Chapter 3

Balanced capacity of wireline multiuser channels

3.1 Introduction

The wireline multiaccess channel is an example of multiuser channel with Inter-Symbol Interference (ISI). It provides simultaneous duplex connections between the cable head-end (access point to external services) and a number of subscribers spatially distributed along the common physical cable. Some practical applications are the outdoor powerline channel and the CATV network. Functionally, it can be modeled as a multiple access channel for the uplink and a broadcast channel for the downlink, and each user channel can be modeled as a linear time-invariant frequency-selective channel with Gaussian noise.

Some simple criteria are needed to characterize the performance of the multiaccess channel. The maximum error-free transmission rate for a given power budget is a commonly adopted criterion, although the minimum required power for a given target rate could also be investigated (see [38]). The list of single user Shannon capacities vs. the position of the remote modem along the cable gives thus a first idea of the channel potential. As pointed out in Chapter 2, the network topology (number of cable derivations before and after the modem under consideration) plays a major role in the determination of the multipath channel characteristics : the capacity profile does not only depend on the physical distance w.r.t. the cable head-end.

Single user capacities are sufficient to characterize the multiaccess channel when time domain multiplexing (TDM) is used in the downlink and time domain multiple access (TDMA) or random access methods are used in the uplink. If more elaborate multiuser signaling techniques are used (involving parallel transmission of the different user signals), the full characterization of the multiacces channel requires the determination of the capacity region. Each point of the capacity region is associated with a given power allocation, and successive decoding (with a specific decoding order) is generally necessary to obtain the corresponding data rates. The boundary of the capacity region can be traced out by maximizing a weighted sum of the individual data rates with variable priority coefficients α_k.

All points of the capacity region, however, are not desirable working points. For instance, many papers focus on the maximum sum-rate, but this solution does generally not guarantee a fair (or even a minimum) data rate to each subscriber. On the opposite, the common rate strategy (meaning that all users transmit or receive at the same rate) is not necessarily a good idea, especially when the attenuation levels associated with the various channels in competition are very different. As a trade-off, the idea of balanced capacity is introduced, and the set of maximum balanced user rates is proposed to characterize the performance of the multiaccess channel.

The **balanced capacity** of a multiuser system is defined as the distribution of maximum simultaneously achievable bit rates that are in proportion with the single-user rates. It is a specific point of the boundary of the capacity region for which the coexistence with the other users has the same relative cost for every user.

The computation of maximum balanced rates basically involves the computation of the right priority coefficients α_k associated with these balanced rates.

The objective of this chapter is to propose practical algorithms for the computation of the uplink/downlink balanced capacity of a multiaccess network with K users, where K may be much larger than two. Different types of power constraints are investigated, which will lead to distinct capacity regions. The problem of optimal duplexing is not considered here : the multiuser channel is supposed to be operated in uplink OR downlink at a given time.

The capacity of multiuser networks has been extensively studied in the past few years, but the issue of a fair rate distribution was generally ignored. In [39], a general expression for the capacity regions of multiaccess channels with memory was obtained. In [40], this result was used to derive the capacity region of a two-user Gaussian multiple access channel with ISI and the well-known single user water-filling algorithm (that provides the optimal power allocations) was extended to the two-user case. The authors of [41] extended these results to channels with an arbitrary number of users in the context of multiaccess fading channels. All these papers assume individual power constraints at the transmitters : the K user channels need to be scaled (concept of *equivalent channel*), so that the multiuser water-filling algorithm provides K power allocations that satisfy the K power constraints simultaneously. The K scaling coefficients have to be computed iteratively. A general algorithm is proposed in [41], while the authors of [42] and [43] propose practical algorithms well suited to specific situations (two-user channel and/or maximum sum rate solution). An alternative method for the computation of the maximum sum rate solution was proposed by [44] and basically involves the iterative application of the single user water-filling algorithm. A detailed analysis of this method is proposed in [45]. [46] addresses the problem of computing the FDMA capacity region (which is useful when successive decoding is undesired at the receiver) and proposes suboptimal algorithms for the two-user case in the context of optimal duplex in DSL. In the general case, the optimal user rates can be achieved through the use of multicarrier multiple access [47] [48] with appropriate coding on each carrier.

The multiuser water-filling process is highly simplified when the individual power constraint is relaxed and a single constraint is put on the transmission power sum. In that case, the uplink capacity region is known to be the same as the downlink one [49]-[50]. The capacity of Gaussian broadcast channels is analyzed in [51] and [52]. The issue of a fair rate allocation in the uplink was addressed in [53] with the same constraint relaxation.

This chapter is organized as follows. In Section 3.2, the wireline multiaccess channel model is revisited. Section 3.3 describes the notion of capacity region and introduces the concept of balanced capacity. In Section 3.4, the optimal power allocation associated with a given set of relative priorities α_k is analyzed in detail, for the K-user Gaussian memoryless broadcast and multiple access channels, respectively. A simple allocation algorithm is given, as well as explicit expressions for the optimal allocated powers and the associated user rates. An iterative method is derived to obtain balanced user rates. The problem of optimal spectrum allocation in multiuser frequency-selective channels is then addressed in Section 3.5. Different types of power constraints are introduced, and the spectrum allocation problem is analyzed in three specific situations. Practical algorithms for the computation of balanced user rates are finally proposed. Section 3.6 addresses the issue of FDMA balanced capacity. Finally, Section 3.7 presents detailed results obtained with a 20 user wireline access network.

3.2 Wireline multiaccess channels

The general multiaccess channel considered in this chapter (see Figure 2.11) is composed of a main cable (the *distribution cable*) and $\bar{K}$ derivations (the *connection cables*) whose terminations are connected to the $\bar{K}$ user modems. The end of the distribution cable is connected to the cable head-end which is supposed to provide a common access to the broadband services. Any subset of $K \leq \bar{K}$ user modems are about to establish a duplex connection with the cable head-end at a given time. Unlike the DSL access networks (where each user has its own twisted pair), the physical medium has to be shared by all the users who wish to establish a connection simultaneously.

The *uplink* (data transmission from the user modems to the cable head-end) corresponds to a *multiple access channel* (MA), while the *downlink* (data transmission from the cable head-end to the user modems) corresponds to a *broadcast channel* (BC). The schemes of these two types of multiuser channels are given in Figure 3.1. The additive noises at all receivers (n and $\{n_1, \cdots, n_K\}$) are supposed to be Gaussian and white, with a double-sided power spectral density $N_0/2$.

In a practical wireline communications system, constraints are put on the *bandwidth*, on the *powers*, and on the *power spectral densities* (PSD) of the transmitted signals. We assume in the sequel a transmission bandwidth $[0, B]$. The power constraint $\bar{P}$, and the PSD constraint $\bar{\gamma}$ will be analyzed separately. While in the downlink, the power constraint actually refers to the sum of all the user signals (located in the unique downlink transmitter), in the uplink, individual power constraints are generally associated with each transmitter. In the sequel, the individual power constraint is relaxed and a single power-sum constraint is considered both for the downlink and the uplink. The obtained results are then extended to the (more complex) individual power constraint scenario in the uplink.

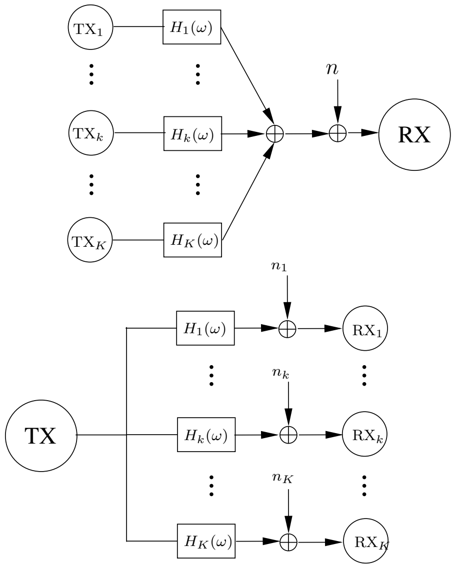

Figure 3.1: *Multiple access and broadcast channels*

3.3 Multiuser capacity region and balanced capacity

A multiuser capacity region is defined as the set of achievable rates $\{R_k\}$ at which the receiver(s) may decode information from the transmitter(s) without error for a given set of (power) constraints. For the Gaussian K-user channel, it is a convex region in the K-dimensional space. This region reflects the trade-off among the individual data rates of the different users competing for the limited resources. The boundary of the capacity region can be traced out by means of a set of relative priority coefficients α_k with $\sum_k \alpha_k = 1$. Each boundary point of the capacity region maximizes the linear combination of the user rates $R_\alpha = \sum_k \alpha_k R_k$.

Let $\mathbf{R}^* = [R_1^*, \cdots, R_k^*]^T$ be the vector of maximum user rates associated with a given vector of relative priorities $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_k]^T$. Actually, the coefficients α_k do not bring any information about the distribution of the maximum user rates R_k^*. Even with a higher relative priority $\alpha_k > 1/K$, a given user k with a poor channel quality could obtain a lower data rate R_k^* that the other users, or even a zero data rate. The only interpretation that can be associated with the relative priorities α_k is that $\mathbf{dR}^* \perp \boldsymbol{\alpha}$, which means that the boundary of the capacity region is normal to the vector $\boldsymbol{\alpha}$ in the neighbourhood of the point $\mathbf{R}^*$. In a two-user case, for example, the maximum rate distribution (R_1^*, R_2^*) associated with equal priorities $\alpha_1 = \alpha_2 = 1/2$ would have a local tangent with a slope of -45 degrees. The maximum rate distribution (R_1^*, R_2^*) associated with unequal priorities $\alpha_1 = 1/3$ and $\alpha_2 = 2/3$ would have a local tangent with a slope of $\arctan(-1/2) = -26.6$ degrees.

To obtain a practical characterization of the access network capacity, it is useful to put forward some specific points of the boundary :

- The *single user rates* R_k^1 are the maximum achievable rates in the single user communication scenario. They correspond to having only one alpha different from 0.
- The *maximum sum rate* corresponds to the setting $\alpha_k = 1/K$. It generally results in unfair situations where the users with the best channels have a much higher rate than the others, which is not desirable in practical applications. It may even happen that some subscribers do not have any bit rate at all.
- The *maximum common rate* [53] is such that $R_1 = \cdots = R_K$. When the single user rates are very different, the common rate constraint is generally a waste of resources as it forces the users with the best channels to lower their rate dramatically to reach the level of the weakest channels.
- The **maximum balanced rate** is such that $\frac{R_1}{R_1^1} = \cdots = \frac{R_K}{R_K^1}$. In other words, all users transmit at a rate R_k which is in the same proportion w.r.t. the potential rate R_k^1 offered by their own channel. The relative cost implied by the coexistence with the other users is the same for all the users.

Figure 3.2 gives an example of a convex capacity region and the corresponding boundary points for $K = 2$. A and F give the single user rates. E gives the maximum sum rate : it corresponds to the point where the tangent has a -45° slope. B gives the maximum common rate and D gives the maximum balanced rates.

As a lower bound on the balanced rates, time-domain multiplexing (TDM) or multiple access (TDMA) can be considered, with time slots of equal duration for each user. This strategy gives a set of rates $R_k = \frac{R_k^1}{K}$ (line AGF on Figure 3.2). Allowing simultaneous transmission of all the users, with

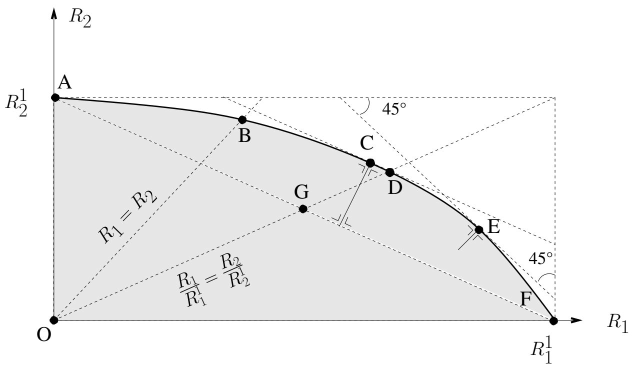

Figure 3.2: *Two-user capacity region and specific points of the boundary*

a smart power and spectrum management, offers higher balanced rates, of course. In any case, the maximum balanced rates can be written as :

$$R_k = g \, \frac{R_k^1}{K} \tag{3.1}$$

where g is the rate gain with respect to the TDM(A) strategy (OD/OG on Figure 3.2). Thanks to the convexity property of the capacity region, we know that $g \geq 1$. This gain will be referred to as the 'multiuser gain' in the sequel. The multiuser gain increases when the user channel frequency responses are very different. An equivalent concept in the context of multiuser fading channels is the concept of multiuser diversity (see [54] for example).

The aim of this chapter is to propose practical algorithms allowing the **computation of the maximum balanced rates for an arbitrary number of users**, and the corresponding (downlink and uplink) power allocations.

The derivation of these algorithms basically involves three steps :

- The computation of the maximum aggregate rate R_α for a fixed power allocation and a fixed sequence of priority coefficients $\{\alpha_k\}$.
- The computation of the optimal power allocation (i. e. maximizing R_α) for a fixed set of priority coefficients $\{\alpha_k\}$.
- The computation of the appropriate sequence $\{\alpha_k\}$ providing balanced user rates.

These three steps will firstly be analyzed in detail for the case of a multiuser memoryless channel (Section 3.4). Extensions will then be provided for a multiuser frequency selective channel with various kinds of power constraints (Section 3.5).

3.4 Memoryless channel

In the memoryless scenario, the channels are fully defined by the set of attenuation factors $\{h_k^2\}$ and the AWGN variance $\sigma^2 = N_0 B$. A total power $\bar{P}$ should be allocated among the different users. The maximum balanced rates $\{R_k^*\}$ are here associated with a specific power allocation $\{P_k\}$, and a specific decoding order.

3.4.1 Maximum aggregate rate for a given power allocation

This section provides explicit expressions for the MA and BC maximum aggregate rates R_α associated with a fixed power allocation $\{P_k\}$ and a fixed set $\{\alpha_k\}$. This allows to formulate the power allocation problem as a constrained optimization problem.

3.4.1.1 MA channel

The capacity region for the MA channel (uplink) is known to be [41] :

$$\left\{ \{R_k\} : \sum_{k \in \mathcal{S}} R_k \leq \mathcal{C}\left(\frac{\sum_{k \in \mathcal{S}} P_k h_k^2}{\sigma^2} \right) \; \forall \mathcal{S} \subset \{1, \cdots, K\} \right\} \tag{3.2}$$

where $\mathcal{C}(x) \triangleq B \log(1+x)$.[1] This region is defined by $2^K - 1$ constraints, each one corresponding to a nonempty subset $\mathcal{S}$ of users. It has $K!$ vertices in the positive quadrant. Each vertex is achievable by a successive decoding using one of the $K!$ possible orderings of the users. In the special case $K = 2$, the capacity region is a pentagon whose two useful vertices correspond to the decoding orders $(1, 2)$ and $(2, 1)$ respectively (see Figure 3.3). The 3-user capacity region is illustrated in Figure 3.4. The region is defined by 7 planes in the positive quadrant. It includes 6 useful vertices associated with the different decoding orders. Returning to the general case $K \geq 2$, and without loss of generality, we assume that *the users are ordered in ascending order of the priority coefficients* $\{\alpha_k\}$. For the choice of priority

[1] The logarithm used in the analytical derivations has a natural base, but a base of 2 is used in Section 3.6 in order to obtain numerical results expressed in Mbits/s.

coefficients $\{\alpha_k\}$, the maximum of R_α is then obtained at a single vertex that corresponds to the decoding order $\{1, \cdots, K\}$. In other words, the messages with the lower priorities are decoded first and subtracted from the received signal before decoding the messages with higher priorities. The maximum weighted rate is then given by:

$$R_{\alpha,\mathrm{MA}} = \sum_{k=1}^{K} \alpha_k \mathcal{C}\left(\frac{P_k h_k^2}{\sigma^2 + \sum_{l=k+1}^{K} P_l h_l^2}\right) \tag{3.3}$$

$$= \sum_{k=1}^{K} (\alpha_k - \alpha_{k-1})\, \mathcal{C}\left(\frac{\sum_{l=k}^{K} P_l h_l^2}{\sigma^2}\right) \tag{3.4}$$

with $\alpha_0 \triangleq 0$.

3.4.1.2 BC channel

In the BC channel (downlink), the decoding strategy is different for each receiver. Here the decoding order is not dictated by the user relative priorities, but by the relative channel gains ([51]-[52]). A given receiver is able to decode the messages intended for the users with lower channel gains (and thus transmitted at a lower rate) before decoding its own message. The remaining messages are considered as noise. For a given set of transmission powers $\{P_k\}$, the capacity region has a single vertex that maximizes the weighted rate R_α for any set $\{\alpha_k\}$ [52] :

$$\left\{ \{R_k\} : \, R_k \leq \mathcal{C}\left(\frac{P_k h_k^2}{\sigma^2 + \left(\sum_{h_k^2 < h_l^2} P_l\right) h_k^2}\right) \right\}. \tag{3.5}$$

Figure 3.3 gives the capacity region (a rectangle) for the two-user case, assuming $h_1^2 \geq h_2^2$. Let π be a permutation on the set $\{1, \cdots, K\}$ such that $h_{\pi(i)}^2 \geq h_{\pi(j)}^2$ for $i < j$. In other words, this permutation is used to sort the users in decreasing order of their channel gains. The maximum weighted rate is now:

$$R_{\alpha,\mathrm{BC}} = \sum_{k=1}^{K} \alpha_{\pi(k)} \mathcal{C}\left(\frac{P_{\pi(k)} h_{\pi(k)}^2}{\sigma^2 + \left[\sum_{l=1}^{k-1} P_{\pi(l)}\right] h_{\pi(k)}^2}\right). \tag{3.6}$$

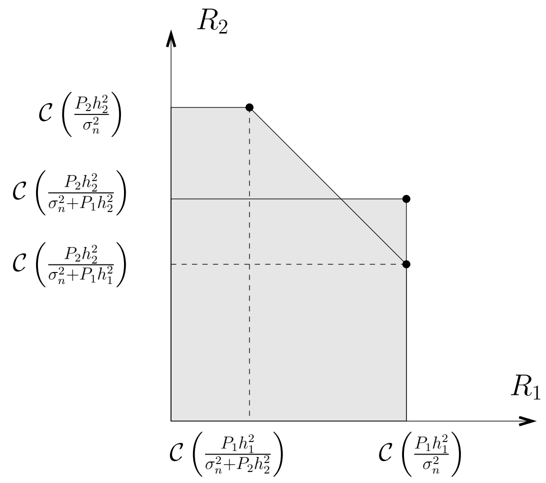

Figure 3.3: *Two-user capacity regions for a fixed power distribution : multiple access channel (pentagon) and broadcast channel (rectangle)*

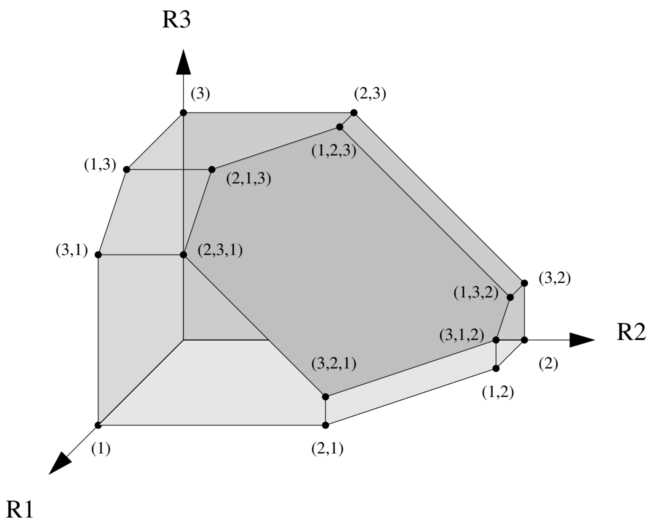

Figure 3.4: *Three-user capacity region for a fixed power distribution (multiple access channel)*

3.4.1.3 Formulation of the optimal power allocation problem

For a given constraint $\bar{P}$ on the total transmitted power, our objective is now to find the optimal power distributions $\{P_k^\circ\}$ and $\{P_k^*\}$ that maximize R_α for the MA and BC channels respectively:

$$\begin{aligned}\{P_k^\circ\} &= \arg\max\left[R_{\alpha,\mathrm{MA}}\left(\{P_k\}\right)\right] &(3.7)\\ \{P_k^*\} &= \arg\max\left[R_{\alpha,\mathrm{BC}}\left(\{P_k\}\right)\right] &(3.8)\end{aligned}$$

with constraints $\sum_k P_k \leq \bar{P}$ and $P_k \geq 0\ \forall k$. In each case, the solution satisfies the Kuhn-Tucker conditions:

$$\frac{1}{B}\frac{\partial R_\alpha}{\partial P_k}\begin{cases} = \mu & P_k > 0 \\ < \mu & P_k = 0\end{cases} \tag{3.9}$$

where μ is a positive Lagrange multiplier.

3.4.2 Optimal power allocation for the BC memoryless channel

(Given $\{\alpha_k\}$)
The users are supposed to be sorted in decreasing order of their channel gains through the permutation π. Starting the allocation process with $P_{\pi(k)} = 0\ \forall k$, the marginal rate gain for each user is given by $\frac{1}{B}\frac{\partial R_\alpha}{\partial P_{\pi(k)}} = D_{\pi(k)}(0)$ where the functions $D_k(x)$ are defined by

$$D_k\left(x\right) \triangleq \frac{\alpha_k h_k^2/\sigma^2}{1 + x h_k^2/\sigma^2}. \tag{3.10}$$

Some power $P_{\pi(k_1)}$ is firstly allocated to the user $\pi(k_1)$ with the maximum $\alpha_{\pi(k)} h_{\pi(k)}^2$.

The users with a higher $h_{\pi(k)}^2$ but a lower $\alpha_{\pi(k)} h_{\pi(k)}^2$ can be discarded. Indeed, equation (3.6) ensures that, for $k < k_1$ (i.e. for users with a higher channel gain),

$$\begin{aligned}\frac{1}{B}\frac{\partial R_\alpha}{\partial P_{\pi(k)}} &= \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{\pi(k_1)}} + \frac{1}{\sigma^2}\left(\alpha_{\pi(k)} h_{\pi(k)}^2 - \alpha_{\pi(k_1)} h_{\pi(k_1)}^2\right)\\ &< \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{\pi(k_1)}}, \end{aligned} \tag{3.11}$$

if

$$P^*_{\pi(k)} = 0, \tag{3.12}$$

which is consistent with the Kuhn-Tucker conditions (3.9).

The marginal gain for the remaining users becomes $D_{\pi(k)}(P_{\pi(k_1)})$ with $k \in \{k_1, \cdots, K\}$. The power $P_{\pi(k_1)}$ is then increased until a second user $\pi(k_2)$ (with $k_2 > k_1$) reaches a marginal gain equal to that of user $\pi(k_1)$ (unless $\bar{P}$ is reached before). As $h^2_{\pi(k_2)} < h^2_{\pi(k_1)}$, this is only possible if $\alpha_{\pi(k_2)} > \alpha_{\pi(k_1)}$. When this happens, the common marginal gain can be shown to be:

$$\begin{aligned} D_{\pi(k_1)}\left(P^*_{\pi(k_1)}\right) &= D_{\pi(k_2)}\left(P^*_{\pi(k_1)}\right) \\ &= \left(\alpha_{\pi(k_2)} h^2_{\pi(k_2)}/\sigma^2\right) F_{\pi(k_1),\pi(k_2)} \end{aligned} \tag{3.13}$$

where the correction factor $F_{i,j} < 1$ is defined as

$$F_{i,j} = \frac{1 - \alpha_i/\alpha_j}{1 - h_i^{-2}/h_j^{-2}}. \tag{3.14}$$

User $\pi(k_2)$ is thus selected as the one with the largest $\left(\alpha_{\pi(k)} h^2_{\pi(k)}\right) F_{\pi(k_1),\pi(k)}$ on the set $k \in \{k_1 + 1, \cdots, K\}$. The power allocated to user $\pi(k_1)$ is then fixed to

$$P^*_{\pi(k_1)} = \sigma^2 G_{\pi(k_1),\pi(k_2)} \tag{3.15}$$

where

$$G_{i,j} \triangleq \frac{\alpha_i h_i^2 - \alpha_j h_j^2}{(\alpha_j - \alpha_i) h_i^2 h_j^2} = h_j^{-2}\left(F_{i,j}^{-1} - 1\right). \tag{3.16}$$

The users with a higher $h^2_{\pi(k)}$ but a lower $\left(\alpha_{\pi(k)} h^2_{\pi(k)}\right) F_{\pi(k_1),\pi(k)}$ can be discarded : from (3.6), it can be checked that

$$\begin{aligned} \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{\pi(k)}} = \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{\pi(k_2)}} + \frac{1}{\sigma^2}\Big(&\alpha_{\pi(k)} h^2_{\pi(k)} F_{\pi(k_1),\pi(k)} \\ &- \alpha_{\pi(k_2)} h^2_{\pi(k_2)} F_{\pi(k_1),\pi(k_2)}\Big) < \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{\pi(k_2)}}, \end{aligned} \tag{3.17}$$

if

$$P^*_{\pi(k)} = 0 \qquad \forall k \in \{k_1 + 1, \cdots, k_2 - 1\}, \tag{3.18}$$

which is again consistent with the Kuhn-Tucker conditions (3.9).

The rate $R^*_{\pi(k_1)}$ for user $\pi(k_1)$ is given by

$$R^*_{\pi(k_1)} = B \log \left(\frac{\alpha_{\pi(k_1)} h^2_{\pi_{k_1}}}{\alpha_{\pi(k_2)} h^2_{\pi(k_2)} F_{\pi(k_1),\pi(k_2)}} \right). \tag{3.19}$$

This rate is left unaffected by the allocation of power to the other users (with $k > k_1$) as those users have a lower channel gain and their signals can be perfectly decoded and subtracted by receiver k_1. If there is still some power to allocate, then the allocation goes on with $P_{\pi(k_2)}$, which increases the rate $R_{\pi(k_2)}$ while leaving $R_{\pi(k_1)}$ unchanged, until a third user $\pi(k_3)$ (with a smaller $h^2_{\pi(k_3)}$ but a higher $\alpha_{\pi(k_3)}$) reaches the same marginal gain, and so on.

At the end of the allocation process, a subset $S_J = \{\pi(k_1), \cdots, \pi(k_J)\} \subset \{1, \cdots, K\}$ of $J \leq K$ users is obtained, who get a non-zero fraction of the total power $\bar{P}$. The users in this subset are sorted in *decreasing order of the channel gains* h^2_k and in *increasing order of the priority coefficients* α_k. As a consequence, the allocation algorithm can work indifferently on the user sequence $\{\pi(1), \cdots, \pi(K)\}$ or on the initial sequence $\{1, \cdots, K\}$. The allocation rule can be stated in very simple terms : *always add power to the user with the maximum* $D_k(P)$ where P is the power already allocated. Mathematically, we get:

$$\begin{aligned} R_\alpha &= B \sum_{i=1}^{J} \int_{P^*_{k_1}+\cdots+P^*_{k_{i-1}}}^{P^*_{k_1}+\cdots+P^*_{k_i}} D_{k_i}(x)dx \\ &= B \int_0^{\bar{P}} \max_{k\in\{1,\cdots,K\}} D_k(x)dx, \end{aligned} \tag{3.20}$$

which provides an intuitive graphical interpretation. Figure 3.5 illustrates the evolution of D_k and the individual powers as a function of the allocated power P for $K = 4$. In this example, $h^2_1 > h^2_2 > h^2_3 > h^2_4$ and $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. The optimal allocation gives $J = 3$ and $S_J = \{1, 2, 4\}$. For decreasing values of $\bar{P}$, the subset S_J can be reduced to $\{1, 2\}$ or $\{1\}$. Obviously, when the power to be allocated is large enough, the last user in the subset S_J is always the user with the highest priority α_k.

Once the subset S_J is known, the set of optimal powers $\left\{P^*_{k_i}\right\}$ can be

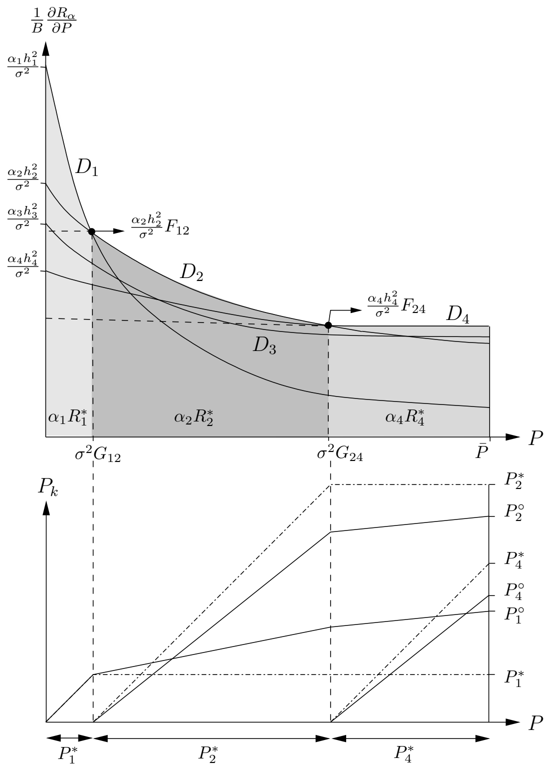

Figure 3.5: *Marginal gain and power distribution during the allocation process ($K = 4$)*

computed by solving the following system :

$$
\begin{aligned}
D_{k_i}\left(\sum_{l=1}^{i} P_{k_l}^*\right) &= D_{k_{i+1}}\left(\sum_{l=1}^{i} P_{k_l}^*\right), \quad i \leq J-1 \\
\sum_{i=1}^{J} P_{k_i}^* &= \bar{P}
\end{aligned}
\tag{3.21}
$$

Finally, the complete allocation procedure and the corresponding user rates are explicitly given as follows :

Algorithm 3.1 *Algorithm for the selection of the user sequence receiving a fraction of the total power $\bar{P}$ (subset S_J) :*

- *Initialization* $(J = 1)$ *:*

$$k_1 = \arg \max_{k \in [1,K]} \left(\alpha_k h_k^2\right). \tag{3.22}$$

- *While $k_J < K$:*
  - *Compute the next candidate :*

$$k^* = \arg \max_{k \in [k_J+1,K]} \left(\alpha_k h_k^2\, F_{k_J,k}\right). \tag{3.23}$$

  - *If there is enough power :*

$$\bar{P} \geq \sigma^2\, G_{k_J,k^*} \tag{3.24}$$

    then: $J = J+1$, $k_J = k^*$.
  - *Else: stop.*

Optimal BC allocated powers P_k^* (with $k \in S_J$) :

$$
\begin{pmatrix} P_{k_1}^* \\ P_{k_2}^* \\ \cdots \\ P_{k_j}^* \\ \cdots \\ P_{k_{J-1}}^* \\ P_{k_J}^* \end{pmatrix}
= \sigma^2
\begin{pmatrix} G_{k_1,k_2} \\ G_{k_2,k_3} - G_{k_1,k_2} \\ \cdots \\ G_{k_j,k_{j+1}} - G_{k_{j-1},k_j} \\ \cdots \\ G_{k_{J-1},k_J} - G_{k_{J-2},k_{J-1}} \\ -G_{k_{J-1},k_J} \end{pmatrix}
+
\begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ \cdots \\ 0 \\ \bar{P} \end{pmatrix}. \tag{3.25}
$$

Optimal user rates R_k^* (with $k \in S_J$) :

$$\begin{pmatrix} R_{k_1}^* \\ R_{k_2}^* \\ \cdots \\ R_{k_j}^* \\ \cdots \\ R_{k_{J-1}}^* \\ R_{k_J}^* \end{pmatrix} = B \log \begin{pmatrix} \dfrac{\alpha_{k_1} h_{k_1}^2}{\alpha_{k_2} h_{k_2}^2 F_{k_1,k_2}} \\ \dfrac{\alpha_{k_2} h_{k_2}^2 F_{k_1,k_2}}{\alpha_{k_3} h_{k_3}^2 F_{k_2,k_3}} \\ \cdots \\ \dfrac{\alpha_{k_j} h_{k_j}^2 F_{k_{j-1},k_j}}{\alpha_{k_{j+1}} h_{k_{j+1}}^2 F_{k_j,k_{j+1}}} \\ \cdots \\ \dfrac{\alpha_{k_{J-1}} h_{k_{J-1}}^2 F_{k_{J-2},k_{J-1}}}{\alpha_{k_J} h_{k_J}^2 F_{k_{J-1},k_J}} \\ \left[1 + \dfrac{\bar{P} h_{k_J}^2}{\sigma^2}\right] F_{k_{J-1},k_J} \end{pmatrix}. \qquad (3.26)$$

Algorithm 3.1 ensures that the obtained powers and rates are positive.

3.4.3 Optimal power allocation for the MA memoryless channel

(Given $\{\alpha_k\}$)
The computation of the optimal allocation is very similar in the MA and the BC channels. These two channels are known to have the same capacity region when the constraint is put on the transmitted power sum [49]-[50]. However, the power distributions corresponding to a given boundary point of the capacity region are distinct. In this section, the expression of the optimal user rates (3.26) is derived again in the MA context, allowing the computation of an explicit expression for the optimal MA power allocation $\{P_k^\circ\}$.

The users are now sorted in increasing order of their priority coefficients α_k. For a given power allocation $\{P_k\}$, we define the sequence $\{x_k\} \leq 1$ as :

$$x_k = \left(1 + \sum_{i=k}^{K} P_i h_i^2 / \sigma^2\right)^{-1}. \qquad (3.27)$$

From (3.3), the marginal rate gains can be obtained iteratively as follows :

$$\begin{aligned} \frac{1}{B}\frac{\partial R_\alpha}{\partial P_1} &= \frac{\alpha_1 h_1^2}{\sigma^2} x_1 \qquad (3.28) \\ \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{k+1}} &= \frac{h_{k+1}^2}{h_k^2}\left(\frac{1}{B}\frac{\partial R_\alpha}{\partial P_k}\right) + (\alpha_{k+1} - \alpha_k)\frac{h_{k+1}^2}{\sigma^2} x_{k+1}. \end{aligned}$$

The initial marginal rate gain for each user ($x_k = 1 \quad \forall k$) is again $\alpha_k h_k^2$. The power is thus firstly allocated to user k_1 with the maximum $\alpha_k h_k^2$.

The lower-indexed users (i.e. the users with a lower priority) are discarded: $P_k^\circ = 0$ for $k < k_1$. From (3.28), it follows that :

$$\begin{aligned} x_k &= x_{k_1} && (3.29) \\ \frac{\partial R_\alpha}{\partial P_k} &= \frac{\alpha_k h_k^2}{\alpha_{k_1} h_{k_1}^2} \frac{\partial R_\alpha}{\partial P_{k_1}} < \frac{\partial R_\alpha}{\partial P_{k_1}}, && (3.30) \end{aligned}$$

for $k < k_1$, which is consistent with the Kuhn-Tucker conditions (3.9).

The marginal gains for the remaining users is again $D_k\left(P_{k_1}\right)$. Power is thus allocated to user k_1 until $P_{k_1} = \sigma^2 G_{k_1,k_2}$ for a given user k_2 with

$$k_2 = \arg \max_{k \in \{k_1+1,\cdots,K\}} \left(\alpha_k h_k^2 F_{k_1,k}\right) \tag{3.31}$$

and G_{k_1,k_2} is defined in (3.16). At that time, the power allocated to user k_1 is equal to $P_{k_1}^*$, that is to say the optimal power that would be allocated in a BC scenario (see Figure 3.5).

Now the allocation process becomes different. While in the BC channel, any power increase dP would be fully allocated to user k_2 (which increases R_{k_2} and leaves R_{k_1} unaffected), in the MA channel dP has to be shared by users k_1 and k_2. The fraction of power dP_{k_2} allocated to user k_2 directly increases the rate R_{k_2}. But this power is seen as noise by user k_1 (which is decoded *before* user k_2) : some power dP_{k_1} is then allocated to user k_1 to compensate for the noise enhancement and leave R_{k_1} unchanged. Some power is thus allocated simultaneously to users k_1 and k_2.

Users with an intermediate priority are discarded: $P_k^\circ = 0$ with $k_1 < k < k_2$. From (3.28), we obtain the system :

$$\begin{aligned} \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{k_1}} &= \mu && (3.32) \\ \frac{1}{B}\frac{\partial R_\alpha}{\partial P_k} &= \frac{h_k^2}{h_{k_1}^2}\mu + \left(\alpha_k - \alpha_{k_1}\right)\frac{h_k^2}{\sigma^2} x_{k_2} && (3.33) \\ \frac{1}{B}\frac{\partial R_\alpha}{\partial P_{k_2}} &= \frac{h_{k_2}^2}{h_{k_1}^2}\mu + \left(\alpha_{k_2} - \alpha_{k_1}\right)\frac{h_{k_2}^2}{\sigma^2} x_{k_2} = \mu. && (3.34) \end{aligned}$$

Solving this system, we get :

$$\frac{\partial R_\alpha}{\partial P_k} = \frac{\partial R_\alpha}{\partial P_{k_1}} \Big(1 - h_k^2 \left(\alpha_k - \alpha_{k_1}\right) \Big[\left(\alpha_k h_k^2 F_{k_1,k}\right)^{-1} - \left(\alpha_{k_2} h_{k_2}^2 F_{k_1,k_2}\right)^{-1}\Big]\Big) < \frac{\partial R_\alpha}{\partial P_{k_1}}, \quad (3.35)$$

for $k_1 < k < k_2$, which is again consistent with the Kuhn-Tucker conditions (3.9).

The optimal rate $R_{k_1}^*$ for user k_1 is found to be the same as (3.19), and the common marginal gain can be shown to be :

$$\mu = D_{k_2}\left(P_{k_1} + P_{k_2}\right). \quad (3.36)$$

The two-user allocation continues until a third user $k_3 \in \{k_2+1, \cdots, K\}$ reaches the same marginal gain. When this happens, we have $P_{k_1} + P_{k_2} = P_{k_1}^* + P_{k_2}^*$, that is to say the total power allocated to users k_1 and k_2 is the same as the optimal power sum that would be allocated in a BC scenario (see Figure 3.5). Power is then allocated simultaneously to users (k_1, k_2, k_3), and so on.

The relation (3.20) given for the BC scenario remains valid for the MA channel. The allocation rule can be restated as : *always increase the rate of the user k_i with the maximum $D_k(P)$ while leaving the other rates unchanged*, where P is the power already allocated. This can be obtained by allocating power simultaneously to all the users in the set $\{k_1, \cdots, k_i\}$. The user subset S_J and the optimal user rates are found to be the same as in the BC channel (see algorithm 3.1 and equation (3.26)).

Once the subset S_J is known, the set of optimal powers $\left\{P_{k_i}^\circ\right\}$ can be computed from (3.28) and (3.9). The sequence x_{k_j} is first computed as

$$\begin{aligned} x_{k_1} &= \mu \left(\frac{\alpha_{k_1} h_{k_1}^2}{\sigma^2}\right)^{-1} \quad (3.37) \\ x_{k_{j+1}} &= \mu \left(\frac{\alpha_{k_{j+1}} h_{k_{j+1}}^2}{\sigma^2} F_{k_j,k_{j+1}}\right)^{-1}, \quad j \in \{1, \cdots, J-1\}. \end{aligned}$$

The optimal powers $P_{k_j}^\circ$ are then given by

$$\begin{aligned} P_{k_j}^\circ &= \sigma^2 h_{k_j}^{-2} \left(x_{k_j}^{-1} - x_{k_{j+1}}^{-1}\right), \quad j \in \{1, \cdots, J-1\} \\ P_{k_J}^\circ &= \sigma^2 h_{k_J}^{-2} \left(x_{k_J}^{-1} - 1\right) \quad (3.38) \end{aligned}$$

Using the power constraint, the marginal rate is computed as

$$\mu = \frac{\alpha_{k_J} h_{k_J}^2/\sigma^2}{1 + \bar{P} h_{k_J}^2/\sigma^2}. \tag{3.39}$$

Finally, the **optimal MA allocated powers** P_k° (with $k \in S_J$) are given explicitly as follows :

$$\begin{pmatrix} P_{k_1}^\circ \\ P_{k_2}^\circ \\ \cdots \\ P_{k_j}^\circ \\ \cdots \\ P_{k_{J-1}}^\circ \\ P_{k_J}^\circ \end{pmatrix} = \frac{1}{\alpha_{k_J}} \left(\bar{P} + \frac{\sigma^2}{h_{k_J}^2} \right) \mathbf{\Pi}(\alpha, \mathbf{h}) - \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ \cdots \\ 0 \\ \frac{\sigma^2}{h_{k_J}^2} \end{bmatrix} \tag{3.40}$$

with vector $\mathbf{\Pi}(\alpha, \mathbf{h})$ defined as :

$$\mathbf{\Pi}(\alpha, \mathbf{h}) = \begin{bmatrix} h_{k_1}^{-2} \left(\alpha_{k_1} h_{k_1}^2 - \alpha_{k_2} h_{k_2}^2 F_{k_1,k_2} \right) \\ h_{k_2}^{-2} \left(\alpha_{k_2} h_{k_2}^2 F_{k_1,k_2} - \alpha_{k_3} h_{k_3}^2 F_{k_2,k_3} \right) \\ \cdots \\ h_{k_j}^{-2} \left(\alpha_{k_j} h_{k_j}^2 F_{k_{j-1},k_j} - \alpha_{k_{j+1}} h_{k_{j+1}}^2 F_{k_j,k_{j+1}} \right) \\ \cdots \\ h_{k_{J-1}}^{-2} \left(\alpha_{k_{J-1}} h_{k_{J-1}}^2 F_{k_{J-2},k_{J-1}} - \alpha_{k_J} h_{k_J}^2 F_{k_{J-1},k_J} \right) \\ h_{k_J}^{-2} \left(\alpha_{k_J} h_{k_J}^2 F_{k_{J-1},k_J} \right) \end{bmatrix} \tag{3.41}$$

Algorithm 3.1 also ensures that the obtained powers are positive.

3.4.4 Maximum balanced rates

The previous section explained how to compute the optimal power allocation $\{P_k\}$ for a given set of priority coefficients $\{\alpha_k\}$ and a total power $\bar{P}$.

To obtain a specific boundary point of the capacity region such that the corresponding user rates satisfy

$$\frac{R_1^*}{\beta_1} = \cdots = \frac{R_K^*}{\beta_k} = \bar{R} \tag{3.42}$$

for a predefined set of normalization coefficients $\{\beta_k\}$, the right set of priority coefficients $\{\alpha_k\}$ has to be found iteratively. Computation of the maximum balanced rates corresponds to $\beta_k = R_k^1$, but the idea is the same to get any other point of the capacity region. Equation (3.42) implies that every user gets a non-zero fraction of the power. Sorting the users in decreasing order of their channel gain h_k^2, the solution of our problem requires the computation of an appropriate set of priority coefficients $\alpha_1 < \cdots < \alpha_K$, such that algorithm 3.1 selects the whole set of users ($J = K$ and $S_K = \{1, \cdots, K\}$), and that the associated user rates in (3.26) satisfy the constraint in (3.42).

Introducing the expression of the optimal user rates (3.26) into the balanced rates constraints (3.42), we get :

$$\mathcal{A}_k = \frac{2^{\beta_k \bar{R}/B}}{\mathcal{H}_k} \tag{3.43}$$

with

$$\boldsymbol{\mathcal{H}} = \begin{pmatrix} \frac{h_2^{-2} - h_1^{-2}}{h_1^{-2}} \\ \frac{h_3^{-2} - h_2^{-2}}{h_2^{-2} - h_1^{-2}} \\ \vdots \\ \frac{h_{k+1}^{-2} - h_k^{-2}}{h_k^{-2} - h_{k-1}^{-2}} \\ \vdots \\ \frac{h_K^{-2} - h_{K-1}^{-2}}{h_{K-1}^{-2} - h_{K-2}^{-2}} \\ \frac{1 + \frac{\bar{P} h_K^{-2}}{\sigma^{-2}}}{1 - \frac{h_{K-1}^{-2}}{h_K^{-2}}} \end{pmatrix} \qquad \boldsymbol{\mathcal{A}} = \begin{pmatrix} \frac{\alpha_1}{\alpha_2 - \alpha_1} \\ \frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_2} \\ \vdots \\ \frac{\alpha_k - \alpha_{k-1}}{\alpha_{k+1} - \alpha_k} \\ \vdots \\ \frac{\alpha_{K-1} - \alpha_{K-2}}{\alpha_K - \alpha_{K-1}} \\ \frac{\alpha_K - \alpha_{K-1}}{\alpha_K} \end{pmatrix} \tag{3.44}$$

The components of the vector $\boldsymbol{\mathcal{A}}$ have to satisfy the relation :

$$\sum_{j=1}^{K} \left(\prod_{i=j}^{K} \mathcal{A}_i \right) = 1. \tag{3.45}$$

There is a single target rate $\bar{R}$ producing a set of $\mathcal{A}_k$ that fulfills equation (3.45). The right value can be found by a simple binary search. The drawback of this simple algorithm is that it can not be extended to the case of multiuser frequency-selective channels.

Actually, the problem of maximum balanced rates computation can be modeled as a nonlinear system with K unknowns α_k as follows :

$$\mathbf{F}\left(\boldsymbol{\alpha}\right) = \begin{pmatrix} \frac{R_2^*}{\beta_2} - \frac{R_1^*}{\beta_1} \\ \vdots \\ \frac{R_K^*}{\beta_K} - \frac{R_1^*}{\beta_1} \\ \sum_k \alpha_k - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.46}$$

A Newton-Raphson iterative search is proposed to find the solution:

Algorithm 3.2 *Maximum balanced rates for a Gaussian memoryless multiuser channel*

- *Start with* $\boldsymbol{\alpha}^{(0)} = [1/R_1^1, \cdots, 1/R_K^1]^T / \sum_k (1/R_k^1)$, *set* $i = 0$.
- *Compute the optimal power allocation with algorithm 3.1. Compute the corresponding user rates* $\{R_k^*\}$ *and the normalized rates* $\{R_k^*/\beta_k\}$.
- *Compute the average normalized rate* μ_{R_β} *and the standard deviation* σ_{R_β}, *defined by :*

$$\mu_{R_\beta} \triangleq \frac{1}{K}\sum_k \left(R_k^*/\beta_k\right) \tag{3.47}$$

$$\sigma_{R_\beta} \triangleq \sqrt{\frac{1}{K}\sum_k \left(R_k^*/\beta_k - \mu_{R_\beta}\right)^2} \tag{3.48}$$

- *While* $\sigma_{R_\beta}/\mu_{R_\beta} > \varepsilon$ *:*
  - *Update the set of priority coefficients as follows:*

$$\boldsymbol{\Delta}_\alpha^{(i)} = -\left(\frac{\partial \mathbf{F}}{\partial \boldsymbol{\alpha}}\right)^{-1}_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^{(i)}} \mathbf{F}\left(\boldsymbol{\alpha}^{(i)}\right) \tag{3.49}$$

$$\boldsymbol{\alpha}^{(i+1)} = \boldsymbol{\alpha}^{(i)} + \lambda_i \, \boldsymbol{\Delta}_\alpha^{(i)}. \tag{3.50}$$

  - *Update* $\{R_k^*\}$, $\{R_k^*/\beta_k\}$, μ_{R_β} *and* σ_{R_β}, *set* $i = i + 1$.

where ε *is a tolerance parameter and the sequence* $\lambda_i \leq 1$ *is used to ensure convergence in the first few iterations.*

The computation of the update direction $\boldsymbol{\Delta}_{\alpha}^{(i)}$ requires the knowledge of the Jacobian matrix $\frac{1}{B}\frac{\partial \mathbf{R}}{\partial \boldsymbol{\alpha}}$. From (3.26), it is a symmetric tridiagonal matrix $\mathbf{T}(\boldsymbol{\alpha})$ with the following entries on the three main diagonals :

$$\begin{aligned}
\mathbf{T}_{k_j,k_j} &= \frac{\alpha_{k_{j+1}} - \alpha_{k_{j-1}}}{(\alpha_{k_j} - \alpha_{k_{j-1}})(\alpha_{k_{j+1}} - \alpha_{k_j})} \\
\mathbf{T}_{k_j,k_{j+1}} &= \frac{-1}{\alpha_{k_{j+1}} - \alpha_{k_j}} = \mathbf{T}_{k_{j+1},k_j}
\end{aligned} \tag{3.51}$$

where $\alpha_{k_0} = \alpha_{k_{J+1}} \triangleq 0$. In particular, the Jacobian matrix is zero on the rows and columns corresponding to users who are not in the subset S_J. As this matrix should be invertible to allow the computation of the update direction, it is important to choose a convenient starting vector $\boldsymbol{\alpha}^{(0)}$, and to control the size of the updating step λ_i in order to keep $\boldsymbol{\alpha}^{(i+1)}$ close to the solution and keep the matrix inversion possible. The convexity property of the capacity region guarantees that every user is included in the allocation subset for the proposed $\boldsymbol{\alpha}^{(0)}$. Figure 3.2 illustrates the initial rates obtained with $\boldsymbol{\alpha}^{(0)}$ (point C) in a two-user problem. After some iterations, the solution converges to the maximum balanced rates (point D).

To conclude this section, it should be noted that the solution proposed here is probably not the most efficient one in the case of a multiuser memoryless channel : however, it can be easily extended to the case of a multiuser frequency selective channel, which is the purpose of the next section.

3.5 Frequency-selective channel

In the case of frequency-selective channels with frequency responses $H_k(\omega)$, the power allocation problem can be discretized by dividing the frequency spectrum into a large number N of frequency bins of width df. As N increases to infinity, a piece-wise constant channel model converges to the actual channels. Each frequency bin $n \in [1, N]$ corresponds to a multiuser memoryless Gaussian channel with channel gains $h_{kn}^2 \triangleq |H_k(\omega_n)|^2$ and a bandwidth B/N. The problem is now to find the K optimal power spectra P_{kn} corresponding to a given boundary point of the capacity region. The user rates R_k are computed by summing the partial rates R_{kn} corresponding to the N parallel subchannels.

3.5.1 Types of constraints

Talking about the capacity region of a multiuser channel may be confusing as this region actually depends on the nature of the power constraints. Various types of constraints can be considered, leading to distinct capacity regions. The next items should be taken in consideration in the definition of the capacity region :

- The constraints can be put on the transmitted powers or on the received powers. The latter scenario won't be investigated here.

- Considering a given user k, there can be one constraint on the total transmitted power $\sum_n (P_{kn})$ or N constraints on the power in each frequency bin P_{kn}. In a wired system, it is usual to impose a transmission mask on the power spectral density (PSD). This makes sense, for example, to solve the problem of electromagnetic compatibility with existing narrowband services (egress problem). Both constraints are also commonly combined : in that case, a total power $\bar{P}$ has to be distributed on the frequency axis, and the power in each frequency bin can not go beyond a given limit $\bar{\gamma}$ with $\bar{\gamma}\, df > \bar{P}/N$. The two types of constraints (power constraint and PSD constraint) will be analyzed separately in the sequel.

- In a multiuser setup, the constraint can be put on the power-sum or on the individual powers. In the uplink, the K different transmitters can be constrained separately or considered as a single 'distributed' entity, with a single constraint on the power-sum or PSD-sum. This makes sense again when the egress problem is considered, as an external receiver gets interference from the K transmitters simultaneously. For example, in a time-division duplex scheme, the signals are produced alternately by the single head-end (during the downstream time slots) and by the K user modems (during the upstream time slots). In the downlink, the sum of the signals intended to the different receivers will be considered as a unique power-constrained signal.

- An FDMA restriction is sometimes required to simplify the design of the communication system and avoid the use of successive decoding. In that case, each frequency bin n is exclusively allocated to a single user. The corresponding capacity region is then reduced as an additional constraint is added to the problem.

- A TDMA solution can also be investigated. This means that single user communications are established for each user in turn. The corresponding capacity region is a hyperplane defined by the K single user rates vectors. The TDMA solution provides a lower bound for the capacity region corresponding to the simultaneous transmission of signals by the different users.

- Even in the case of simultaneous transmission, timesharing can be used between different power allocations to reach specific points of the capacity region. These solutions can provide useful indications on the properties of the capacity region, but solutions corresponding to a fixed power distribution across the users will always be considered in the sequel.

- Finally, the upstream capacity region (MA channel) is not always the same as the downstream capacity region (BC channel). Both are identical in the following situations :

  – When the same power (or PSD) constraint is put on the LT transmitter (on one hand) and on the K NT transmitters considered as a single signal source (on the other hand). This was shown in section 3.4.

  – In the TDMA and FDMA scenarios, as the MA and BC channels are then reduced to K parallel single user channels.

Let us illustrate these considerations on a qualitative example with $K = 2$. Figure 3.6 gives the boundaries of various capacity regions. All these regions are valid for both the MA and BC channels, except when the opposite is explicitly mentioned.

PSD-constrained capacity regions :
Points A and B give the single user rates for a flat PSD : $\left(P_{1n} = 0, P_{2n} = \frac{\bar{P}}{N}\right)$ and $\left(P_{1n} = \frac{\bar{P}}{N}, P_{2n} = 0\right)$, respectively. The line ACB bounds the flat-PSD TDMA capacity region : every point on this line can be obtained by timesharing between the single-user solutions. The line OCE is the locus of balanced rates solutions. Pentagon AFGB bounds the MA capacity region for a flat PSD assigned to both users simultaneously : $\left(P_{1n} = \frac{\bar{P}}{N}, P_{2n} = \frac{\bar{P}}{N}\right)$. Every point on the segment FG is obtained by timesharing between the two possible decoding orders. The curve AEB gives the capacity region for a constraint on the PSD-sum : $P_{1n} + P_{2n} = \frac{\bar{P}}{N}$. Point S gives the maximum sum rate while point E gives the maximum balanced rate. The multiuser

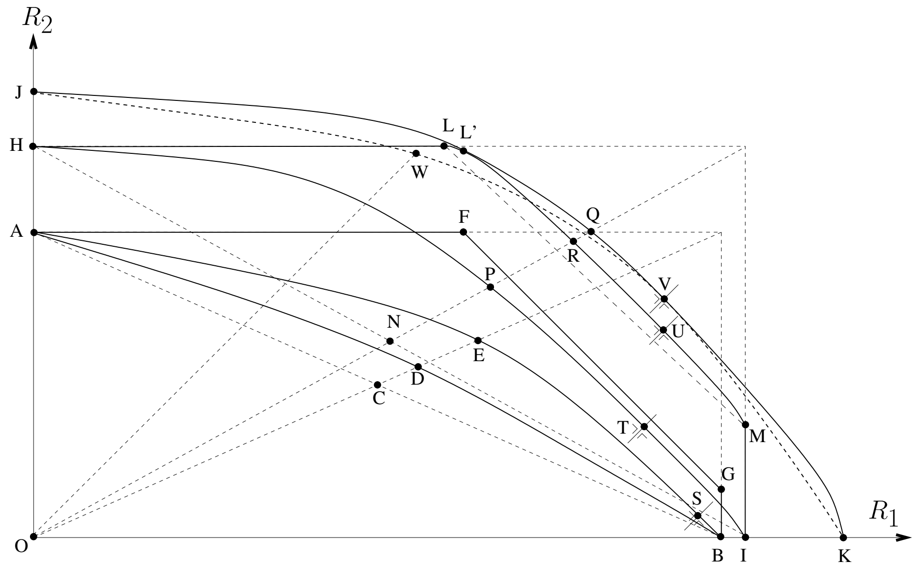

Figure 3.6: *Various capacity regions for ISI channels*

gain is obtained here by the ratio $\frac{\mathrm{OE}}{\mathrm{OC}}$. Finally, the curve ADB bounds the MA capacity region for a constraint on the PSD-sum and a flat PSD allocated to each user : $\left(P_{1n} = \gamma, P_{2n} = \frac{\bar{P}}{N} - \gamma\right)$. The ratio $\frac{\mathrm{OD}}{\mathrm{OC}}$ appears as a basic multiuser gain, which does not require channel knowledge at the transmitters. The ratio $\frac{\mathrm{OE}}{\mathrm{OD}}$ gives the additional gain obtained by a smart spectrum allocation, which requires knowledge of the channels at the transmitters.

Power-constrained capacity regions :
Points H and I give the single user rates for $\left(\sum_{n=1}^{N} P_{1n} = \bar{P}, P_{2n} = 0\right)$ and $\left(P_{1n} = 0, \sum_{n=1}^{N} P_{2n} = \bar{P}\right)$, respectively. These rates are of course higher than rates A and B as the flat PSD constraint is relaxed. The line HNI bounds the unconstrained-PSD TDMA capacity region. The locus of balanced rates solutions is here given by the line ONQ. The curve HPI gives the capacity region with a single constraint on the transmitted power sum : $\sum_n (P_{1n} + P_{2n}) = \bar{P}$. The multiuser gain is given here by $\frac{\mathrm{OP}}{\mathrm{ON}}$. The curve HLMI bounds the MA capacity region with constraints on the individual powers : $\left(\sum_{n=1}^{N} P_{1n} = \bar{P}, \sum_{n=1}^{N} P_{2n} = \bar{P}\right)$. Only the LM portion of the curve is useful. The multiuser gain is here $\frac{\mathrm{OR}}{\mathrm{ON}}$. JQVK gives the capacity

region obtained when the total power $2\bar{P}$ can be redistributed arbitrarily to the users : in other words, the constraint on the power sum is here $\sum_n (P_{1n} + P_{2n}) = 2\bar{P}$. This curve is tangent to the HLMI capacity curve at some point L' where the optimal power allocation happens to share the available power $2\bar{P}$ in two equal parts. On the JL' segment, the power allocated to user 1 is lower, while on the L'K segment it is higher. Finally, the dashed line JVK represents the same MA capacity region, but with an additional FDMA constraint (i.e. each frequency bin is exclusively allocated to one user). Both curves have a common point V with a tangent at 45 degrees : it is well known, indeed, that the maximum sum rate solution has the FDMA property [40]. The authors of [53] proposed a method to compute the maximum common rate of multiple access channels with that FDMA constraint, i.e. point W in the figure.

The object of the following sections is to provide efficient algorithms to compute the balanced rates solutions E, P (or Q) and R, as well as the associated power allocations, for an arbitrary number of users K. Note that these algorithms could be used to compute other boundary points of the capacity regions.

3.5.2 Multiuser diversity

If the constraint is put on the power-sum (or PSD-sum), the multiuser gain g defined in (3.1) is equal to 1 in the special case where all the users have identical channel responses ($H_k(\omega) = H_{k'}(\omega)\ \forall k, k'$). In other words, allowing simultaneous transmission by the different users does not increase the size of the capacity region if the channels of all users are identical. For this reason, g can be seen as a *multiuser diversity gain*.

If the constraint is put on the individual power, [40] showed that the capacity region is a polytope (a pentagon for $K = 2$) when the channels are identical. Some gain is thus available with respect to the balanced TDMA rates, as the users are not only allowed to transmit simultaneously, but also with the same power as in the single-user situation : the total power transmitted on the channel at a given time is K times higher than in the TDMA solution. However, the multiuser gain g increases when the channel responses are different. The origin of the multiuser gain is thus twofold :

- The gain brought by the transmission power increase. For K identical channels, this gain is given by $\frac{R_k^1(K\bar{P})}{R_k^1(\bar{P})} \leq 1 + \frac{B\log(K)}{R_k^1(\bar{P})}$ where $R_k^1(P)$ stands for the maximum single-user rate with a transmitted power P. The last inequality corresponds to the high-snr approximation.

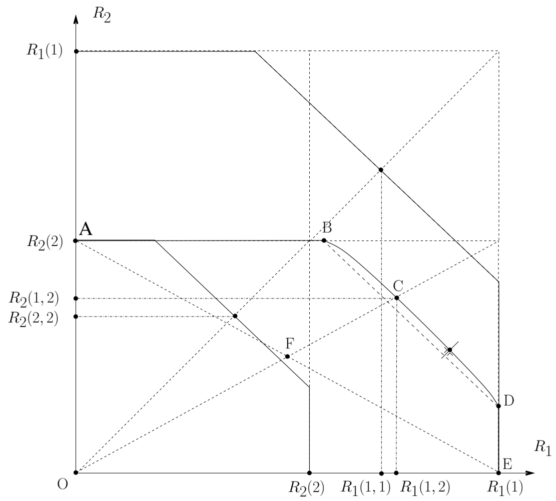

Figure 3.7: *Capacity regions for identical and different two-user channels*

- The multiuser diversity.

The maximum gain g is obtained when the user channels are sufficiently different to provide single-user optimal power allocations in K mutually exclusive sets of frequency bins. In that case, the users are not in competition for the best frequencies and they can all transmit at their SU maximum rate : $R_k = R_k^1\ \forall k$. An upper bound for the multiuser gain is thus : $g \leq K$.

Figure 3.7 illustrates these ideas for $K = 2$. The bound of the capacity region for channels $(1, 2)$ is given by the curve ABCDE. The total multiuser gain is here $\frac{OC}{OF}$. The two pentagons give the capacity regions for identical channels (channels $(1, 1)$ or channels $(2, 2)$). The maximum balanced rates with different channels ($R_1(1, 2)$ and $R_2(1, 2)$) are higher than the corresponding balanced rates with identical channels ($R_1(1, 1)$ and $R_2(2, 2)$), which denotes the effect of multiuser diversity.

3.5.3 Maximum aggregate rate for a given power allocation

As the total rates are simply obtained by summing the partial rates in each frequency bin, the maximum aggregate rate R_α for a given power allocation

$\{P_{kn}\}$ is given by :

$$R_\alpha = \sum_{k=1}^{K} \alpha_k \left(\sum_{n=1}^{N} R_{kn} \right) \tag{3.52}$$

with

$$(R_{kn})_{\mathrm{MA}} = \frac{1}{N}\,\mathcal{C}\left(\frac{P_{kn} h_{kn}^2}{\sigma_n^2 + \sum_{l=k+1}^{K} P_{ln} h_{ln}^2} \right) \tag{3.53}$$

$$(R_{kn})_{\mathrm{BC}} = \frac{1}{N}\,\mathcal{C}\left(\frac{P_{kn} h_{kn}^2}{\sigma_n^2 + \left(\sum_{h_k^2 < h_l^2} P_{ln} \right) h_{kn}^2} \right). \tag{3.54}$$

3.5.4 Optimal power allocation

(Given $\{\alpha_k\}$)
We wish now to compute the power allocations $\{P_{kn}^*\}$ and $\{P_{kn}^\circ\}$ maximizing the aggregate rate R_α for the MA and BC frequency selective channels, respectively. The general strategy used to solve this problem is made of two steps :

1. Find the solution along the frequency axis, i.e. compute the optimal spectral allocation of the power sum $\bar{P}_n = \sum_k P_{kn}$.

2. Find the solution along the multiuser axis, i.e. allocate the power $\bar{P}_n$ to the K users separately in each frequency bin n by using the results of Section 3.4.

Let us analyze in more details the three different scenarios.

3.5.4.1 PSD-sum constraint

The constraints are : $\sum_k \gamma_k(\omega) = \bar{\gamma}(\omega)$.

The spectrum allocation is trivially given by $\bar{P}_n = \sum_k P_{kn} = \bar{\gamma}(\omega_n) df$. The Kuhn-Tucker conditions are :

$$\frac{N}{B} \frac{\partial R_\alpha}{\partial P_{kn}} \begin{cases} = \mu_n & P_{kn} > 0 \\ < \mu_n & P_{kn} = 0 \end{cases} \tag{3.55}$$

The power allocation algorithm is then applied separately in the N frequency bins. A constraint on the PSD-sum actually generates independent constraints on the power-sum inside every frequency bin.

3.5.4.2 Power-sum constraint

The constraint is : $\sum_k \sum_n P_{kn} = \bar{P}$.

The Kuhn-Tucker conditions are now :

$$\frac{N}{B}\frac{\partial R_\alpha}{\partial P_{kn}} \begin{cases} = \mu & P_{kn} > 0 \\ < \mu & P_{kn} = 0 \end{cases} \tag{3.56}$$

Using (3.20), the partial derivative for each frequency bin is $\max_k \left\{ D_{kn}(\bar{P}_n) \right\}$ (the KN functions $D_{kn}(x)$ are defined in the same way as equation (3.10)), which provides the solution :

$$\mu \bar{P}_n = \max_{k \in [1,K]} \left(1 - \left[\frac{\mu}{h_{kn}^2/\sigma^2} + (1-\alpha_k) \right] \right)_+ \tag{3.57}$$

where $(x)_+$ denotes $\max(x, 0)$. The spectral allocation is the multiuser extension of the well-known water-filling algorithm ([52]-[40]). The solution can be represented in a single water-filling diagram with a common water level fixed to 1 and K containers $\mu\sigma^2/h_{kn}^2$ modified by the correction terms $(1-\alpha_k)$ reflecting the effect of the user relative priorities. The bottom of the resulting multiuser container is the minimum of the K modified individual containers. A simple binary search must be performed to find the right coefficient μ that satisfies the total power constraint. The power allocation algorithm is then applied separately for the N frequency bins.

3.5.4.3 Individual power constraint (MA channel only)

The K power constraints are : $\sum_n P_{kn} = \bar{P}_k$.

The Kuhn-Tucker conditions are :

$$\frac{N}{B}\frac{\partial R_\alpha}{\partial P_{kn}} \begin{cases} = \mu_k & P_{kn} > 0 \\ < \mu_k & P_{kn} = 0 \end{cases} \tag{3.58}$$

which is not directly compatible with (3.9) as the marginal gain for each user is now different. The solution lies in the observation that $R_\alpha = f\left(P_{kn}h_{kn}^2\right) = f\left(\mu_k P_{kn}\,\frac{h_{kn}^2}{\mu_k}\right)$, i.e. transmitting a signal with power P_{kn} in a channel with gain h_{kn}^2 is equivalent to transmitting a signal with power $\mu_k P_{kn}$ in a channel with gain $\frac{h_{kn}^2}{\mu_k}$ as far as the resulting capacity is concerned. The product $\mu_k P_{kn}$ appears as an 'equivalent power' allocated to user k for transmission on the 'equivalent channel' $\frac{h_{kn}^2}{\mu_k}$. If the equivalent power sum $\bar{P}_n^\mu \triangleq \sum_k \mu_k P_{kn}$ to be allocated on a given frequency bin n is known, then the optimal equivalent power allocation to the users $\{\mu_k P_{kn}\}$ verifies (3.9) with $\mu = 1$ and algorithm 3.1 can be used on the set of equivalent channels $\frac{h_{kn}^2}{\mu_k}$. Using equation (3.39), we get :

$$\frac{N}{B}\frac{\partial R_\alpha}{\partial\left(\mu_k P_{kn}\right)}\begin{cases} = 1 = \max_k\left(\frac{\frac{\alpha_k h_{kn}^2}{\mu_k\sigma^2}}{1+\bar{P}_n^\mu\frac{h_{kn}^2}{\mu_k\sigma^2}}\right) & P_{kn} > 0 \\ < 1 & P_{kn} = 0 \end{cases} \tag{3.59}$$

For a given set $\{\mu_k\}$, the distribution of the equivalent power sum $\bar{P}_n^\mu$ along the frequency axis is finally obtained as follows :

$$\bar{P}_n^\mu = \max_{k\in[1,K]}\left(1 - \left[\frac{\mu_k}{h_{kn}^2/\sigma^2} + (1-\alpha_k)\right]\right)_+ . \tag{3.60}$$

The power allocation algorithm is then applied separately in the N frequency bins.

Equation (3.60) can be associated with a multiuser water-filling diagram as in [40]. Figure 3.8 is an example of a three-user water-filling diagram. The common water level is 1. The three containers (bold curves) are associated with the equivalent channels h_{kn}^2/μ_k, and are modified by $(1-\alpha_k)$ to reflect the different user relative priorities. The total area defined by the water level and the bottom of the composite container represents the equivalent power sum $\mu_1\bar{P}_1+\mu_2\bar{P}_2+\mu_3\bar{P}_3$. After the application of the power allocation algorithm in each equivalent subchannel, the water area can be subdivided in three parts (represented with three different patterns in Figure 3.8). The area of each part should be equivalent to the individual equivalent power $\mu\bar{P}_k$.

There is a single set of coefficients $\{\mu_k\}$ satisfying the K power constraints simultaneously. These coefficients have to be computed iteratively. [41] proposed a simple algorithm to compute the optimum μ_k : at iteration $i+1$, each $\mu_k^{(i+1)}$ is computed separately to satisfy the k^{th} power constraint while the $K-1$ other μ_k's remain fixed at $\mu_k^{(i)}$. We propose here an algorithm updating the K coefficients μ_k simultaneously, based on a careful analysis of the optimal power allocation in each frequency bin. Rewriting equation (3.40) with the equivalent powers $\mu_{k_j}P_{k_j n}$ and channels $h_{k_j n}^2/\mu_{k_j}$, and using (3.60), the vector of optimal powers allocated to the users in the subset $S_J(n)$ in the n^{th} frequency bin, and the corresponding optimal user rates

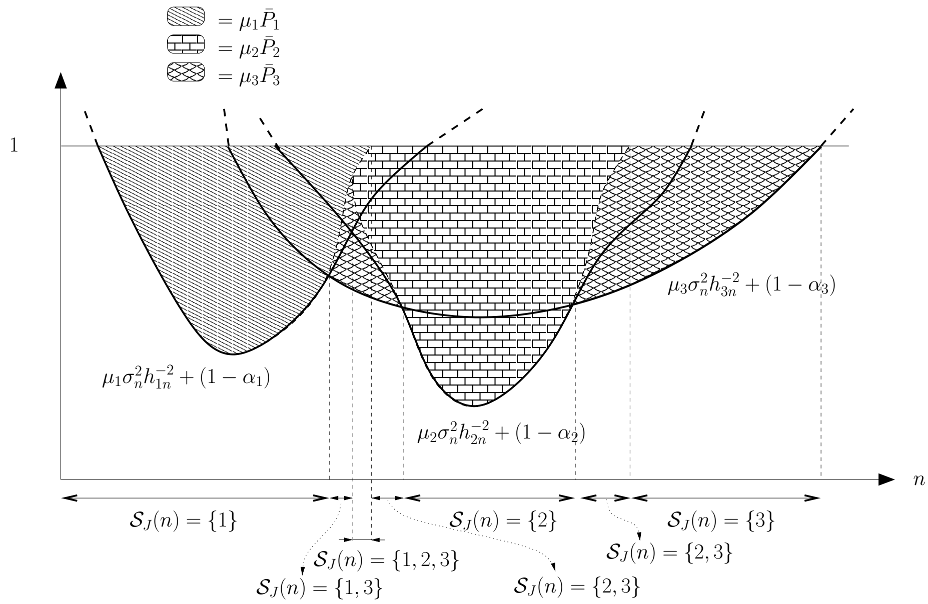

Figure 3.8: *Multiuser water-filling diagram*

are given explicitly as :

$$
\begin{pmatrix} P^{\circ}_{k_1 n} \\ P^{\circ}_{k_2 n} \\ \vdots \\ P^{\circ}_{k_j n} \\ \vdots \\ P^{\circ}_{k_{J-1} n} \\ P^{\circ}_{k_J n} \end{pmatrix} = \begin{bmatrix}
h^{-2}_{k_1 n}\left(\frac{\alpha_{k_1}}{\mu_{k_1} h^{-2}_{k_1 n}} - \frac{\alpha_{k_2}-\alpha_{k_1}}{\mu_{k_2} h^{-2}_{k_2 n} - \mu_{k_1} h^{-2}_{k_1 n}}\right) \\
h^{-2}_{k_2 n}\left(\frac{\alpha_{k_2}-\alpha_{k_1}}{\mu_{k_2} h^{-2}_{k_2 n} - \mu_{k_1} h^{-2}_{k_1 n}} - \frac{\alpha_{k_3}-\alpha_{k_2}}{\mu_{k_3} h^{-2}_{k_3 n} - \mu_{k_2} h^{-2}_{k_2 n}}\right) \\
\vdots \\
h^{-2}_{k_j n}\left(\frac{\alpha_{k_j}-\alpha_{k_{j-1}}}{\mu_{k_j} h^{-2}_{k_j n} - \mu_{k_{j-1}} h^{-2}_{k_{j-1} n}} - \frac{\alpha_{k_{j+1}}-\alpha_{k_j}}{\mu_{k_{j+1}} h^{-2}_{k_{j+1} n} - \mu_{k_j} h^{-2}_{k_j n}}\right) \\
\vdots \\
h^{-2}_{k_{J-1} n}\left(\frac{\alpha_{k_{J-1}}-\alpha_{k_J}}{\mu_{k_{J-1}} h^{-2}_{k_{J-1} n} - \mu_{k_J} h^{-2}_{k_J n}} - \frac{\alpha_{k_J}-\alpha_{k_{J-1}}}{\mu_{k_J} h^{-2}_{k_J n} - \mu_{k_{J-1}} h^{-2}_{k_{J-1} n}}\right) \\
h^{-2}_{k_J n}\left(\frac{\alpha_{k_J}-\alpha_{k_{J-1}}}{\mu_{k_J} h^{-2}_{k_J n} - \mu_{k_{J-1}} h^{-2}_{k_{J-1} n}} - \sigma^2_n\right)
\end{bmatrix} \tag{3.61}
$$

and

$$
\begin{pmatrix} R^{*}_{k_1 n} \\ R^{*}_{k_2 n} \\ \vdots \\ R^{*}_{k_j n} \\ \vdots \\ R^{*}_{k_{J-1} n} \\ R^{*}_{k_J n} \end{pmatrix} = \frac{B}{N}\log \begin{bmatrix}
\frac{\alpha_{k_1}}{\mu_{k_1} h^{-2}_{k_1 n}} \Big/ \frac{\alpha_{k_2}-\alpha_{k_1}}{\mu_{k_2} h^{-2}_{k_2 n} - \mu_{k_1} h^{-2}_{k_1 n}} \\
\frac{\alpha_{k_2}-\alpha_{k_1}}{\mu_{k_2} h^{-2}_{k_2 n} - \mu_{k_1} h^{-2}_{k_1 n}} \Big/ \frac{\alpha_{k_3}-\alpha_{k_2}}{\mu_{k_3} h^{-2}_{k_3 n} - \mu_{k_2} h^{-2}_{k_2 n}} \\
\vdots \\
\frac{\alpha_{k_j}-\alpha_{k_{j-1}}}{\mu_{k_j} h^{-2}_{k_j n} - \mu_{k_{j-1}} h^{-2}_{k_{j-1} n}} \Big/ \frac{\alpha_{k_{j+1}}-\alpha_{k_j}}{\mu_{k_{j+1}} h^{-2}_{k_{j+1} n} - \mu_{k_j} h^{-2}_{k_j n}} \\
\vdots \\
\frac{\alpha_{k_{J-1}}-\alpha_{k_J}}{\mu_{k_{J-1}} h^{-2}_{k_{J-1} n} - \mu_{k_J} h^{-2}_{k_J n}} \Big/ \frac{\alpha_{k_J}-\alpha_{k_{J-1}}}{\mu_{k_J} h^{-2}_{k_J n} - \mu_{k_{J-1}} h^{-2}_{k_{J-1} n}} \\
\frac{\alpha_{k_J}-\alpha_{k_{J-1}}}{\mu_{k_J} h^{-2}_{k_J n} - \mu_{k_{J-1}} h^{-2}_{k_{J-1} n}} \Big/ \sigma^2_n
\end{bmatrix}. \tag{3.62}
$$

Starting from an initial set of coefficients $\boldsymbol{\mu}^{(0)}$ the proposed updating strategy is given by :

$$
\begin{aligned}
\boldsymbol{\Delta}^{(i)}_{\mu} &= -\left[\sum_{n=1}^{N}\left(\frac{\partial \boldsymbol{P}^{\circ}_n}{\partial \boldsymbol{\mu}}\right)_{\boldsymbol{\mu}=\boldsymbol{\mu}^{(i)}}\right]^{-1}\left[\left(\sum_{n=1}^{N}\boldsymbol{P}^{\circ}_n\left(\boldsymbol{\mu}^{(i)}\right)\right) - \bar{\boldsymbol{P}}\right] && (3.63)\\
\boldsymbol{\mu}^{(i+1)} &= \boldsymbol{\mu}^{(i)} + \lambda_i \boldsymbol{\Delta}^{(i)}_{\mu}. && (3.64)
\end{aligned}
$$

From (3.61), each Jacobian matrix $\frac{\partial \boldsymbol{P}^{\circ}_n}{\partial \boldsymbol{\mu}}$ can be computed explicitly : the $J(n) \times J(n)$ submatrix corresponding to indexes in $S_J(n)$ is a tridiagonal matrix, and the remaining entries are all zero. The main difficulty with this algorithm is to find a starting point $\boldsymbol{\mu}^{(0)}$ such that the Jacobian matrix sum is invertible, and to keep this matrix invertible at each iteration. In particular, the total power allocated to each user has to be non-zero for each $\boldsymbol{\mu}^{(i)}$. The size of the updating step (coefficient λ_i) should be adapted accordingly.

3.5.5 Maximum balanced rates

Algorithm 3.2 can be extended to compute the maximum balanced rates of frequency selective multiuser channels. Let us analyze in more details the three different scenarios.

3.5.5.1 PSD-sum constraint

The Jacobian matrix $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}$ is now the sum of Jacobian matrices corresponding to each frequency bin. For each term, the $J(n) \times J(n)$ submatrix corresponding to the users active on the frequency bin n is a symmetric tridiagonal matrix $\mathbf{T}_n(\boldsymbol{\alpha})$ with non-zero entries defined in (3.51) :

$$\frac{N}{B}\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}} = \sum_{n=1}^{N} \mathbf{T}_n(\boldsymbol{\alpha}). \tag{3.65}$$

The resulting Jacobian matrix is generally not tridiagonal. The sum should be invertible to allow the computation of the update direction, which requires that each user gets some power in at least one frequency bin.

3.5.5.2 Power-sum constraint

The total power to be allocated in each frequency bin $\bar{P}_n$ is now dependent on the user relative priorities. From (3.26), it follows that a correction term $\mathbf{Q}_n(\boldsymbol{\alpha})$ should be included in the expression of $\frac{\partial \mathbf{R}_n}{\partial \boldsymbol{\alpha}}$:

$$\frac{N}{B}\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}} = \sum_{n=1}^{N} \left[\mathbf{T}_n(\boldsymbol{\alpha}) + \mathbf{Q}_n(\boldsymbol{\alpha})\right]. \tag{3.66}$$

Let us define N_k as the number of frequency bins n such that $k_J(n) = k$ (i.e. the last user in the allocation process (algorithm 3.1) is user k), with $\sum_k N_k = N$. Combining the water-filling solution (3.57) with the power-sum constraint, the marginal rate μ and the total power allocation $\bar{P}_n$ can be expressed as :

$$\mu = \frac{\sum_{k=1}^{K} \alpha_k N_k}{\bar{P} + \sigma^2 \sum_{n=1}^{N} h_{k_J(n)}^{-2}} \quad (3.67)$$

$$\bar{P}_n = \frac{\alpha_{k_J(n)}}{\sum_{k=1}^{K} \alpha_k N_k} \left(\bar{P} + \sigma^2 \sum_{l=1}^{N} h_{k_J(l)}^{-2} \right) - \sigma^2 h_{k_J(n)}^{-2} \quad (3.68)$$

Using (3.26) and (3.68), the correction term $\mathbf{Q}_n(\boldsymbol{\alpha})$ has the following entries on its k_J^{th} row :

$$\begin{aligned} \left[\mathbf{Q}_n(\boldsymbol{\alpha})\right]_{k_J(n),k} &= \left(\frac{h_{k_J n}^2/\sigma^2}{1 + P_n h_{k_J n}^2/\sigma^2} \right) \frac{\partial P_n}{\partial \alpha_k} \quad (3.69) \\ &= \frac{1}{\alpha_{k_J(n)}} \left(\delta_{k_J(n),k} - \frac{\alpha_{k_J(n)} N_k}{\sum_{l=1}^{K} \alpha_l N_l} \right). \end{aligned}$$

The $K-1$ other rows of $\mathbf{Q}_n(\boldsymbol{\alpha})$ are all zero. The algorithm for the computation of the maximum balanced rates is then identical to the original algorithm, except that a new water level μ and a new spectral allocation $\{\bar{P}_n\}$ have to be computed by (3.57) for each new set of priority coefficients $\boldsymbol{\alpha}^{(i+1)}$.

3.5.5.3 Individual power constraint (MA channel only)

The computation of maximum balanced rates is much more complex in this case as the optimal vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\mu}$ have to be found, that satisfy simultaneously the K balanced rates equations and the K individual power constraints :

$$\boldsymbol{R}^*(\boldsymbol{\alpha}, \boldsymbol{\mu}) = \sum_{n=1}^{N} \boldsymbol{R}_n^*(\boldsymbol{\alpha}, \boldsymbol{\mu}) = \left(\frac{g}{K}\right) \boldsymbol{R}^{\mathbf{1}} \quad (3.70)$$

$$\boldsymbol{P}^\circ(\boldsymbol{\alpha}, \boldsymbol{\mu}) = \sum_{n=1}^{N} \boldsymbol{P}_n^\circ(\boldsymbol{\alpha}, \boldsymbol{\mu}) = \bar{\boldsymbol{P}} \quad (3.71)$$

where the non-zero elements of $\boldsymbol{P}_n^\circ$ and $\boldsymbol{R}_n^*$ are given by equations (3.61) and (3.62).

We propose the following algorithm, that updates alternately the α_k's (to obtain balanced rates) and the μ_k's (to obtain balanced powers) until both sets of constraints are satisfied.

Algorithm 3.3 *Maximum balanced rates with individual power constraints (exact solution)*

- *Set* $i = 0$, $\boldsymbol{\mu}^{*(0)} = [1, \cdots, 1]^T$ *and* $\boldsymbol{\alpha}^{*(0)} = [1, \cdots, 1]^T / K$
- *While both sets of constraints (3.70) and (3.71) are not satisfied simultaneously :*
  - ***Rates balancing step***
    - *Consider the problem of allocating an equivalent power-sum* $\bar{P}^{\mu} = \sum_{k=1}^{K} \mu_k^{*(i)} \bar{P}_k$ *to the* K *users with equivalent channels* $h_{kn}^2 / \mu_k^{*(i)}$.
    - *Compute the set* $\left\{R_k^1\right\}$ *of SU rates for the current equivalent power* $\bar{P}^{\mu}$ *and the current equivalent channels.*
    - *Use algorithm 3.2 to compute* $\boldsymbol{\alpha}^{*(i+1)}$ *that satisfies the balanced rates constraints (3.70).*
    - *Start with* $\boldsymbol{\alpha}^{(0)} = [1/R_1^1, \cdots, 1/R_K^1]^T / \sum_k (1/R_k^1)$.
    - *Compute the scalar parameter* $\mu^{(i+1)}$ *from the multiuser water-filling equation (3.57) for the vector* $\boldsymbol{\alpha}^{*(i+1)}$ *obtained at convergence.*
    - *The pair* $\left[\boldsymbol{\alpha}^{*(i+1)}, \mu^{(i+1)} . \boldsymbol{\mu}^{*(i)}\right]$ *now satisfies the balanced rates constraints. But the individual power constraints are generally not satisfied : the power allocation is unbalanced.*
  - ***Powers balancing step***
    - *Compute the vector* $\boldsymbol{\mu}^{*(i+1)}$ *that satisfies the power constraints (3.71) with the updating equations (3.63) and (3.64).*
    - *Start with* $\boldsymbol{\mu}^{(0)} = \mu^{(i+1)} \boldsymbol{\mu}^{*(i)}$. *The corresponding optimal power allocation is known to provide balanced rates : each user gets some power and the sum of Jacobian matrices in (3.63) should be invertible.*
    - *The pair* $\left[\boldsymbol{\alpha}^{*(i+1)}, \boldsymbol{\mu}^{*(i+1)}\right]$ *now satisfies the individual power constraints, but the updated rates are generally unbalanced.*
  - *Set* $i = i + 1$

The complexity of the last algorithm is very high as two levels of iterations are involved. Its main utility lies in the fact that it provides the *exact solution*, which is helpful in the evaluation of alternative algorithms providing suboptimal solutions.

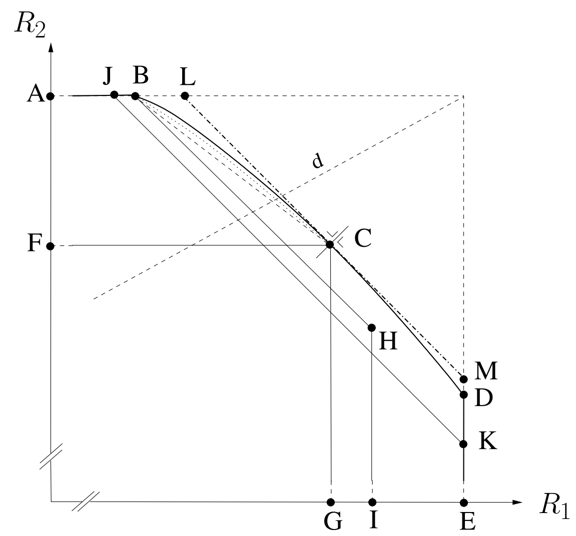

Figure 3.9: *Two-user capacity region and IMWF algorithm*

The alternative method we propose is based on the Iterative Multiuser Water-Filling algorithm (IMWF)[44]. The IMWF algorithm is used to compute the maximum sum-rate of a multiuser frequency selective channel, with individual power constraints. It is based on the observation that each power allocation $\boldsymbol{P}_k$ from the set of optimal power allocations $[\boldsymbol{P}_1, \cdots, \boldsymbol{P}_K]$ maximizing the rate-sum is obtained by single-user water-filling (SUWF) with an equivalent noise $\sigma_n'^2 = \sigma_n^2 + \sum_{k' \neq k} P_{k'n} h_{k'n}^2$. In other words, the set of K optimal power allocations should satisfy the *simultaneous water-filling* condition. The IMWF starts from arbitrary power allocations $\boldsymbol{P}_k$ and updates them successively by applying the SUWF algorithm with the $K - 1$ other allocations fixed. The obtained solution can be shown to converge quickly to the simultaneous WF solution.

Figure 3.9 gives a two-user capacity region with individual power constraints (curve ABCDE). Line d is the locus of balanced rates pairs. The IMWF algorithm can be used to obtain the power allocation corresponding to point C ($\alpha_1 = \alpha_2 = \frac{1}{2}$). As this allocation is FDMA, the associated capacity region is the FCG rectangle. The rates obtained with this solution are unbalanced : a higher priority should be given to user 2 ($\alpha_2 > \alpha_1$), which involves that the decoding order would be $(1, 2)$. When the highest priority is given to user 2 ($\alpha_1 = 0, \alpha_2 = 1$), the solution consists of *successive water-filling* : $\boldsymbol{P}_1$ is first computed with the SUWF algorithm ignoring the presence of user 2, $\boldsymbol{P}_2$ is then computed with the same algorithm and an equivalent noise including the effect of user 1 signal. The capacity region associated with the successive WF algorithm is the ABHI pentagon, and B represents the useful rate pair. Obviously, the maximum balanced rates pair ($BC \cap d$) results from a compromise between the simultaneous WF solution and the successive WF solution. A lower bound for the BC curve can be found by considering the allocation $\boldsymbol{P}_1(x) = x\boldsymbol{P}_1^B + (1-x)\boldsymbol{P}_1^C$ for user 1, and the SUWF allocation $\boldsymbol{P}_2(x)$ for user 2, with noise including the effect of $\boldsymbol{P}_1(x)$. The rate pairs obtained with this allocation policy are given by the dotted line BC. For some x in the range $[0,1]$, balanced rates are obtained that are very close to the maximum balanced rates. A comparison with the rates obtained by algorithm 3.3 confirmed that the proposed solution is a tight lower bound for the optimal solution. Simple and useful lower and upper bounds (g_-, g_+) on the multiuser gain g can also be obtained as follows. The ALCME pentagon is obviously an upper bound for the convex capacity region. The corresponding balanced rates pair ($LC \cap d$) is $\frac{g_+}{2}(R_1^1, R_2^1)$. The AJKE pentagon is the capacity region obtained when the SUWF algorithm is applied separately for every user. This pentagon appears to be a lower bound for the true capacity region. The corresponding balanced rates pair ($JK \cap d$) is $\frac{g_-}{2}(R_1^1, R_2^1)$.

Generalizing these ideas to multiuser channels with an arbitrary number of users K is not easy as the optimal decoding order ($K!$ different combinations) is a priori unknown. The next algorithm is based on the idea that a higher priority should be given to the users to whom the IMWF algorithm is unfair.

Algorithm 3.4 *Maximum balanced rates with individual power constraints (approximate solution)*

- *Apply the SUWF and IMWF algorithms on the set* $\mathcal{S}_K = \{1, \cdots, K\}$. *Compute the corresponding power allocations, denoted by* $\left\{\boldsymbol{P}_k^1\right\}$ *and* $\left\{\boldsymbol{P}_k^{\mathcal{S}_K}\right\}$, *respectively. Compute the associated user rates, denoted by* $\left\{R_k^1\right\}$ *and* $\left\{R_k^{\mathcal{S}_K}\right\}$, *respectively.*
- *Compute initial bounds for the multiuser gain* g_- *and* g_+ *:*

$$\frac{g_-}{K} = \frac{\sum_n \mathcal{C}\left(\sum_k P_{kn}^1 h_{kn}^2/\sigma_n^2\right)}{\sum_k R_k^1} \tag{3.72}$$

$$\frac{g_+}{K} = \frac{\sum_k R_k^{\mathcal{S}_K}}{\sum_k R_k^1} \tag{3.73}$$

- *Set* $k_K = \arg\min_{k \in \mathcal{S}_K} R_k^{\mathcal{S}_K} / R_k^1$ *: the highest priority is given to the user whose normalized rate would be the lowest if the same priority was given to all users.*

- *While the balanced rates constraints are not satisfied :*

  – *Set* $i = K$ *and* $g^* = \frac{1}{2}(g_- + g_+)$.

  – *Set* $\boldsymbol{P}_{k_K}(x_K) = x_K \boldsymbol{P}^1_{k_K} + (1 - x_K)\boldsymbol{P}^{\mathcal{S}_K}_{k_K}$. *Compute* x_K *such that :*

$$R_{k_K} = \sum_n \mathcal{C}\left(\frac{P_{k_K n}(x_K) h^2_{k_K n}}{\sigma_n^2}\right) = \frac{g^*}{K} R^1_{k_K} \tag{3.74}$$

  x_K *can be found by a simple binary search in the interval* $[0, 1]$.

  – *For* $i = K - 1$ *down to* 2 *:*

    ∗ *Set* $\mathcal{S}_i = \mathcal{S}_{i+1} \setminus \{k_{i+1}\}$

    ∗ *Compute the equivalent noise* $\boldsymbol{I}_i$ *including the additive noise* $\boldsymbol{\sigma}^2$ *and the interference from higher-priority users :* $\boldsymbol{I}_i = \boldsymbol{\sigma}^2 + \sum_{j=i+1}^{K} \boldsymbol{P}^T_{k_j} \boldsymbol{h}^2_{k_j}$.

    ∗ *Apply the IMWF algorithm on the set of remaining users* $\mathcal{S}_i$*, with an equivalent noise power* $\boldsymbol{I}_i$*. The obtained power allocations and the associated user rates are denoted* $\left\{\boldsymbol{P}^{\mathcal{S}_i}_k(\boldsymbol{I}_i)\right\}_{k \in \mathcal{S}_i}$ *and* $\left\{R^{\mathcal{S}_i}_k(\boldsymbol{I}_i)\right\}_{k \in \mathcal{S}_i}$*. The last IMWF allocation* $\left\{\boldsymbol{P}^{\mathcal{S}_{i+1}}_k(\boldsymbol{I}_{i+1})\right\}_{k \in \mathcal{S}_i}$ *can be updated to make the computation faster.*

    ∗ *Set* $k_i = \arg\min_{k \in \mathcal{S}_i} R^{\mathcal{S}_i}_k(\boldsymbol{I}_i)/R^1_k$ *: the next higher priority is given to the user in* $\mathcal{S}_i$ *whose normalized rate would be the lowest if the same priority was given to all users in* $\mathcal{S}_i$.

    ∗ *Apply the SUWF algorithm on user* k_i*, with an equivalent noise power* $\boldsymbol{I}_i$*. The obtained power allocation is* $\boldsymbol{P}^1_{k_i}(\boldsymbol{I}_i)$ *and the associated rate is* $R^1_{k_i}(\boldsymbol{I}_i)$.

    ∗ *If* $R^{\mathcal{S}_i}_{k_i}(\boldsymbol{I}_i) > \frac{g^*}{K} R^1_{k_i}$*, set* $g_- = g^*$*, return to* $i = K$.

    ∗ *Else, if* $R^1_{k_i}(\boldsymbol{I}_i) < \frac{g^*}{K} R^1_{k_i}$*, set* $g_+ = g^*$*, return to* $i = K$.

    ∗ *Else, set* $\boldsymbol{P}_{k_i}(x_i) = x_i \boldsymbol{P}^1_{k_i}(\boldsymbol{I}_i) + (1 - x_i)\boldsymbol{P}^{\mathcal{S}_i}_{k_i}(\boldsymbol{I}_i)$. *Compute* x_i *such that :*

$$R_{k_i} = \sum_n \mathcal{C}\left(\frac{P_{k_i n}(x_i) h^2_{k_i n}}{I_{in}}\right) = \frac{g^*}{K} R^1_{k_i} \tag{3.75}$$

x_i can be found by a simple binary search in the interval $[0, 1]$.

- *Set* $k_1 = \mathcal{S}_2 \setminus \{k_2\}$.
- *Compute the equivalent noise* $\boldsymbol{I}_1 = \boldsymbol{\sigma}^2 + \sum_{j=2}^{K} \boldsymbol{P}_{k_j}^T \boldsymbol{h}_{k_j}^2$
- *Compute* $\boldsymbol{P}_{k_1}$ *by applying the SUWF algorithm on user* k_1*, with an equivalent noise power* $\boldsymbol{I}_1$*. The associated rate is* R_{k_1}.
- *If* $R_{k_1} > \frac{g^*}{K} R_{k_1}^1$ *: set* $g_- = g^*$*; else set* $g_+ = g^*$

In Section 3.7, the power allocation obtained with this suboptimal algorithm, and the corresponding balanced rates, are compared to the optimal solution provided by algorithm 3.3.

3.6 FDM(A) capacity

The purpose of this section is to investigate methods for the computation of maximum balanced rates with an FDM(A) constraint. The key to solving this class of problems is to find the optimal partition of the frequency spectrum among the different users. Once the frequency band assignment is fixed, the optimal spectra for each user are determined separately, depending on the type of power constraint. The three types of power constraints are considered again in the sequel. In the FDM(A) scenario, the uplink and downlink capacity regions are identical as successive decoding is not necessary any more. The results presented in this section are thus valid as well for the broadcast channel and for the multiple access channel.

3.6.1 PSD-sum constraint

With a constraint $\bar{\gamma}$ on the transmitted PSD, the powers P_{kn} assigned to the different users in every subchannel n can be written as :

$$P_{kn} = \rho_{kn} \cdot \bar{\gamma}\, \mathrm{d}f \qquad \text{with} \left\{ \begin{array}{ccc} \rho_{kn} & = & 0 \text{ or } 1 \\ \rho_{kn}\rho_{k'n} & = & 0 \quad \forall\, k \neq k' \end{array} \right. \tag{3.76}$$

The aggregate rate R_α is then given by

$$R_\alpha = \sum_{n=1}^{N} \sum_{k=1}^{K} \rho_{kn} \, \left(\alpha_k R_{kn}\right), \qquad \text{with } R_{kn} \triangleq \mathcal{C}\left(\frac{\bar{\gamma}\mathrm{d}f}{\sigma_n^2}\right). \tag{3.77}$$

The maximization of the aggregate rate R_α for a given set of relative priorities $\{\alpha_k\}$ is trivial :

$$\rho_{k^*n} = 1, \qquad k^* = \arg\max_k (\alpha_k R_{kn}) \tag{3.78}$$

$$\rho_{kn} = 0, \qquad \forall\, k \neq k^* \tag{3.79}$$

The balanced rate constraint requires the computation of allocation coefficients $\{\rho_{kn}\}$ providing identical normalized rates $\frac{1}{\beta_k}\sum_n \rho_{kn} R_{kn}$ for every user, with $\beta_k = \sum_n R_{kn}$. A balanced rate solution generally does not exist as it is highly probable that none of the K^N possible allocations strictly satisfies the balanced rate constraint. The balanced rate constraint can be relaxed, according to a given tolerance. The computation of maximum quasi-balanced rates, however, is quite complex. This problem actually belongs to the class of integer programming problems, which usually require an exhaustive search.

One possible simplification is to approximate the problem by its continuous relaxation. The difficulty in solving an integer programming problem lies in the fact that the constraint set is a set of isolated points ($\rho_{kn} = 0$ or 1). The idea of continuous relaxation is to enlarge the constraint set to include all convex combinations of the original points, thus generating a convex constraint set. Instead of forcing the assignment variables ρ_{kn} to be either 0 or 1, the constraint can be relaxed to :

$$0 \leq \rho_{kn} \leq 1, \qquad \sum_{k=1}^{K} \rho_{kn} = 1. \tag{3.80}$$

The interpretation of equation (3.80) is that each subchannel n of width $\mathrm{d}f$ can be subdivided arbitrarily into K narrower subchannels. For this reason, the continuous coefficients ρ_{kn} are called 'sharing coefficients' in the sequel. As the optimum frequency allocation usually involves only a few frequency bands, most of the sharing coefficients corresponding to the solution are 0 or 1. Only the few subchannels including the boundaries of those frequency bands have to be shared by two or more users. With the proposed relaxation, the subchannel allocation rule for a given set of relative priorities becomes :

Algorithm 3.5 *Subchannel allocation rule for a given set* $\{\alpha_k\}$

- $\forall\, n \in \{1, \cdots, N\}$, *do :*
  - *Compute the maximum weighted rate* $t_n \triangleq \max_k (\alpha_k R_{kn})$

– *Separate the users in two disjoint sets* $\mathcal{V}_n$ *and* $\bar{\mathcal{V}}_n$ *:*

$$\mathcal{V}_n = \{k \mid \alpha_k R_{kn} = t_n\} \qquad \bar{\mathcal{V}}_n = \{k \mid \alpha_k R_{kn} < t_n\}$$

– *Set the sharing coefficients* ρ_{kn} *as follows :*

$$\rho_{kn} = 0 \quad \forall\, k \in \bar{\mathcal{V}}_n, \qquad \sum_{k \in \mathcal{V}_n} \rho_{kn} = 1$$

- *If some of the subsets* $\mathcal{V}_n$ *contain more than one element, a number of positive coefficients* ρ_{kn} *are still undetermined. They are fixed by the introduction of the following additional relations :*

$$\frac{R_{k_1}}{\beta_{k_1}} = \frac{R_{k_2}}{\beta_{k_2}} \quad \forall\, k_1, k_2 \in \mathcal{U}^+, \qquad R_k = \sum_n \rho_{kn} R_{kn}$$

where $\mathcal{U}^+$ *is a well-chosen subset of users.*

The derivation of maximum balanced rates requires the computation of an appropriate set of coefficients $\{\alpha_k\}$ such that the allocation rule given by algorithm 3.5 provides only *positive sharing coefficients* ρ_{kn}, with $\mathcal{U}^+ = \{1, \cdots, K\}$. Let $\mathcal{U}^-$ and $\mathcal{U}^+$ be two user sets with low and high normalized rates $\frac{R_k}{\beta_k}$, respectively. Starting from initial priorities $\alpha_k^{(0)}$, the relative priorities of the users in $\mathcal{U}^-$ are progressively increased, allowing the transfer of selected subchannels from $\mathcal{U}^+$ to $\mathcal{U}^-$, until the balanced rates constraints are all satisfied. The next algorithm specifies how these transfers should be handled to reach the final solution after a finite number of steps.

Algorithm 3.6 *Optimal subchannel assignment with balanced rates constraints*

- *Set* $i = 0$, *start with an arbitrary set of priority coefficients* $\{\alpha_k^{(0)}\}$
- *Compute the initial sharing coefficients* $\left\{\rho_{kn}^{(0)}\right\}$ *according to algorithm 3.5, with* $\mathcal{U}^+ = \emptyset$. *The initial sharing coefficients are all 0 or 1. The maximum weighted rates for each subchannel are* $\left\{t_n^{(0)}\right\}$.
- *Compute the total user rates* $\left\{R_k^{(0)}\right\}$ *and the maximum normalized user rate* $\overline{R}_{max}^{(0)}$ *:*

$$\left\{R_k^{(0)}\right\} = \sum_n \rho_{kn}^{(0)} R_{kn} \qquad \overline{R}_{max}^{(0)} = \max_k \left(\frac{R_k^{(0)}}{\beta_k}\right)$$

- *Separate the users in two disjoint sets* $\mathcal{U}^+$ *and* $\mathcal{U}^-$ *:*

$$\mathcal{U}^+ = \left\{ k \mid \frac{R_k^{(0)}}{\beta_k} = \overline{R}_{max}^{(0)} \right\} \qquad \mathcal{U}^- = \left\{ k \mid \frac{R_k^{(0)}}{\beta_k} < \overline{R}_{max}^{(0)} \right\}$$

- *While* $\mathcal{U}^+ \neq \{1, \cdots, K\}$*, do :*
  - *Subdivide the user set* $\mathcal{U}^+$ *in two disjoint subsets* $\mathcal{U}_1^+$ *and* $\mathcal{U}_2^+$ *where :*
    * $\mathcal{U}_1^+$ *is the larger subset of* $\mathcal{U}^+$ *whose members share only subchannels with other members of* $\mathcal{U}_1^+$*.*
    * *The members of* $\mathcal{U}_2^+$ *share one or more subchannels with a member of* $\mathcal{U}^-$ *or with other members of* $\mathcal{U}_2^+$*.*
  - *Define* $\mathcal{U} = \mathcal{U}^- \bigcup \mathcal{U}_2^+$ *and* $\mathcal{S}$ *as the set of subchannels allocated to the users in* $\mathcal{U}_1^+$*, and compute* $\pi^{(i)}$ *as :*

$$\pi^{(i)} = \min_{(n \in \mathcal{S}),\, (k \in \mathcal{U})} \left(\frac{t_n^{(i)}}{\alpha_k^{(i)} R_{kn}} \right) > 1$$

    where the minimum is found for user k^* *and subchannel* n^**.*
  - *Update the priority coefficients as follows :*

$$\alpha_k^{(i+1)} = \pi^{(i)}\, \alpha_k^{(i)}\, (k \in \mathcal{U}) \qquad \alpha_k^{(i+1)} = \alpha_k^{(i)}\, (k \in \mathcal{U}_1^+)$$

  - *Increase* $\rho_{k^*n^*}$ *as much as possible, while adapting the sharing coefficients* ρ_{kn} *of the other subchannels to maintain the equality of the normalized user rates in* $\mathcal{U}^+$*, until one of the following events happen :*

    case 1 $\rho_{k^*n^*} = 1$*. In that case, subchannel* n^* *can be fully transferred to user* k^**. The remaining sharing coefficients just need to be updated to maintain equal normalized rates* $\frac{R_k}{\beta_k}$ *for the users in* $\mathcal{U}^+$*.*

    case 2 *One of the other non-zero sharing coefficients* ρ_{kn} *is decreased down to zero. A further increase of* $\rho_{k^*n^*}$ *would then break the equality of the normalized rates* $\frac{R_k}{\beta_k}$ *for the users in* $\mathcal{U}^+$*. The members of* $\mathcal{U}^+$ *sharing subchannel* n^* *with user* k^* *are placed in the subset* $\mathcal{U}_2^+$*.*

    case 3 *The normalized rate* $\frac{R_{k^*}}{\beta_k}$ *reaches the same value as that of the users in* $\mathcal{U}^+$*. User* k^* *is then incorporated into the set* $\mathcal{U}^+$*.*

– *Update* $\left\{\rho_{kn}^{(i+1)}\right\}$, $\left\{t_n^{(i+1)}\right\}$, $\left\{R_k^{(i+1)}\right\}$, $\overline{R}_{max}^{(i+1)}$, *and set* $i = i + 1$.

An example is illustrated in Figure 3.10 to clarify the mechanisms associated with algorithm 3.6. A set of $N = 8$ subchannels should be allocated to $K = 4$ users. The rates R_{kn} are given in the first matrix. The normalization coefficients β_k are all set to 1, which corresponds to a common rate constraint. The initial priority coefficients $\alpha_k^{(0)}$ are chosen arbitrarily. With these priority coefficients, the allocation rule (algorithm 3.5) assigns each subchannel to a single user : the sharing coefficients $\rho_{kn}^{(0)}$ are all 0 or 1. The total rates $R_k^{(0)}$ are all different, with a maximum for user 2. The user subsets are thus $\mathcal{U}^+ = \mathcal{U}_1^+ = \{2\}$ and $\mathcal{U}^- = \{1, 3, 4\}$. In step 1, the coefficients α_1, α_3 and α_4 are increased simultaneously until $\alpha_1^{(1)} R_{15} = \alpha_2^{(0)} R_{25}$. The allocation rule allows then the transfer of subchannel 5 from user 2 to user 1. The transfer is full here ('case 1' in algorithm 3.6), i.e. ρ_{25} goes from 1 to 0 and ρ_{15} goes from 0 to 1. The new total rates $R_k^{(1)}$ are again all different, with a maximum for user 2. The user subsets are unchanged. In step 2, the same coefficients α_1, α_3 and α_4 are increased together until $\alpha_1^{(2)} R_{17} = \alpha_2^{(1)} R_{27}$. Now the transfer of subchannel 7 from user 2 to user 1 is partial : with sharing coefficients $\rho_{17}^{(2)} = 0.13$ and $\rho_{27}^{(2)} = 0.87$, the updated total rates $R_1^{(2)}$ and $R_2^{(2)}$ are found to be equal ('case 3' in algorithm 3.6). The user subsets are now $\mathcal{U}^+ = \mathcal{U}_1^+ = \{1, 2\}$ and $\mathcal{U}^- = \{3, 4\}$. In step 3, coefficients α_3 and α_4 are increased until $\alpha_4^{(3)} R_{44} = \alpha_1^{(2)} R_{14}$. Subchannel 4 is then partially transferred from user 1 to user 4 while the sharing coefficients associated with subchannel 7 are updated to maintain equal total rates for users 1 and 2. The user subsets are then $\mathcal{U}^+ = \mathcal{U}_1^+ = \{1, 2, 4\}$ and $\mathcal{U}^- = \{3\}$. In step 4, the coefficient α_3 is increased until $\alpha_3^{(4)} R_{31} = \alpha_2^{(3)} R_{21}$. Subchannel 1 is then partially transferred from user 2 to user 3. The sharing coefficients associated with subchannels 4 and 7 are updated to maintain equal total rates for users 1, 2 and 4. For $\rho_{31} = 0.6$, the coefficient ρ_{17} is reduced to 0 and can not be reduced further ('case 2' in algorithm 3.6). Higher values of ρ_{31} would compromise the equality of the total rates R_k for users 1, 2 and 4. The user subsets become $\mathcal{U}_1^+ = \{1, 4\}$ and $\mathcal{U}_2^+ = \{2\}$. In step 5, coefficients α_2 and α_3 are increased until $\alpha_3^{(5)} R_{33} = \alpha_1^{(2)} R_{13}$. Subchannel 3 is then partially transferred from user 1 to user 3, until subchannel 4, that was previously shared by users 1 and 4, is fully transferred to user 1 in order to maintain the equality of the total rates for users 1, 2 and 4 ('case 2' again). The subsets become $\mathcal{U}_1^+ = \{4\}$ and $\mathcal{U}_2^+ = \{1, 2\}$. Finally, in step 6, coefficients

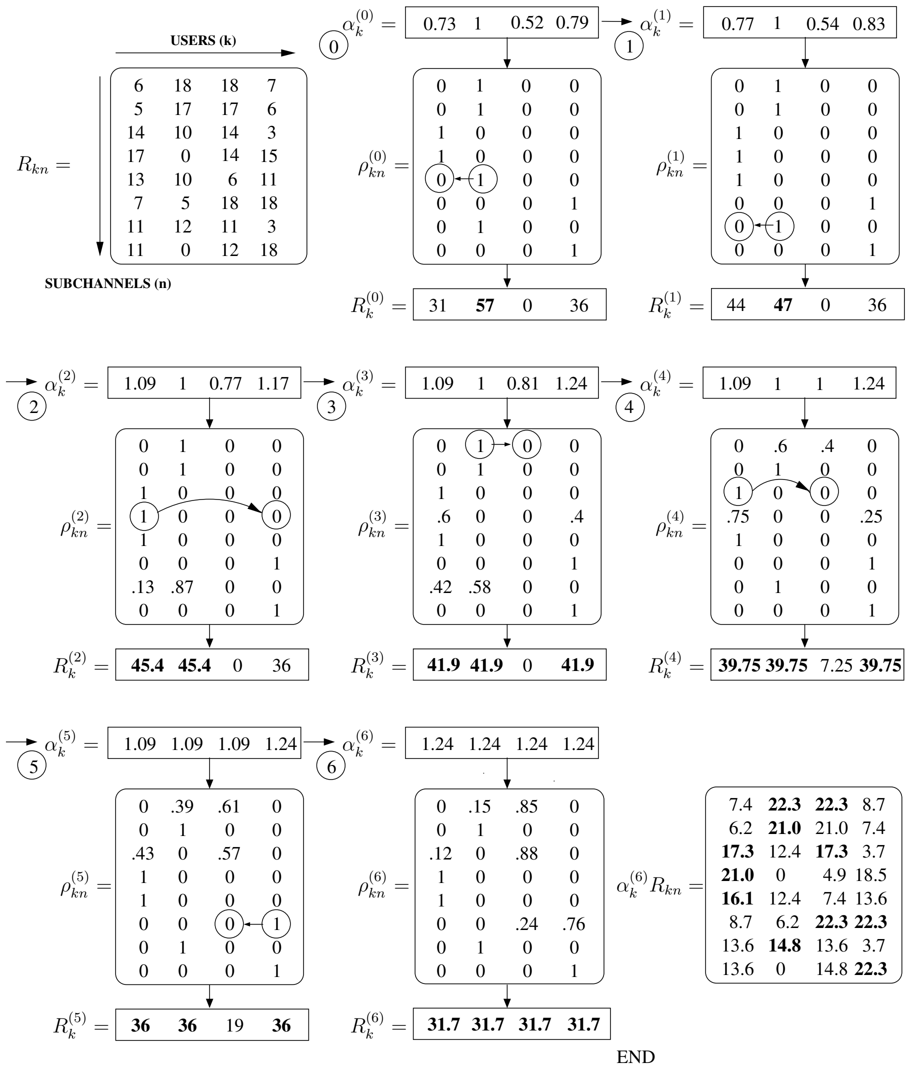

Figure 3.10: *Derivation of the optimal assignment of 8 subchannels to 4 users, with an equal rate constraint ($\beta_k = 1\,\forall\, k$), using algorithm 3.6*

α_1, α_2 and α_3 are increased until $\alpha_3^{(6)} R_{36} = \alpha_4^{(2)} R_{46}$. Subchannel 6 is then partially transferred to user 3. With well-chosen sharing coefficients for subchannels 1, 3 and 6, all total rates $R_k^{(6)}$ are now equal ('case 3'). User 3 is added to the set $\mathcal{U}^+$ and the algorithm is terminated. The last matrix gives the weighted rates $\alpha_k^{(6)} R_{kn}$ associated with the final solution.

The algorithm proposed here is actually a special case of the well-known *simplex algorithm* used in linear optimization. Its convergence properties are thus identical to those of the simplex algorithm. The simplex algorithm moves along the edges of the polyhedron defined by the constraints, from one vertex to another, while decreasing the value of the objective function at each step. The exact solution is obtained after a finite number of steps. The implementation proposed here is much more efficient than a general-purpose simplex algorithm as the specific structure of the problem (objective function and constraints) has been taken into account.

The number of iterations required to reach the exact solution can be prohibitive, however, if the total number of subchannels is very high ($N \gg 1$) and if the initial priority coefficients $\boldsymbol{\alpha}^{(0)}$ are far from the solution. For this reason, it may be desirable to use an alternative method to obtain an approximate solution $\hat{\boldsymbol{\alpha}}$ that will serve as initial step for algorithm 3.6. Actually, for a large N, the total rates R_k can be considered on a large scale as approximately continuous functions of the priority coefficients α_k. For each vector $\boldsymbol{\alpha}^{(i)}$, a Jacobian matrix $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}$ can be computed, that approximates the behavior of the total rate functions R_k in the neighbourhood of $\boldsymbol{\alpha}^{(i)}$. A Newton-Raphson updating algorithm, similar to algorithm 3.2, can then be used to obtain a new vector $\boldsymbol{\alpha}^{(i+1)}$ that should be closer to the desired solution. After a few iterations of the Newton-Raphson updating strategy, the obtained priority coefficients $\boldsymbol{\alpha}^{(i)}$ are close to the solution, but can not get closer because of the discontinuity of the allocation rule. Further updates of the priority coefficients are then generated by algorithm 3.6, which provides the exact solution with a small number of iterations. The complete procedure for the computation of maximum balanced rates is finally given as follows :

1. Initialization : set $\boldsymbol{\alpha}^{(0)} = [1/R_1^1, \cdots, 1/R_K^1]^T / \sum_k (1/R_k^1)$.

2. Rough solution : update the relative priorities according to

$$\boldsymbol{\alpha}^{(i+1)} = \boldsymbol{\alpha}^{(i)} - \lambda_i \left(\frac{\partial \mathbf{F}}{\partial \boldsymbol{\alpha}} \right)^{-1}_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^{(i)}} \mathbf{F}\left(\boldsymbol{\alpha}^{(i)}\right) \tag{3.81}$$

where the vector function $\mathbf{F}(\boldsymbol{\alpha})$ is defined in (3.46). Continue while convergence is observed. Set $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}^{(i+1)}$.

3. Final solution : apply algorithm 3.6 with $\boldsymbol{\alpha}^{(0)} = \hat{\boldsymbol{\alpha}}$.

The Jacobian matrix $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}$ can be obtained as follows. Let $\mathcal{S}_k$ be the set of subchannels allocated to user k by algorithm 3.5 and let $n_{k,k'}$ be the subchannel index defined by :

$$n_{k,k'} = \arg\min_{n \in \mathcal{S}_k} \frac{\alpha_k R_{kn}}{\alpha_{k'} R_{k'n}}. \tag{3.82}$$

In other words, increasing the relative priority of user k' with respect to that of user k will firstly prompt the transfer of subchannel $n_{k,k'}$ from user k to user k'. With this definition, the entries of the Jacobian matrix $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}$ can be shown to be :

$$\frac{\partial R_k}{\partial \alpha_k} \approx \frac{1}{2} \sum_{k \neq k'} \left[\frac{R_{kn_{k,k'}} + R_{kn_{k',k}}}{\alpha_{k'} \left(\frac{R_{k'n_{k,k'}}}{R_{kn_{k,k'}}} - \frac{R_{k'n_{k',k}}}{R_{kn_{k',k}}} \right)} \right] \tag{3.83}$$

$$\frac{\partial R_{k'}}{\partial \alpha_k} \approx -\frac{1}{2} \frac{R_{k'n_{k,k'}} + R_{k'n_{k',k}}}{\alpha_{k'} \left(\frac{R_{k'n_{k,k'}}}{R_{kn_{k,k'}}} - \frac{R_{k'n_{k',k}}}{R_{kn_{k',k}}} \right)} \quad \forall k' \neq k \tag{3.84}$$

3.6.2 Power-sum constraint

The authors of [53] propose an original method to compute the maximum common rate in multiple access channels with a constraint on the power sum $\bar{P}$. Individual power budgets $\{\bar{P}_k^{(0)}\}$ with $\sum_k \bar{P}_k = \bar{P}$ are fixed arbitrarily. The IMUWF algorithm [45] is then used to maximize the sum-rate $\sum_k R_k$ with the selected individual power constraints. The obtained power allocation is known to be of the FDMA type, i.e. power is allocated to the different users in separate frequency bands. The obtained user rates, however, can be very different. The total power $\bar{P}$ is then redistributed in parts $\{\bar{P}_k^{(i+1)}\}$ so as to enforce the same rate to all users, and the IMUWF algorithm is run again with the new set of individual power budgets. With an appropriate individual power updating strategy, the algorithm provides identical rates to all users after a few iterations. The extension of this method to the derivation of maximum balanced rates is trivial.

3.6.3 Individual power constraint

This problem can be solved easily once the optimal partition of the frequency spectrum among the different users is known : the optimal spectrum for each user is just the water-filling spectrum within the assigned band. Unfortunately, the derivation of the optimal frequency bands is extremely complex : K^N different partitions are possible and should be tested in turn. In [46], the maximization of the aggregate data rate R_α with both individual power constraints and an FDMA constraint is formulated as follows :

$$\{P_{kn}, \rho_{kn}\} = \arg \max_{P_{kn} \geq 0, \rho_{kn} \geq 0} \left[\sum_{k=1}^{K} \alpha_k \sum_{n=1}^{N} \rho_{kn} \log \left(1 + \frac{P_{kn} h_{kn}^2}{\rho_{kn} \sigma_n^2} \right) \right] \tag{3.85}$$

subject to

$$\sum_n P_{kn} = \bar{P}_k \quad \forall\, k \qquad\qquad \sum_k \rho_{kn} = 1 \quad \forall\, n \tag{3.86}$$

This formulation corresponds again to the continuous relaxation of the initial integer programming problem. As demonstrated in [46], the objective function in (3.85) is a concave function in (P_{kn}, ρ_{kn}). As the constraints in (3.86) are linear, this optimization problem is a standard convex programming problem and the global optimum can be obtained with a standard convex programming algorithm.

As usual, the derivation of maximum balanced rates requires the computation of relative priorities α_k such that the optimal power allocation in (3.85) offers balanced rates to the different users. This problem is extremely difficult and is not further developed in this work.

3.7 Applications to the powerline channel

3.7.1 Regular-pattern network with matched terminations

The case of a regular-pattern wireline access network with matched terminations is firstly considered. The basic network structure is given in Figure 2.11. The example considered here includes $\bar{K} = 20$ derivations, identical cable segments of length $d_k = d'_k = 15$ m, and ideal matched terminations. The lossy cables parameters are $\kappa = 2.3$, $\delta_d = 0.02$ and $\delta_{c1} = 85\sqrt{\text{Hz}}$ (see Chapter 2). The top of Figure 3.11 gives the channel frequency responses $|H_k(\omega)|^2$ in the bandwidth $[0, B]$ with $B = 10$ MHz. Through the combined effect of cable losses and multiple reflections on the cable derivations, the

Table 3.1: *Maximum balanced rates (in Mbits/s) for* $\mathcal{C}_4 = \{5, 10, 15, 20\}$

Constraint	User 5	User 10	User 15	User 20	%
Flat SU	127.41	59.82	13.12	4.05	100
PSD-sum	71.52	33.58	7.36	2.27	56
Flat FDMA	55.77	26.19	5.74	1.77	44
Flat TDMA	31.85	14.96	3.28	1.01	25
SU	127.41	59.85	16.04	7.06	100
Power-sum ($4\bar{P}$)	84.47	39.68	10.63	4.68	66
Individual power	77.94	36.61	9.81	4.32	61
Power-sum ($\bar{P}$)	65.99	31.00	8.31	3.66	52
TDMA	31.85	14.96	4.01	1.77	25

channel gains for the remote users can go below -100 dB at some frequencies. The noise level is chosen as -120 dBm/Hz.

Two kinds of power constraints are put on the SU transmission power : a maximum transmission PSD of $\overline{\gamma} = -60$ dBm/Hz ('flat PSD'), or a maximum transmission power of $\bar{P} = \overline{\gamma}\, B = 10$ mW ('water-filling PSD'). Figure 3.13 gives the evolution of the SU capacities R_k^1 for the two kinds of power constraints. The SU capacity decreases rapidly for the remote users. As expected, the water-filling solution provides a higher rate with respect to the flat PSD solution only for the users with a low received SNR, i.e. the most remote users. The SU water-filling PSD's (normalized with respect to $\overline{\gamma}$) are illustrated on the top of Figure 3.12 for users 5, 10, 15 and 20.

In a multiuser scenario, a subset $\mathcal{C}_K \subset \left\{1, \cdots, \bar{K}\right\}$ of $K \leq \bar{K}$ active users transmit their signals simultaneously. Table 3.1 gives the maximum balanced rates for $\mathcal{C}_4 = \{5, 10, 15, 20\}$ and different types of constraints, and compares them with the SU and TDMA rates. Figure 3.14 gives the multiuser gain g vs. K for various sets of active users $\mathcal{C}_K$. The main curves correspond to $\mathcal{C}_K = \{1, 2, \cdots, K\}$, i.e. the $K \leq \bar{K}$ closest modems are supposed to be active. The multiuser gain increases with K, and is much larger when the user channels in $\mathcal{C}_K$ are very different. Comparing these different curves, we get some insight in the respective gains that can be expected from :

- The simultaneous transmission by all users (with the same total power $\bar{P}$) instead of transmission in separate time slots.

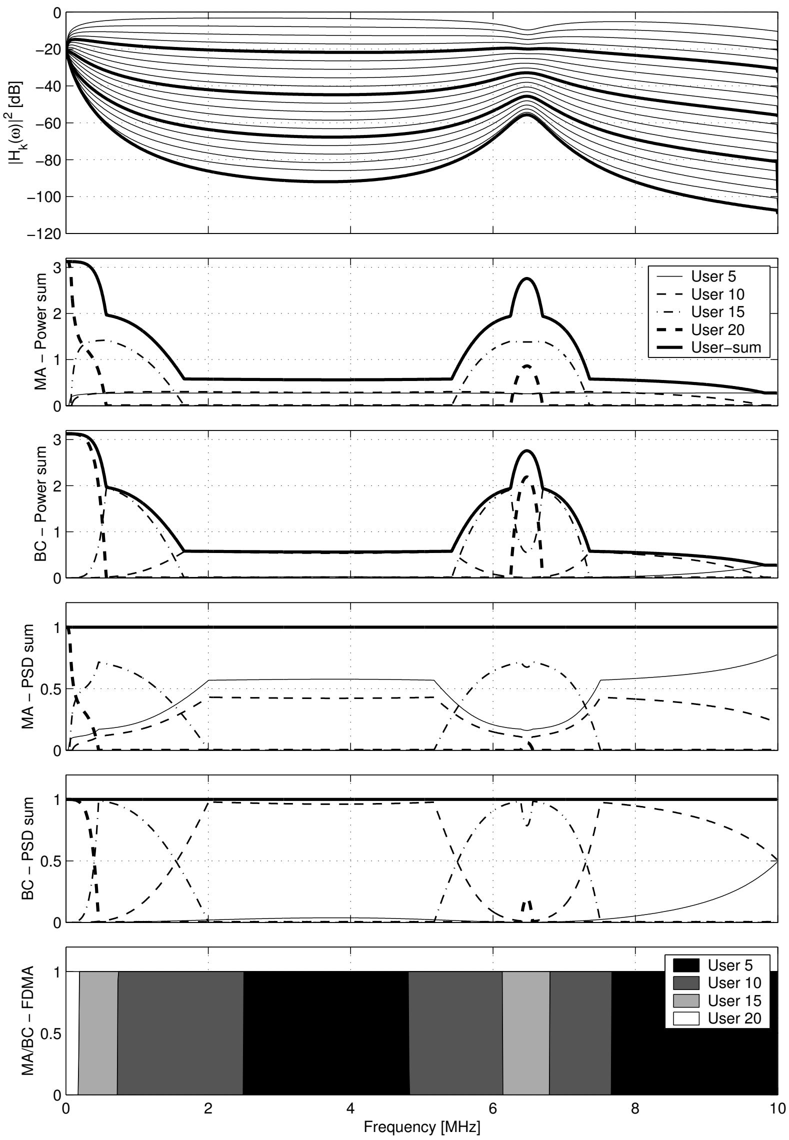

Figure 3.11: *Channel frequency responses (figure 1), MA and BC power allocations for $\mathcal{C}_4 = \{5, 10, 15, 20\}$: power-sum constraint (figures 2-3), PSD-sum constraint (figures 4-5), and flat-FDMA power allocation (figure 6).*

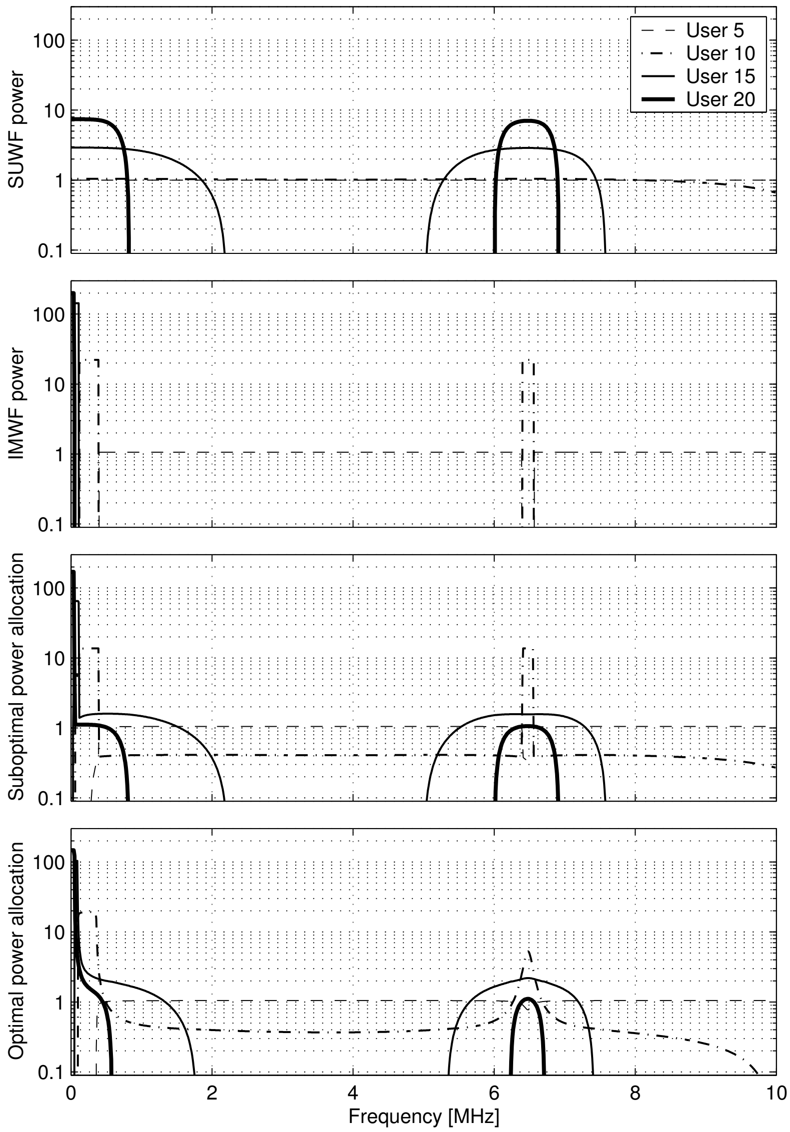

Figure 3.12: *Power allocations for* $\mathcal{C}_4 = \{5, 10, 15, 20\}$ *and individual power constraints*

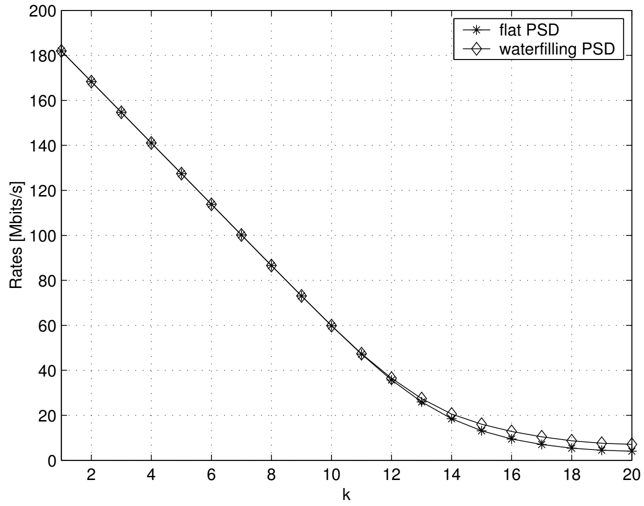

Figure 3.13: *Single user rates vs. user index k*

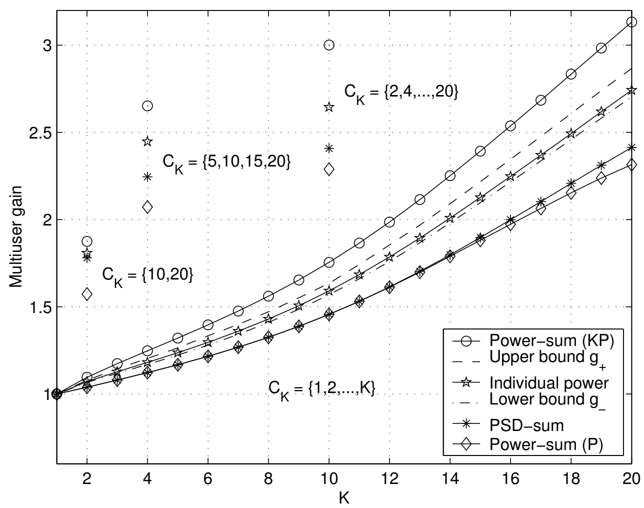

Figure 3.14: *Multiuser gain vs. number of active users K*

- The allocation of $K\bar{P}$ to the users ($\bar{P}$ for each) instead of $\bar{P}$.
- The redistribution of the total power $K\bar{P}$ in unequal parts.

The dashed lines give the lower and upper bounds g_- and g_+ obtained by equations (3.72) and (3.73).

Figure 3.11 also gives the results of the optimal MA and BC power allocations (normalized with respect to $\overline{\gamma}$) for $\mathcal{C}_4 = \{5, 10, 15, 20\}$, and three kinds of constraints (power-sum, PSD-sum, and flat-FDMA). The normalized optimal PSD-sum is also given (upper lines). As expected, the users with the weakest channels concentrate their transmission power in the best parts of the frequency spectrum while the users with the best channels (and lowest priorities) can make use of the remaining frequencies. The remote users require more power in the BC channel (where they are decoded first) than in the MA channel (where they are decoded last).

The lower part of Figure 3.12 gives the optimal MA power allocation for the same user set, but with constraints on the individual powers. As the remote users tend to concentrate their power $\bar{P}$ in a small fraction of the available bandwidth, a logarithmic scale was chosen for the vertical axis. This power allocation was obtained by algorithm 3.3. The suboptimal power allocation obtained by algorithm 3.4 is given in the third graph. It appears as a compromise between the SUWF power allocation (first graph) and the IMWF allocation (second graph). The rate loss associated with the suboptimal power allocation is negligible (less than 0.1 %). A smart redistribution of the total power $4\bar{P}$ between the 4 users would allocate less power and more bandwidth to the remote users (who benefit from the best frequencies), leading to improved balanced rates as illustrated in Table 3.1. Forcing each user to transmit with the same power $\bar{P}$ actually decreases the 'power price' associated with the remote users, and the excess power that needs to be allocated to these remote users is accumulated in a narrow frequency band, giving rise to very concentrated power allocations.

3.7.2 Complex network with open terminations

Similar results are now given for the case of the example powerline access network presented in Figure 2.12. This network has the same total length (300 meter) and the same number of derivations ($\bar{K} = 20$) as in the previous example, but the lengths of the different cable segments are not uniform, and the unused terminations are assumed to be open. The $K = 5$ active users are in positions $2, 5, 10, 15$ and 20. The 5 end-to-end channel transfer

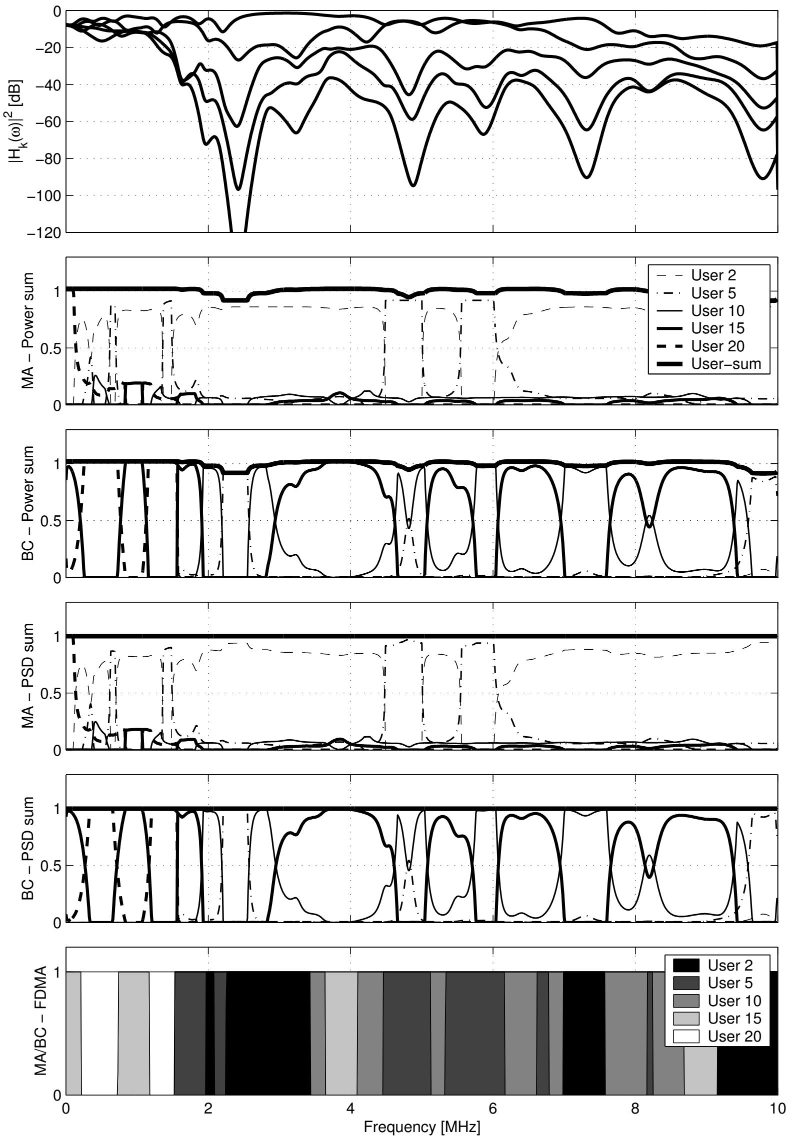

Figure 3.15: *Channel frequency responses (figure 1), MA and BC power allocations for* $\mathcal{C}_4 = \{5, 10, 15, 20\}$*: power-sum constraint (figures 2-3), PSD-sum constraint (figures 4-5), and flat-FDMA power allocation (figure 6).*

Table 3.2: *Maximum balanced rates (in Mbits/s) for* $\mathcal{C}_5 = \{2, 5, 10, 15, 20\}$

Constraint	User 2	User 5	User 10	User 15	User 20	%
Flat SU	175.09	149.36	107.03	73.33	49.63	100
PSD-sum	54.76	46.71	33.47	22.93	15.52	31
Flat FDMA	47.56	40.57	29.07	19.92	13.48	27
Flat TDMA	35.02	29.87	21.41	14.67	9.93	20
SU	175.09	149.36	107.13	74.29	52.64	100
Power-sum ($5\bar{P}$)	61.94	52.84	37.90	26.28	18.62	35
Individual power	58.77	50.14	35.96	24.94	17.67	34
Power-sum ($\bar{P}$)	54.33	46.34	33.24	23.05	16.33	31
TDMA	35.02	29.87	21.43	14.86	10.53	20

functions (see Figure 2.23) are reported on the top of Figure 3.15, in the bandwidth $[0, B]$ with $B = 10$ MHz. The noise level is again assumed to be -120 dBm/Hz in the whole bandwidth.

Table 3.2 gives the maximum balanced rates associated with the different types of power constraints. Figure 3.15 (to be compared with Figure 3.11) gives the optimal MA and BC power allocations (normalized with respect to $\overline{\gamma}$) for three kinds of constraints (power-sum, PSD-sum, and flat-FDMA). Finally, Figure 3.16 gives the optimal SUWF and IMWF power allocations, and the MA balanced power allocations with individual power constraints as given by Algorithms 3.3 and 3.4.

3.8 Conclusion

This chapter introduces the concept of *balanced capacity* to characterize the performance of multiuser channels. This concept appears as a smart compromise between the global system efficiency and fairness requirements among the users. The balanced capacity of a multiuser channel is actually a specific point on the boundary of the capacity region, corresponding to a specific (but a priori unknown) set of user relative priorities α_k. The uplink (multiple access channel) and downlink (broadcast channel) balanced capacities of wireline multiaccess networks are computed and compared *for an arbitrary number of users.*

A detailed analysis of the optimal power allocation (for a given set of relative priorities) in a *memoryless multiuser channel* is first given. The power

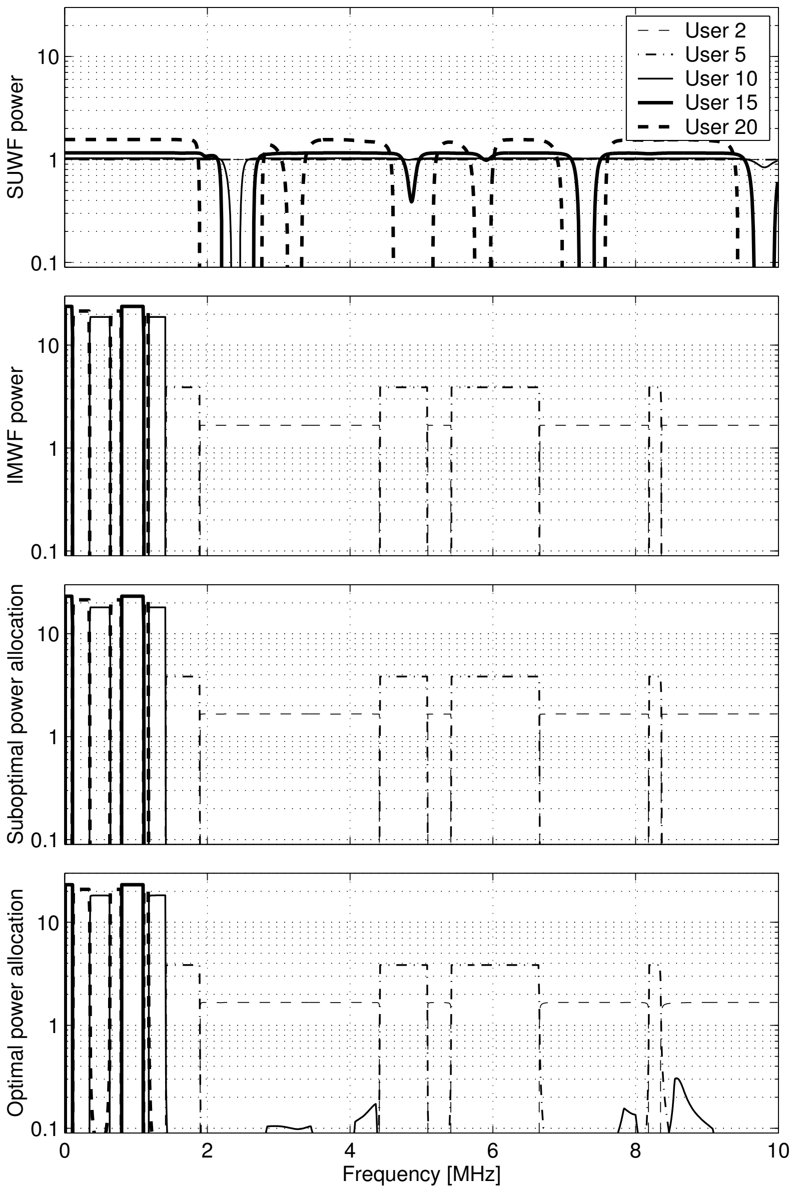

Figure 3.16: *Power allocations for* $\mathcal{C}_5 = \{2, 5, 10, 15, 20\}$ *and individual power constraints*

allocation algorithm is shown to be extremely simple, and an explicit expression of the corresponding powers and rates is available, as a function of the priorities α_k. These maximum rates are identical for the uplink and the downlink, but the respective power allocations are distinct. An iterative algorithm for the computation of the maximum balanced rates has been proposed : the user relative priorities α_k are updated at each iteration, which moves the rate distribution along the boundary of the capacity region until the balanced rate constraints are satisfied. The balanced capacity of the memoryless multiuser channel is found to be obtained within a few iterations.

This algorithm is then extended to the case of *frequency selective multiuser channels*. Three kinds of power constraints (with increasing complexity) are considered : a constraint on the total transmitted power spectral density, a constraint on the total transmitted power, and individual constraints on the transmitted powers. In the first scenario, the extension of the initial algorithm is trivial. The second scenario requires an additional step (*multiuser water-filling*) at each iteration, in order to specify the optimal distribution of the total allocated power along the frequency axis. In the third scenario, the extension of the initial algorithm is still possible by using the concept of *equivalent channel*, but the obtained method appeared to converge very slowly. An alternative algorithm, based on the iterative multiuser water-filling algorithm, has been proposed to compute a suboptimal power allocation, and the rate loss was found to be negligible.

Finally, the specific problem of FDMA balanced capacity computation with a PSD-sum constraint was analyzed separately and found to be equivalent to an optimal subchannel assignment problem. A fast (but tricky) algorithm was given, that is able to find the solution after a finite number of steps.

Results have been provided for a powerline access network with $K = 20$ users. A significant rate gain was found with respect to the trivial TDMA solution, which definitely proves the importance of a smart power allocation in a multiuser frequency selective channel.