Tracers in a river: simple one-dimensional analytical expressions

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Consider a river whose main channel cross-sectional area is denoted $A(t,x)$, where t and x are the time and the along-flow coordinate, respectively. Let $U(t,x)$ represent the cross section-averaged velocity component that is normal to the latter, so that the volumetric flow is the product $A(t,x)U(t,x)$. Then, the continuity equation reads

$$(1) \qquad \frac{\partial A}{\partial t} + \frac{\partial (AU)}{\partial x} = 0 .$$

Now assume that a quantity M of a tracer is released into the main channel of the river at time t=0 and location x=0. Then, the average over the cross-section of the tracer concentration, $C(t,x)$, obeys the equation

$$(2) \qquad \frac{\partial (AC)}{\partial t} = M\delta(t-0)\delta(x-0) - \frac{\partial}{\partial x}\left(AUC - AK\frac{\partial C}{\partial x}\right) - A\lambda C - A\alpha(C - C^s) ,$$

where δ denotes the Dirac function; $K(t,x)$ (>0) is the diffusion coefficient; the constant λ (≥0) is the decay rate of the tracer under study; $\alpha(t,x)$ (≥0) represents the storage zone exchange coefficient, while $C^s(t,x)$ is the concentration is the storage zone. The latter satisfies the equation

$$(3) \qquad \frac{\partial (A^s C^s)}{\partial t} = -\alpha A(C^s - C) ,$$

where $A^s(t,x)$ is the cross-sectional area of the storage zone. Clearly, the dimension of concentrations C and C^s is $[M]/\text{length}^3$, where [M] denotes the dimension of M.

This working note addresses the question as to how to infer the value of some of the coefficients that appear in equations (2) and (3) from time series of the concentration measured in the main channel of the river.

1. The zero decay and negligible storage case

Assume that the tracer is conservative ($\lambda = 0$) and the storage is negligible ($\alpha = 0$). Let A, U and K be positive constants. Then, the concentration in the main channel reads

(4) $$C(t,x) \;=\; \frac{M\,e^{-(x-Ut)^2/(4Kt)}}{A\sqrt{4\pi Kt}} \;.$$

Since the tracer does not decay and is not stored laterally, the amount of tracer present in the river remains constant:

(5) $$\int_{-\infty}^{\infty} A\,C(t,x)\,dx \;=\; M \;.$$

If time series of the concentration are available at various locations it is possible to infer the values of some or all of the coefficients *A*, *U* and *K* by selecting the values that achieve the smallest discrepancies between the idealised solution (4) and the measurements. This could be done from one time series only. Another approach consists in estimating the unknown coefficient values from the moments of the concentration, i.e.

(6) $$\mu_n(x) \;=\; \int_0^{\infty} t^n\,C(t,x)\,dt \;, \quad n = 0,1,2,\ldots$$

The first four moments of (4) may be obtained rather easily with the help of a symbolic calculation software:

(7) $$\mu_0(x) \;=\; \frac{M}{AU}\,e^{U(x-|x|)/(2K)} \;,$$

(8) $$\mu_1(x) \;=\; \frac{M}{AU^3}\,e^{U(x-|x|)/(2K)}\,(2K + U|x|) \;,$$

(9) $$\mu_2(x) \;=\; \frac{M}{AU^5}\,e^{U(x-|x|)/(2K)}\,(12K^2 + 6KU|x| + U^2x^2) \;,$$

(10) $$\mu_3(x) \;=\; \frac{M}{AU^7}\,e^{U(x-|x|)/(2K)}\,(120K^3 + 60K^2U|x| + 12KU^2x^2 + U^3|x|^3) \;.$$

It is noteworthy that that downstream of the release point (x>0) the expression $e^{U(x-|x|)/(2K)}$ actually is equal to unity, while it simplifies to $e^{Ux/K}$ upstream of the release point (x<0).

The moments above are related to physically-meaningful quantities, i.e. the mean age of the tracer

$$(11) \qquad a(x) = \frac{\int_0^\infty t\,C\,dt}{\int_0^\infty C\,dt} = \frac{\mu_1}{\mu_0} = \frac{2K}{U^2} + \frac{|x|}{U} ,$$

the variance of the concentration distribution

$$(12) \qquad \sigma^2(x) = \frac{\int_0^\infty (t-a)^2\,C\,dt}{\int_0^\infty C\,dt} = \frac{\mu_2 - 2a\mu_1 + a^2\mu_0}{\mu_0} = \frac{8K^2}{U^4} + \frac{2K|x|}{U^3} ,$$

and a measure of the skewness of the concentration distribution

$$(13) \qquad \gamma^3(x) = \frac{\int_0^\infty (t-a)^3\,C\,dt}{\int_0^\infty C\,dt} = \frac{\mu_3 - 3a\mu_2 + 3a^2\mu_1 - \mu_0 a^3}{\mu_0} = \frac{64K^3}{U^6} + \frac{12K^2|x|}{U^5} .$$

It must be underscored that the three quantities estimated above are symmetric with respect to the release point. Though this property may come as a surprise, the symmetry of the age has already been noticed and, to a certain extent, explained in other problems (Beckers et al. 2001, Deleersnijder et al. 2001, Deleersnijder and Delhez 2004, Hall and Haine 2004).

Physical intuition suggests that the net amount of tracer that crosses a section of the river, $\Phi(x)$, must satisfy

$$(14) \qquad \Phi(x) = \begin{cases} M\ , & x > 0 \\ 0\ , & x < 0 \end{cases} .$$

The flux $\Phi(x)$ has an advective and a diffusive components, $\Phi_a(x)$ and $\Phi_d(x)$, with

$$(15) \qquad \Phi(x) = \Phi_a(x) + \Phi_d(x) .$$

The advective and diffusive fluxes are

$$(16) \qquad \Phi_a = \int_0^\infty AUC\,dt = \begin{cases} M\ , & x > 0 \\ Me^{-U|x|/K}\ , & x < 0 \end{cases}$$

and

$$(17) \qquad \Phi_a = \int_0^\infty \left(-AK\frac{\partial C}{\partial x}\right) dt = \begin{cases} 0\ , & x > 0 \\ -Me^{-U|x|/K}\ , & x < 0 \end{cases} .$$

So, downstream of the release point (x>0), the net transport is due to advection only, i.e. the net diffusive flux is zero, whereas the net advective and diffusive components of the transport balance each other upstream of the release point (x<0), leading to a zero net flux in this region.

2. The decaying tracer without storage

Now consider a slightly more general problem, in which the tracer is assumed to exhibit a non zero decay rate ($\lambda > 0$), while the storage is still assumed to be negligible ($\alpha = 0$). In this case, the tracer concentration is

$$(18) \qquad C(t,x) = \frac{M\,e^{-\lambda t-(x-Ut)^2/(4Kt)}}{A\sqrt{4\pi Kt}} .$$

Since the tracer decays progressively, the amount of tracer present in the river decreases exponentially as time progresses:

$$(19) \qquad \int_{-\infty}^{\infty} A\,C(t,x)\,dx = M\,e^{-\lambda t} .$$

The first four moments of (18) are

$$(20) \qquad \mu_0(x) = \frac{M}{AV}\,e^{(Ux-V|x|)/(2K)} ,$$

$$(21) \qquad \mu_1(x) = \frac{M}{AV^3}\,e^{(Ux-V|x|)/(2K)}\,(2K + V|x|) ,$$

$$(22) \qquad \mu_2(x) = \frac{M}{AV^5}\,e^{(Ux-V|x|)/(2K)}\,(12K^2 + 6KV|x| + V^2x^2) ,$$

$$\mu_3(x) \;=\; \frac{M}{AV^7}\, e^{(Ux-V|x|)/(2K)}\,(120K^3 + 60K^2V|x| + 12KV^2x^2 + V^3|x|^3)\ , \tag{23}$$

with

$$V \;=\; \sqrt{U^2 + 4K\lambda}\ . \tag{24}$$

It is readily seen that $V > U$ as long as $\lambda > 0$. If the decay rate is set to zero ($\lambda = 0$), then the expressions (20)-(23) transform to their counterparts obtained in the previous section. This is reassuring as to the validity of the present calculations.

The net advective and diffusive fluxes are

$$\Phi_a(x) \;=\; \begin{cases} M\, e^{-(V-U)x/(2K)}\,(U/V)\ , & x > 0 \\ M\, e^{-(V+U)|x|/(2K)}\,(U/V)\ , & x < 0 \end{cases} \tag{25}$$

and

$$\Phi_d(x) \;=\; \begin{cases} M\, e^{-(V-U)x/(2K)}\,\dfrac{1-U/V}{2}\ , & x > 0 \\ M\, e^{-(V+U)|x|/(2K)}\,\dfrac{-1-U/V}{2}\ , & x < 0 \end{cases} \tag{26}$$

so that the total flux is

$$\Phi(x) \;=\; \begin{cases} M\, e^{-(V-U)x/(2K)}\,\dfrac{1+U/V}{2}\ , & x > 0 \\ M\, e^{-(V+U)|x|/(2K)}\,\dfrac{-1+U/V}{2}\ , & x < 0 \end{cases}\ . \tag{27}$$

Clearly, the total flux is smaller than M downstream of the point where the tracer is released ($x > 0$) and is negative upstream of this point ($x < 0$).

3. Release of a passive tracer into a flow with lateral storage

Now assume that the tracer is passive ($\lambda = 0$) and that lateral storage is present and characterised by the constant exchange coefficient α (> 0). To the best of my knowledge, it is not possible to obtained a closed-form expression of the concentrations in the main channel of the river and in the storage zone. However, as the tracer is passive, the concentrations must satisfy the integral constraint

$$(28) \qquad \int_{-\infty}^{\infty} (AC + A^s C^s)\, dx \;=\; M \;.$$

On the other hand, by manipulating equation (2) and (3), one may derive equations for the moments of the concentration in the main channel

$$(29) \qquad \begin{cases} K\dfrac{d^2\mu_0}{dx^2} \;-\; U\dfrac{d\mu_0}{dx} \;-\; \alpha(\mu_0 - \mu_0^s) \;=\; -\dfrac{M}{S}\delta(x-0) \\ K\dfrac{d^2\mu_n}{dx^2} \;-\; U\dfrac{d\mu_n}{dx} \;-\; \alpha(\mu_n - \mu_n^s) \;=\; -n\mu_{n-1} \;, \quad n = 1,2,\ldots \end{cases}$$

and in the storage zone

$$(30) \qquad \begin{cases} -\beta(\mu_0^s - \mu_0) \;=\; 0 \\ -\beta(\mu_n^s - \mu_n) \;=\; -n\mu_{n-1}^s \;, \quad n = 1,2,\ldots \end{cases}$$

with

$$(31) \qquad \beta \;=\; \frac{\alpha A}{A^s} \;.$$

The relations above immediately imply that the zero-th order moments are equal in the main channel and in the storage, and that these moments are equivalent to those obtained in the case of a passive tracer released into a flow without storage. Therefore, as there is no transport in the storage zone, the net fluxes are those obtained in section (1), i.e. relations (14)-(17). This is somewhat surprising; so far I have no physical explanation thereof.

The first moments of the storage zone concentration satisfy

(32)
$$\begin{cases} \mu_0^s = \mu_0 \\ \mu_1^s = \mu_1 + \mu_0/\beta \\ \mu_2^s = \mu_2 + 2\mu_1/\beta + 2\mu_0/\beta^2 \\ \mu_3^s = \mu_3 + 3\mu_2/\beta + 6\mu_1/\beta^2 + 6\mu_0/\beta^3 \end{cases}$$

with

(33)
$$\begin{cases} \mu_0 = \xi_0 \\ \mu_1 = (1+\alpha/\beta)\xi_1 \\ \mu_2 = (1+\alpha/\beta)^2\xi_2 + 2\alpha\xi_1/\beta^2 \\ \mu_3 = (1+\alpha/\beta)^3\xi_3 + 3\alpha(3+\alpha/\beta)\xi_2/\beta^3 + 6\alpha\xi_1/\beta^3 \end{cases}$$

where

(34)
$$\begin{cases} \xi_0(x) = \dfrac{M}{AU} e^{U(x-|x|)/(2K)} \\ \xi_1(x) = \dfrac{M}{AU^3} e^{U(x-|x|)/(2K)}(2K+U|x|) \\ \xi_2(x) = \dfrac{M}{AU^5} e^{U(x-|x|)/(2K)}(12K^2+6KU|x|+U^2x^2) \\ \xi_3(x) = \dfrac{M}{AU^7} e^{U(x-|x|)/(2K)}(120K^3+60K^2U|x|+12KU^2x^2+U^3|x|^3) \end{cases}$$

References

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