^{[2]})\sum_{i=2}^{j-1}\min\{t^{[i]}+d^{[i]},\lambda\} \\
&\quad+\sum_{i=j}^{|S|}(t^{[2]}+d^{[2]})\sum_{i=2}^{j-1}\min\{t^{[i]}+d^{[i]},\lambda\} \\
&\geq \sum_{i=2}^{j-1}\min\{t^{[i]}+d^{[i]},\lambda\}\sum_{i=2}^{j-1}(t^{[2]}+d^{[2]}) \\
&\quad+\sum_{i=j}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}\sum_{i=2}^{j-1}(t^{[2]}+d^{[2]}) \\
&= \sum_{i=2}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}\sum_{i=2}^{j-1}(t^{[2]}+d^{[2]}) \\
&\geq \sum_{i=2}^{|S|}\min\{t^{[i]}+d^{[i]},\lambda\}\sum_{i=2}^{j-1}(t^{[i]}+d^{[i]}).
\end{aligned}$$

So (31) holds, and $(\bar{x}, \bar{y}, \bar{s})$ satisfies (25), when $\bar{x}^{i'} = \mu^1 + \sum_{i=2}^{j-1}(t^{[i]} + d^{[i]}) - t^{i'}$. Given this, it is not difficult to see that $(\bar{x}, \bar{y}, \bar{s})$ still satisfies (25) when

$$\bar{x}^{i'} > \mu^1 + \sum_{i=2}^{j-1}(t^{[i]} + d^{[i]}) - t^{i'},$$

because the left hand side of (31) increases one unit for every unit of $\bar{x}^{i'}$ above $\mu^1 + \sum_{i=2}^{j-1}(t^{[i]} + d^{[i]}) - t^{i'}$, but the right hand side increases less than one unit per unit increase of $\bar{x}^{i'}$. Moreover, when

$$\bar{x}^{i'} < \mu^1 + \sum_{i=2}^{j-1}(t^{[i]} + d^{[i]}) - t^{i'},$$

one can see that $(\bar{x}, \bar{y}, \bar{s})$ still satisfies (25), because the left hand side of (31) remains constant as $\bar{x}^{i'}$ decreases below $\mu^1 + \sum_{i=2}^{j-1}(t^{[i]} + d^{[i]}) - t^{i'}$, while the right hand side decreases.

When there is more than one element in T'^1, the proof proceeds analogously. Having more than one element in T'^1 weakens (25), however, and there are no tight points in which $|T'^1| > 1$. □

Example 1 (continued)

Again choose $S = \{1, 2, 3\}$, and recall that $\lambda = 3$. Then $t^{[1]} + d^{[1]} = t^1 + d^1 = 7$, and $\mu^1 = 4$. Also note that $t^{[2]} + d^{[2]} = t^2 + d^2 = t^3 + d^3 = 6$. Take $T' = \{4\}$, then

$$\frac{\sum_{i \in S} \min\{t^i + d^i, \lambda\} - \lambda}{(|S| - 1)(t^{[2]} + d^{[2]})} = \frac{3 + 3 + 3 - 3}{2 * 6} = \frac{1}{2},$$

and (25) yields

$$\begin{aligned} s^1 + s^2 + s^3 &\geq 3 + 2(1 - y^1) + (1 - y^2) + 2(1 - y^3) + \frac{1}{2}(x^4 - y^4) \\ &= 8 - 2y^1 - y^2 - 2y^3 + \frac{1}{2}(x^4 - y^4), \end{aligned} \tag{32}$$

a valid inequality for X^{PI}. □

In addition to the fractional dominant solutions mentioned in Proposition 14, under certain conditions, inequalities of the form (25) cut off *all* fractional dominant solutions of X^{PI}_{LP}. For example, we have

Proposition 17 *For PI, if setup times and demand are constant for all items, then every dominant fractional solution of X^{PI}_{LP} of the form (17) is cut off by an inequality of the form (25).*

To prove this proposition, first observe again that $Q_l = \emptyset$. Then to define (25) let $T' = \{r\}$, and let S be any set such that $|S| = \lceil \frac{c}{t+d} \rceil$ and $S \supset Q_u$. Thus, given Proposition 9, in this case cover inequalities cut off all fractional dominant solutions of X^{PI}_{LP}. It is again interesting to recall at this point that X^{PI}_{LP} is an LP relaxation of X^{PI} that has already been strengthened with the family of facet–inducing inequalities (13). It is also interesting to note that the family of reverse cover inequalities is stronger than the family of cover inequalities, in the sense that the former *always* cuts off all possible fractional optimal solutions to X^{PI}_{LP}.

There are facets induced by inequalities of the form (25) with $T' \neq \emptyset$. We can extend Proposition 15 to establish conditions under which these inequalities induce facets. For example, we have

Proposition 18 *If $\lambda > 0$, $t^{[2]}+d^{[2]} > \lambda$, $t^i < \mu^1, i \in T$, and $t^i+d^i = t^{[2]}+d^{[2]}, \forall i \in S\backslash[1]$, then (25) induces a facet of conv(X^{PI}).*

A proof of this proposition can be found in Miller [1999].

Example 1 (continued)

The inequality (32) satisfies the conditions of Proposition 18, and is therefore a facet of conv(X^{PI}). □

4.2.3 Lifted Cover Inequalities

In order to expand further this set of inequalities, we need to extend (25) by lifting into it variables associated with some subset $U \subset \mathcal{P} \setminus \{S \cup T'\}$. Expanding (25) to include coefficients on variables associated with items not in $S \cup T'$ is important, because even in the simplest cases (for example, the polynomially solvable case of Proposition 1), there are facets defined by such inequalities.

To lift s^i and y^i, $i \in U \subset \mathcal{P} \setminus \{S \cup T'\}$, into (25), we use a superadditive function to ensure that the lifting coefficients are sequence independent. However, unlike the usual application of lifting, the function we use does not immediately yield the coefficients of the variables lifted into (25). A second, related difference is that the constant term in the inequality changes as variables are lifted in. That is, the constant term in (25) is

$$\lambda + \sum_{i \in S} \max\{-t^i, d^i - \lambda\},$$

but for each item $i \in U$ for which we lift s^i and y^i into (25), this term will be increased by d^i.

Let

$$k' = |\{i \in S : t^i + d^i > \lambda\}|, \tag{33}$$

and define

$$\phi_S(u) = \begin{cases} 0 & \text{if} \quad u \le \mu^1, \\ (j-1)\lambda + (u - [\mu^1 + \sum_{i=2}^{j}(t^{[i]} + d^{[i]})]) & \text{if} \quad \begin{array}{c} \mu^1 + \sum_{i=2}^{j}(t^{[i]} + d^{[i]}) \\ \le u \le \\ \mu^1 + \lambda + \sum_{i=2}^{j}(t^{[i]} + d^{[i]}), \\ 1 \le j \le k' - 1, \end{array} \\ j\lambda & \text{if} \quad \begin{array}{c} \mu^1 + \lambda + \sum_{i=2}^{j}(t^{[i]} + d^{[i]}) \\ \le u \le \\ \mu^1 + \sum_{i=2}^{j+1}(t^{[i]} + d^{[i]}), \\ 1 \le j \le k' - 1, \end{array} \\ (k'-1)\lambda + (u - [\mu^1 + \sum_{i=2}^{k'}(t^{[i]} + d^{[i]})]) & \text{if} \quad u \ge \mu^1 + \sum_{i=2}^{k'}(t^{[i]} + d^{[i]}). \end{cases} \tag{34}$$

It can be shown that the function $\phi_S(\cdot)$ is superadditive on $\mathbb{R}_+$. A graph of an example of the function $\phi_S(\cdot)$ appears in Figure 2; in this example $k' = 3$.

Intuitively, for each $i' \in \mathcal{P} \setminus S$, $\phi_S(t^{i'} + d^{i'})$ is defined to be the maximum amount by which the right hand side of (25) can be increased if $s^{i'}$ is added to the left hand side, provided that $y^{i'} = 1$ and $T' = \emptyset$. That is, we have

Lemma 19 *Given an instance of PI and a cover S, then*

$$\phi_S(t^{i'} + d^{i'}) = \min_{(x,y,s) \in X^{PI}, y^{i'}=1} \left\{ \sum_{S \cup i'} s^i - (\lambda + \sum_{i \in S} \max\{-t^i, d^i - \lambda\}(1 - y^i)) \right\},$$

where $\phi_S(\cdot)$ is defined as in (34).

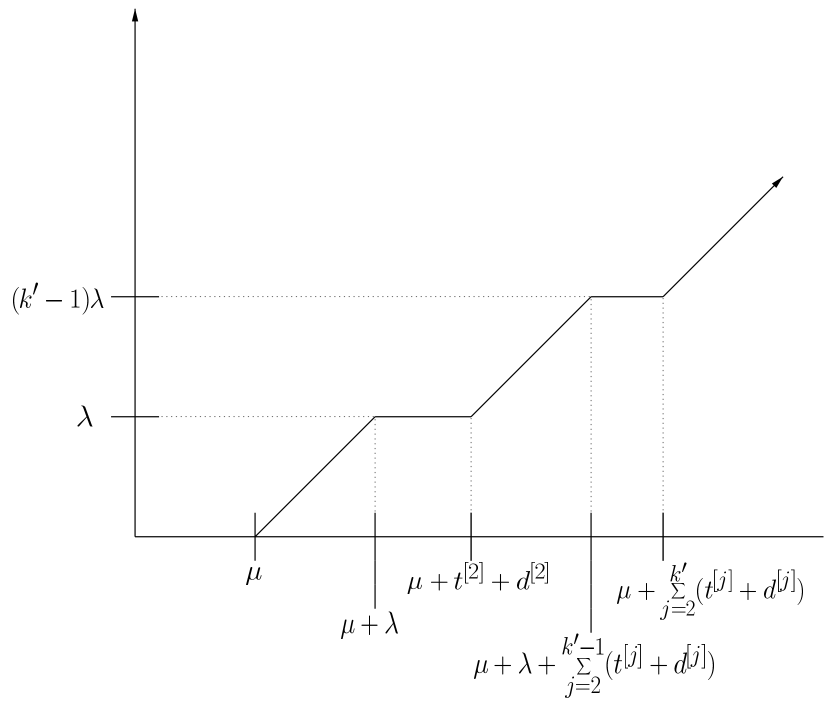

Figure 2: Graph of the function $\phi_S(\cdot)$

The proof of this Lemma is long and tedious. Moreover, it resembles the development of a similar lifting function in Marchand and Wolsey [1999]; therefore it is omitted here.

For each $i' \in \mathcal{P} \setminus S$, if $y^{i'} = 0$ and $T' = \emptyset$, it is clear that the maximum amount that the right hand side can be increased is $d^{i'}$, if $s^{i'}$ is added to the left hand side; that is,

$$\min_{(x,y,s)\in X^{PI}, y^{i'}=0} \left\{ \sum_{S\cup i'} s^i - (\lambda + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}(1-y^i)) \right\} = d^{i'}.$$

Therefore we define a function $f(\cdot,\cdot) : \mathbb{R}_+^2 \to \mathbb{R}$ to represent the difference between these two quantities:

$$f(t^0, d^0) = \phi_S(t^0 + d^0) - d^0.$$

It can be verified that

$$f(t^0,d^0)=\begin{cases} -d^0 & \text{if} \quad t^0+d^0\le \mu^1, \\ t^0+(j-1)\lambda-[\mu^1+\sum_{i=2}^{j}(t^{[i]}+d^{[i]})] & \text{if} \quad \mu^1+\sum_{i=2}^{j}(t^{[i]}+d^{[i]})\le t^0+d^0\le \\ & \quad \mu^1+\lambda+\sum_{i=2}^{j}(t^{[i]}+d^{[i]}), j\le k'-1 \\ j\lambda-d^0 & \text{if} \quad \mu^1+\lambda+\sum_{i=2}^{j}(t^{[i]}+d^{[i]})\le t^0+d^0\le \\ & \quad \mu^1+\sum_{i=2}^{j+1}(t^{[i]}+d^{[i]}), j\le k'-1 \\ t^0+(k'-1)\lambda-[\mu^1+\sum_{i=2}^{k'}(t^{[i]}+d^{[i]})], & \text{if} \quad t^0+d^0\ge \mu^1+\sum_{i=2}^{k'}(t^{[i]}+d^{[i]}). \end{cases} \tag{35}$$

Now we can state the complete class of cover inequalities:

Proposition 20 (Lifted Cover Inequalities) *Given a cover S of PI, order the $i\in S$ as in Proposition 16. Also, define μ^1 as in Proposition 16. Let (T,U) be any partition of $\mathcal{P}\setminus S$, let (T',T'') be any partition of T, and define*

$$D'=\max\{t^{[2]}+d^{[2]},\max_{i\in U}\{t^i+d^i\}\}$$

If $\mu^1\ge 0$ and $|S|\ge 2$, then

$$\begin{gathered}\textstyle\sum_{i\in S\cup U}s^i\ge \lambda+\sum_{i\in S}\max\{-t^i,d^i-\lambda\}(1-y^i)+ \\ \textstyle\sum_{i\in U}f(t,d)y^i+\sum_{i\in U}d^i+ \\ \frac{\sum_{i\in S}\min\{t^i+d^i,\lambda\}+\sum_{i\in U}\phi(t^i+d^i)-\lambda}{(|S|+|U|-1)D'}\textstyle\sum_{i\in T'}(x^i-(\mu^1-t^i)y^i)\end{gathered} \tag{36}$$

is valid for X^{PI}.

Proof: Consider any point $(\bar{x},\bar{y},\bar{s})\in X^{PI}$. Let $U^1=\{i\in U:\bar{y}^i=1\}$, and let $U^0=\{i\in U:\bar{y}^i=0\}$. First, consider the case when $T'^1=\{i\in T':\bar{y}^i=1\}=\emptyset$. Let $t^{U^1}=\sum_{i\in U^1}t^i$, and let $d^{U^1}=\sum_{i\in U^1}d^i$. Then Lemma 19 and the fact that $\phi_S(\cdot)$ is superadditive yield

$$\begin{aligned}\sum_{i\in U^1}\phi_S(t^i+d^i) &\le \phi_S(t^{U^i}+d^{U^i}) \\ &= \min_{(x,y,s)\in X^{PI};y^i=1,i\in U^1}\left\{\sum_{S\cup U^1}s^i-(\lambda+\sum_{i\in S}\max\{-t^i,d^i-\lambda\}(1-y^i))\right\} \\ &\le \sum_{S\cup U^1}\bar{s}^i-(\lambda+\sum_{i\in S}\max\{-t^i,d^i-\lambda\}(1-\bar{y}^i)).\end{aligned}$$

This, the fact that $\bar{s}^i \geq d^i, i \in U^0$, and the fact that $\phi_S(t^i + d^i) = f(t^i, d^i)\bar{y}^i + d^i, i \in U$, imply that (36) holds for $(\bar{x}, \bar{y}, \bar{s})$.

If $T'^1 \neq \emptyset$, then validity can be proven by considering a minimization problem similar to that described in the proof of Proposition 16. $\square$

Example 1 (continued)
Again take $S = \{1, 2, 3\}$ as our cover, so that $\lambda = 3$, $t^{[1]} + d^{[1]} = t^1 + d^1 = 7$, and $\mu^1 = 4$. Then $k' = 3$, and

$$\phi_S(u) = \begin{cases} 0 & \text{if} \quad u \leq 4, \\ u - 4 & \text{if} \quad 4 \leq u \leq 7 \\ 3 & \text{if} \quad 7 \leq u \leq 10 \\ 3 + (u - 10) & \text{if} \quad 10 \leq u \leq 13 \\ 6 & \text{if} \quad 13 \leq u \leq 16 \\ 6 + (u - 16) & \text{if} \quad u \geq 16, \end{cases} \tag{37}$$

Let $U = \{4\}$. Then $\phi_S(t^4 + d^4) = \phi_S(13) = 6$, and so $f(t^4, d^4) = \phi_S(t^4 + d^4) - d^4 = 6 - 10 = -4$. Then (25) yields

$$\begin{aligned} s^1 + s^2 + s^3 + s^4 &\geq 3 + 2(1 - y^1) + (1 - y^2) + 2(1 - y^3) - 4y^4 + 10 \\ &= 18 - 2y^1 - y^2 - 2y^3 - 4y^4, \end{aligned} \tag{38}$$

a valid inequality for X^{PI}. $\square$

Under certain conditions, cover inequalities define facets of conv(X^{PI}). For example, we have

Proposition 21 *Given an inequality of the form (36), let $\lambda > 0$, $t^{[2]} + d^{[2]} > \lambda$, $t^i < \mu^1, i \in T$, and $T' = \emptyset$, Also, let k' be defined as in (33); for each $i \in U$, assume that $t^i + d^i = \sum_{j=1}^{k}(t^{[j]} + d^{[j]})$, for some $k : 1 \leq k \leq k' - 1$. Then (36) induces a facet of conv(X^{PI}).*

Proof: Again it suffices to exhibit $3P - T$ points that are in the intersection of the hyperplane defined by (36) and of X^{PI}, and that are linearly independent of each other and of the $|T|$ rays $s^i = 1, i \in T$,

To simplify notation we first assume that that $d^i + d^{[1]} \geq \lambda, i \in \{S \backslash [1]\} \cup U$. Note that if we add the qualification $x^i = y^i = 0, s^i = d^i, i \in U$, to each of the points presented in the proof of Proposition 15, then we have $3|S| + 2|T|$ points that are linearly independent of each other and of the extreme directions, and they all satisfy (36) at equality. (This is true because of the condition, implied in Proposition 15 and included in the statement of

Proposition 21, that $T' = \emptyset$.) Therefore, it suffices to show $3|U|$ points in the hyperplane defined by (36) that are linearly independent of each other, of the $3|S| + 2|T|$ points just mentioned, and of the $|T|$ extreme rays. That is, for each item $i' \in U$, it suffices to show three points in the hyperplane defined by (36) that are linearly independent of each other and the points and rays already mentioned.

Let $k' = |\{i \in S : t^i + d^i > \lambda\}|$. Recall that, for each $i' \in U$,

$$t^{i'} + d^{i'} = \sum_{i=1}^{k} (t^{[i]} + d^{[i]}) = \mu^1 + \lambda + \sum_{i=2}^{k} (t^{[i]} + d^{[i]}),$$

for some k such that $1 \leq k \leq k' - 1$. So for each $i' \in U$, consider the points

1.
$$\begin{gathered} x^{i'} = (d^{i'} - \lambda)^+, y^{i'} = 1, s^{i'} = \min\{t^{i'} + d^{i'} + \textstyle\sum_{i=k+1}^{|S|} (t^{[i]} + d^{[i]}) - c, d^{i'}\} = \min\{\lambda, d^{i'}\} \\ x^{[i]} = y^{[i]} = 0, s^{[i]} = d^{[i]}, i = 1, ..., k \\ x^{[k+1]} = d^{[k+1]} - (\lambda - d^{i'})^+, y^{[i]} = 1, s^{[i]} = (\lambda - d^{i'})^+, \\ x^{[i]} = d^{[i]}, y^{[i]} = 1, s^{[i]} = 0, i = k+2, ..., |S| \\ x^i = y^i = 0, s^i = d^i, i \notin S \cup i'; \end{gathered}$$

2.
$$\begin{gathered} x^{i'} = d^{i'}, y^{i'} = 1, s^{i'} = 0 \\ x^{[i]} = y^{[i]} = 0, s^{[i]} = d^{[i]}, i = 1, ..., k+1 \\ x^{[i]} = d^{[i]}, y^{[i]} = 1, s^{[i]} = 0, i = k+2, ..., |S| \\ x^i = y^i = 0, s^i = d^i, i \notin S \cup i'; \end{gathered}$$

3.
$$\begin{gathered} x^{i'} = d^{i'} + t^{[k+1]} + d^{[k+1]} - \lambda, y^{i'} = 1, s^{i'} = 0 \\ x^{[i]} = y^{[i]} = 0, s^{[i]} = d^{[i]}, i = 1, ..., k+1 \\ x^{[i]} = d^{[i]}, y^{[i]} = 1, s^{[i]} = 0, i = k+2, ..., |S| \\ x^i = y^i = 0, s^i = d^i, i \notin S \cup i'. \end{gathered}$$

It can be checked that the $3|U|$ points defined in this manner are linearly independent of each other and of the previously given points and rays.

As before, when the assumption $d^i + d^{[1]} \geq \lambda, i \in \{S \setminus [1]\} \cup U$, does not hold, the proof is similar. □

There are facets induced by inequalities of the form (36) in which $T' \neq \emptyset$; conditions can be stated that are similar to those in Proposition 18.

Although the conditions in Proposition 21 on the elements in U (and T') are restrictive, it is not difficult to show that (36) induces faces of dimension at least $3P - |U| - |T'|$, if the restrictions on items in U and T' do not hold but the other conditions of the proposition do. The proof of this proceeds along the same lines as that given for Proposition 21, and uses many of the same linearly independent points. Moreover, when setup times and demand are constant for all items, under mild conditions *all* cover inequalities (including those with $T' \neq \emptyset$) induce facets, since $t^i + d^i = t^{[1]} + d^{[1]}, i \in \mathcal{P}$.

Example 1 (continued)

The inequality (38) satisfies the conditions of Proposition 21; in particular, note that $t^4 + d^4 = t^1 + d^1 + t^2 + d^2$. Therefore (38) defines a facet of conv(X^{PI}). $\square$

4.3 Completeness Issues

An interesting question is what other classes of inequalities define facets of conv(X^{PI}). In certain special cases, it may be possible to find a complete description of conv(X^{PI}). The question of defining a set of inequalities that characterizes conv(X^{PI}) in the polynomially solvable case mentioned in Proposition 1 remains open. We can, however, state a similar result that, given the cost conditions $q^i > 0, h^i \geq 0, i = 1, ..., P$, has the same algorithmic implications as a complete description would have.

Proposition 22 *Given that $q^i > 0, h^i \geq 0, i = 1, ..., P$, and given an instance of PI in which setup times and demand are constant for all items, optimizing (1) over the polyhedron defined by (2)–(5), (13), (18), (36), and the bounds $y^i \leq 1, i = 1, ..., P$, solves PI.*

The extensive development necessary to prove this result can be found in Miller et al. [2000a]. Thus, the set of inequalities that we have defined for PI suffices to solve it by linear programming in this polynomially solvable case. We remark here that the extensions of the class of cover inequalities accomplished in Propositions 16 and 20 are essential for obtaining this result.

5 Conclusions and Future Directions

In this paper we have introduced a model PI that provides a relaxation of a number of lot–sizing and more general mixed integer models. We have stated results concerning the structure and complexity of this model; among these, we have given three nontrivial classes of facet–inducing valid inequalities for PI. The first of these is analogous to the well–known path inequalities; the other two are new. For one of the new classes, we use the concepts of superadditivity and lifting in order to obtain the complete class. We have shown conditions under which both of these classes of inequalities define facets of the convex hull. Moreover, these classes cut off all fractional extreme optimal solutions of X^{PI}_{LP}, and they are sufficient to allow X^{PI} to be solved by LP in a polynomial solvable case.

In Miller [1999], a number of extensions to PI are considered. For example, when variable lower bounds are added to the formulation of MCL, it is possible to derive an extension of PI that is analogous to a single period of this new lot–sizing model. These extensions provide potentially useful relaxations for even more general models than those for which PI is a relaxation. Analysis of these other generalizations of PI is one area of future work.

Computational experience using the inequalities presented for PI in solving MCL confirms the expectation that they should be effective in solving MIP problems that have substructures of the form PI (Miller et al. [2000b]). We expect that additional computational experience will show that valid inequalities for PI and similar models prove useful in solving instances of the extended lot–sizing models mentioned above. We also expect that they will be useful in solving more general problems such as the fixed charge network flow model mentioned in Section 2.

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