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Efficient hedging of life insurance portfolio for loss-averse insurers

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Abstract

This paper investigates the hedging of equity-linked life insurance portfolio for loss-averse insurers. We consider a general arbitrage-free financial market and an actuarial market composed of $n-$independent policyholders. As the combined market is incomplete, perfect hedging of any actuarial-financial payoff is not possible. Instead, we study the efficient hedging of $n-$size equity-linked life insurance portfolio for insurers who are only concerned with their losses. To this end, we consider stochastic control problems (under the real-world measure) in order to determine the optimal hedging strategies that either maximize the probability of successful hedge (called quantile hedging) or minimize the expectation for a class of shortfall loss functions (called shortfall hedging). We show that the optimal strategies depend both on actuarial and financial risks. Moreover, these strategies adapt not only to the size of the insurance portfolio but also to the risk-aversion of the insurer. Under the additional assumption of complete financial market, we derive the explicit forms of the optimal hedging strategies. The numerical results show that, for loss-averse insurers, the optimal strategies outperform the optimal mean-variance hedging strategy, demonstrating the relevance of adopting the optimal strategy according to the insurers' risk aversion and portfolio size.

Keywords: *Efficient hedging, Super hedging, Incomplete market, Insurance portfolio, Loss-averse.*

1 Introduction

Equity-linked life insurance contracts are characterized by their dependence on both hedgeable and non-hedgeable contingent claims. While the pricing and the fair-valuation of these contracts have been widely discussed in the literature, see among others [1, 10, 12, 27], the question of hedging, especially for loss-averse insurers, receives less attention. Nevertheless, this question is an essential one and is very important, if not more important than the question of pricing, as the insurance market is by definition incomplete. A perfect hedging strategy for any financial-insurance payoff does not exist, and therefore insurers are exposed to losses. The life insurers' profit and loss (P&L) depends both on the observed mortality of their portfolio and on how they invest the premiums received, which is the only thing they can control. The insurers must therefore think about how they will invest all the premiums received so as to honour their obligations to their policyholders and to minimize their losses, whatever adverse situations arise.

Different hedging approaches for equity-linked life insurance portfolios have been proposed in the literature. [21, 22] first introduced risk-minimizing hedging strategies based on quadratic hedging criteria. These risk-minimizing hedging results were then extended to other models and frameworks: [29] considered an incomplete financial market, [9] assumed systematic mortality risk, and [6] considered partially observable market models. Other approaches for hedging equity-linked life portfolios were also examined: [23] studied hedging strategies associated with financial

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variance and standard deviation principles; [3, 12] introduced convex hedging strategies; [2] studied robust hedging strategies for an insurer seeking to maximize its profit; and more recently, [4, 7] investigated loss-averse convex hedging approaches for insurers giving more importance to losses than to gains. Note that, as hedging of insurance portfolios is closely related to asset and liability management (ALM), we can also refer to [16, 19] that study optimal ALM strategies in expected utility frameworks. Nevertheless, the question of the optimal hedging of life insurance portfolios, considered as a whole, for loss-averse insurers that only focus on losses has not yet been addressed. This paper, therefore, aims to fill this gap. For this purpose, we will consider insurers who give absolutely no satisfaction to profits and whose sole aim is to minimize their losses. This framework can be seen as the limiting case where the coefficient of loss aversion tends towards infinity. For this type of insurer, on the one hand, it is important to consider the global hedging of the portfolio and not individual hedging for each contract composing the portfolio; on the other hand, it is important to determine the optimal strategy by formulating the problem in terms of the real measure P, so as to include all possible situations, even the most severe, that insurers may have to face.

To reflect the profile of insurers who are concerned with the downside only, we consider the partial hedging of equity-linked life insurance portfolios using shortfall loss functions. The partial hedging strategies studied in this paper therefore aim to either maximize the probability of successful hedge (called quantile hedging) or minimize a function of the shortfall loss (called shortfall hedging) under the real measure P. These hedging techniques were primarily studied by [14, 15] and then extended to several fields and settings. In incomplete financial markets, [18] extended the strategy in a stochastic volatility environment, and [24] extended it to rough volatility models. In defaultable markets, such as the insurance market, [25, 30] considered the partial hedging of defaultable claims, while in actuarial sciences, [20] studied the efficient hedging of life insurance contracts. However, these different approaches, in defaultable markets, essentially consider only one default risk and not multiple default risks. In practice, insurers mutualize their risks by considering sufficiently large portfolios. The study of optimal hedging for portfolios subject to default therefore seems very relevant.

To the best of our knowledge, this paper is the first to investigate the optimal hedging of n-size life portfolio for loss-averse insurers that only focus on their losses. To this end, we consider quantile and shortall hedging approaches applied in a general financial-actuarial mathematical framework. Although [20] discussed the risk-management of a life insurance portfolio in the context of quantile hedging, the optimal strategy is never deduced. Moreover, as the quantile hedging approach does not control the severity of the losses, it is quite interesting to also consider shortfall hedging strategies for insurers seeking to control the size of the loss. The hedging problem considered is thus formulated as a stochastic control problem with a stochastic target that depends both on financial risk (the value of the financial payoff) and actuarial risk (the number of alive policyholders). Based on the theory of super-replication, initially introduced by [8, 13], we show that optimal strategies are equivalent to (super) hedging strategies of modified payoffs that depend on both financial and actuarial risks. Moreover, we prove that the optimal strategies adapt to the size of the portfolio as well as to the insurers' aversion. Assuming, additionally, that the financial market is complete, we deduce explicit forms for the hedging strategies. Finally, we demonstrate through numerical results that optimal strategies outperform the mean-variance hedging strategy, showing the relevance of using the right strategy depending on the size of the portfolio and the risk aversion of the insurers, especially if the size of the portfolio is large.

The paper is outlined as follows. In Section 2, the mathematical framework for financial and actuarial markets is presented, and the hedging problem is posed. In Section 3, we consider the quantile hedging approach, deduce the optimal (super) hedging strategy, and provide the explicit form of the optimal strategy in complete financial markets. Then, in Section 4, we consider the class of shortfall hedging problems in order to determine hedging strategies that account for the size of the losses. As for the quantile hedging case, we characterize the optimal strategies and deduce, in complete financial markets, the explicit forms of strategies for shortfall and quadratic shortfall losses. Finally, in Section 5, we provide numerical applications in a Black and Scholes framework and compare the optimal strategies with the mean-variance hedging strategy.

2 Mathematical framework

In this paper, we work with a financial market composed of two traded assets: a risky asset $S := (S_t)_{t\geq 0}$ and a risk-free asset $B := (B_t)_{t\geq 0}$. The financial market probability space is given by $(\Omega^f, \mathcal{F}^f, P^f)$ equipped with a filtration $\left(\mathcal{F}_t^f\right)_{t\geq 0}$ satisfying the usual conditions. We make the assumption that the financial market is arbitrage-free. Under this assumption, we know that there exists a set of equivalent measures under which the discounted value of the risky asset is a martingale. This set is denoted by $\mathcal{Q}^f$ such that

$$\mathcal{Q}^f := \{Q^f : \textit{equivalent to } P^f, \textit{ and } \frac{S}{B} \textit{ is a martingale under } Q^f\}.$$

Our aim in the following is to consider the optimal hedging of insurance-financial payoffs by trading in the financial market. In this sense, we consider a self-financing hedging portfolio denoted by $V := (V_t)_{t\geq 0}$ and defined, for $t \geq 0$, by

$$\begin{aligned} dV_t &= (V_t - \xi_t S_t)\frac{dB_t}{B_t} + \xi_t dS_t, \\ V_0 &= v_0, \end{aligned} \tag{1}$$

on $(\Omega^f, \mathcal{F}^f, P^f)$, where $\xi := (\xi_t)_{t\geq 0}$ is an admissible dynamic hedging strategy i.e. ξ is a $\mathcal{F}_t^f-$progressively measurable process such that, for all $t \geq 0$, $E(V_t^2) < +\infty$. At this early stage, we can first consider the hedging of purely financial payoffs of the form $H_T \in \mathcal{F}_T^f$ with $T > 0$ a fixed maturity. As the financial market can be incomplete, not every contingent claim $H_T \in \mathcal{F}_T^f$ can be perfectly replicated by a hedging portfolio, but super-replicating hedging strategies exist. As shown, for example, in [8, 13], for a $T-$maturity given claim $H_T \in \mathcal{F}_T^f$, if

$$v_0 = \sup_{Q^f \in \mathcal{Q}^f} E^{Q_f}\left(\frac{H_T}{B_T}\right),$$

then there exists a super-replicating hedging strategy i.e.

$$\exists \xi \in \mathcal{A} \; s.t. \; V_T \geq H_T \; P^f - a.s.$$

where $\mathcal{A}$ is the set of all admissible hedging strategies valued in $\mathbb{R}$. In particular, if in addition the financial market is assumed to be complete, such that the set of equivalent measures reduces to a singleton, i.e. $\mathcal{Q}^f = \left\{\tilde{Q}^f\right\}$, then for

$$v_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T}\right),$$

a perfect hedging strategy exists i.e.

$$\exists \xi \in \mathcal{A} \; s.t. \; V_T = H_T \; P^f - a.s.$$

and the value at time $t \in [0, T]$ of the perfect hedging portfolio is given by

$$V_t = E^{\tilde{Q}^f}\left(\frac{B_t}{B_T} H_T | \mathcal{F}_t^f\right).$$

We now introduce the life insurance market, described by the probability space $(\Omega^a, \mathcal{F}^a, P^a)$ equipped with a filtration $(\mathcal{F}^a_t)_{t\geq 0}$. We assume that the remaining lifetime of an $x-$year-old individual is a random variable denoted by τ_x and defined on $(\Omega^a, \mathcal{F}^a, P^a)$. The mortality rate is noted $(\mu_{x+t})_{t\geq 0}$ and the probability of survival from age x till age $x+t$, for $t \geq 0$, is noted ${}_tp_x := E^{P^a}\Bigg(\exp\left(-\int_0^t \mu_{x+s} ds\right)\Bigg)$. We assume that the insurer has a $n-$size portfolio with policyholders from the same cohort, such that for $i = 1, ..., n$, $x_i = x$ and $\mu_{x_i+t} = \mu_{x+t}$. In this case, the remaining lifetime of the i^{th} policyholder is denoted by τ_i and $\tau_1, ..., \tau_n$ are assumed to be independent. If we define, for $i = 1, ..., n$,

$$N^i_t := 1_{\{\tau_i > t\}},$$

then the total number of policyholders remaining at time t is defined by

$$N_t := \sum_{i=1}^{n} N^i_t.$$

In the following, we will focus on the combined model denoted $(\Omega, \mathcal{F}, P)$, which is defined as the product space of the two individual spaces, financial and actuarial. We let $\Omega := \Omega^f \times \Omega^a$ and $P := P^f \otimes P^a$. The $\sigma-$algebra $\mathcal{F}$ is generated by all subsets of null-sets from $\mathcal{F}^a \otimes \mathcal{F}^f$ and the associated filtration is denoted by $(\mathcal{F}_t)_{t\geq 0}$. We make the standard assumption that actuarial and financial risks are independent. In this setting, $\mathcal{F}^f$ and $\mathcal{F}^a$ martingales are martingales in $(\Omega, \mathcal{F}, P)$. Even if the financial market is assumed to be complete, the combined market is incomplete since the mortality risk cannot be perfectly hedged. Therefore, we know that the pricing measure is not unique and that there exists a class of equivalent martingale measures denoted by $\mathcal{Q}$ such

$$\mathcal{Q} := \{Q : \textit{equivalent to } P, \textit{ and } \frac{S}{B} \textit{ is a martingale under } Q\}.$$

For any contingent claim of the form $g(S_T, N_T) \in \mathcal{F}_T$, with $T > 0$ a fixed maturity and $g(x, y)$ a positive function increasing in y, we can characterize the super-replication price as the super-replication price of a $\mathcal{F}^f_T-$measurable payoff. This is the purpose of the next proposition.

Proposition 1. *Let $g(x, y)$ a positive function increasing in y, then*

$$\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{g(S_T, N_T)}{B_T}\right) = \sup_{Q^f\in\mathcal{Q}^f} E^{Q_f}\left(\frac{g(S_T, n)}{B_T}\right).$$

Proof. The proof relies on Propositions 12 and 13 in Appendix 7.1. First, by independence between financial and actuarial probability spaces, we observe that

$$\sup_{Q\in\mathcal{Q}} E^Q\left(\frac{g(S_T, N_T)}{B_T}\right) = \sup_{Q^f\in\mathcal{Q}^f} \sup_{Q^a\in\mathcal{Q}^a} E^{Q^f\otimes Q^a}\left(\frac{g(S_T, N_T)}{B_T}\right),$$

where $\mathcal{Q}^a$ is the set of measures equivalent to P^a. Then, on the one hand, since $g(x, y)$ is growing in y and $N_T \leq n$, we observe that, for all $Q^f \in \mathcal{Q}^f$,

$$\sup_{Q^a\in\mathcal{Q}^a} E^{Q^f\otimes Q^a}\left(\frac{g(S_T, N_T)}{B_T}\right) \leq E^{Q_f}\left(\frac{g(S_T, n)}{B_T}\right).$$

On the other hand, using Proposition 13, for $\varepsilon > -1$, for all $\mathcal{F}^a_t-$adapted process $(h_t)_{t\geq 0}$ such that for $\forall t \geq 0$, $h_t > -1$ almost surely and for all $Q^f \in \mathcal{Q}^f$, there exists an equivalent martingale measure $Q_{h^\varepsilon} \in \mathcal{Q}$ such that

$$\lim_{\varepsilon\to -1} E^{Q_{h^\varepsilon}}\left(\frac{g(S_T, N_T)}{B_T}\right) = E^{Q_f}\left(\frac{g(S_T, n)}{B_T}\right).$$

Therefore, we have that, $\forall Q^f \in \mathcal{Q}^f$,

$$\sup_{Q^a \in \mathcal{Q}^a} E^{Q^f \otimes Q^a}\left(\frac{g(S_T, N_T)}{B_T}\right) = E^{Q_f}\left(\frac{g(S_T, n)}{B_T}\right)$$

and we easily conclude that

$$\sup_{Q \in \mathcal{Q}} E^{Q}\left(\frac{g(S_T, N_T)}{B_T}\right) = \sup_{Q^f \in \mathcal{Q}^f} E^{Q_f}\left(\frac{g(S_T, n)}{B_T}\right).$$

□

Suppose now that the insurer guarantees to its policyholders a stochastic positive amount $H_T \in \mathcal{F}_T^f$ if they are alive at a predefined maturity $T > 0$. In this case, the total liability of the insurer at maturity is given by

$$H_T^{Total} := N_T \times H_T \in \mathcal{F}_T.$$

To meet its liabilities, the insurer constructs a hedging portfolio denoted by V and driven by an admissible hedging strategy ξ that evolves as (1). In the best of all possible worlds, the insurer would like to find a strategy that perfectly replicates its liabilities. Due to the incompleteness of the global market, perfect hedging strategy does not exist for any contingent claim $H_T^{Total} \in \mathcal{F}_T$. However, if

$$v_0 = \sup_{Q \in \mathcal{Q}} E^{Q}\left(\frac{H_T^{Total}}{B_T}\right) = \begin{cases} \sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(n \times \frac{H_T}{B_T}\right), & \\ E^{\tilde{Q}^f}\left(n \times \frac{H_T}{B_T}\right) & if\ \mathcal{Q}^f = \left\{\tilde{Q}^f\right\}. \end{cases} \tag{2}$$

then there exists a super-replicating hedging strategy i.e.

$$\exists \xi \in \mathcal{A}\ s.t.\ V_T \geq H_T^{Total}\ P - a.s.$$

The super-replication price of the total liability (2) is actually too expensive, even if the financial market is complete, to be competitive in practice, which is why the insurer is not going to build the super hedging strategy. In fact, the premium that the insurer has to charge its policyholders for super hedging is the price of the same product without the mortality risk. Policyholders would therefore have no interest in choosing the insurance product over the financial product since the financial product is less risky for them and, consequently, the insurer would not have any business.

In what follows, we will assume that the insurer sells its life product at a total price $\bar{V}_0$ such that $\bar{V}_0 \leq \sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(n \times \frac{H_T}{B_T}\right)$ and that this agent is loss-averse. So, instead of seeking to maximize the final value of its P&L and therefore its profit, its main aim is to limit the events leading to losses and also to control the size of these losses. Based on its own risk aversion, the insurer can define an efficient hedging strategy.
On one hand, the insurer can seek to maximize the probability of successful hedge. Thus, in this case, the insurer constructs a strategy to maximize the probability (under P) that the terminal value of the hedging portfolio is above the total payoff. From a mathematical point of view, the problem reduces to the following stochastic control problem:

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}} P(V_T \geq H_T^{Total}), \tag{3}$$

under the initial constraint

$$v_0 \leq \bar{V}_0.$$

On the other, the insurer can seek to minimize the loss resulting from the hedging operation. In this case, the insurer constructs a strategy to minimize the expectation of a class shortfall loss functions. This problem can be written as stochastic control problem of the form:

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}} E^P\Big(L\Big((H_T^{Total} - V_T)_+\Big)\Big), \tag{4}$$

with $L \in C^1$, under the initial constraint

$$v_0 \leq \bar{V}_0.$$

Such hedging strategies are called quantile hedging or shortfall hedging strategies and were initially studied by Föllmer and Leukert in [14, 15]. The difference with the Föllmer and Leukert setting is that here we have both financial and actuarial risks. Note that [20] has already studied efficient hedging of equity-linked insurance contracts and has shown that the optimal quantile hedging strategy, when $n = 1$, is the quantile hedging strategy of the purely financial payoff $H_T \in \mathcal{F}_T^f$ (without actuarial risk). However, although the risk management of portfolios is discussed, the optimal hedging strategies are not deduced when $n > 1$.
In the following, we will study quantile and shortfall hedging strategies applied in the context of hedging a $n-$size life insurance portfolio for $n \geq 1$. Under suitable hypotheses, the stochastic control problems studied in this paper can be solved using dynamic programming techniques like solving the Hamilton-Jacobi-Bellman (HJB) equation as it was done in [18, 24]. Nevertheless, instead of applying those techniques, we will use the super hedging theory to characterize the optimal strategies. In this sense, we will transform stochastic control problems into static optimization problems. This allows us to derive that optimal hedging portfolios are actually the (super) replication hedging portfolios of $\mathcal{F}_T^f-$measurable modified payoffs. Moreover, when the inital super-replication prices of modified payoffs are attainable for an equivalent measure in $\mathcal{Q}^f$, then we can deduce explicit forms for the optimal strategies. The advantage of this approach, compared with dynamic programming techniques, is that it allows to easily characterize the strategies, and as soon as the financial market is complete, explicit forms of optimal hedging strategies can be deduced even if the financial framework is not Markovian[1].

3 Optimal quantile hedging strategy

In this section, we solve the introduced quantile hedging problem (3). To this end, we introduce the sets of sucessfull financial hedge $A_1, ..., A_n \in \mathcal{F}_T^f$ defined as

$$A_i := \Big\{V_T \geq iH_T\Big\},\ i = 1, ..., n,$$

and we show that the dynamic optimization problem (3) is equivalent to a static optimization problem that allows to characterize the optimal strategy.

Theorem 2. *Let $p_i := P(N_T = i)$ and $A_i^* \in \mathcal{F}_T^f$, $i = 1, ..., n$ the solutions of the following problem*

$$\max_{A_1,...,A_n \in \mathcal{F}_T^f,\, A_{i+1} \subseteq A_i} E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i\right) + p_0, \tag{5}$$

under the constraint

$$\sup_{Q \in \mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \times 1_{N_T \geq 1} \sum_{i=1}^{N_T} \mathbb{1}_{A_i}\right) = \sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right) \leq \bar{V}_0.$$

[1]There exist different non-Markovian complete financial markets, such as, for example, fully-endogenous rough volatility or path-dependent volatility models.

Let ξ^{QH} denote the (super) replicating hedging strategy for the modified payoff

$$H_T^{QH} := H_T \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*}. \tag{6}$$

Then $(\bar{V}_0, \xi^{QH})$ solves the stochastic control problem defined by (3), and

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \text{ } s.t. \text{ } v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}) = E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i\right) + p_0.$$

Proof. 1) Let us denote a solution to the Problem (5) by $A_i^* \in \mathcal{F}_T^f$, $i = 1, ..., n$ such that $A_{i+1}^* \subseteq A_i^*$. For any admissible hedging strategy with $v_0 \leq \bar{V}_0$, we first observe that the associated success set is given by

$$\mathbb{1}_{\{V_T \geq H_T^{Total}\}} = 1_{N_T=0} + \sum_{i=1}^{n} \mathbb{1}_{\{V_T \geq i\, H_T\}} \mathbb{1}_{\{N_T=i\}}.$$

Moreover, we have that, $P-$almost surely,

$$\begin{aligned} V_T &\geq H_T^{Total}\, 1_{\{V_T \geq H_T^{Total}\}} \\ &= H_T \times N_T\, 1_{\{V_T \geq H_T \times N_T\}} \\ &= H_T \times 1_{N_T \geq 1} \sum_{i=1}^{N_T} 1_{\{V_T \geq i\, H_T\}}. \end{aligned}$$

Therefore, for $A_i = \Big\{V_T \geq i H_T\Big\} \in \mathcal{F}_T^f$ such that $A_{i+1} \subseteq A_i$, we observe that,

$$\begin{aligned} \bar{V}_0 \geq v_0 \geq \sup_{Q \in \mathcal{Q}} E^Q\left(\frac{V_T}{B_T}\right) &\geq \sup_{Q \in \mathcal{Q}} E^Q\left(\frac{H_T}{B_T} \times 1_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} \mathbb{1}_{A_i}\right), \\ &= \sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right) \end{aligned}$$

for any $Q \in \mathcal{Q}$, since nonnegative local $Q-$local martingale V is super martingale, this implies

$$\begin{aligned} E^P\left(\mathbb{1}_{\{V_T \geq H_T^{Total}\}}\right) &= E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times \mathbb{1}_{\{N_T=i\}} + \mathbb{1}_{\{N_T=0\}}\right) \\ &\leq E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}} + \mathbb{1}_{\{N_T=0\}}\right) \\ &= E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i\right) + p_0, \end{aligned} \tag{7}$$

and thus

$$\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \text{ } s.t. \text{ } v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}) \leq E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i\right) + p_0. \tag{8}$$

2) Let now consider the hedging strategy $\xi^{QH} \in \mathcal{A}$ associated the (super) hedging for the modified payoff $H_T^{QH} = H_T \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*}$ such that $\sup_{Q^f \in \mathcal{Q}^f} E^{Q_f}\left(\frac{H_T^{QH}}{B_T}\right) \leq \bar{V}_0$. In this case, if we denote $\bar{V}_T$ the portfolio with initial value

$\bar{V}_0$ that evolves according to the hedging strategy ξ^{QH}, we have that

$$\bar{V}_T \geq H_T^{QH},$$

$P-$almost surely.
The corresponding maximal "success set" satisfies

$$\begin{aligned}
E^P\left(\mathbb{1}_{\{\bar{V}_T \geq H_T^{Total}\}}\right) &\geq E^P\left(\mathbb{1}_{\{H_T^{QH} \geq H_T^{Total}\}}\right) \\
&= E^P\left(\mathbb{1}_{\{N_T=0\}} + \mathbb{1}_{\{N_T \geq 1\}}\mathbb{1}_{\{H_T \times \sum_{i=1}^n 1_{A_i^*} \geq N_T H_T\}}\right) \\
&= E^P\left(\mathbb{1}_{\{N_T=0\}} + \sum_{i=1}^n \mathbb{1}_{\{N_T=i\}} \times \mathbb{1}_{\{H_T 1_{A_i^*} \geq H_T\}}\right) \\
&= E^P\left(\sum_{i=1}^n \mathbb{1}_{\{H_T \mathbb{1}_{A_i^*} \geq H_T\}} \times p_i\right) + p_0
\end{aligned}$$

For $i = 1, ..., n$, let define $\tilde{A}_i \in \mathcal{F}_T^f$ such that

$$\tilde{A}_i := \left\{H_T \mathbb{1}_{A_i^*} \geq H_T\right\} \supset A_i^*$$

and $\tilde{A}_{i+1} \subseteq \tilde{A}_i$.
Thus, we deduce that

$$\begin{aligned}
E^P\left(\mathbb{1}_{\{\bar{V}_T \geq H_T^{Total}\}}\right) &\geq E^P\left(\sum_{i=1}^n \mathbb{1}_{\tilde{A}_i} \times p_i\right) + p_0 \\
&\geq E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times p_i\right) + p_0.
\end{aligned} \tag{9}$$

Combining (7), (8) and (9), we have that

$$\begin{aligned}
\max_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \ s.t. \ v_0 \leq \bar{V}_0} P(V_T \geq H_T^{Total}) =& E^P\left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times p_i\right) + p_0 \\
=& E^P\left(\mathbb{1}_{\{\bar{V}_T \geq H_T^{Total}\}}\right),
\end{aligned}$$

and the optimality of the strategy $(\bar{V}_0, \xi^{QH})$ in Problem (3) follows. □

Remark 3. The existence of a solution to the problem (5) follows from the Proposition 14 in Appendix 7.2.

The previous theorem shows that the optimal quantile hedging of a $n-$size life insurance portfolio is linked to a (super) replication strategy of a modified financial payoff that takes into account the actuarial risk through the probabilities $p_i := P(N_T = i)$, $i = 1, ..., n$. In general, for an incomplete financial market, we cannot unfortunately deduce the explicit form of the optimal sets A_i^*, $i = 1, ..., n$ solving problem (5). However, if the financial market is assumed to be complete, or more generally, if there exists an equivalent martingale measure $\tilde{Q}^f \in \mathcal{Q}^f$ such that, for all sets $A_1, ..., A_n \in \mathcal{F}_T^f$, $A_{i+1} \subseteq A_i$,

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i}\right) = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i}\right),$$

then we can deduce the explicit form of the optimal sets A_i^* associated with the optimal quantile hedging strategy. This is the purpose of the next proposition.

Proposition 4. *Supppose that* $\exists \tilde{Q}^f \in \mathcal{Q}^f$ *such that,* $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$ *and for all sets* $A_1, ..., A_n \in \mathcal{F}_T^f,\ A_{i+1} \subseteq A_i$*,*

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right) = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right),$$

then the optimal solution to the problem (3) is such that

$$\max_{v_0 \in \mathbb{R}_+,\ \xi \in \mathcal{A}\ s.t.\ v_0 \le \bar{V}_0} P(V_T \ge H_T^{Total}) = E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i\right) + p_0$$

where

$$A_i^* = \left\{\alpha_i \ge \lambda^* \times \tilde{Z}_T \frac{H_T}{B_T}\right\},\ i = 1, ..., n,$$

with:

- $\alpha_n = \min\left(p_n, \frac{p_n + p_{n-1}}{2}, ..., \frac{\sum_{j=1}^{n} p_j}{n}\right)$,
- for $i = 1, ..., n-1$, $\alpha_i = \max\left(\alpha_{i+1}, \min_{k=1,..,i}\left(\frac{\sum_{j=k}^{i} p_j}{i-k+1}\right)\right)$,
- λ^* is such that
$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*}\right).$$

Moreover, the optimal hedging strategy ξ^{QH} is the admissible strategy associated to the (super) hedging of the modified payoff $H_T \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*}$.

Proof. By Slater's condition (see for example Theorem 3.11.2 in [28]), strong duality holds and the constrained optimization problem (5) is equivalent to

$$\min_{\lambda \in \mathbb{R}_+} \max_{A_1, ..., A_n \in \mathcal{F}_T^f,\ A_{i+1} \subseteq A_i} E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i\right) + p_0 + \lambda\left(\bar{V}_0 - E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right)\right)$$

or

$$\min_{\lambda \in \mathbb{R}_+} \max_{A_1, ..., A_n \in \mathcal{F}_T^f\ A_{i+1} \subseteq A_i} E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i\right) - \lambda\, E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right) + p_0 + \lambda\, \bar{V}_0$$

Furthermore, as $A_1, ..., A_n \in \mathcal{F}_T^f$ such that for $i = 1, ..., n-1,\ A_{i+1} \subseteq A_i$ and $\lambda \in \mathbb{R}_+$, we observe that,

$$\begin{aligned}
&E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i\right) - \lambda\, E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right) \\
=&E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i - \lambda\, \tilde{Z}_T \frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\right) \\
=&E^P\left(\mathbb{1}_{A_1}\left(p_1 - \lambda\, \tilde{Z}_T \frac{H_T}{B_T} + 1_{A_2}\left(p_2 - \lambda\, \tilde{Z}_T \frac{H_T}{B_T} + ... + 1_{A_n}\left(p_n - \lambda\, \tilde{Z}_T \frac{H_T}{B_T}\right)...\right)\right)\right) \\
\le&E^P\left(\mathbb{1}_{\tilde{A}_1(\lambda)}\left(p_1 - \lambda\, \tilde{Z}_T \frac{H_T}{B_T} + \mathbb{1}_{\tilde{A}_2(\lambda)}\left(p_2 - \lambda\, \tilde{Z}_T \frac{H_T}{B_T} + ... + \mathbb{1}_{\tilde{A}_n(\lambda)}\left(p_n - \lambda\, \tilde{Z}_T \frac{H_T}{B_T}\right)...\right)\right)\right),
\end{aligned}$$

with

$$\tilde{A}_n(\lambda) = \left\{p_n \ge \lambda\, \tilde{Z}_T \frac{H_T}{B_T}\right\} \tag{10}$$

and for $i = 1, ..., n-1$,

$$\tilde{A}_i(\lambda) = \left\{ p_i + \sum_{j=i+1}^{n} \mathbb{1}_{\tilde{A}_j(\lambda)} p_j \geq \lambda \, \tilde{Z}_T \frac{H_T}{B_T} \left(1 + \sum_{j=i+1}^{n} \mathbb{1}_{\tilde{A}_j(\lambda)} \right) \right\}. \tag{11}$$

In this case, by taking $A_1^*(\lambda), ..., A_n^*(\lambda) \in \mathcal{F}_T^f$ such that

$$A_i^*(\lambda) := \bigcap_{j=1}^{i} \tilde{A}_j(\lambda), \tag{12}$$

as for $i = 1, ..., n-1$, $A_{i+1}^*(\lambda) \subseteq A_i^*(\lambda)$, we deduce that, for $\lambda \in \mathbb{R}_+$,

$$\begin{aligned} \max_{A_1,...,A_n \in \mathcal{F}_T^f,\, A_{i+1} \subseteq A_i} & E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i \right) - \lambda \, E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i} \right) \\ &= E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda)} \times p_i \right) - \lambda \, E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda)} \right). \end{aligned}$$

Moreover, by duality, we obtain that

$$\begin{aligned} & E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i \right) + p_0 \\ &= \min_{\lambda \in \mathbb{R}_+} E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda)} \times p_i \right) + p_0 + \lambda \left(\bar{V}_0 - E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda)} \right) \right) \\ &= E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda^*)} \times p_i \right) + p_0 + \lambda^* \left(\bar{V}_0 - E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda^*)} \right) \right) \\ &\geq E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i \right) + p_0 + \lambda^* \left(\bar{V}_0 - E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*} \right) \right) \\ &\geq E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i \right) + p_0, \end{aligned}$$

then, we have that $A_i^* = A_i^*(\lambda^*)$, $i = 1, ..., n$ with the optimal constant λ^* that satisfies[2]

$$\bar{V}_0 = E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda^*)} \right).$$

We conclude that

$$\begin{aligned} \min_{\lambda \in \mathbb{R}_+} \max_{A_1,...,A_n \in \mathcal{F}_T,\, A_{i+1} \subseteq A_i} & E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i \right) + p_0 + \lambda \left(\bar{V}_0 - E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i} \right) \right) \\ &= E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda^*)} \times p_i \right) + p_0 + \lambda^* \left(\bar{V}_0 - E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*(\lambda^*)} \right) \right) \\ &= E^P\left(\sum_{i=1}^{n} \mathbb{1}_{A_i^*} \times p_i \right) + p_0. \end{aligned}$$

[2]The constraint $\bar{V}_0 \leq E^{\tilde{Q}^f}\left(n \times \frac{H_T}{B_T} \right)$ ensures that $\lambda^* \in \mathbb{R}_+$

Finally, using together (10), (11) and (12), by defining the constants

$$\alpha_n := \min\left(p_n, \frac{p_n + p_{n-1}}{2}, ..., \frac{\sum_{j=1}^{n} p_j}{n}\right),$$

and for $i = 1, ..., n-1$,

$$\alpha_i := \max\left(\alpha_{i+1}, \min\left(p_i, \frac{p_i + p_{i-1}}{2} ..., \frac{\sum_{j=2}^{i} p_j}{i-1}, \frac{\sum_{j=1}^{i} p_j}{i}\right)\right),$$

we can easily deduce that

$$A_n^* = \left\{\alpha_n \geq \lambda^* \times \tilde{Z}_T \frac{H_T}{B_T}\right\},$$

and for $i = 1, ..., n-1$,

$$\begin{aligned} A_i^* &= \left\{\max\left(\alpha_{i+1}, \min\left(p_i, \frac{p_i + p_{i-1}}{2} ..., \frac{\sum_{j=2}^{i} p_j}{i-1}, \frac{\sum_{j=1}^{i} p_j}{i}\right)\right) \geq \lambda^* \times \tilde{Z}_T \frac{H_T}{B_T}\right\} \\ &= \left\{\max\left(\alpha_{i+1}, \min_{k=1,..,i}\left(\frac{\sum_{j=k}^{i} p_j}{i-k+1}\right)\right) \geq \lambda^* \times \tilde{Z}_T \frac{H_T}{B_T}\right\}. \end{aligned}$$

In addition, the optimal hedging strategy ξ^{QH} is the admissible strategy associated to the (super) hedging of the modified payoff $H_T \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*}$. □

Remark 5. The result of Proposition 4 can obviously be applied as soon as the financial market is complete since, in this case, $\mathcal{Q}^f = \{\tilde{Q}^f\}$, but also in some incomplete markets if, for all $A_1, ..., A_n \in \mathcal{F}_T^f$, $A_{i+1} \subseteq A_i$, the modified payoff $H_T \times \sum_{i=1}^{n} \mathbb{1}_{A_i}$ is almost surely upper bounded and the set $\mathcal{Q}^f$ is weakly compact as mentioned in [5].

From Proposition 4, we deduce that, if the financial market is complete, then the optimal quantile hedging strategy of insurance-financial payoffs is equivalent to a perfect hedging strategy of a sum of knock-out financial options. This strategy therefore generalizes the "pure" quantile hedging strategy[3], taking into account the actuarial risk. In particular, if $n = 1$, the two strategies coincide, and we recover the result deduced in [20]. We can also observe that, when $n > 1$, the strategy associated with the quantile hedging of $n \times H_T$ is optimal only if $\forall i \in \{1, ..., n\}$, $\alpha_i = \alpha$, which is very unlikely to occur, especially when n is large. Therefore, in general, the quantile hedging of $n \times H_T$ is not the optimal strategy, and it is preferable for the insurer to adopt the optimal quantile hedging strategy deduced for a $n-$size portfolio. Moreover, the possible mean-variance hedging approach (see [3, 12]), which consists of perfectly hedge the contingent claim $n \times_T p_x \times H_T$, is not optimal either on the basis of the criterion of maximizing the probability of successful hedge. In the numerical results, we compare the different hedging strategies to quantify the difference between them.

4 Optimal shortfall hedging strategy

In the previous section, we showed how to deduce the optimal quantile hedging strategy associated with the hedging of a life insurance portfolio. However, the main drawback of such hedging strategy is that it does not control the severity of the loss when the final value of the hedging portfolio is not sufficient to totally hedge the total payoff H_T^{Total}. This quantile hedging approach therefore suffers from the same issue as optimal dynamic asset allocations under value-at-risk constraint (see [17]), and this can lead to very large losses. Therefore, in order to control the size of the shortfall loss, the insurer might be motivated to minimize the expectation of a class shortfall loss functions.

[3] Strategy associated to the quantile hedging of $H_T \in \mathcal{F}_T^f$

To this end, we will solve in this section the introduced shortfall hedging problem (4). As in the previous section, we introduce auxiliary $\mathcal{F}_T^f-$measurable variables $\phi_1, ..., \phi_n$ defined such that

$$\phi_i H_T := \begin{cases} 0, & if\ V_T \leq (i-1)H_T \\ H_T & if\ V_T \geq iH_T \\ V_T - (i-1)H_T & if\ (i-1)H_T < V_T < iH_T, \end{cases} \tag{13}$$

where ϕ_i can be interpreted as the success ratio for the hedging of the i^{th} financial contingent claim H_T. Using these introduced variables, we show that the dynamic optimization problem (4) is equivalent to a static optimization problem that enables us to characterise the class of optimal shortfall hedging strategies.

Theorem 6. *Let $p_i := P(N_T = i)$ and $\phi_i^* \in \mathcal{F}_T^f$, $i = 1, ..., n$ the solutions of the following problem*

$$\min_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\ \phi_0=1} E^P\Bigg(\sum_{i=1}^n p_i \times L\Bigg(\sum_{j=1}^i H_T(1-\phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\Bigg)\Bigg), \tag{14}$$

with

$$\mathcal{R} = \Big\{\phi : \Omega \to [0,1], \mathcal{F}_T^f - measurable\Big\}$$

under the constraint

$$\sup_{Q \in \mathcal{Q}} E^Q\Bigg(\frac{H_T}{B_T}\sum_{i=1}^{N_T} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\Bigg) = \sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\Bigg(\frac{H_T}{B_T}\sum_{i=1}^{n} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\Bigg) \leq \bar{V}_0.$$

Let ξ^{SH} denote the (super) replicating hedging strategy of the modified payoff

$$H_T^{SH} = H_T \times \sum_{i=1}^n \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}. \tag{15}$$

Then $(\bar{V}_0, \xi^{SH})$ solves the stochastic control problem defined by (4), and

$$\min_{v_0 \in \mathbb{R}_+,\ \xi \in \mathcal{A}\ s.t.\ v_0 \leq \bar{V}_0} E^P\Bigg(L\Big((H_T^{Total} - V_T)_+\Big)\Bigg) = E^P\Bigg(\sum_{i=1}^n p_i \times L\Bigg(\sum_{j=1}^i H_T(1-\phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}})\Bigg)\Bigg).$$

Proof. 1) Let us denote a solution to the Problem (14) by $\phi_i^* \in \mathcal{R}$, $i = 1, ..., n$ such that $\phi_{i+1}^* \leq \phi_i^*$. Firsly, for any admissible hedging strategy with $v_0 \leq \bar{V}_0$, we can observe that

$$\begin{aligned}(H_T^{Total} - V_T)_+ &= H_T N_T - \min\Big\{V_T, N_T H_T\Big\} \\ &= \sum_{i=1}^{N_T} H_T - \sum_{i=1}^n \min\Big\{V_T, iH_T\Big\}\mathbb{1}_{\{N_T=i\}}.\end{aligned}$$

For $i = 1, ..., n$, we postulate that

$$\Phi_i H_T := \min\Big\{V_T, iH_T\Big\},$$

with

$$\Phi_i \in \Big\{\phi : \Omega \to [0,i],\ \mathcal{F}_T^f - measurable\Big\}$$

such that $P-$almost surely

$$\Phi_{i+1} \geq \Phi_i,$$

and then

$$(H_T^{Total} - V_T)_+ = \sum_{i=1}^{N_T} H_T - \sum_{i=1}^{n} \Phi_i H_T \mathbb{1}_{\{N_T=i\}}.$$

Moreover, for $i = 1, ..., n$, we observe that, for ϕ_i defined by (13),

$$(\Phi_i - \Phi_{i-1})H_T = \phi_i H_T$$

such that

$$\phi_i \in \left\{ \phi : \Omega \to [0,1],\ \mathcal{F}_T^f - measurable \right\}$$

and $P-$almost surely

$$\phi_{i+1} \leq \phi_i.$$

Thus, for $i = 1, ..., n$, we can rewritte Φ_i as

$$\Phi_i = \sum_{j=1}^{i} \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}},$$

with $\phi_0 := 1$. In this case, we deduce that

$$\begin{aligned} L\Big((H_T^{Total} - V_T)_+ \Big) &= L\Big(\sum_{i=1}^{N_T} H_T - \sum_{i=1}^{n} \Phi_i H_T \mathbb{1}_{\{N_T=i\}} \Big) \\ &= L\Big(\sum_{i=1}^{N_T} H_T (1 - \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}) \Big). \end{aligned}$$

Moreover, as we have that

$$\min \left\{ V_T, H_T N_T \right\} = \sum_{i=1}^{N_T} H_T \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}},$$

then

$$V_T \geq \sum_{i=1}^{N_T} H_T \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}},$$

and we observe that

$$\begin{aligned} \bar{V}_0 \geq v_0 \geq \sup_{Q \in \mathcal{Q}} E^Q \Big(\frac{V_T}{B_T} \Big) &\geq \sup_{Q \in \mathcal{Q}} E^Q \Big(\frac{H_T}{B_T} \sum_{i=1}^{N_T} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \Big), \\ &= \sup_{Q^f \in \mathcal{Q}^f} E^{Q^f} \Big(\frac{H_T}{B_T} \sum_{i=1}^{n} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \Big) \end{aligned}$$

for any $Q \in \mathcal{Q}$, since nonnegative local $Q-$local martingale V is super martingale. This implies that

$$\begin{aligned} E^P\Big(L\Big((H_T^{Total} - V_T)_+ \Big) \Big) &\geq E^P\Big(L\Big(\sum_{i=1}^{N_T} H_T (1 - \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}) \Big) \Big) \\ &= E^P\Big(\sum_{i=1}^{n} p_i \times L\Big(\sum_{j=1}^{i} H_T (1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}}) \Big) \Big), \end{aligned}$$

and thus

$$\min_{v_0\in\mathbb{R}_+,\,\xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0} E^P\Big(L\Big((H_T^{Total}-V_T)_+\Big)\Big) \geq E^P\Big(\sum_{i=1}^{n} p_i\times L\Big(\sum_{j=1}^{i} H_T(1-\phi_j^*\mathbb{1}_{\{\phi_{j-1}^*=1\}})\Big)\Big). \tag{16}$$

2) Let now consider the hedging strategy $\xi^{SH}\in\mathcal{A}$ associated to the (super) replicating hedging of the modified payoff $H_T^{SH}=H_T\times\sum_{i=1}^{n}\phi_i^*\mathbb{1}_{\{\phi_{i-1}^*=1\}}$ such that $\sup_{Q^f\in\mathcal{Q}^f}E^{Q_f}\left(\frac{H_T^{QH}}{B_T}\right)\leq\bar{V}_0$. In this case, if we denote $\bar{V}_T$ the portfolio with initial value $\bar{V}_0$ that evolves according to the hedging strategy ξ^{SH}, we have that

$$\bar{V}_T\geq H_T^{SH},$$

$P-$almost surely.
The expected shortfall loss of this hedging strategy satisfies

$$\begin{aligned} E^P\Big(L\Big((H_T^{Total}-\bar{V}_T)_+\Big)\Big) &\leq E^P\Big(L\Big((H_T^{Total}-H_T^{SH})_+\Big)\Big) \\ &= E^P\Big(L\Big(\sum_{i=1}^{N_T} H_T(1-\phi_i^*\mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)\Big) \\ &= E^P\Big(\sum_{i=1}^{n} p_i\times L\Big(\sum_{j=1}^{i} H_T(1-\phi_j^*\mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)\Big) \end{aligned} \tag{17}$$

Combining (16) and (17), we have that

$$\begin{aligned} \min_{V_0,\,\xi\in\mathcal{A}\ s.t.\ V_0\leq\bar{V}_0} E^P\Big(L\Big((H_T^{Total}-V_T)_+\Big)\Big) &=E^P\Big(\sum_{i=1}^{n} p_i\times L\Big(\sum_{j=1}^{i} H_T(1-\phi_j^*\mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)\Big) \\ &=E^P\Big(L\Big((H_T^{Total}-\bar{V}_T)_+\Big)\Big), \end{aligned}$$

and the optimality of the strategy $(\bar{V}_0,\xi^{SH})$ in Problem (4) follows. □

Remark 7. From Proposition 15 in Appendix 7.2, the existence of a solution to the problem (14) is guaranteed.

As for the quantile hedging problem, the previous theorem shows that the optimal shortfall hedging of a $n-$size life insurance portfolio is linked to a (super) replication strategy of a modified financial payoff that takes into account the actuarial risk through the probabilities $p_i=P(N_T=i),\ i=1,...,n$. In general, we cannot unfortunately deduce the explicit form of the optimal $\mathcal{F}_T^f-$measurable variables $\phi_i^*,\ i=1,...,n$ solving problem (14). However, as for the quantile hedging problem, if there exists an equivalent martingale measure $\tilde{Q}^f\in\mathcal{Q}^f$ such that, for all $\phi_1,...,\phi_n\in\mathcal{R},\phi_{i+1}\leq\phi_i,\ \phi_0=1$,

$$\sup_{Q^f\in\mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T}\sum_{i=1}^{n}\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}}\right)=E^{\tilde{Q}^f}\left(H_T\times\sum_{i=1}^{n}\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}}\right), \tag{18}$$

then we can deduce the explicit form of $\phi_i^*,\ i=1,...,n$ in some particular optimal cases. This is the purpose of the next sections. In the following, we will solve the problem (14) when the condition (18) holds. Firstly, we will consider the linear case where $L(x)=x$ and then we will generalize the strategy for a general convex function L.

4.1 Minimizing the expected shortfall

In this section, we consider the linear case where $L(x) = x$. In this particular case, the hedging problem reduces to

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \ s.t.\ v_0 \leq \bar{V}_0} E^P\Big((H_T^{Total} - V_T)_+ \Big),$$

under the constraint

$$v_0 \leq \bar{V}_0.$$

that is equivalent to the optimization problem (14) with $L(x) = x$. The next proposition states the optimal hedging strategy that minimizes the expected shortfall when the (super) replication price of the modified payoff is attainable.

Proposition 8. *Supppose that $\exists \tilde{Q}^f \in \mathcal{Q}^f$ such that, $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$ and for all $\phi_1, ..., \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\ \phi_0 = 1$,*

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right) = E^{\tilde{Q}^f}\left(H_T \times \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right),$$

then the optimal solution to the problem (4) for

$$L(x) = x,$$

is such that

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \ s.t.\ v_0 \leq \bar{V}_0} E^P\Big((H_T^{Total} - V_T)_+ \Big) = E^P\left(\sum_{i=1}^n p_i \left(\sum_{j=1}^i (1 - \mathbb{1}_{A_j^*}) H_T \right) \right)$$

where

$$A_i^* = \left\{ \sum_{j=i}^n p_j \geq \lambda^* \frac{\tilde{Z}_T}{B_T} \right\},\ i = 1, ..., n,$$

with λ^* such that

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right).$$

Moreover, the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the (super) hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. By Slater's condition, strong duality holds and the constrained optimization problem is equivalent to

$$\max_{\lambda \in \mathbb{R}_+} \min_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\, \phi_0 = 1} E^P\left(\sum_{i=1}^n p_i \times \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right) + \lambda \left(E^{Q^f}\left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} - \bar{V}_0 \right) \right),$$

or

$$E^P\left(H_T \sum_{i=1}^n p_i \times i \right) - \min_{\lambda \in \mathbb{R}_+} \max_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\, \phi_0 = 1} E^P\left(H_T \sum_{i=1}^n p_i \sum_{j=1}^i \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}} - \lambda\, \tilde{Z}_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right) + \lambda \bar{V}_0$$

$$= E^P\left(H_T \sum_{i=1}^n p_i \times i \right) - \min_{\lambda \in \mathbb{R}_+} \max_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\, \phi_0 = 1} E^P\left(H_T \left(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \left(\sum_{j=i}^n p_j - \lambda \frac{\tilde{Z}_T}{B_T} \right) \right) \right) + \lambda \bar{V}_0.$$

Let $\phi_1,...,\phi_n \in \mathcal{R}$ with $\phi_0 = 1$ and $\phi_{i+1} \leq \phi_i$. In this case, for any $\lambda \in \mathbb{R}_+$, we have that

$$
\begin{aligned}
&E^P\bigg(H_T\bigg(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\bigg(\sum_{j=i}^n p_j - \lambda\,\frac{\tilde{Z}_T}{B_T}\bigg)\bigg)\\
=&E^P\bigg(H_T\bigg(\phi_1\bigg(\sum_{i=1}^n p_i - \lambda\,\frac{\tilde{Z}_T}{B_T} + \phi_2 \mathbb{1}_{\{\phi_1=1\}}\bigg(\sum_{i=2}^n p_i - \lambda\,\frac{\tilde{Z}_T}{B_T} + ... + \phi_n \mathbb{1}_{\{\phi_{n-1}=1\}}(p_n - \lambda\,\frac{\tilde{Z}_T}{B_T})...\bigg)\bigg)\bigg)\\
\leq&E^P\bigg(H_T\bigg(\mathbb{1}_{A_1(\lambda)}\bigg(\sum_{i=1}^n p_i - \lambda\,\frac{\tilde{Z}_T}{B_T} + \mathbb{1}_{A_2(\lambda)}\bigg(\sum_{i=2}^n p_i - \lambda\,\frac{\tilde{Z}_T}{B_T} + ... + \mathbb{1}_{A_n(\lambda)}(p_n - \lambda\,\frac{\tilde{Z}_T}{B_T})...\bigg)\bigg)\bigg)\\
=&E^P\bigg(H_T\sum_{i=1}^n \phi_i^*(\lambda)\,\bigg(\sum_{j=i}^n p_j - \lambda\,\frac{\tilde{Z}_T}{B_T}\bigg)\bigg),
\end{aligned}
$$

with

$$A_n(\lambda) = \bigg\{p_n \geq \lambda\,\frac{\tilde{Z}_T}{B_T}\bigg\}, \tag{19}$$

for $i = 1,...,n-1$,

$$A_i(\lambda) = \bigg\{\sum_{j=i}^n p_i + \sum_{j=i+1}^n \mathbb{1}_{A_j}\bigg(\sum_{k=j}^n p_k\bigg) \geq \lambda\,\bigg(1 + \sum_{j=i+1}^n \mathbb{1}_{A_j}\bigg)\frac{\tilde{Z}_T}{B_T}\bigg\}, \tag{20}$$

such that

$$A_i(\lambda) \subseteq A_{i-1}(\lambda),$$

and

$$\phi_i^*(\lambda) = \prod_{j=1}^i \mathbb{1}_{A_i(\lambda)} \tag{21}$$

Therefore, for $\lambda \in \mathbb{R}_+$, since $\phi_i^*(\lambda) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda) \leq \phi_i^*(\lambda)$, we easily deduce that

$$
\begin{aligned}
\max_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1}\leq\phi_i,\,\phi_0=1} &E^P\bigg(H_T\bigg(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\bigg(\sum_{j=i}^n p_j - \lambda\,\frac{\tilde{Z}_T}{B_T}\bigg)\bigg)\\
&=E^P\bigg(H_T\sum_{i=1}^n \phi_i^*(\lambda)\,\bigg(\sum_{j=i}^n p_j - \lambda\,\frac{\tilde{Z}_T}{B_T}\bigg)\bigg).
\end{aligned}
$$

Moreover, using similar arguments than in the proof of Proposition 4, $\phi_i^* = \phi_i^*(\lambda^*)$ where the optimal constant λ^* satisfies[4]

$$\bar{V}_0 = E^{\tilde{Q}^f}\bigg(\frac{H_T}{B_T} \times \sum_{i=1}^n \phi_i^*(\lambda^*)\bigg).$$

Hence, we conclude that

$$
\begin{aligned}
&\max_{\lambda\in\mathbb{R}_+}\min_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1}\leq\phi_i,\,\phi_0=1} E^P\bigg(\sum_{i=1}^n p_i \times \bigg(\sum_{j=1}^i H_T(1-\phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\bigg)\bigg) + \lambda\,\bigg(E^{Q^f}\bigg(\frac{H_T}{B_T}\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} - \bar{V}_0\bigg)\bigg)\\
&=E^P\bigg(\sum_{i=1}^n p_i \times \bigg(\sum_{j=1}^i H_T(1-\phi_j^*(\lambda^*)\mathbb{1}_{\{\phi_{j-1}^*(\lambda^*)=1\}})\bigg)\bigg)\\
&=E^P\bigg(\sum_{i=1}^n p_i \times \bigg(\sum_{j=1}^i H_T(1-\phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}})\bigg)\bigg).
\end{aligned}
$$

[4]The constraint $\bar{V}_0 \leq E^{\tilde{Q}^f}\left(n \times \frac{H_T}{B_T}\right)$ ensures that $\lambda^* \in \mathbb{R}_+$

Finally, using (19), (20) and (21), we can deduce that, for $i = 1, ..., n$,

$$\phi_i^* = \mathbb{1}_{A_i^*},$$

with

$$A_i^* = \left\{ \sum_{j=i}^{n} p_j \geq \lambda^* \frac{\tilde{Z}_T}{B_T} \right\},$$

and therefore

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \; s.t. \; v_0 \leq \bar{V}_0} E^P\Big((H_T^{Total} - V_T)_+ \Big) = E^P\left(\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i} (1 - \mathbb{1}_{A_j^*}) H_T \right) \right).$$

Furthermore, the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the (super) hedging of the modified payoff $H_T \times \sum_{i=1}^{n} \mathbb{1}_{A_i^*}$. □

As for the quantile hedging problem, we observe that the optimal shortfall strategy for a $n-$size portfolio generalizes the "pure" shortfall hedging strategy deduced in [15]. In particular, if $n = 1$, the two strategies coincide. When $n > 1$, we prove that the optimal shortfall hedging strategy should depend on both financial and actuarial risks and that the insurer has a strong interest in following the optimal strategy.

4.2 Minimizing the expectation of the class of convex shortfall loss functions

In this section, we consider a more general case where the function L is assumed to be convex. The next proposition states the optimal hedging strategy that minimizes the expectation of a convex function of the shortfall.

Proposition 9. *Supppose that* $\exists \tilde{Q}^f \in \mathcal{Q}^f$ *such that,* $\tilde{Z}_T = \frac{d\tilde{Q}^f}{dP^f}|_T$ *and for all* $\phi_1, ..., \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i,\ \phi_0 = 1$,

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(H_T \times \sum_{i=1}^{n} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right) = E^{\tilde{Q}^f}\left(H_T \times \sum_{i=1}^{n} \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right).$$

If there exists $\phi_1^*, ..., \phi_n^* \in \mathcal{R}$ *such that* $\phi_{i+1}^* \leq \phi_i^*$ *that solve the system of* $n-$*equations given by*

$$E^P\left(H_T \mathbb{1}_{\{\phi_{i-1}^*=1\}} \mathbb{1}_{\{H_T \phi_i^* > 0\}} \left(\sum_{j=i}^{n} p_j \times L'\left(H_T \sum_{k=1}^{j} (1 - \phi_k^* \mathbb{1}_{\{\phi_{k-1}^*=1\}}) \right) - \lambda^* \frac{\tilde{Z}_T}{B_T} \right) \right) = 0,\ i = 1, ..., n, \tag{22}$$

with $\phi_0^* = 1$ *and* $\lambda^* \in \mathbb{R}_+$ *such that*

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}} \right),$$

Then, the optimal solution to the problem (4), for a convex function L*, is such that*

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A} \; s.t. \; v_0 \leq \bar{V}_0} E^P\Big(L\Big((H_T^{Total} - V_T)_+ \Big) \Big) = E^P\left(\sum_{i=1}^{n} p_i\, L\left(H_T \sum_{j=1}^{i} (1 - \phi_j^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}) \right) \right)$$

Proof. By Slater's condition, strong duality holds and the constrained static optimization problem (14) is equivalent to

$$\max_{\lambda\in\mathbb{R}_+}\min_{\phi_1,...,\phi_n\in\mathcal{R},\phi_{i+1}\leq\phi_i,\,\phi_0=1}\underbrace{E^P\left(\sum_{i=1}^n p_i\times L\Big(H_T\sum_{j=1}^i(1-\phi_j\mathbb{1}_{\{\phi_{i-1}=1\}})\Big)+\lambda\Big(\tilde{Z}_T\left(\frac{H_T}{B_T}\sum_{i=1}^n\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}}\right)-\bar{V}_0\Big)\right)}_{=\mathcal{L}(\phi_1,...,\phi_n;\lambda)}$$

For a given $\lambda\in\mathbb{R}_+$, since L is convex, using a first order condition, if there exists $\phi_1^*(\lambda),...,\phi_n^*(\lambda)\in\mathcal{R}$ such that $\phi_{i+1}^*(\lambda)\leq\phi_i^*(\lambda)$ that satisfy

$$\frac{\partial\mathcal{L}(\phi_1^*(\lambda),...,\phi_n^*(\lambda);\lambda)}{\partial\phi_i}=0,\ i=1,...,n, \tag{23}$$

then

$$\min_{\phi_1,...,\phi_n\in\mathcal{R},\phi_{i+1}\leq\phi_i,\,\phi_0=1}\mathcal{L}(\phi_1,...,\phi_n;\lambda)=\mathcal{L}(\phi_1^*(\lambda),...,\phi_n^*(\lambda);\lambda).$$

Moreover, we observe that, for $\phi_1^*(\lambda),...,\phi_n^*(\lambda)\in\mathcal{R}$ such that $\phi_{i+1}^*(\lambda)\leq\phi_i^*(\lambda)$, the condition (23) is equivalent to

$$E^P\left(H_T\mathbb{1}_{\{\phi_{i-1}^*(\lambda)=1\}}\mathbb{1}_{\{H_T\phi_i^*(\lambda)>0\}}\left(\sum_{j=i}^n p_j\times L^{'}\Big(H_T\sum_{k=1}^j(1-\phi_k^*(\lambda)\mathbb{1}_{\{\phi_{k-1}^*(\lambda)=1\}})\Big)-\lambda\frac{\tilde{Z}_T}{B_T}\right)\right)=0,\ i=1,...,n. \tag{24}$$

Finally, we have that, if there exists $\phi_1^*(\lambda),...,\phi_n^*(\lambda)\in\mathcal{R}$ such that $\phi_{i+1}^*(\lambda)\leq\phi_i^*(\lambda)$ that satisfy (24), by strong duality,

$$\begin{aligned}\max_{\lambda\in\mathbb{R}_+}\min_{\phi_1,...,\phi_n\in\mathcal{R},\phi_{i+1}\leq\phi_i,\,\phi_0=1}&E^P\left(\sum_{i=1}^n p_i\times L\Big(H_T\sum_{j=1}^i(1-\phi_j\mathbb{1}_{\{\phi_{j-1}=1\}})\Big)+\lambda\Big(\tilde{Z}_T\left(\frac{H_T}{B_T}\sum_{i=1}^n\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}}\right)-\bar{V}_0\Big)\right)\\&=\mathcal{L}(\phi_1^*(\lambda^*),...,\phi_n^*(\lambda^*);\lambda^*)\\&=E^P\left(\sum_{i=1}^n p_i\times L\Big(H_T\sum_{j=1}^i(1-\phi_j^*\mathbb{1}_{\{\phi_{j-1}^*=1\}})\Big)\right).\end{aligned}$$

Thus, $\phi_i^*=\phi_i^*(\lambda^*)$ with the constant $\lambda^*\in\mathbb{R}_+$ satisfying

$$\bar{V}_0=E^{Q^f}\left(\frac{H_T}{B_T}\times\sum_{i=1}^n\phi_i^*(\lambda^*)\mathbb{1}_{\{\phi_{i-1}^*(\lambda^*)=1\}}\right),$$

and the Theorem (6) guarantees that

$$\min_{v_0\in\mathbb{R}_+,\,\xi\in\mathcal{A}\ s.t.\ v_0\leq\bar{V}_0}E^P\left(L\Big((H_T^{Total}-V_T)_+\Big)\right)=E^P\left(\sum_{i=1}^n p_i\times L\Big(H_T\sum_{j=1}^i(1-\phi_j^*\mathbb{1}_{\{\phi_{j-1}^*=1\}})\Big)\right).$$

□

The previous proposition gives a system of equations characterizing the optimal solution for a convex loss function. In general, this system is not solvable. However, if $n=1$ or $L'(.)$ is linear then we can deduce the explicit form of the optimal solution. As the case $n=1$ is similar to the classical shortfall hedging problem treated by [15], we will only consider the linear case $L'(x)=x$.

Corollary 10. *Supppose that* $\exists\tilde{Q}^f\in\mathcal{Q}^f$ *such that,* $\tilde{Z}_T=\frac{d\tilde{Q}^f}{dP^f}|_T$ *and for all* $\phi_1,...,\phi_n\in\mathcal{R},\phi_{i+1}\leq\phi_i,\ \phi_0=1$,

$$\sup_{Q^f\in\mathcal{Q}^f}E^{Q^f}\left(H_T\times\sum_{i=1}^n\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}}\right)=E^{\tilde{Q}^f}\left(H_T\times\sum_{i=1}^n\phi_i\mathbb{1}_{\{\phi_{i-1}=1\}}\right).$$

For a quadratic function $L(x) = \frac{1}{2}x^2$, the optimal solution to the problem (4) is such that

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in A \ s.t.\ v_0 \leq \bar{V}_0} E^P\left(\frac{1}{2}\Big((H_T^{Total} - V_T)_+\Big)^2\right) = E^P\left(\frac{1}{2}\sum_{i=1}^{n} p_i \left(H_T \sum_{j=1}^{i}(1-\phi_j^*)\right)^2\right)$$

where:

- $\phi_0^* = 1$ and

$$\phi_n^* = 1 - \left(\frac{\lambda^*}{p_n}\frac{\tilde{Z}_T}{B_T H_T} \wedge 1\right) \ on \ \{H_T > 0\},$$

- For $i \in [1, ..., n-1]$,

$$\phi_i^* = \mathbb{1}_{A_i} + \mathbb{1}_{A_i^c}\left(1 - \left(\frac{1}{\sum_{j=i}^n p_j}\left(\frac{\lambda^* \tilde{Z}_T}{B_T H_T} - \sum_{j=i+1}^{n} p_j\right) \wedge 1\right)\right) \ on \ \{H_T > 0\},$$

  with

$$A_i = \left\{\frac{\lambda^* \tilde{Z}_T}{B_T H_T} \leq \sum_{j=i+1}^{n}\sum_{k=j}^{n} p_k\right\}.$$

- λ^* such that

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \phi_i^*\right).$$

Moreover, the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the (super) replicating hedging strategy of the modified payoff $H_T \times \sum_{i=1}^n \phi_i^*$.

Proof. This result is a direct consequence of Proposition 9. Consider the $\mathcal{F}_T^f-$measurable variables $\phi_1^*, ..., \phi_n^*$ defined such that

$$\phi_n^* := 1 - \frac{\lambda^*}{p_n}\frac{\tilde{Z}_T}{B_T H_T} \wedge 1 \ on \ \{H_T > 0\}$$

and for $i \in [1, ..., n-1]$,

$$\phi_i^* := 1 - \frac{1}{\sum_{j=i}^n p_j}\left(\frac{\lambda^* \tilde{Z}_T}{B_T H_T} - \sum_{j=i+1}^{n}\left((1-\phi_j^*) \times \sum_{k=j}^{n} p_k\right)\right) \wedge 1\Big) \ on \ \{H_T > 0\}. \tag{25}$$

In this case, we have that $\phi_1^*, ..., \phi_n^* \in \mathcal{R}$ and $\phi_{i+1}^* \leq \phi_i^*$. Moreover, using the recursive form (25) of ϕ_i^*, we can check that $\phi_1^*, ..., \phi_n^*$ solve the system of $n-$equations given by

$$E^P\left(H_T \mathbb{1}_{\{\phi_{i-1}^*=1\}}\mathbb{1}_{\{H_T\phi_i^*>0\}}\left(\sum_{j=i}^{n} p_j \times \left(H_T \sum_{k=1}^{j}(1-\phi_k^* \mathbb{1}_{\{\phi_{k-1}^*=1\}})\right) - \lambda^* \frac{\tilde{Z}_T}{B_T}\right)\right) = 0, \ i = 1, ..., n,$$

with $\phi_0^* = 1$. In addition, the optimal constant λ^* is such that[5]

$$\begin{aligned}\bar{V}_0 &= E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}\right) \\ &= E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \phi_i^*\right).\end{aligned}$$

[5]The constraint $\bar{V}_0 \leq E^{\tilde{Q}^f}\left(n \times \frac{H_T}{B_T}\right)$ ensures that $\lambda^* \in \mathbb{R}_+$

Thus, we find $\phi_1^*, ..., \phi_n^* \in \mathcal{R}$ and $\phi_{i+1}^* \le \phi_i^*$ and that satisfy (22) when $L'(x) = x$. From Proposition 9, we conclude that

$$\min_{v_0 \in \mathbb{R}_+,\, \xi \in \mathcal{A}\ s.t.\ v_0 \le \bar{V}_0} E^P\left(\frac{1}{2}\Big((H_T^{Total} - V_T)_+\Big)^2\right) = E^P\left(\sum_{i=1}^n p_i\, \frac{1}{2}\Big(H_T \sum_{j=1}^i (1 - \phi_j^* \mathbb{1}_{\{\phi_{i-1}^*=1\}})\Big)^2\right)$$
$$= E^P\left(\sum_{i=1}^n p_i\, \frac{1}{2}\Big(H_T \sum_{j=1}^i (1 - \phi_j^*)\Big)^2\right)$$

Finally, we can go a step further and deduce the explicit form of ϕ_i^*. By using the recursive relation (25) and the fact that, for $i = 1, ..., n-1$:

- if $\phi_i^* = 1$, then $\phi_{i-1}^* = 1$,
- if $0 < \phi_i^* < 1$ then $\phi_{i-1}^* = 1$ and $\phi_j^* = 0$ for $j = i+1, .., n$,
- if $\phi_i^* = 0$ then $\phi_{i-1}^* < 1$,

we can deduce that for $i = 1, ..., n-1$,

$$\phi_i^* = \mathbb{1}_{A_i} + \mathbb{1}_{A_i^c}\left(1 - \left(\frac{1}{\sum_{j=i}^n p_j}\left(\frac{\lambda^* \tilde{Z}_T}{B_T H_T} - \sum_{j=i+1}^n p_j\right) \wedge 1\right)\right)\ on\ \{H_T > 0\},$$

with

$$A_i = \left\{\frac{\lambda^* \tilde{Z}_T}{B_T H_T} \le \sum_{j=i+1}^n \sum_{k=j}^n p_k\right\}.$$

Finally, the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the (super) hedging of the modified payoff $H_T \times \sum_{i=1}^n \phi_i^*$. □

Once again, we observe that the shortfall hedging strategy with a quadratic loss function generalizes the classical quadratic shortfall hedging strategy when $n = 1$ (see [15]) and that the optimal strategy adapts to the size of the portfolio n.

5 Application in Black and Scholes model

The various optimal strategies deduced in this paper, for loss-averse insurers, admit some explicit forms as soon as the financial market is assumed complete. We now illustrate these strategies through numerical results. For the sake of simplicity, we consider a simple Black and Scholes model for the financial market, but the optimal strategies can be applied in more sophisticated models, such as, for example, local volatility models, fully-endogenous (rough) stochastic volatility models, or path-dependent volatility models. Thus, in this section, we consider a financial market composed of the two traded assets that satisfy, for $t \in [0, T]$, the following dynamics:

$$dB_t = rB_t\, dt,$$

and

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$

on $(\Omega^f, \mathcal{F}^f, P^f)$, with $r \in \mathbb{R}$ the risk-free interest rate, $\mu > r$, $\sigma \in \mathbb{R}_+$ respectively the return and the volatility of the risky asset, and $(W_t)_{0 \le t \le T}$ is a standard Brownian motion defined on P^f. This financial market is abritrage-free and complete. Therefore, we know that there exists a unique equivalent martingale measure denoted by $\tilde{Q}^f$ and characterized by the following change of measure, for $t \in [0, T]$,

$$\begin{aligned}\tilde{Z}_t &:= \frac{d\tilde{Q}^f}{dP}|_t \\ &= \exp\Big(- \frac{(\mu - r)}{\sigma} W_t - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} t \Big). \\ &= K \times S_t^{-(\mu - r)/\sigma^2}.\end{aligned}$$

As the Black and Scholes market is complete, we also know that for any contingent claim of the form $H_T = f(S_T) \in \mathcal{F}_T^f$, we can construct a hedging portfolio $(V_t^{BS})_{0 \le t \le T}$ driven by a hedging strategy $(\xi_t^{BS})_{0 \le t \le T}$ such, P^f-almost surely,

$$V_T^{BS} = f(S_T)$$

with for $t \in [0, T]$,

$$V_t^{BS} = E^{\tilde{Q}^f} \Big(e^{-r(T-t)} H_T | \mathcal{F}_t^f \Big)$$

and

$$\xi_t^{BS} = \frac{\partial V_t^{BS}}{\partial S_t}.$$

For the insurance market, we assume that the mortality rate μ_{x+t} is deterministic such that ${}_T p_x = \exp\Big(- \int_0^T \mu_{x+t} dt \Big)$, $N_T \sim Bin(n, {}_T p_x)$ and for $i = 1, ..., n$, $p_i = P(N_T = i) = \binom{n}{i} {}_T p_x^i \times (1 - {}_T p_x)^{n-i}$. In the numerical results, we will vary the survival probability ${}_T p_x$ in order to analyze its effect on the hedging strategies.
We assume that the insurer has sold a life insurance product guaranteeing a stochastic positive amount at maturity given by a call payoff

$$H_T = (S_T - K_{call})_+,$$

and at an initial price[6]

$${}_T p_x \times E^{\tilde{Q}^f} (e^{-rT} (S_T - K_{call})_+),$$

In this case, the initial constraint on the value of the hedging portfolio is given by

$$\bar{V}_0 = n \times {}_T p_x \times E^{\tilde{Q}^f} (e^{-rT} (S_T - K_{call})_+).$$

We now construct the efficient hedging strategies (quantile/shortfall) for this type of $n-$size portfolio in a Black and Scholes setting. To this end, we consider three different loss-averse insurers:

1. A risk-averse insurer who seeks to avoid as much as possible the events of losses, but who does not particularly intend to control the size of thoses losses. For this agent, we consider the quantile hedging approach.

2. An insurer who controls the size of the losses and minimizes the expected shortfall loss. For this agent, we consider the shortfall hedging approach.

3. An insurer who minimizes the expected shortfall loss while also controlling the variance of this loss. For this agent, we consider the quadratic shortfall hedging approach.

[6] Without loss of generality, we assume that the insurer does not consider safety-loading for the mortality risk.

Note that for the numerical results, without loss of generality, we consider the following parameters for the Black and Scholes model:

$$S_0 = 100,\ r = 0.02,\ \mu = 0.06,\ \sigma = 0.2.$$

5.1 Quantile hedging

Let us consider the quantile hedging of a $n-$size life insurance portfolio with call payoff as stochastic maturity value. From Proposition 4, we know that the success sets $A_i^*,\ i = 1, ..., n$, are given by

$$\begin{aligned} A_i^* &= \left\{ \alpha_i \geq \lambda^* \times \tilde{Z}_T \frac{H_T}{B_T} \right\} \\ &= \left\{ e^{rT} \times \alpha_i \times \frac{S_T^{(\mu-r)/\sigma^2}}{K} \geq \lambda^* (S_T - K_{call})_+ \right\}, \end{aligned}$$

In general, we cannot obtain a more explicit expression for the modified payoff $H_T \sum_{i=1}^n \mathbb{1}_{A_i^*}$. Nevertheless, assuming that $\mu - r = \sigma^2$ holds, we can rewrite $A_i^*, i = 1, ..., n$ such that

$$A_i^* = \left\{ \alpha_i \times C^* \times S_T \geq (S_T - K_{call})_+ \right\}$$

with $C^* := \frac{e^{rT}}{K\lambda^*}$ satisfying

$$\bar{V}_0 = E^{\tilde{Q}^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right).$$

In this particular case, the modified payoff becomes

$$\begin{aligned} H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*} &= (S_T - K_{call})_+ \sum_{i=1}^n \mathbb{1}_{A_i^*} \\ &= \sum_{i=1}^n (S_T - K_{call}) \times \mathbb{1}_{\{S_T \geq K_{call}\}} \mathbb{1}_{\{K_{call} \geq S_T(1 - C^* \times \alpha_i)\}} \\ &= \sum_{i=1}^n (S_T - K_{call})_+ \times 1_{\{S_T \leq \frac{K_{call}}{1 - C^* \times \alpha_i}\}}. \end{aligned}$$

We therefore observe that the modified payoff associated with the quantile hedging of the $n-$size portfolio with call option is equal to a sum of up-and-out call options with out strikes larger than K_{call}. Moreover, as in Black and Scholes model, the price of up-and-out call options admits a closed form, the initial value of the modified payoff as well as the optimal quantile hedging loss are obtained by closed formula, and we can therefore easily deduce the optimal constant C^*.

Remark 11. In the case $\mu - r \neq \sigma^2$, we can use a stochastic algorithm (see Appendix 7.3) to deduce the optimal constant C^*.

To illustrate the efficiency of the optimal quantile hedging strategy with numerical results, we will benchmark it against the optimal mean-variance hedging strategy that is well known in actuarial sciences (see for example [3, 12]). This benchmark strategy consists of a perfect hedge of the financial payoff given by $n \times {}_T p_x \times H_T$. Note that the probability of sucessful hedge for the benchmark strategy can be easily deduced and is given by

$$P(V_T^{Bench} \geq H_T^{Total}) = P(N_T \leq n \times {}_T p_x)\, P(H_T^{Total} > 0) + P(H_T^{Total} = 0).$$

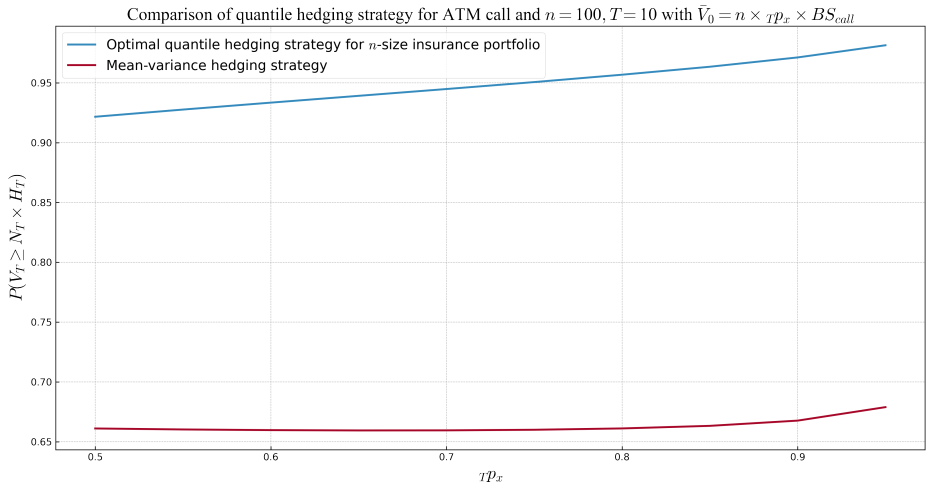

Figure 1: Success probability for the optimal quantile hedging strategy and the the benchmark strategy for ATM call payoff with respect to ${}_Tp_x$ with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

	$P(V_T \geq H_T^{Total})$	
${}_Tp_x$	Optimal quantile hedging	Mean-variance hedging
0.9	0.9794	0.6544
0.8	0.9695	0.6503
0.7	0.9611	0.6493

Table 1: Comparison of success probability for the optimal quantile hedging strategy and the benchmark strategy for ATM call payoff with $n = 250$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Figure 1 and Table 1 compare the probability of successful hedge with respect to the survival probability ${}_Tp_x$ for ATM call. We observe that, in terms of success probability, the optimal quantile hedging strategy outperforms the mean-variance hedging strategy. We observe that it is therefore in the insurer's interest to implement the optimal strategy instead of the benchmark strategy if it wants to maximize its probability of successful hedge. The optimal strategy is actually the best trade-off between actuarial and financial risks.

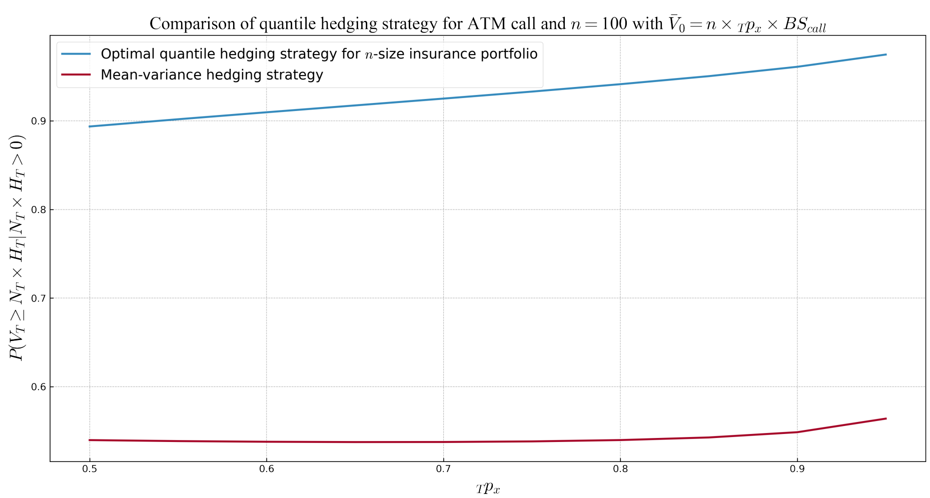

Figure 2: $P(V_T \geq H_T^{Total} | H_T^{Total} > 0)$ for the optimal quantile hedging strategy and the the benchmark strategy for ATM call payoff with respect to ${}_Tp_x$ with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

	$P(V_T \geq H_T^{Total} \mid H_T^{Total} > 0)$	
${}_Tp_x$	Optimal quantile hedging	Mean-variance hedging
0.9	0.9721	0.5308
0.8	0.9586	0.5252
0.7	0.9471	0.5238

Table 2: Comparison of $P(V_T \geq H_T^{Total} \mid H_T^{Total} > 0)$ for the optimal quantile hedging strategy and the benchmark strategy for ATM call payoff with $n = 250$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Notice that the payoff H_T^{Total} can be zero, for example if $S_T \leq K_{call}$, the probability of successful hedge is automatically greater than $P(S_T \leq K_{call})$. It is therefore interesting to look at the probability of successful hedge when the insurer has to pay an amount to its policyholders, i.e. $P(V_T \geq H_T^{Total} \mid H_T^{Total} > 0)$. Note that it is easy to deduce that the optimal hedging strategy maximizing $P(V_T \geq H_T^{Total})$ is the same as the one that maximizes $P(V_T \geq H_T^{Total} \mid H_T^{Total} > 0)$. Figure 2 and Table 2 compare this probability for different survival probabilities. These success probabilities, knowing that $H_T^{Total} > 0$, further underline the outperformance of the optimal strategy over the benchmark strategy. We observe that when the insurer has to pay something, i.e. $H_T^{Total} > 0$, then with the mean-variance approach, in almost 50% of cases, the insurer will experience a loss, whereas with the optimal approach, this probability of loss is considerably reduced.

We can also analyze the effect of the portfolio size n. Figure 3 compares the effect of the size of the portfolio on the hedging strategy for a fixed survival probability ${}_Tp_x = 0.9$. We observe that by increasing the number of policyholders in the portfolio, the gap between the optimal strategy and the benchmark strategy increases. As the gain from adopting the optimal strategy compared to the benchmark strategy increases with the size of the portfolio, it is in the insurer's interest to follow the optimal strategy, especially if it has a sufficiently large portfolio. In addition, the larger the portfolio, the higher the probability of successful hedge from the optimal strategy. It is therefore in the insurer's interest to have a large portfolio.

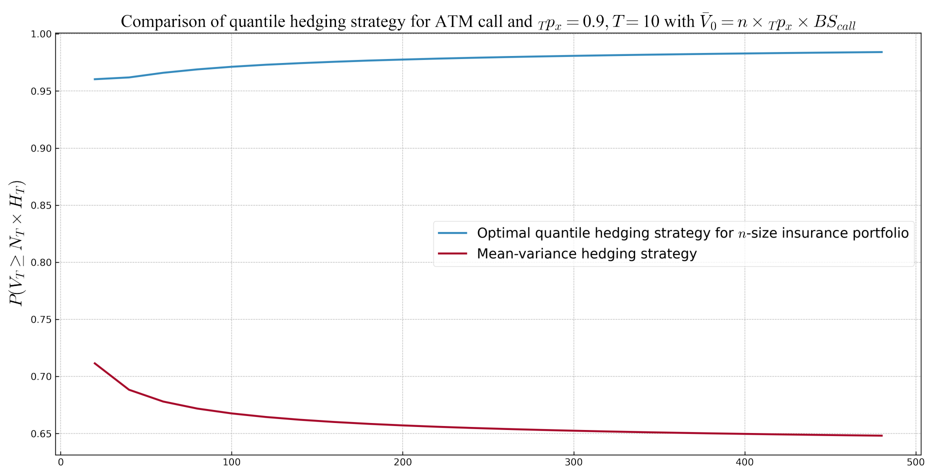

Figure 3: Success probability for the optimal quantile hedging strategy and the benchmark strategy for ATM call payoff with respect to n with ${}_Tp_x = 0.9$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

5.2 Shortfall hedging

Let now consider that the insurer minimizes the expected shortfall of its $n-$size portfolio with call payoff. In this case, from Proposition 8, the optimal sets A_i^*, for $i = 1, ..., n$, are given by

$$\begin{aligned} A_i^* &= \left\{ \sum_{j=i}^{n} p_j \geq \lambda^* \frac{\tilde{Z}_T}{B_T} \right\} \\ &= \left\{ \sum_{j=i}^{n} p_j \geq \frac{\lambda^* K}{e^{rT}} S_T^{-(\mu-r)/\sigma^2} \right\} \\ &= \left\{ S_T \geq K_i \right\}, \end{aligned}$$

with $m := \frac{\mu - r}{\sigma^2}$, $K_i = \frac{C^*}{\left(\sum_{j=i}^{n} p_j\right)^{1/m}}$ for $i = 1, ..., n$ and C^* satisfying

$$\bar{V}_0 = E^{\tilde{Q}^f}\left(\frac{H_T}{B_T} \sum_{i=1}^{n} \mathbb{1}_{A_i^*} \right).$$

The modified payoff becomes

$$\begin{aligned} \sum_{i=1}^{n} H_T 1_{A_i^*} &= (S_T - K_{call})_+ \sum_{i=1}^{n} \mathbb{1}_{A_i^*} \\ &= \sum_{i=1}^{n} (S_T - K_{call})_+ \times \mathbb{1}_{\{S_T \geq K_i\}} \end{aligned}$$

We observe that the modified payoff associated with the optimal expected shortfall hedging is decomposed as a sum of up-and-in call options. If we compare the modified payoff with that for quantile hedging, we can see that they are very similar. Instead of sum of up-and-out call options for the quantile hedging strategy, we have a sum of up-and-in call options for the shortfall hedging strategy. This is simply explained by the fact that having up-and-in options instead of up-and-out makes it possible to limit losses in the event of default, which is the aim of the shortfall strategy.

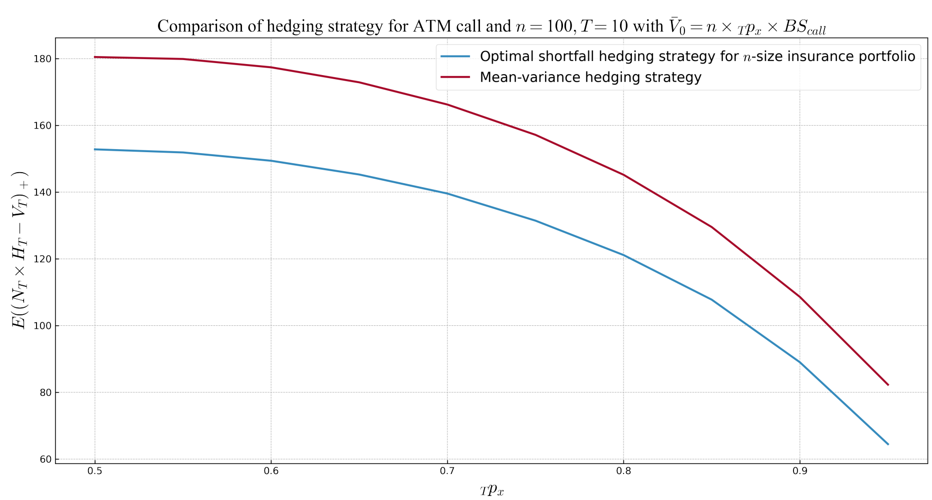

Figure 4: Expected shortfall for the optimal shortfall hedging strategy and the benchmark strategy for ATM call payoff with respect to ${}_Tp_x$ with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

	$E^P\Big((H_T^{Total} - V_T)_+\Big)$	
${}_Tp_x$	Optimal shortfall hedging	Mean-variance hedging
0.9	141.89	169.37
0.8	190.22	226.19
0.7	219.09	259.27

Table 3: Comparison of the expected shortfall for the optimal shortfall hedging strategy and the benchmark strategy for ATM call payoff with $n = 250$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Let's now illustrate, with numerical results, the optimal shortfall hedging strategy for a $n-$size life insurance portfolio with $H_T = (S_T - K_{call})_+$. We consider the same parameters as for the quantile hedging results. Figure 4 and Table 3 compare the expected shortfall with respect to the survival probability ${}_Tp_x$ for ATM call. We observe that, in terms of expected shortfall, the optimal shortfall hedging strategy also outperforms the benchmark strategy (mean-variance approach), and the gap is more prononced for low values of ${}_Tp_x$.. We conclude that it is therefore in the insurer's interest to choose the optimal strategy instead of the benchmark strategy if it wants to minimize the expected shortfall loss.

We can also analyze the effect of the portfolio size n. Figure 5 compares the effect of the size n of the portfolio on the hedging strategy for a fixed survival probability ${}_Tp_x = 0.9$ in terms of expected shortfall loss. We observe that by increasing the number of policyholders in the portfolio, the gap between the optimal strategy and the mean-variance strategy increases. As the gain from adopting the optimal strategy compared with the benchmark strategy increases with the size of the portfolio, an insurer with an aversion to shortfall losses should follow the optimal shortfall strategy, especially if it has a sufficiently large portfolio.

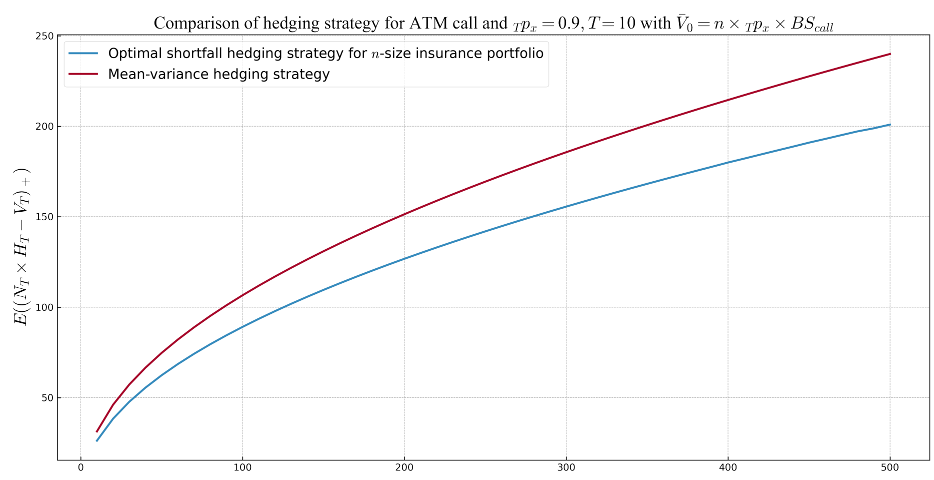

Figure 5: Expected shortfall for the optimal shortfall hedging strategy and the benchmark strategy for ATM call payoff with respect to n with ${}_Tp_x = 0.9$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

5.3 Quadratic shortfall hedging

Finally, let us consider that the insurer minimizes the expected quadratic shortfall loss of its $n-$size portfolio. From the Corrolary 10, we know that the optimal $\mathcal{F}_T^f-$measurable variables ϕ_i^*, $i = 1, ..., n$, satisfy

$$\phi_n^* H_T = H_T - \left(\frac{C^*}{p_n} S_T^{-(\mu-r)/\sigma^2} \wedge H_T\right) \; on \; \{H_T > 0\},$$

and for $i \in [1, ..., n-1]$,

$$\phi_i^* H_T = \mathbb{1}_{A_i} H_T + \mathbb{1}_{A_i^c} \left(H_T - \left(\frac{1}{\sum_{j=i}^n p_j} \left(C^* \times S_T^{-(\mu-r)/\sigma^2} - \sum_{j=i+1}^n p_j H_T \right) \wedge H_T \right) \right) \text{ on } \{H_T > 0\},$$

with

$$A_i = \left\{ C^* \times S_T^{-(\mu-r)/\sigma^2} \leq \sum_{j=i+1}^n \sum_{k=j}^n p_k H_T \right\}$$

and C^* such that

$$\bar{V}_0 = E^{\tilde{Q}^f} \left(e^{-rT} \times \sum_{i=1}^n \phi_i^* H_T \right).$$

Unfortunately, even if $\mu = r + \sigma^2$, we cannot obtain a more explicit form of the modified payoff. Therefore, to deduce the optimal constant C^*, we use a stochastic algorithm (see Appendix 7.3) and the optimal quadratic shortfall loss is computed via Monte Carlo method. Let's now compare the optimal quadratic shortfall hedging strategies with the mean-variance hedging strategy. To this end, we consider the same setting as before.

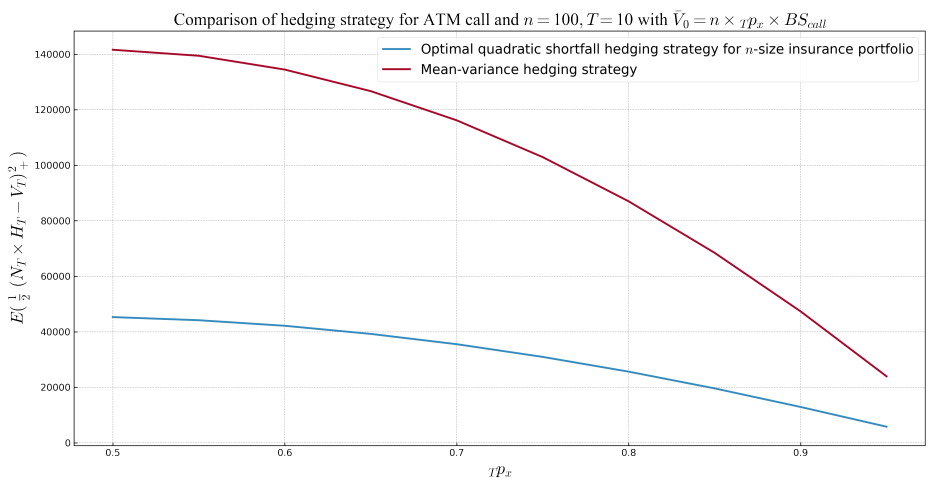

Figure 6: Expected quadratic shortfall for the optimal quadratic shortfall hedging strategy and the benchmark strategy for ATM call payoff with respect to ${}_Tp_x$ with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

	$E^P\left(\frac{1}{2}(H_T^{Total} - V_T)^2_+\right)$	
${}_Tp_x$	Optimal quadratic shortfall hedging	Mean-variance hedging
0.9	44 854.92	122 567.58
0.8	63 370.48	221 346.01
0.7	83 314.77	294 329.04

Table 4: Comparison of the expected quadratic shortfall for the optimal quadratic shortfall hedging strategy and the benchmark strategy for ATM call payoff with $n = 250$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Figure 6 and Table 4 compare the expected quadratic shortfall with respect to the survival probability ${}_Tp_x$ for ATM call. We observe that, in terms of quadratic shortfall loss, the optimal strategy outperforms the benchmark strategy. As for the two other hedging strategies, we conclude that it is therefore in the insurer's interest to implement the optimal strategy instead of the benchmark strategy if it wants to minimize the expected quadratic shortfall loss. Note that the discrepancy between the 2 strategies is even more significant than for the shortfall hedging case, especially for low values of ${}_Tp_x$.

Figure 7 compares the effect of the size n of the portfolio on the expected quadratic shortfall loss for a fixed survival probability ${}_Tp_x = 0.9$. Once again, we notice that the gap increases with the size of the portfolio, and thus, if the portfolio is large, the insurer should therefore opt for the optimal strategy if it is averse to quadratic shortfall losses.

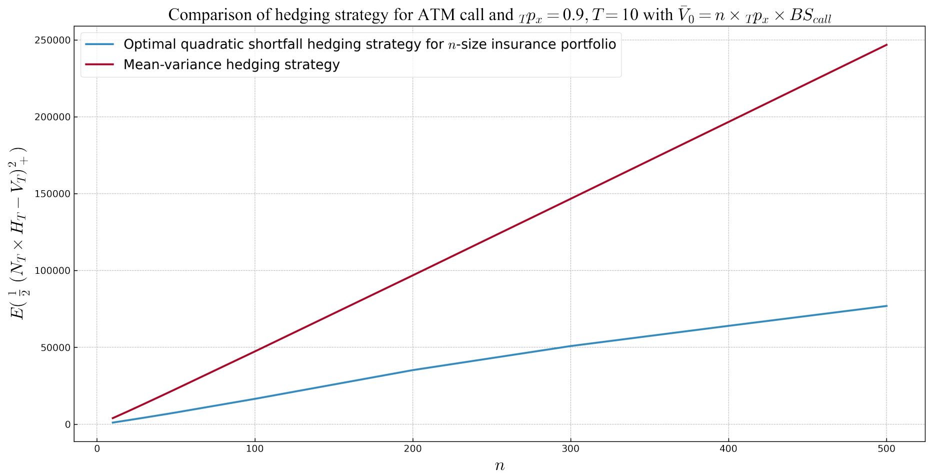

Figure 7: Expected quadratic shortfall for the optimal quadratic shortfall hedging strategy and the benchmark strategy for ATM call payoff with respect to n with ${}_Tp_x = 0.9$, $T = 10$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Finally, we compare the different hedging strategies. To better analyze the quadratic shortfall hedging strategy, we can split the expected quadratic shortfall loss in terms of the quadratic mean and variance of the loss. Table 5 compares the different strategies using several metrics for fixed portfolio size n and survival probability ${}_Tp_x$. Firstly, we clearly observe that even if the optimal quantile hedging strategy maximizes the probability of successful hedge, this strategy does not control the size of losses at all, and by following this strategy, the insurer is exposed to infrequent but very large losses, as revealed by comparing the expected shortfall and quadratic shortfall against other strategies. Next, the optimal shortfall strategy is effectively the one that minimizes the expected shortfall, but we can see that the variance of this shortfall can be significant. At last, we observe that, as expected, the optimal quadratic hedging strategy appears to be the strategy leading to the best low mean-variance trade-off for the shortfall loss.

Hedging strategy	$P(\pi_T \leq 0)$	$E^P\left((\pi_T)_+\right)$	$V^P\left((\pi_T)_+\right)$	$E^P\left((\pi_T)^2_+\right)$
Opt. quantile	**0.9794**	4691	528 328 227	550 335 619
Opt. quad. shortfall	0.5661	164	**62 763**	**89 709**
Opt. shortfall	0.5945	**141**	82 936	103 071
Mean-variance	0.6844	169	214 432	245 135

Table 5: Comparison of the hedging strategies for ATM call payoff with $\pi_T := H_T^{Total} - V_T$, ${}_Tp_x = 0.9$, $T = 10$, $n = 250$ and $\bar{V}_0 = n \times {}_Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

6 Conclusion

This paper investigates the efficient hedging of equity-linked life insurance portfolio. To this end, we consider loss-averse insurers and a general financial-actuarial framework. As the insurers are assumed to be loss-averse, their main goal is to minimize either the events leading to losses or the size of these losses. Therefore, we focus on optimal quantile and shortfall hedging strategies and posed the hedging problems as stochastic control problems. Based on the theory of super-replication, we are able to characterize the optimal strategies and show that these strategies depend not only on financial and actuarial risks but also on the size of the portfolio as well as the risk aversion of the insurers. Moreover, for complete financial markets, we deduce explicit forms for these optimal strategies that can be easily interpreted. Numerical results show satisfying results since, depending on the risk-aversion of the insurers, the optimal strategies outperform the mean-variance hedging strategy. Hence, it demonstrates the relevance of using the right optimal strategy for a given risk-aversion and portfolio size. Nevertheless, in the case of incomplete financial markets, we are not able to derive the explicit forms for optimal strategies. As mentioned in different papers [18, 24], it is actually difficult to derive explicit forms for partial hedging strategies in this type of market. Future research could therefore study possible approximations of the optimal strategies in incomplete financial market setting as well as deep hedging techniques applied in the context of efficient hedging of insurance portfolios.

7 Appendix

7.1 Equivalent Martingale Measures in combined market

Proposition 12. *For $i = 1, ..., n$, suppose that*

$$M_t^i = (1 - N_t^i) - \int_0^t N_u^i \, \mu_{x+u} \, du,$$

$$M_t := \sum_{i=1}^n M_t^i.$$

and $(L_t^h)_{t\geq 0}$ such that

$$\begin{aligned} dL_t^h &= L_{t^-}^h \, h_t \, dM_t, \\ L_0^h &= 1, \end{aligned} \tag{26}$$

for a given $\mathcal{F}_t^a-$adapted process $(h_t)_{t\geq 0}$ such that for all $t \geq 0$, $h_t > -1$ almost surely. Then, for all $\mathcal{F}_t^a-$adapted processes $(h_t)_{t\geq 0}$ and $Q^f \in \mathcal{Q}^f$ there exists a probability measure denoted by Q^h such that

$$\frac{dQ^h}{dP}|_t = L_t^h Z_t, \tag{27}$$

with $Z_t^f = \frac{dQ^f}{dP^f}|_t$ and

$$Q^h \in \mathcal{Q}.$$

Proof. Let $(h_t)_{t\geq 0}$ be a $\mathcal{F}_t^a-$adapted process such that for all $t \geq 0$, $h_t > -1$ almost surely and $Q^f \in \mathcal{Q}^f$ such that $Z_t^f = \frac{dQ^f}{dP^f}|_t$. For $i = 1, ..., n$, using the Doob-Meyer decomposition of $(N_t^i)_{t\geq 0}$, we observe that $(M_t^i)_{t\geq 0}$ is a $\mathcal{F}_t^a-$martingale and, as $M_t = \sum_{i=1}^n M_t^i$, M_t is also a $\mathcal{F}_t^a-$martingale. Since, $(L_t^h)_{t\geq 0}$ with $L_0^h = 1$ is a strictly positive $\mathcal{F}_t^a-$martingale and $(Z_t^f)_{t\geq 0}$ with $Z_0^f = 1$ is a strictly positive $\mathcal{F}^f-$martingale, we have that, for $0 \leq t \leq s$,

$$\begin{aligned} E^P(L_t^h Z_t | \mathcal{F}_s) &= E^{P_a}(L_t^h | \mathcal{F}_s^a) \, E^{P_f}(Z_t^f | \mathcal{F}_s^f) \\ &= L_s^h Z_s^f, \end{aligned}$$

and in particular,

$$E^P(L_t^h Z_t^f) = 1.$$

Hence the change of measure (27) is well defined and as $(L_t^h Z_t)_{t\geq 0}$ is a strictly positive process, the probability measure Q^h is equivalent to P. Moreover, we observe that, for $0 \leq s \leq t$,

$$\begin{aligned}
E^{Q^h}\left(\frac{S_t}{B_t}|\mathcal{F}_s\right) &= \frac{E^P\left(L_t^h Z_t^f \frac{S_t}{B_t}|\mathcal{F}_s\right)}{L_s^h Z_s^f} \\
&= \frac{E^{P_a}(L_t^h|\mathcal{F}_s^a)}{L_s^h}\frac{E^{P_f}\left(Z_t^f \frac{S_t}{B_t}|\mathcal{F}_s^f\right)}{Z_s^f} \\
&= E^{Q_f}\left(\frac{S_t}{B_t}|\mathcal{F}_s^f\right) \\
&= \frac{S_s}{B_s}.
\end{aligned}$$

We conclude that $\left(\frac{S_t}{B_t}\right)_{t\geq 0}$ is a martingale under Q^h and then, as $Q^h \sim P$, $Q^h \in \mathcal{Q}$. □

Proposition 13. *For $\epsilon > -1$, for all $\mathcal{F}_t^a-$adapted process $(h_t)_{t\geq 0}$ such that $\forall t \geq 0$, $h_t > -1$ almost surely and for all financial martingale measure $Q^f \in \mathcal{Q}^f$, consider the probability measure Q_{h^ε} such that*

$$\frac{dQ_{h^\varepsilon}}{dP}|_t := L_t^{h^\varepsilon} Z_t^f,$$

with $Z_t^f := \frac{dQ^f}{dP^f}|_t$, $(L_t^{h_\varepsilon})_{t\geq 0}$ defined by (26) and $(h_t^\varepsilon)_{t\geq 0} := ((1+\varepsilon)h_t + \varepsilon)_{t\geq 0}$. Then,

$$Q_{h^\varepsilon} \in \mathcal{Q},$$

and

$$\lim_{\varepsilon\to -1} E^{Q_{h^\varepsilon}}\Big(f(S_T, N_T)\Big) = E^{Q_f}\Big(f(S_T, n)\Big).$$

Proof. Let $\varepsilon > -1$, consider a given $\mathcal{F}_t^a-$adapted processes $(h_t)_{t\geq 0}$ such that for all $t \geq 0$, $h_t > -1$ almost surely and a financial martingale measure $Q^f \in \mathcal{Q}^f$. As $(h_t^\varepsilon)_{t\geq 0} = ((1+\varepsilon)h_t + \varepsilon)_{t\geq 0}$ is $\mathcal{F}_t^a-$adapted such that for all $t \geq 0$, $h_t^\varepsilon > -1$, using Proposition 12, we have that

$$Q_{h^\varepsilon} \in \mathcal{Q}.$$

We observe that, by indepedence between actuarial and financial risks,

$$\begin{aligned}
E^{Q_{h^\varepsilon}}\Big(f(S_T, N_T)\Big) &= \sum_{i=0}^n E^{Q_{h^\varepsilon}}\Big(f(S_T, i)\Big) \times Q_{h^\varepsilon}(N_T = i) \\
&= \sum_{i=0}^n E^{Q_f}\Big(f(S_T, i)\Big) \times Q_{h^\varepsilon}(N_T = i).
\end{aligned}$$

Using Ito's lemma, we obtain that

$$L_t^{h_\varepsilon} = \prod_{i=1}^n (1 + h_{\tau_i}^\varepsilon)^{1-N_t^i} \exp\Big(-\int_0^t h_u^\varepsilon \mu_{x+u} N_u du\Big).$$

Therefore, under Q_{h^ε}, for $i = 0, ..., n$, we have

$$\begin{aligned}
Q_{h^\varepsilon}(N_T = i) =& E^{Q_{h^\varepsilon}}(\mathbb{1}_{\{N_T=i\}}) \\
=& E^P(L_T^{h_\varepsilon} Z_T \mathbb{1}_{\{N_T=i\}}) \\
=& E^P\Big(\mathbb{1}_{\{N_T=i\}} \prod_{j=1}^{n}(1+h^\varepsilon_{\tau_j})^{1-N_t^j} \ \exp\Big(-\int_0^T h^\varepsilon_u \mu_{x+u} N_u du\Big)\Big).
\end{aligned}$$

In particular, for $i = n$, we obtain

$$\begin{aligned}
Q_{h^\varepsilon}(N_T = n) =& E^P\Big(\mathbb{1}_{\{N_T=n\}} \prod_{j=1}^{n}(1+h^\varepsilon_{\tau_j})^{1-N_t^j} \ \exp\Big(-\int_0^T h^\varepsilon_u \mu_{x+u} N_u du\Big)\Big) \\
=& E^P\Big(\mathbb{1}_{\{N_T=n\}} \exp\Big(-n\int_0^T \mu_{x+u} h^\varepsilon_u) du\Big)\Big) \\
=& E^P\Big(\mathbb{1}_{\{N_T=n\}} \exp\Big(-n\int_0^T \mu_{x+u}((1+\varepsilon)h_u + \varepsilon) du\Big)\Big).
\end{aligned}$$

By monotone convergence, we have

$$\begin{aligned}
\lim_{\varepsilon\to -1} Q_{h^\varepsilon}(N_T = n) &= E^P\Big(\mathbb{1}_{\{N_T=n\}} \exp\Big(n\int_0^T \mu_{x+u} du\Big)\Big) \\
&= 1,
\end{aligned}$$

and thus for $i = 0, ..., n-1$,

$$\lim_{\varepsilon\to -1} Q_{h^\varepsilon}(N_T = i) = 0,$$

We easily conclude that

$$\begin{aligned}
\lim_{\varepsilon\to -1} E^{Q_{h^\varepsilon}}\Big(f(S_T, N_T)\Big) &= \sum_{i=0}^{n} E^{Q_f}\Big(f(S_T, i)\Big) \times \lim_{\varepsilon\to -1} Q_{h^\varepsilon}(N_T = i) \\
&= E^{Q_f}\Big(f(S_T, n)\Big).
\end{aligned}$$

□

7.2 Existence results for solutions to static optimization problems

Proposition 14. *There exists a solution $\tilde{A} = (\tilde{A}_1, ..., \tilde{A}_n) \in \mathcal{F}_T^f$ such that $\tilde{A}_{i+1} \subseteq \tilde{A}_i$, $i = 1, ..., n-1$ to the problem*

$$\max_{A_1,...,A_n \in \mathcal{F}_T^f,\ A_{i+1}\subseteq A_i} E^P\Big(\sum_{i=1}^{n} \mathbb{1}_{A_i} \times p_i\Big) + p_0, \tag{28}$$

under the constraint

$$\sup_{Q^f\in\mathcal{Q}^f} E^{Q^f}\Big(\frac{H_T}{B_T} \times \sum_{i=1}^{n} \mathbb{1}_{A_i}\Big) \leq \bar{V}_0. \tag{29}$$

Proof. The proof is just an adaptation of the proof of Proposition 3.1. in [15]. Let define $\mathcal{R}^n := \Big\{(\mathbb{1}_{A_1}, ..., \mathbb{1}_{A_n})|\ A_i \in$

$\mathcal{F}_T^f,\ A_{i+1} \subseteq A_i,\ i = 1, ..., n-1\Big\}$ and . In this case, we can rewritte the problem as

$$\max_{g \in \mathcal{R}^n} E^P\left(\sum_{i=1}^n g^i \times p_i\right) + p_0,$$

under the constraint

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n g^i\right) \leq \bar{V}_0. \tag{30}$$

Let $\mathcal{R}_0^n$ the elements of $\mathcal{R}^n$ satisfying (30) and consider $(g_m)_{m\geq 1}$ a maximizing sequence for (28) in $\mathcal{R}_0^n$. Using Lemma A.1.1. in [11], we can choose $\tilde{g}_m \in conv(g_m, g_{m+1}, ...)$ such that $(\tilde{g}_m)_{m\geq 1}$ converges $P-$almost surely to $\tilde{g} \in \mathcal{R}^n$. Therefore, we have that, by dominated convergence theorem[7],

$$E^P\left(\sum_{i=1}^n \tilde{g}_m^i \times p_i\right) + p_0 \to \max_{g\in\mathcal{R}^n} E^P\left(\sum_{i=1}^n g^i \times p_i\right) + p_0 = E^P\left(\sum_{i=1}^n \tilde{g}^i \times p_i\right) + p_0$$

as m tends to infinity. Moreover, by Fatou's lemma, for all $Q^f \in \mathcal{Q}^f$, we observe that

$$E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \tilde{g}^i\right) \leq \liminf E^{Q^f}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \tilde{g}_m^i\right) \leq \bar{V}_0,$$

then

$$\tilde{g} \in \mathcal{R}_0^n$$

and the existence of a solution to the problem (28) with constraint (29) follows. □

Proposition 15. *There exists a solution $\tilde{\phi} = (\tilde{\phi}_1, ..., \tilde{\phi}_n)$ such that $\tilde{\phi}_i \in \mathcal{R}$ and $\tilde{\phi}_i \geq \tilde{\phi}_{i-1}$, $i = 1, ..., n$, to the problem*

$$\min_{\phi_1,...,\phi_n \in \mathcal{R}, \phi_{i+1}\leq\phi_i,\ \phi_0=1} E^P\left(\sum_{i=1}^n p_i \times L\left(\sum_{j=1}^i H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)\right), \tag{31}$$

with

$$\mathcal{R} = \left\{\phi : \Omega \to [0,1], \mathcal{F}_T^f - measurable\right\}$$

under the constraint

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\right) \leq \bar{V}_0. \tag{32}$$

Proof. The proof is just an adaptation of the proof of Proposition 3.1. in [15]. Let define $\mathcal{R}^n := \Big\{\phi = (\phi_1, ..., \phi_n) | \phi_i \in \mathcal{R},\ \phi_i \leq \phi_{i-1},\ i = 1, ..., n\Big\}$. In this case, the problem is equivalent to

$$\min_{\phi\in\mathcal{R}^n} E^P\left(\sum_{i=1}^n p_i \times L\left(\sum_{j=1}^i H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)\right),$$

under the constraint

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f}\left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\right) \leq \bar{V}_0. \tag{33}$$

Let $\mathcal{R}_0^n$ the elements of $\mathcal{R}^n$ satisfying (33) and consider $(g_m)_{m\geq 1}$ a minimizing sequence for (31) in $\mathcal{R}_0^n$. By using

[7] Since $\forall m \geq 1, \forall i = 1, ..., n,\ 0 \leq \tilde{g}_m^i \leq 1$

similar arguments than in the proof of Proposition 14, we can easily deduce the existence of a solution to the problem (31) with constraint (32). □

7.3 Stochastic algorithm for root finding

Assume that we want to find the root for a function of the form $h(\lambda) = E^P(H(\lambda, Z))$ with $\lambda \in \mathbb{R}$ and Z a random variable defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. To this end, we can determine a stochastic algorithm that will converge (from any strating point) to λ^*. The stochastic algorithm can be seen as an extension of the Newton-Raphson algorithm.

Algorithm 16. *Let $(Z_n)_{n\geq 1}$ simulations of Z and $(\gamma_n)_{n\geq 1}$ a deterministic sequence such that $\sum_{n\geq 1} \gamma_n = +\infty$, $\sum_{n\geq 1} \gamma_n^2 < +\infty$. We consider a stochastic sequence $(\lambda_n)_{n\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ and propose the following algorithm for root finding of $h(\lambda) = E^P(H(\lambda, Z))$:*

$$\begin{aligned} \lambda_{n+1} &= \lambda_n - \gamma_{n+1} H(\lambda_n, Z_{n+1}), \\ \lambda_0 &\in L^2(P). \end{aligned} \tag{34}$$

Proposition 17. *(Robbins-Monro algorithm*, see Corollary 6.1 in [26])
Assume that the mean function $h(\lambda) = E^P(H(\lambda, Z))$ is continuous and satisfies

$$\forall \lambda \in \mathbb{R},\ \lambda \neq \lambda^*,\ \ (\lambda - \lambda^*)h(\lambda) > 0$$

Suppose furthermore that $\lambda_0 \in L^2(P)$ and that H satisfies

$$\forall \lambda \in \mathbb{R}, ||H(\lambda, Z)||_{2,P} \leq C(1 + |\lambda|).$$

Assume that $(\gamma_n)_{n\geq 1}$ satisfies

$$\sum \gamma_n = +\infty,\ \sum \gamma_n^2 < +\infty.$$

Then $(\lambda_n)_{n\geq 1}$ defined in (34) is such that

$$\lambda_n \xrightarrow{a.s.} \lambda^*,$$

and in every $L^p(P)$, $p \in (0, 2)$.

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