

GODUNOV VARIABLES AND CONVEX ENTROPY FOR RELATIVISTIC FLUID DYNAMICS WITH BULK VISCOSITY

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ABSTRACT. Based on the conservation-dissipation formalism proposed by Zhu and collaborators [ZHY15] we formulate a general version of the Israel-Stewart theory for relativistic fluid dynamics with bulk viscosity. Our generalization consists in allowing for a wide range of dependence of the entropy density on the bulk viscosity. We show the existence of Godunov-Boillat variables for this model. By known properties of systems possessing such variables, this provides an alternative proof of the recently established existence of solutions for the Israel-Stewart theory locally in time, and a proof that entropy production is positive across weak Lax shocks.

1. INTRODUCTION

The theories of relativistic fluid dynamics including viscosity by Müller [Mül67] and Israel & Stewart [IS79] are based on the idea to introduce thermodynamic fluxes as independent dynamic variables. For each of the new variables, one equation of motion has to be added to the conservation laws of energy-momentum and particle density. There are several variants of the theory, depending on the number of additional variables one includes, and other modeling choices; we will call all of these different theories Müller-Stewart-Israel (MIS) type theories. They have a long and successful history as a tool in simulations [RR19]. The mathematical underpinning of MIS theories is less developed. Only recently statements about well-posedness have been proved: For the theory including bulk viscosity the existence of solutions in Sobolev spaces has been shown [BDN19], and for the theory including bulk and shear viscosity as well as heat fluxes in Gevrey spaces [Dis14]¹. The proof in [BDN19] is based on writing the equations as a first order symmetric hyperbolic system, for which the existence of solutions locally in time is guaranteed by classical results by Friedrichs [Fri54] and Lax [Lax57] (see also the textbooks [Daf05, Tay11]).

In the present article, we will formulate a MIS-type theory with bulk viscosity following the so-called conformal-dissipation formalism (CDF) by Zhu, Hong, Yang & Yong [ZHY15]. The CDF is based on elements from extended irreversible thermodynamics [JCVL96], rational extended thermodynamics [MR13] and the GENERIC (general equations for non-equilibrium reversible irreversible coupling) formalism [Ött05]. It consists in enlarging the state space of thermodynamic variables by initially non-specified dynamic non-equilibrium variables. The formalism is set up in a way to guarantee that the equations of motions are a hyperbolic relaxation system, i.e., of the form

$$\partial_\alpha F^\alpha(U) = Q(U),$$

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¹In these references, the equations of motion for the fluid are coupled with the Einstein equations, and existence is shown for this larger system. We will limit ourselves here to a fixed Minkowskian spacetime.

where U is an unknown m -dimensional vector valued function, and Q, F^α are given m -dimensional vector valued functions, and we are using the convention that double greek indices are being summed over ($\partial_\alpha F^\alpha(U) \equiv \sum_{\alpha=0}^d \frac{\partial}{\partial x^\alpha} F^\alpha(U)$). In particular, this form allows for a straightforward definition of a concept of weak solutions, including discontinuous shocks to which Lax's classic analysis [Lax73] applies. Dissipative relativistic fluid equations in "divergence form" have been considered in [GL91, GL90], but to the best of our knowledge this is the first time that it is pointed out that the MIS equations with bulk viscosity can be cast in this form.

To set up our model, we will introduce only one dynamic non-equilibrium variable $\mathcal{C}$ in addition to the dynamic variables present in the three-dimensional relativistic Euler equations. This variable will be associated with the bulk viscosity π . The specific entropy density s will then depend on the equilibrium variables (on the internal energy per particle ε and the specific volume per particle ν , say) and the additionally introduced variable $\mathcal{C}$. The relation of $\mathcal{C}$ to the bulk viscosity π will be fixed through Legendre-Fenchel duality, and the equation of motion for $\mathcal{C}$ will be fixed by the requirement that the entropy production shall be non-negative. The standard MIS equations for bulk viscosity become a special case of our setting, defined by $s(\varepsilon, \nu, \mathcal{C}) = s^{\text{eq}}(\varepsilon, \nu) - \frac{\mathcal{C}^2}{2\theta^{\text{eq}}(\varepsilon, \nu)}$, where $s^{\text{eq}}, \theta^{\text{eq}}$ denote the equilibrium entropy density per particle and temperature, respectively.

We then prove that the generalized MIS equations with bulk viscosity possess the structure of a *causal covariant Godunov-Boillat system*. This is the relativistic version of the structure identified by Godunov [God61] that provides natural field variables that render the equations symmetric hyperbolic. In the covariant setting, the theory has been developed by Ruggeri & Strumia [RS81]. The proof that our model is a Godunov-Boillat system will follow the same lines as in that reference. It is also proved there that entropy production is positive across weak Lax shocks in Godunov-Boillat systems. By our theorem, this applies in particular the generalized MIS model with bulk viscosity, a fact that we note as a corollary.

The rest of this paper is structured as follows: In Section 2, we will construct our model based on the CDF. In Section 3 we show that this model is a causal covariant Godunov-Boillat system (Theorem 1), and state the corollary on entropy production across weak Lax shocks.

Notation. We will use throughout the Minkowskian metric $g = \text{diag}(-1, 1, 1, 1)$ on $\mathbb{R}^4$. Greek indices $\alpha, \beta, \dots$ will denote space-time components, and we will sum over repeated indices. The inverse of $g_{\alpha\beta}$ is denoted by $g^{\alpha\beta}$. Indices can be lifted or lowered by $g^{\alpha\beta}$ and $g_{\alpha\beta}$ respectively. On $\mathbb{R}^4$ we will consider vector fields with values in $\mathbb{R}^6$ (i.e., we consider a relativistic 6-field theory). Indices of the components of these vectors will be Latin letters, $i, j, k, \dots \in \{0, \dots, 5\}$, and summations will be indicated for these. In some equations, we will only consider field components carrying indices $0, \dots, 3$ - in these cases, we allow for greek indices, and summation will be indicated.

2. CONSTRUCTION OF ISRAEL-STEWART THEORY WITH BULK VISCOSITY VIA THE CONSERVATION-DISSIPATION FORMALISM

Consider a relativistic viscous fluid with stress energy tensor $T^{\alpha\beta}$ and four-current J^α given by

$$\begin{aligned} T^{\alpha\beta} &= n\varepsilon u^\alpha u^\beta + (p + \pi)\Delta^{\alpha\beta} \\ J^\alpha &= nu^\alpha \end{aligned}$$

where $n = \nu^{-1}$ is the particle density, ε the internal energy per particle, u the hydrodynamic velocity satisfying $u^\alpha u_\alpha = -1$, $\Delta^{\alpha\beta}$ the symmetric spatial projection $\Delta^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$, p the equilibrium pressure, π the bulk viscous pressure. We will also use the notation $\vec{u} = (u^1, u^2, u^3)^T$.

Now we adapt the conservation-dissipation formalism by Zhu, Hong, Yang, Yong [ZHY15] to the relativistic setting. One introduces a new field variable $\mathcal{C}$, that is supposed to be conjugate to the bulk viscosity π (in the sense of (1) below) and supposes that the equation of state is given in the form

$$s = s(\varepsilon, \nu, \mathcal{C}),$$

supposing that s is a convex function of its arguments. The variable $\mathcal{C}$ is conjugate to π in the sense that

$$\frac{\pi}{\theta} = -\partial_{\mathcal{C}} s. \tag{1}$$

As usual, the relation of θ, p to the other variables is fixed by requiring $\theta^{-1} = \partial_\varepsilon s$, $\theta^{-1}p = \partial_\nu s$.

Let $\Omega \subset \mathbb{R}^6$ denote the “field manifold”, which for definiteness we write as

$$\Omega = \{U = (\vec{u}^T, \varepsilon, \nu, \mathcal{C})^T \in \mathbb{R}^3 \times (0, \infty)^3\}.$$

From now on, u^β , $\beta = 0, \dots, 3$, as well as $\varepsilon, n, \pi, p, \theta, s, \nu$ will be considered as real-valued smooth functions on Ω (that will be partly determined by the equation of state relating the different thermodynamic quantities). Whenever a 6-tuple of functions $W = (w_0, \dots, w_5)^T : \Omega \rightarrow \mathbb{R}^6$ satisfies $\det D_U w(U) \neq 0$ throughout Ω , we will say that w provides a set of coordinates (on Ω). In such a case, we define partial derivatives with respect to w_j in the obvious way by $\partial_{U_j} f = \sum_{i=0}^5 \partial_{U_j} w_i \partial_{w_i} f$.

Our aim is to complete the set of equations $\partial_\alpha T^{\alpha\beta} = \partial_\alpha J^\alpha = 0$ with an equation of motion for $\mathcal{C}$ such that the resulting system becomes symmetric hyperbolic. For the four-dimensional entropy flux

$$S^\alpha = nu^\alpha s(\varepsilon, \nu, \mathcal{C}), \quad \alpha = 0, \dots, 3$$

we require that $\nabla_\alpha S^\alpha \geq 0$, which is the relativistic formulation of the second law of thermodynamics; this requirement will restrict the choice of evolution equation for $\mathcal{C}$. We

have that

$$\begin{aligned}
\partial_\alpha(nu^\alpha s(\nu, \varepsilon, \mathcal{C})) &= \frac{\partial s}{\partial \varepsilon} \partial_\alpha(nu^\alpha \varepsilon) + \frac{\partial s}{\partial \nu} \partial_\alpha(nu^\alpha \nu) + \frac{\partial s}{\partial \mathcal{C}} \partial_\alpha(nu^\alpha \mathcal{C}) \\
&= -\theta^{-1} u_\beta \partial_\alpha(n\varepsilon u^\alpha u^\beta) + p\theta^{-1} \partial_\alpha u^\alpha - nu^\alpha \theta^{-1} \pi \partial_\alpha \mathcal{C} \\
&= \theta^{-1} u_\beta \partial_\alpha \left((p + \pi) \Delta^{\alpha\beta} \right) + p\theta^{-1} \partial_\alpha u^\alpha - nu^\alpha \theta^{-1} \pi \partial_\alpha \mathcal{C} \\
&= -\theta^{-1} \pi \partial_\alpha u^\alpha - nu^\alpha \theta^{-1} \pi \partial_\alpha \mathcal{C}.
\end{aligned}$$

Here we have used identities like $u_\beta \partial_\alpha \Delta^{\alpha\beta} = -\partial_\alpha u^\alpha$, etc.

Grouping together the terms in π and requiring $\partial_\alpha S^\alpha \geq 0$ leads to imposing the equation of motion for $\mathcal{C}$,

$$nu^\alpha \partial_\alpha \mathcal{C} + \partial_\alpha u^\alpha = -M(U)\pi,$$

where M is strictly positive, but otherwise arbitrary.

Summarizing, the generalized MIS equations with bulk viscosity are

$$\partial_\alpha F^\alpha(U) = Q(U) \quad (2)$$

with

$$F^\alpha(U) = \begin{pmatrix} n\varepsilon u^\alpha u^\beta + (p + \pi) \Delta^{\alpha\beta} \\ nu^\alpha \\ (n\mathcal{C} + 1)u^\alpha \end{pmatrix}, \quad Q(U) = \begin{pmatrix} 0 \\ 0 \\ -M(U)\pi \end{pmatrix}.$$

Remark 1. A treatment of the non-relativistic analog of this theory can be found in [Rug16, RS15].

3. GODUNOV-BOILLAT VARIABLES

Following [RS81, Fre19], we will show that the generalized MIS equations with bulk viscosity are a convex density system:

Definition 1. A set of PDE

$$\partial_\alpha F_i^\alpha = Q_i \quad i = 0, \dots, n-1 \quad (3)$$

is called a causal covariant Godunov-Boillat system and $\psi = (\psi_0, \dots, \psi_{n-1})^T$ Godunov variables or main field, as well as $X^\alpha = X^\alpha(\psi)$ a potential for (3) if

$$F_i^\alpha = \partial_{\psi_i} X^\alpha(\psi) \quad (4)$$

and

$$\left(\frac{\partial^2 X^\beta(\psi)}{\partial \psi_j \partial \psi_i} \right)_{i,j=0,\dots,n-1} \xi_\beta \quad \text{is definite for each fixed vector } \xi_\beta \text{ with } \xi_\beta \xi^\beta < 0. \quad (5)$$

Remark 2. Any causal covariant Godunov-Boillat system can be written as a first order symmetric hyperbolic system,

$$\partial_\alpha F^\alpha = (D_\psi^2 X^\alpha) \partial_\alpha \psi = \bar{Q}.$$

The properties (4) and (5) supply an alternative proof of the symmetrizability of the (generalized) MIS equations with bulk viscosity that has been stated in [BDN19]. Symmetrizability in turn implies existence of solutions in Sobolev spaces locally in time (see [Lax57, Fri54, Daf05, Tay11]).

Now following the convex covariant density formalism from [RS81], the Godunov variables ψ are obtained as a solution of

$$\psi^T D_U F^\alpha = D_U S^\alpha, \quad \alpha = 0, \dots, 3.$$

Obviously this system is overdetermined. We will now exploit the necessary conditions above and will see at the end of these calculations that there exists indeed a unique solution.

It is convenient to introduce yet another set of field coordinates: Let $v^\alpha = nu^\alpha$, and $V = (v^0, \vec{v}, \varepsilon, \mathcal{C})^T$, where $\vec{v} = (v^1, v^2, v^3)$. The vector F^α comprising energy-momentum, particle density flux and the expression whose differentiation yields the additional equation of motion reads in the V -variables as

$$F^\alpha = \begin{pmatrix} \varepsilon \nu v^\alpha v^\beta + (p + \pi) \Delta^{\alpha\beta} \\ v^\alpha \\ (\mathcal{C} + \nu) v^\alpha \end{pmatrix}$$

This yields

$$D_V F^\alpha = \begin{pmatrix} (A_\mu^{\alpha\beta})_{\beta,\mu=0,\dots,3} & (B^{\alpha\beta})_{\beta=0,\dots,3} & \partial_{\mathcal{C}}(p + \pi)(\Delta^{\alpha\beta})_{\beta=0,\dots,3} \\ (\delta_\mu^\alpha)_{\mu=0,\dots,3} & 0 & 0 \\ (\delta_\mu^\alpha(\mathcal{C} + \nu) + \nu^3 v^\alpha v_\mu)_{\mu=0,\dots,3} & 0 & v^\alpha \end{pmatrix}$$

with

$$\begin{aligned} A_\mu^{\alpha\beta} &= \frac{\partial}{\partial v^\mu} \left((\nu \varepsilon + \nu^2(p + \pi)) v^\alpha v^\beta + (p + \pi) g^{\alpha\beta} \right) \\ &= \left((\varepsilon + 2\nu(p + \pi)) v^\alpha v^\beta + \partial_\nu(p + \pi) \Delta^{\alpha\beta} \right) \nu^3 v_\mu + (\varepsilon \nu + \nu^2(p + \pi)) (\delta_\mu^\alpha v^\beta + \delta_\mu^\beta v^\alpha) \end{aligned}$$

and

$$\begin{aligned} B^{\alpha\beta} &= \partial_\varepsilon \left((\nu \varepsilon + \nu^2(p + \pi)) v^\alpha v^\beta + (p + \pi) g^{\alpha\beta} \right) \\ &= \nu v^\alpha v^\beta + \partial_\varepsilon(p + \pi) \Delta^{\alpha\beta}. \end{aligned}$$

Furthermore we have

$$D_V S^\alpha = ((\delta_\mu^\alpha s + \nu^3 p \theta^{-1} v^\alpha v_\mu)_{\mu=0,\dots,3}, \theta^{-1} v^\alpha, -v^\alpha \theta^{-1} \pi)$$

We insert these formulae in $D_V S^\alpha = \psi^T D_V F^\alpha$, and have

$$\begin{aligned} B^{\alpha\beta} \psi_\beta &= \theta^{-1} v^\alpha \Rightarrow \psi_\gamma = -\theta^{-1} u_\gamma, \quad \gamma = 0, \dots, 3 \\ v^\alpha \psi_5 &= -v^\alpha \theta^{-1} \pi \Rightarrow \psi_5 = -\theta^{-1} \pi. \end{aligned}$$

We write $\psi^T = -\theta^{-1}(u_0, \vec{u}, g, \pi)$, where g has to satisfy the identity

$$\begin{aligned} -\delta_\mu^\alpha s - \nu p \theta^{-1} u^\alpha u_\mu &= g \delta_\mu^\alpha + \theta^{-1} u_\beta \left((\varepsilon + 2\nu(p + \pi)) u^\alpha u^\beta u_\mu \right. \\ &\quad \left. + (\varepsilon + \nu(p + \pi)) (\delta_\mu^\alpha u^\beta + \delta_\mu^\beta u^\alpha) \right) \\ &\quad + \theta^{-1} \pi (\delta_\mu^\alpha (\mathcal{C} + \nu) + \nu u^\alpha u_\mu). \end{aligned} \tag{6}$$

We calculate

$$\begin{aligned} & \theta^{-1} u_\beta \left((\varepsilon + 2\nu(p + \pi)) u^\alpha u^\beta u_\mu + (\varepsilon + \nu(p + \pi)) (\delta_\mu^\alpha u^\beta + \delta_\mu^\beta u^\alpha) \right) \\ &= -\theta^{-1} \nu(p + \pi) u^\alpha u_\mu - \theta^{-1} (\varepsilon + \nu(p + \pi)) \delta_\mu^\alpha \end{aligned}$$

and we see that (6) holds true for $\alpha = 0, \dots, 3$ provided that

$$g = \varepsilon - \theta s + \nu p - \mathcal{C}\pi, \quad (7)$$

which is the free enthalpy with natural variables θ, p, π .

Above we have made the assumption that s is a strictly concave function of its arguments $\varepsilon, \nu, \mathcal{C}$. This is equivalent to the requirement that g is a strictly concave function of θ, p, π . In particular, this implies that θ, g, π provide coordinates for the thermodynamical state space (by that we mean that $(u^1, u^2, u^3, \theta, g, \pi)^T$ provide a set of coordinates on Ω). Hence ψ also provides a set of coordinates.

Following the general arguments from [RS81], we now set

$$X^\alpha(\psi) = \psi^T F^\alpha(\psi) - S^\alpha(\psi)$$

and will show in the upcoming theorem that this defines a Godunov-Boillat system under suitable assumptions on the state function. We use the following notation in the statement: Whenever z_1, z_2, z_3 provide coordinates for the thermodynamical state space, we write

$$\left. \frac{\partial h}{\partial z_1} \right|_{z_2, z_3}$$

for the partial derivative with respect to z_1 of the function $(z_1, z_2, z_3) \mapsto h(z_1, z_2, z_3)$, and when h_1, h_2, h_3 are functions depending only on the thermodynamic state variables, we write

$$\frac{D(h_1, h_2, h_3)}{D(z_1, z_2, z_3)}$$

for the Jacobian determinant $\det \left(\frac{\partial h_i}{\partial z_j} \right)_{i,j=1,2,3}$. We also write

$$f = \varepsilon + \nu(p + \pi).$$

Theorem 1. *Suppose that $(\theta, p, \pi) \mapsto g(\theta, p, \pi)$ is strictly concave and that additionally*

$$\begin{aligned} & 1 + \frac{\nu^2}{f} \left. \frac{\partial p}{\partial \nu} \right|_{\theta, \mathcal{C}} \geq 0 \\ & 1 + \frac{\nu^2}{f} \left(- \left. \frac{\partial \pi}{\partial \mathcal{C}} \right|_{\theta, p} + \left. \frac{\partial p}{\partial \nu} \right|_{\theta, \pi} + 2 \left. \frac{\partial \pi}{\partial \nu} \right|_{\theta, p} \right) \geq 0 \end{aligned} \quad (8)$$

on Ω . Then the function $X^\alpha(\psi)$ defines a Godunov-Boillat system.

Proof. We need to verify (4) and (5). Firstly,

$$\frac{\partial X^\alpha}{\partial \psi_i} = (F^\alpha)_i + \psi^T \frac{\partial F^\alpha}{\partial \psi_i} - \frac{\partial S^\alpha}{\partial \psi_i}.$$

Obviously

$$\psi^T \frac{\partial F^\alpha}{\partial \psi_i} - \frac{\partial S^\alpha}{\partial \psi_i} = \sum_{j=0}^5 \left(\psi^T \frac{\partial F^\alpha}{\partial U_j} - \frac{\partial S^\alpha}{\partial U_j} \right) \frac{\partial U_j}{\partial \psi_i} = 0,$$

and hence (4) is proved.

For an arbitrary future-directed 4-vector ξ^β with $\xi_\beta \xi^\beta = -1$, we set

$$\begin{aligned}\bar{X} &= \xi_\alpha X^\alpha \\ \bar{F} &= \xi_\alpha F^\alpha\end{aligned}$$

In order to verify (5), it suffices to prove

$$\begin{aligned}0 &< \sum_{i,j=0}^5 v_i v_j \partial_{\psi^i} \partial_{\psi^j} \bar{X}(\psi) \\ &= \sum_{i,j=0}^5 v_i v_j \partial_{\psi^i} \bar{F}_j(\bar{\psi}) \\ &= \sum_{i,j,k=0}^5 v_i v_j \partial_{\psi^j} \psi_k \partial_{\psi^i} \bar{F}_k(\bar{\psi}) \\ &= \sum_{i,j,k=0}^5 \tilde{v}_i \tilde{v}_j \partial_{U_j} \psi_k \partial_{U_i} \bar{F}_k\end{aligned} \tag{9}$$

for every $v \in T_P \Omega \equiv \mathbb{R}^6$, $P \in \Omega$, where we have used the notation

$$\tilde{v}_j = \sum_{l=0}^5 \frac{\partial U_j}{\partial \psi_l} v_l.$$

The vector $v \in T_P \Omega$ will be fixed from now on, and for any (smooth) function $z : \Omega \rightarrow \mathbb{R}$ we will write

$$\delta z = \sum_{l=0}^5 \tilde{v}_l \partial_{U_l} z.$$

We set

$$\begin{aligned}\mathcal{Q} &:= \sum_{i,j,k=0}^5 \tilde{v}_i \tilde{v}_j \partial_{U_i} \psi_k \partial_{U_j} \bar{F}_k \\ &= \sum_{k=0}^5 \delta \psi_k \delta \bar{F}_k\end{aligned}$$

and write $\bar{v} = \xi_\alpha v^\alpha$, $\bar{u} = \xi_\alpha u^\alpha$. We have by $\psi = -\theta^{-1}(u_0, \vec{u}^T, g, \pi)^T$,

$$\theta^2 \mathcal{Q} = \sum_{\beta=0}^4 (u_\beta \delta \theta - \theta \delta u_\beta) \delta \bar{F}_\beta + (g \delta \theta - \theta \delta g) \delta \bar{v} + (\pi \delta \theta - \theta \delta \pi) \delta (\bar{v}(\mathcal{C} + \nu))$$

In a lengthy but straightforward calculation, in which we use the identities (7), and

$$\begin{aligned}u^\beta \delta u_\beta &= 0 \\ \delta g &= -s \delta \theta + \nu \delta p - \mathcal{C} \delta \pi \\ \delta^2 g &= -\delta s \delta \theta + \delta \nu \delta p - \delta \mathcal{C} \delta \pi\end{aligned}$$

we obtain

$$\theta \mathcal{Q} = \bar{v}(\delta^2 g - f \delta u_\beta \delta u^\beta) - 2\delta(p + \pi) \delta \bar{u} \tag{10}$$

We now introduce the notation $\delta \vec{u} = (\delta u^1, \delta u^2, \delta u^3)^T$ and use the relations

$$\begin{aligned}\delta u^0 &= \sum_{i=1}^3 \frac{u_i \delta u^i}{u^0} \equiv \frac{\vec{u} \cdot \delta \vec{u}}{u^0} \\ \delta u_\beta \delta u^\beta &= \sum_{i,j=1}^3 \delta u^i (\delta_{ij} - (u^0)^{-2} u_i u_j) \delta u^j \\ &= \delta \vec{u}^T (\text{Id}_{3 \times 3} + \vec{u} \otimes \vec{u})^{-1} \delta \vec{u} \\ \delta \bar{u} &= \left(\vec{\xi} - \frac{\xi^0}{u^0} \vec{u} \right) \cdot \delta \vec{u}\end{aligned}$$

and rewrite (10) as

$$\theta \mathcal{Q} = \bar{v} \delta^2 g - \bar{v} f \delta \vec{u}^T (\text{Id}_{3 \times 3} + \vec{u} \otimes \vec{u})^{-1} \delta \vec{u} - 2\delta(p + \pi) \delta \vec{u} \cdot \left(\vec{\xi} - \frac{\xi^0}{u^0} \vec{u} \right). \quad (11)$$

To “complete the square”, we add and subtract

$$\begin{aligned}& \left(\vec{\xi} - \frac{\xi^0}{u^0} \vec{u} \right)^T (\text{Id}_{3 \times 3} + \vec{u} \otimes \vec{u}) \left(\vec{\xi} - \frac{\xi^0}{u^0} \vec{u} \right) \frac{(\delta(p + \pi))^2}{\bar{v} f} \\ &= (\xi_\beta \xi^\beta + |u_\beta \xi^\beta|^2) \frac{(\delta(p + \pi))^2}{\bar{v} f} \\ &= (\bar{u}^2 - 1) \frac{(\delta(p + \pi))^2}{\bar{v} f}\end{aligned}$$

on the right hand side in (11) and obtain

$$\theta \mathcal{Q} = \bar{v} \delta^2 g - \bar{v} f \vec{w}^T (\text{Id}_{3 \times 3} + \vec{u} \otimes \vec{u})^{-1} \vec{w} + (\bar{u}^2 - 1) \frac{(\delta(p + \pi))^2}{\bar{v} f},$$

where

$$\vec{w} = \delta \vec{u} + (\text{Id}_{3 \times 3} + \vec{u} \otimes \vec{u}) \left(\vec{\xi} - \frac{\xi^0}{u^0} \vec{u} \right)^T \frac{\delta(p + \pi)}{\bar{v} f}.$$

Multiplying with $\nu^2 \bar{v}$ (which is negative since both ξ^μ and v^μ are future directed time-like), we obtain finally

$$\begin{aligned}\bar{v} \nu^2 \theta \mathcal{Q} &= \bar{u}^2 \delta^2 g - \bar{u}^2 f \vec{w}^T (\text{Id}_{3 \times 3} + \vec{u} \otimes \vec{u})^{-1} \vec{w} + (\bar{u}^2 - 1) \frac{\nu^2}{f} (\delta(p + \pi))^2 \\ &= -(\bar{u}^2 - 1) \left(-\delta g^2 - \frac{\nu^2}{f} (\delta(p + \pi))^2 \right) + \underbrace{\delta g^2 - \bar{u}^2 f \vec{w}^T (\text{Id}_{3 \times 3} + \vec{u} \otimes \vec{u})^{-1} \vec{w}}_{< 0 \forall v \in T_p \Omega \setminus \{0\}}\end{aligned} \quad (12)$$

We have that $\bar{u}^2 = |\xi_\beta u^\beta| \geq 1$ and hence the right hand side is negative for every $v \in T_p \Omega \setminus \{0\}$ if the matrix

$$\tilde{G} = \begin{pmatrix} -\partial_\theta^2 g & -\partial_\theta \partial_p g & -\partial_\theta \partial_\pi g \\ -\partial_\theta \partial_p g & -\partial_p^2 g - f^{-1} \nu^2 & -\partial_p \partial_\pi g - f^{-1} \nu^2 \\ -\partial_\theta \partial_\pi g & -\partial_p \partial_\pi g - f^{-1} \nu^2 & -\partial_\pi^2 g - f^{-1} \nu^2 \end{pmatrix}$$

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is non-negative definite. By the strict concavity of g , we have $-\partial_\theta^2 g > 0$. We claim that the two conditions in (8) are precisely the non-negativity of the other two leading principal minors of $\tilde{G}$, which proves negativity of the right hand side in (12). Indeed,

$$\begin{aligned} \det \begin{pmatrix} -\partial_\theta^2 g & -\partial_\theta \partial_p g \\ -\partial_\theta \partial_p g & -\partial_p^2 g - f^{-1} \nu^2 \end{pmatrix} &= \frac{D(-s, \nu, -\mathcal{C})}{D(\theta, p, -\mathcal{C})} \left(1 + \frac{\nu^2}{f} \frac{D(\theta, p, -\mathcal{C})}{D(-s, \nu, -\mathcal{C})} \frac{D(-s, \nu, -\mathcal{C})}{D(\theta, \nu, -\mathcal{C})} \right) \\ &= \frac{D(-s, \nu, -\mathcal{C})}{D(\theta, p, -\mathcal{C})} \left(1 + \frac{\nu^2}{f} \frac{\partial p}{\partial \nu} \Big|_{\theta, \mathcal{C}} \right) \geq 0 \end{aligned}$$

and

$$\begin{aligned} \det \tilde{G} &= -\frac{D(-s, \nu, -\mathcal{C})}{D(\theta, p, \pi)} - \frac{\nu^2}{f} \left(\frac{D(-s, \nu, -\mathcal{C})}{D(\theta, p, -\mathcal{C})} + \frac{D(-s, \nu, -\mathcal{C})}{D(\theta, \nu, \pi)} - 2 \frac{D(-s, \nu, -\mathcal{C})}{D(\theta, \nu, p)} \right) \\ &= -\frac{D(-s, \nu, -\mathcal{C})}{D(\theta, p, \pi)} \left(1 + \frac{\nu^2}{f} \left(\frac{D(\theta, p, \pi)}{D(\theta, p, -\mathcal{C})} + \frac{D(\theta, p, \pi)}{D(\theta, \nu, \pi)} - 2 \frac{D(\theta, p, \pi)}{D(\theta, \nu, p)} \right) \right) \\ &= -\det D_{(\theta, p, \pi)}^2 g \left(1 + \frac{\nu^2}{f} \left(-\frac{\partial \pi}{\partial \mathcal{C}} \Big|_{\theta, p} + \frac{\partial p}{\partial \nu} \Big|_{\theta, \pi} + 2 \frac{\partial \pi}{\partial \nu} \Big|_{\theta, p} \right) \right) \geq 0. \end{aligned}$$

This completes the proof. $\square$

In the introduction we have noted that the generalized MIS model with bulk viscosity allows for discontinuous shock solutions. Lax's entropy condition guarantees that weak shocks will have positive entropy production across the shock, see e.g. [Daf05]. A proof of this fact in the relativistic setting (i.e., for causal covariant Godunov-Boillat systems) can be found in [RS81]. We obtain the straightforward corollary:

Corollary 1. *For weak Lax shocks of (weak) solutions of the generalized MIS model with bulk viscosity (2), the entropy production is positive across the shock.*

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