Transportation Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Single-Period Cutting Planes for Inventory Routing Problems

Pasquale Avella, Maurizio Boccia, Laurence A. Wolsey

informs
http://pubsonline.informs.org/journal/trsc/

TRANSPORTATION SCIENCE
Articles in Advance, pp. 1–12
ISSN 0041-1655 (print), ISSN 1526-5447 (online)

Single-Period Cutting Planes for Inventory Routing Problems

Pasquale Avella,[a] Maurizio Boccia,[a] Laurence A. Wolsey[b]

[a]Dipartimento di Ingegneria, Università del Sannio, 82100 Benevento, Italy; [b]Center for Operations Research and Econometrics, Université catholique de Louvain, 1348 Louvain-la-Neuve, Belgium

Contact: avella@unisannio.it (PA); maboccia@unisannio.it (MB); laurence.wolsey@uclouvain.be (LAW)

Received: January 2015
Revised: March 2015, October 2015
Accepted: July 2016
Published Online in Articles in Advance: March 15, 2017

https://doi.org/10.1287/trsc.2016.0729

Abstract. The Inventory Routing Problem (IRP) involves the distribution of one or more products from a supplier to a set of clients over a discrete planning horizon. Each client has a known demand to be met in each period and can only hold a limited amount of stock. The product is shipped through a distribution network by one or more vehicles of limited capacity. The objective is to find replenishment decisions minimizing the sum of the storage and distribution costs. In this paper, we present IRP reformulations under the Maximum Level replenishment policy, derived from a single-period substructure. We define a generic family of valid inequalities, and then introduce two specific subclasses for which the separation problem of generating violated inequalities can be effectively solved. A basic Branch-and-Cut algorithm has been implemented to demonstrate the strength of the single-period reformulations. We present computational results for the benchmark instances with 50 clients and 3 periods and 30 clients and 6 periods.

Funding: This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming [contract P7/36], Combinatorial Optimization: Metaheuristics and Exact Methods.
Supplemental Material: The online appendix is available at https://doi.org/10.1287/trsc.2016.0729.

Keywords: inventory routing • valid inequalities • cutting planes

1. Introduction

The Inventory Routing Problem (IRP) arises from the integration of two basic components of the logistic supply chain, i.e., Inventory Management and Vehicle Routing. IRP involves the distribution of one or more products from a supplier to a set of clients over a discrete planning horizon. Each client has a known demand to be met in each period and can only hold a limited amount of stock. Replenishment policies are defined by the supplier, the most common being the order-up (OU) and maximum level (ML). Under the OU policy, if a client is visited in a period then the amount shipped to the client must bring the stock level up to the upper bound. The ML policy treated here only requires that the maximum stock level at the end of each period cannot be exceeded.

The product is shipped through a distribution network by one or more vehicles of limited capacity. IRP has applications in several contexts, such as maritime logistics and the distribution of gas, perishable items, groceries, etc. The objective is to find replenishment decisions minimizing the sum of the storage and distribution costs.

1.1. Literature Review

Archetti et al. (2007) were the first to propose a Branch-and-Cut algorithm for the IRP. They addressed the single-vehicle case and the replenishment policy variants cited above. Then Solyali and Süral (2011) proposed a shortest-path network reformulation of the single-vehicle problem for the OU replenishment policy. Avella, Boccia, and Wolsey (2015) presented tight reformulations for the lot-sizing subproblems that arise for each client designed to strengthen the dual lower bounds, and showing computational results for the single vehicle problem.

Exact algorithms for multi-vehicle IRP have been proposed by Adulyasak, Cordeau, and Jans (2014), Coelho and Laporte (2013, 2014); and Desaulniers, Rakke, and Coelho (2016). Adulyasak, Cordeau, and Jans (2014) proposed reformulations for the OU and the ML policies using vehicle indices and extended some of the inequalities introduced by Archetti et al. (2007) to the multi-vehicle case. Coelho and Laporte (2013) presented a Branch-and-Cut algorithm to address several IRP variants, including the case with multiple vehicles, under the ML policy. In a subsequent paper, Coelho and Laporte (2014) presented new valid inequalities based on the minimum number of visits each client must receive. They showed that modifying the order of input data can significantly affect the running time and that symmetry breaking constraints can be effective in tightening the formulation. Desaulniers, Rakke, and Coelho (2016) have introduced an extended IRP formulation under the ML replenishment policy, based on route delivery patterns, presenting a Branch-and-Price algorithm that outperforms the Branch-and-Cut algorithm of Coelho and Laporte (2014) on the test instances with four and five vehicles. All of the algorithms proposed in the above papers have been tested on the same set of benchmark instances. For a more general review on algorithms for IRP, see Coelho, Cordeau, and Laporte (2014).

1.2. Outline of the Paper

IRP naturally decomposes into two families of highly structured subproblems, i.e., a lot-sizing problem for each client, discussed by the authors in detail in Avella, Boccia, and Wolsey (2015) and a single-period inventory routing problem (SIRP) for each time period. In this paper, we present IRP reformulations, under the ML replenishment policy, essentially derived from this single-period substructure. We define a generic family of valid inequalities for SIRP, and then introduce two specific subclasses for which the separation problem of generating violated inequalities can be effectively solved.

A basic Branch-and-Cut algorithm, without any special-purpose primal heuristics, has been implemented to demonstrate the strength of the valid inequalities proposed. Computational results are presented for benchmark instances with 50 clients and 3 periods and with 30 clients and 6 periods.

The remainder of the paper is organized as follows. In Section 2 we give a more formal definition of IRP and an initial mixed integer programming (MIP) formulation. In Section 3 we introduce a simple version of the generic family of valid inequalities for the single-period IRP. In Section 4 we describe in detail a family of "disjoint route inequalities" as well as the two subfamilies that we use as cutting planes. In Section 5 we also describe a separation procedure for another family of inequalities obtained by aggregating demands over several time periods. In Section 6 we describe the components of our Branch-and-Cut algorithm. Finally, in Section 7 we present detailed results for instances with 50 clients and 3 periods and 30 clients and 6 periods, showing the effectiveness of the proposed inequalities. Detailed results for the instances with 15, 20, and 25 clients and 6 periods are given in the online appendix.

2. Problem Definition and Formulation

Let $T = \{1, 2, \ldots, T_{\max}\}$ be a discrete time horizon. In each period $t \in T$, D_t^0 units of a single item are delivered to the supplier 0, and the supplier then uses a fleet of K vehicles of capacity C to supply a set of clients $I = \{1, 2, \ldots, n\}$. The deliveries must be planned so that the demand D_t^i of each client $i \in I$ in each period t is satisfied and the storage capacity is not exceeded.

Let $T_0 = T \cup \{0\}$, $sinit^0$ be the initial stock of the supplier and $sinit^i$ the initial stock of client i. Let h^0 and h^i be the unit storage costs for the supplier and for the clients $i \in I$, respectively. Under the ML replenishment policy, there is an upper bound W^i on the stock held by customer $i \in I$ at any moment. Under the assumption that a delivery to client i (if any) in period t occurs before the demand D_t^i is met, it follows that the upper bound on the stock at the end of the period is $W^i - D_t^i$ and the maximum value of a shipment to client i in period t is W^i.

Besides selecting delivery quantities, the vehicle routes in each period $t \in T$ must be determined. The distribution network in each period $t \in T$ is represented by a directed graph $G(I_0, A)$ where $I_0 = \{0\} \cup I$ denotes the set of nodes corresponding to the supplier (depot) and to the clients and $A = \{ij \in I_0 \times I_0 \colon i \neq j\}$. A travel cost c^{ij} is associated with each arc $ij \in A$.

An IRP solution involves decisions as to which clients are served by which vehicles in period t, the order in which they are served (i.e., the route at time t), and the amounts delivered. The objective is to minimize the sum of the storage and routing costs over the time horizon.

The following variables are used:

- x_t^i the amount shipped to client $i \in I$ in period $t \in T$;
- s_t^i the stock of client $i \in I$ at the end of time period $t \in T_0$;
- s_t^0 the stock level of the supplier at the end of time period $t \in T_0$;
- z_t^i a binary variable that is 1 if the client $i \in I$ is visited at time $t \in T$, 0 otherwise;
- z_t^0 an integer variable giving the number of vehicles delivering to one or more clients in period $t \in T$;
- y_t^{ij} a binary variable that is 1 if the arc $ij \in A$ belongs to the route of some vehicle at time $t \in T$, 0 otherwise.

A basic IRP formulation is

$$\min \quad \left\{ \sum_{t \in T_0} \sum_{i \in I_0} h_t^i s_t^i + \sum_{t \in T} \sum_{ij \in A} c^{ij} y_t^{ij} \right\}$$

$$s_t^0 = s_{t-1}^0 + D_t^0 - \sum_{i \in I} x_t^i, \quad t \in T, \tag{1}$$

$$s_t^i = s_{t-1}^i + x_t^i - D_t^i, \quad i \in I, t \in T, \tag{2}$$

$$s_0^i = sinit^i, \quad i \in I_0, \tag{3}$$

$$x_t^i \le W^i z_i^t \quad i \in I, t \in T, \tag{4}$$

$$s_t^i \le W^i - D_t^i, \quad i \in I, t \in T, \tag{5}$$

$$z_t^i \le z_t^0 \le K, \quad i \in I, t \in T, \tag{6}$$

$$\sum_{i \in I} x_t^i \le C z_t^0, \quad t \in T, \tag{7}$$

$$\sum_{j \in I_0} y_t^{ji} = \sum_{j \in I_0} y_t^{ij} = z_t^i, \quad i \in I_0, t \in T, \tag{8}$$

$$\sum_{ij \in (I_0 \setminus S : S)} y_t^{ij} \ge z_t^{\ell}, \quad \ell \in S \subset I, t \in T, \tag{9}$$

$$\sum_{ij \in (I_0 \setminus S : S)} C y_t^{ij} \ge \sum_{i \in S} x_t^i, \quad S \subseteq I, t \in T, \tag{10}$$

$$\sum_{t\in T} z_t^0 \geq \left\lceil \frac{\sum_{i\in I}(D^i_{1,T_{\max}} - sinit^i)}{C} \right\rceil, \quad (11)$$
$$z_t^i \in \{0,1\}, \quad i \in I, t \in T, \quad (12)$$
$$z_t^0 \in \mathbb{Z}^+, \quad t \in T, \quad (13)$$
$$y_t^{ij} \in \{0,1\}, \quad ij \in A, t \in T, \quad (14)$$
$$s_t^i \geq 0, \quad i \in I_0, t \in T_0, \quad (15)$$
$$x_t^i \geq 0, \quad i \in I, t \in T, \quad (16)$$

where $(E\colon F)$ denotes the set of arcs $ij \in A$ with $i \in E$ and $j \in F$, and $D^i_{uv} \equiv \sum_{t=u}^{v} D^i_t$.

Constraints (1) and (2) are the balance constraints for the inventory levels of the supplier and the clients, respectively. Constraints (3) give the initial stock levels for the supplier and the clients. The variable upper bounds (4) bound the amount delivered to client i and enforce $z_t^i = 1$ if the client is served at time t. Constraints (5) give the stock upper bounds of each client. Constraints (6) enforce $z_t^0 \geq 1$ if at least one client is served in period t and limit the number of vehicles available in each period. Constraints (7) impose that the total amount shipped from the supplier to clients at t does not exceed the vehicle capacity C times the number of vehicles in use. Constraints (8) ensure that the number of vehicles leaving and arriving at the depot corresponds to the number of vehicles in use in each period and that a vehicle arrives at and leaves the client j in period t if and only if $z_t^j = 1$. Constraints (9) are classical subtour elimination constraints that are dynamically added to the formulation. The separation algorithm consists of solving a min-cut problem between 0 and i, for each $i \in V$ and $t \in T$ on the graph $G(I_0, A)$, where the arcs A are weighted with the fractional values attained by the variables y_t^{ij} in the current linear programming (LP) relaxation, see, for instance, Naddef and Rinaldi (2001). Constraints (10) are Capacitated Subtour Elimination constraints (Adulyasak, Cordeau, and Jans 2014) that are dynamically added to the formulation. It again follows from Picard and Ratliff (1973) that the separation algorithm is polynomially solvable as a min-cut problem. Constraint (11) provides a lower bound on the number of vehicle tours needed to satisfy demand over the whole time horizon.

Formulation (1)–(16) can be tightened by noting that $s^i_{t-1} \geq sinit^i - D^i_{1,t-1}$ and so an upper bound on the delivery quantity in t is $W^i - (sinit^i - D^i_{1,t-1})^+$. In addition one can use the single-item reformulations of the lot-sizing problem with stock upper bounds discussed in Avella, Boccia, and Wolsey (2015).

3. The Single-Period Subproblem: Introducing Disjoint Route Inequalities

The Single-Period Inventory Routing Problem is defined by constraints (1)–(10) of the formulation, for a fixed t. Specifically, we consider the set (SIRP)

$$\sum_{i\in I_0} y_t^{ij} = \sum_{i\in I_0} y_t^{ji} = z_t^j, \quad j \in I_0, \quad (17)$$
$$\sum_{ij\in(I_0\setminus S: S)} y_t^{ij} \geq z_t^{\ell}, \quad \ell \in S \subseteq I, \quad (18)$$
$$\sum_{ij\in(I_0\setminus S: S)} C y_t^{ij} \geq \sum_{i\in S} x_t^i, \quad S \subseteq I, \quad (19)$$
$$z_t^i \leq z_t^0 \leq K, \quad i \in I, \quad (20)$$
$$0 \leq x_t^i \leq W_t^i z_t^i, \quad i \in I, \quad (21)$$
$$z_t^0 \in \mathbb{Z}^1_+,\ z_t^i \in \{0,1\}\ i \in I,\ y_t^{ij} \in \{0,1\}\ ij \in A, \quad (22)$$
$$s^i_{\tau-1} + x^i_\tau = D^i_\tau + s^i_\tau, x^i_\tau \geq 0 \quad \tau \in \{t, t+1\}, i \in I, \quad (23)$$
$$0 \leq s^i_\tau \leq W^i_\tau - D^i_\tau, \quad \tau \in \{t-1, t, t+1\}, i \in I. \quad (24)$$

Constraints (17)–(22) define the basic single period model (SIRP). The additional constraints (23)–(24) will allow us to generate inequalities involving the stock variables thereby linking consecutive periods. Note that constraints (17)–(22) with the additional equations $x_t^i = D^i z_t^i$ give the feasible region of a "Routing Problem with Profits" discussed in Archetti, Bianchessi, and Speranza (2013); with $x_t^i = D^i z_t^i, z_t^i = 1$ one obtains a formulation of the well known capacitated vehicle routing problem (CVRP).

Now we introduce the Disjoint Route (DR) Inequalities for SIRP. The proof of validity is given for a more general case in Section 4.

Proposition 1. *Given a period $t \in T$, $S \subseteq I$ and $\mu^{ij} \geq 0$ for $ij \in A$, the inequality*

$$\sum_{ij\in A} \mu^{ij} y_t^{ij} \geq \sum_{i\in S} x_t^i$$

is valid for SIRP ((17)–(22)) if

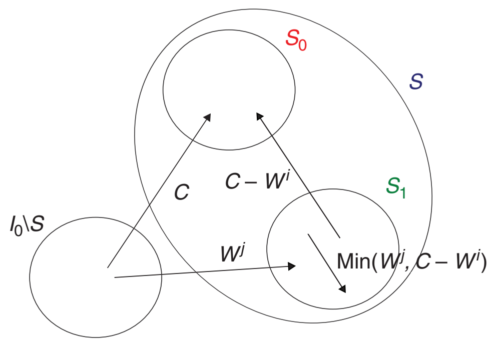
$$\sum_{ij\in R} \mu^{ij} \geq \min\left(C, \sum_{i\in V(R)\cap S} W^i\right) \quad (25)$$

for every route (subtour) R starting and ending at the depot, where $V(R)$ is the set of nodes in I of route R.

In Figure 1 we show the μ^{ij} values leading to a disjoint route inequality. Specifically, we have a partition S_0, S_1 of S. We set $\mu^{ij} = C$ for $ij \in (I_0\setminus S\colon S_0)$, $\mu^{ij} = W^j$ for $ij \in (I_0\setminus S\colon S_1)$, $\mu^{ij} = C - W^i$ for $ij \in (S_1\colon S_0)$, $\mu^{ij} = \min(C - W^i, W^j)$ for $ij \in (S_1\colon S_1)$, and $\mu^{ij} = 0$ otherwise. It is easily checked that the condition (25) holds for every route R and every $S \subseteq I$.

Note that with $\mu^{ij} = C$ for $ij \in (I_0\setminus S\colon S)$ and $\mu^{ij} = 0$ for $ij \notin (I_0\setminus S\colon S)$ one obtains the capacitated subtour constraint (19).

Figure 1. (Color online) A DR Inequality



4. The DR Inequalities

Now we present a more general family of DR Inequalities for SIRP ((17)–(24)) taking into account the demands and stocks from the periods $t-1$ and $t+1$.

Theorem 1. *Given a period $t \in T$, a subset $S \subseteq I$, a partition (S_1, S_2, S_3, S_4) of S, and $\mu^{ij} \geq 0$ for $ij \in A$, the inequality*

$$\sum_{ij \in A} \mu^{ij} y_t^{ij} \geq \sum_{i \in S_1} x_t^i + \sum_{i \in S_2} (x_t^i - s_{t+1}^i) + \sum_{i \in S_3} (x_t^i - s_t^i) + \sum_{i \in S_4} (s_t^i - (W^i - D_{t-1,t}^i)) \tag{26}$$

is valid for SIRP if

$$\sum_{ij \in R} \mu^{ij} \geq \min\left(C, \sum_{i \in V(R) \cap S_1} W^i + \sum_{i \in V(R) \cap S_2} D_{t,t+1}^i + \sum_{i \in V(R) \cap S_3} D_t^i + \sum_{i \in V(R) \cap S_4} D_{t-1}^i\right), \tag{27}$$

for every route R starting and ending at the depot.

Proof. Let (x_t, y_t, z_t, s) be a feasible solution of SIRP ((17)–(24)) for period t and $R_1, \ldots, R_Q$ with $Q \leq K$ be the corresponding routes. As the routes are arc-disjoint

$$\sum_{ij \in A} \mu^{ij} y_t^{ij} = \sum_{q=1}^{Q} \sum_{ij \in R_q} \mu^{ij}.$$

Let R be one such route, $T_\ell = V(R) \cap S_\ell$ and $T = \bigcup_{\ell=1}^4 T_\ell$.

By hypothesis, $\sum_{ij \in R} \mu^{ij} \geq \min(C, \sum_{i \in T_1} W^i + \sum_{i \in T_2} D_{t,t+1}^i + \sum_{i \in T_3} D_t^i + \sum_{i \in T_4} D_{t-1}^i)$.

As each vehicle has capacity C, we have $C \geq \sum_{\ell=1}^4 \sum_{i \in T_\ell} x_t^i$. Therefore

$$\sum_{ij \in R} \mu^{ij} \geq \sum_{i \in T_1} \min(x_t^i, W^i) + \sum_{i \in T_2} \min(x_t^i, D_{t,t+1}^i) + \sum_{i \in T_3} \min(x_t^i, D_t^i) + \sum_{i \in T_4} \min(x_t^i, D_{t-1}^i)$$

as $\min(a_1 + a_2, b_1 + b_2) \geq \min(a_1, b_1) + \min(a_2, b_2)$.

For $i \in T_1$, we have $W^i \geq x_t^i$ and thus $\min(x_t^i, W^i) \geq x_t^i$.

For $i \in T_2$, we have $x_t^i \geq x_t^i - s_{t+1}^i$ as $s_{t+1}^i \geq 0$. Also $D_{t,t+1}^i \geq x_t^i - s_{t+1}^i$ as $x_t^i \leq D_t^i + D_{t+1}^i + s_{t+1}^i$. Thus $\min(x_t^i, D_{t,t+1}^i) \geq x_t^i - s_{t+1}^i$.

For $i \in T_3$, we have $x_t^i \geq x_t^i - s_t^i$ as $s_t^i \geq 0$. Also $D_t^i \geq x_t^i - s_t^i$ as $x_t^i \leq D_t^i + s_t^i$. Thus $\min(x_t^i, D_t^i) \geq x_t^i - s_t^i$.

For $i \in T_4$, we have $x_t^i = s_t^i + D_t^i - s_{t-1}^i$. As $s_{t-1}^i \leq W^i - D_{t-1}^i$; this gives $x_t^i \geq s_t^i + D_t^i - (W^i - D_{t-1}^i)$.

Also as $s_t^i \leq W^i - D_t^i$, $D_{t-1}^i \geq D_{t-1}^i + s_t^i - (W^i - D_t^i) = s_t^i - (W^i - D_{t-1,t}^i)$. Thus $\min(x_t^i, D_{t-1}^i) \geq s_t^i - (W^i - D_{t-1,t}^i)$.

So we have shown that

$$\sum_{ij \in R} \mu_t^{ij} \geq \sum_{i \in T_1} x_t^i + \sum_{i \in T_2} (x_t^i - s_{t+1}^i) + \sum_{i \in T_3} (x_t^i - s_t^i) + \sum_{i \in T_4} (s_t^i - (W^i - D_{t-1,t}^i)).$$

Summing over the Q disjoint routes gives the inequality (26). □

Even for a fixed set $S \subseteq I$, the separation problem for the DR Inequalities appears to be intractable, so we consider two subfamilies, i.e., Simple DR and h-DR Inequalities, whose separation problem can be efficiently solved in practice.

4.1. Simple DR Inequalities

We assume for simplicity that $C \geq W^i \geq D_{t,t+1}^i$ for all $i \in I$. The following proposition introduces the *Simple Disjoint Route* (*Simple DR*) Inequalities.

Proposition 2. *Let $S \subseteq I$, $(S_0, S_1, S_2, S_3, S_4)$ be a partition of S. Let $v^i = C$ for $i \in S_0$, $v^i = W_t^i$ for $i \in S_1$, $v^i = D_{t,t+1}^i$ for $i \in S_2$, $v^i = D_t^i$ for $i \in S_3$, $v^i = D_{t-1}^i$ for $i \in S_4$, and $v^i = 0$ for $i \in I_0 \backslash S$.*

Set $\mu^{ij} = \min(C - v^i, v^j)$ for all $ij \in A$.

Then the inequality

$$\sum_{ij \in A} \mu^{ij} y_t^{ij} \geq \sum_{j \in S_0 \cup S_1} x_t^j + \sum_{j \in S_2} (x_t^j - s_{t+1}^j) + \sum_{j \in S_3} (x_t^j - s_t^j) + \sum_{j \in S_4} (s_t^j - (W^j - D_{t-1,t}^j)) \tag{28}$$

is a DR inequality valid for SIRP.

Proof. Consider any initial subpath P of a route R starting at the depot. We claim that

$$\sum_{ij \in P} \mu^{ij} \geq \min\left(C, \sum_{i \in V(P) \cap S} v^i\right). \tag{29}$$

The claim clearly holds for paths of length $p = 1$. Now consider a subpath P of length p ending at node k and its extension $P' = P \cup (k\ell)$. It suffices to consider the case where $\sum_{ij \in P} \mu^{ij} < C$ and thus we assume $\sum_{ij \in P} \mu^{ij} \geq \sum_{i \in V(P) \cap S} v^i$.

(i) If $\mu^{k\ell} = v^\ell$, then $\sum_{ij \in P'} \mu^{ij} \geq \sum_{i \in V(P) \cap S} v^i + v^\ell = \sum_{i \in V(P') \cap S} v^i$.

(ii) If $\mu^{k\ell} = C - v^k$, then $\sum_{ij \in P'} \mu^{ij} = \sum_{ij \in P} \mu^{ij} + (C - v^k) \geq v^k + (C - v^k) = C$.

Thus the condition (29) holds for any subpath P' of length $p+1$, and thus (27) holds for all routes. It follows that (28) is a DR inequality. □

Note that when $S_2 = S_3 = S_4 = \emptyset$, one obtains the inequality of Proposition 1 and Figure 1.

4.1.1. Separation of Simple DR Inequalities. The separation problem for the Simple DR Inequalities amounts to solving a 0-1 Quadratic Programming problem. Let $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$ be the current fractional solution. Let $S_5 = I_0 \backslash S$ with $v^5 = 0$. Let α_k^i be a binary variable which is 1 if $i \in S_k$ and 0 otherwise, and let v_k^i be the value of v^i when $i \in S_k$. Then the separation problem for (28) can be written as

$$\min \left\{ \sum_{ij\in A} \sum_{k=0}^{5} \sum_{l=0}^{5} \min(C - v_k^i, v_l^j) \bar{y}^{ij} \alpha_k^i \alpha_l^j - \sum_{j\in I_0} \bar{x}_t^j (\alpha_0^j + \alpha_1^j) - \sum_{j\in I_0} (\bar{x}_t^j - \bar{s}_{t+1}^j) \alpha_2^j - \sum_{j\in I_0} (\bar{x}_t^j - \bar{s}_t^j) \alpha_3^j - \sum_{j\in I_0} (\bar{s}_t^j - (W^j - D_{t-1,t}^j)) \alpha_4^j \right\},$$

$$\sum_{k=0}^{5} \alpha_k^j = 1, \quad j \in I_0,$$

$$\alpha_k^j \in \{0,1\} \quad j \in I_0, \quad k \in 0, \dots, 5, \quad \alpha_5^0 = 1.$$

Here the 0-1 variables α_k^j define the partition $(S_1, S_2, S_3, S_4, S_5)$ of I_0 and the objective function is the difference between the left-hand side (lhs) and the right-hand side (rhs) of (28).

Standard linearization gives an MIP that is easy in practice. In general the resulting linear program is not integral due to the constraints $\sum_{k=0}^{5} \alpha_k^j = 1$ for $j \in I_0$.

4.2. h-DR Inequalities

By considering paths of maximum length $h \ge 2$, we get a different subfamily of DR Inequalities, the *h-Disjoint Route* (*h-DR*) *Inequalities*. As before $V(R)$ denotes the set of nodes in I of a route R. Let i_R denote the first node of route R after leaving the depot.

Proposition 3. *Let $S \subseteq I$ and let (S_1, S_2, S_3, S_4) be a partition of S. Let Π_h be the set of routes R starting and ending at the depot with $|V(R)| = k \le h$. The h-DR Inequality*

$$\sum_{ij\in(I_0:S)\backslash(S_1:S_1)} \ell_j y_t^{ij} + \sum_{ij\in(S_1:S_1)} m_{ij} y_t^{ij} \ge \sum_{i\in S_1} x_t^i + \sum_{i\in S_2} (x_t^i - s_{t+1}^i) + \sum_{i\in S_3} (x_t^i - s_t^i) + \sum_{i\in S_4} (s_t^i - (W^i - D_{t-1,t}^i)) \tag{30}$$

is a valid DR Inequality for SIRP if
(i) $W^j \le \ell_j \le C$, $j \in S_1$;
(ii) $D_{t,t+1}^j \le \ell_j \le C$, $j \in S_2$;
(iii) $D_t^j \le \ell_j \le C$, $j \in S_3$;
(iv) $D_{t-1}^j \le \ell_j \le C$, $j \in S_4$;
(v) $\ell_{i_R} + \sum_{ij\in R} m_{ij} \ge \min(C, \sum_{i\in V(R)} W^i)$, $R \in \Pi_{h-1}$ *with* $V(R) \subseteq S_1$;
(vi) $\ell_{i_R} + \sum_{ij\in R} m_{ij} \ge C$, $R \in (\Pi_h \backslash \Pi_{h-1})$ *with* $V(R) \subseteq S_1$;
(vii) $m_{ij} \ge 0$, $ij \in (S_1 \colon S_1)$.

Proof. We show that inequality (30) is a DR inequality. Setting $\mu_{ij} = \ell_j$ for $ij \in (I_0 \colon S) \backslash (S_1 \colon S_1))$, $\mu_{ij} = m_{ij}$ for $ij \in (S_1 \colon S_1)$, and $\mu_{ij} = 0$ otherwise, we need to show that condition (27) holds on every route.

First, we prove the proposition for the case $S_2 \cup S_3 \cup S_4 = \emptyset$ and then in the general case.

Case $S_2 \cup S_3 \cup S_4 = \emptyset$, $(S = S_1)$.

Let R be a route beginning and ending at the depot. We break up the route into subpaths with nodes $T_1, \dots, T_L$ with $L \ge 1$ where each subpath T_ℓ consists of a path in $I_0 \backslash S$ followed by a path in S, except for the last subpath T_L that just consists of a subpath in $I_0 \backslash S$ ending at the depot. Let A_ℓ be the set of arcs of T_ℓ. We will show that

$$\sum_{ij\in A_\ell} \mu^{ij} \ge \min\left(C, \sum_{i\in T_\ell \cap S} W^i\right) \tag{31}$$

for all ℓ. Summing up all ℓ will then give

$$\sum_{ij\in R} \mu^{ij} = \sum_{\ell=1}^{L} \min\left(C, \sum_{i\in T_\ell\cap S} W^i\right) \ge \min\left(C, \sum_{i\in V(R)\cap S} W^i\right),$$

where the inequality follows as $\min(a,b) + \min(a,c) \ge \min(a, b+c)$ when $a, b, c \ge 0$ and the arcs linking subpaths all have heads in $I_0 \backslash S$ and $\mu^{ij} = 0$.

Thus it suffices to show that LHS(31) $\ge$ RHS(31) on each subpath T_ℓ, where LHS(31) denotes the expression on the LHS of the inequality sign in (31).

Since each subpath T_ℓ has the same μ-value as the route $(0, T_\ell, 0)$ of length $|T_\ell|$, the conditions (iv) and (v) ensure that LHS(31) $\ge$ RHS(31) holds for each T_ℓ.

Case $S_2 \cup S_3 \cup S_4 \neq \emptyset$.

LHS(30) can be rewritten as the sum of three terms

$$\sigma_1 \equiv \sum_{ij\in(I_0\backslash S_1:S_1)} \ell_j y_t^{ij} + \sum_{ij\in(S_1:S_1)} m_{ij} y_t^{ij},$$
$$\sigma_2 \equiv \sum_{ij\in(I_0:S_2)} \ell_j y_t^{ij},$$
$$\sigma_{34} \equiv \sum_{ij\in(I_0:S_3\cup S_4)} \ell_j y_t^{ij}.$$

For each route R, the μ-value of T_1 is the same as that of a route R' in which each visit to a node of $S_2 \cup S_3 \cup S_4$ is replaced by a node in $I_0 \backslash S$. The value of T_1 for this route R' was treated above and it was shown that

$$\sum_{ij\in(I_0\backslash S_1:S_1)} \ell_j y_t^{ij} + \sum_{ij\in(S_1:S_1)} m_{ij} y_t^{ij} \ge \min\left(C, \sum_{i\in V(R)\cap S_1} W^i\right). \tag{32}$$

Now we consider the term σ_2. By condition (ii), $\ell_i \ge D_{t,t+1}^i$ for each $i \in S_2$. So, for each route starting and ending at the depot, we have

$$\sum_{ij\in(I_0:S_2)} \ell_j y_t^{ij} \ge \sum_{i\in V(R)\cap S_2} D_{t,t+1}^i. \tag{33}$$

Finally, we consider the term σ_{34}. By condition (iii), $\ell_i \geq D_t^i$ for each $i \in S_3$. So, for each route starting and ending at the depot, we have

$$\sum_{ij \in (I_0: S_3)} \ell_j y_t^{ij} \geq \sum_{i \in V(R) \cap (S_3)} D_t^i. \tag{34}$$

The treatment of the last term is similar.

Summing (32)–(34), we have

$$\sum_{ij \in R} \mu^{ij} \geq \min\Bigg(C, \sum_{i \in V(R) \cap S_1} W^i + \sum_{i \in V(R) \cap S_2} D_{t,t+1}^i + \sum_{i \in V(R) \cap (S_3)} D_t^i + \sum_{i \in V(R) \cap (S_4)} D_{t+1}^i\Bigg)$$

for each route. So condition (27) is satisfied and the claim follows. □

4.2.1. Separation of h-DR Inequalities. As the separation problem for the h-DR inequalities appears to be hard, we consider separation for two subfamilies. In the first case, we assume that S and its partition are given; in the second we assume that $S_2 = S_3 = S_4 = \emptyset$, but we select $S = S_1$ optimally.

Separation Algorithm 1. Let $S \subseteq I$ and (S_1, S_2, S_3, S_4) be a given partition of S and let $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$ be the current fractional solution. We assign values 0 to the arcs ij with $j \in I_0 \backslash S$, a variable value m_{ij} to arcs ij with $ij \in (S_1: S_1)$, and variable value ℓ_j to all other arcs in $(I_0: S) \backslash (S_1: S_1)$.

This leads to the following linear program where the objective corresponds to the lhs of (30) and the rhs of (30) is constant:

$$v^* = \sum_{ij \in (I_0: S) \backslash (S_1: S_1)} \bar{y}_t^{ij} \ell_j + \sum_{ij \in (S_1: S_1)} \bar{y}_t^{ij} m_{ij}, \tag{35}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq \min\Bigg(C, \sum_{i \in V(R)} W^i\Bigg), \quad R \in \Pi_{h-1}, \ V(R) \in S_1, \tag{36}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq C, \quad R \in \Pi_h \backslash \Pi_{h-1}, \ V(R) \in S_1, \tag{37}$$

$$W^i \leq \ell_i \leq C, \quad i \in S_1, \tag{38}$$

$$D_{t,t+1}^i \leq \ell_i \leq C, \quad i \in S_2, \tag{39}$$

$$D_t^i \leq \ell_i \leq C, \quad i \in S_3, \tag{40}$$

$$D_{t-1}^i \leq \ell_i \leq C, \quad i \in S_4, \tag{41}$$

$$m_{ij} \geq 0, \quad ij \in (S_1, S_1).$$

The inequality (30) is violated if we have $v^* < \sum_{i \in S_1} \bar{x}_t^i + \sum_{i \in S_2} (\bar{x}_t^i - \bar{s}_{t+1}^i) + \sum_{i \in S_3} (\bar{x}_t^i - \bar{s}_t^i) + \sum_{i \in S_4} (\bar{s}_t^i - (W^i - D_{t-1,t}^i))$.

This separation LP becomes hard to solve when $|S_1| \geq 20$ because of the huge number of constraints (36) and (37). To overcome this problem, we adopt the following procedure:

- Solve the separation LP on a reduced network $(I_0, \bar{A})$ where $\bar{A} = \{ij: \bar{y}_t^{ij} > 0\}$. Let $\bar{\ell}_i$ for $i \in I$ and $\bar{m}_{ij}$ for $ij \in \bar{A}$ be its solution.
- Extend the solution by setting $\bar{m}_{ij} = 0$ for all $ij \in A \backslash \bar{A}$.
- Enumerate all of the routes $R \in \Pi_h$. As long as a route R is encountered for which (36) or (37) is violated, let $uv \notin \bar{A}$ be an arc of R. Set

$$\bar{m}_{uv} \leftarrow \min\Bigg(C, \sum_{i \in V(R) \cap S_1} W^i + \sum_{i \in V(R) \cap S_2} D_{t,t+1}^i + \sum_{i \in V(R) \cap (S_3)} D_t^i + \sum_{i \in V(R) \cap (S_4)} D_{t-1}^i\Bigg) - \sum_{ij \in R} \bar{m}_{ij} - \bar{\ell}_i.$$

In case of ties, select uv corresponding to the variable y_t^{uv} with the maximum reduced cost in the IRP relaxation providing the current fractional solution $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$.

Note that in the enumeration routes R are often quickly truncated because $\bar{\ell} i + \sum_{ij \in R} \bar{m}_{ij} > C$.

Finally, to choose the set S and its partition, we take the partition obtained by solving the quadratic 0-1 separation problem for the simple DR inequalities presented in Section 4.1.1.

Separation Algorithm 2. With $S_2 = S_3 = S_4 = \emptyset$, the exact separation problem for the h-DR inequalities can be formulated as a (big-M) MIP. Specifically, let α^i be a binary variable such that $\alpha^i = 1$ if $i \in S = S_1$ and 0 otherwise. This leads to the formulation

$$\min\Bigg\{\sum_{ij \in A} \bar{y}_t^{ij} m^{ij} - \sum_{i \in I} \bar{x}_t^i \alpha^i\Bigg\}, \tag{42}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq \min\Bigg(C, \sum_{i \in V(R)} W^i\Bigg)\Bigg(\sum_{j \in V(R)} \alpha^j - (|V(R)| - 1)\Bigg), \quad R \in \Pi_{h-1}, \tag{43}$$

$$\ell_{i_R} + \sum_{ij \in R} m_{ij} \geq C\Bigg(\sum_{j \in V(R)} \alpha^i - (h-1)\Bigg), \quad R \in \Pi_h \backslash \Pi_{h-1}, \tag{44}$$

$$m_{ij} \geq \ell_j - C\alpha^i - C(1 - \alpha^j), \quad i \in I_0, \ j \in I, \tag{45}$$

$$W^i \alpha^i \leq \ell_i \leq C\alpha^i, \quad i \in I, \tag{46}$$

$$m_{ij} \geq 0, \ ij \in A, \ \alpha^i \in \{0, 1\}, \ i \in I.$$

Note that the inequalities (43) and (44) enforce (36) and (37) when the path lies in S. The (45) and the nonnegative coefficient of m_{ij} in the objective function (42) lead to $m_{ij} = \ell_j$ for $ij \in (I_0 \backslash S: S)$. Finally, (46) implies (38) when $i \in S$.

This MIP is also iteratively solved as in Separation Algorithm 1. The MIP is first solved over the reduced network $(I_0, \bar{A})$ providing a set S and an initial solution $(\bar{\ell}, \bar{m})$. The conversion to a feasible solution follows the same procedure as above. Surprisingly the above MIP can be solved in a reasonable time on the benchmark instances.

5. Multiperiod Capacitated Subtour Inequality

Coelho and Laporte (2014) introduced a family of multiperiod inequalities that we call *Multiperiod Capacitated Subtours*.

Let $H = [t_1, t_2]$ be a time interval within T. As $s^i_{t_1-1} \le W^i - D^i_{t_1-1}$, $ND^i_H = D^i_{t_1,t_2} - (W^i - D^i_{t_1-1})$ is a lower bound on the amount that must be delivered to client i during the interval H. Let $S \subseteq I$ be a subset of clients.

Proposition 4. *The Multiperiod Capacitated Subtour Inequality*

$$\sum_{t\in H}\sum_{ij\in(I_0\setminus S:S)} y^{ij}_t \ge \left\lceil \frac{\sum_{i\in S} ND^i_H}{C} \right\rceil \tag{47}$$

is valid for IRP.

Here we generate an (almost exact) separation algorithm for these inequalities.

5.1. Separation of Multiperiod Capacitated Subtours

Given the current fractional solution $(\bar{\mathbf{z}}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{s}})$, the separation problem we wish to solve is

$$v^* = \min_{S\subseteq I} \sum_{t\in H}\sum_{ij\in(I_0\setminus S:S)} \bar{y}^{ij} - \left\lceil \frac{\sum_{i\in S} ND^i_H}{C} \right\rceil, \tag{48}$$

which can be rewritten as

$$v^* = \min \sum_{t\in H}\sum_{ij\in A} \bar{y}^{ij}(1-\alpha^i)\alpha^j - \gamma, \tag{49}$$

$$\gamma \le \left\lceil \frac{\sum_{i\in I} ND^i_H}{C}\alpha^i \right\rceil, \tag{50}$$

$$\alpha \in \{0,1\}^I,\ \gamma \in \mathbb{Z}, \tag{51}$$

where $\alpha^i = 1$ if $i \in S$ and 0 otherwise for $i \in I$ and γ represents the integer value of the rhs of (48).

To linearize the nonlinear integer round up expression, we replace the inequality (50) by the linear constraint

$$C\gamma \le \sum_{i\in I} ND^i_H \alpha^i + C - \epsilon, \tag{52}$$

where ϵ is a very small positive constant. This constraint imposes that $\gamma \le \lceil (\sum_{i\in I} ND^i_H\alpha^i)/C \rceil$ when the fractional part f of $(\sum_{i\in I} ND^i_H\alpha^i)/C$ lies in $\{0\} \cup [\epsilon/C, 1)$ and $\gamma \le \lceil (\sum_{i\in I} ND^i_H\alpha^i)/C \rceil - 1$ otherwise. Thus, the violation of the latter ϵ-small set of points is underestimated by one unit.

Finally, a standard linearization of the 0-1 quadratic terms gives a block-structured MIP problem, having a block for each period. The linear problem is not integral because of the rounding constraint (52), which links all of the blocks. However the problem is very efficiently solved in practice by an MIP solver.

6. Implementation Details

The separation procedures for the Subtour, DR, and the Multiperiod Capacitated Subtour Inequalities have been embedded into the cutting planes generation callbacks of the commercial MIP solver FICO Xpress 7.6. The code is written in ANSI C using the Microsoft Visual C++ 2013 compiler. The resulting Branch-and-Cut algorithm is run in two versions, first as a heuristic to find an initial feasible solution (upper bounding phase), then we change the settings to get provably good solutions (optimality phase).

In the upper bounding phase we adopt a simple MIP-based "fixing" heuristic: After cutting plane generation at the root node, (i) we fix to zero all of the routing variables that are at zero in the current fractional solution and then (ii) solve the restricted MIP by a truncated version of our Branch-and-Cut algorithm with a depth-first branching strategy aimed at finding good feasible solutions.

In both phases we set Xpress parameters to run the code with a single thread. We set a time limit of 1,800 seconds for the upper bounding phase and 3,600 seconds for the optimality phase. The Xpress preprocessing, heuristics, and cut generation procedures were disabled to allow use of callbacks.

6.1. Preprocessing

Reduced cost fixing on the routing variables y^{ij}_t is performed at the end of the root node.

6.2. A Priori Reformulation

The upper bounds on deliveries W^i_t are tightened if the stock at the end of the previous period has a positive lower bound. The tight formulations for single item lot-sizing with stock upper bounds given in Avella, Boccia, and Wolsey (2015) are added for each item/client.

6.3. Cutting Strategy

Separation procedures are performed in the following order:

(i) Subtour Elimination Constraints (at every node of the search tree).

(ii) Capacitated Subtour Constraints (at every node of the search tree).

(iii) Multiperiod Capacitated Subtour Inequalities (only at the root node).

(iv) Simple DR Inequalities (only at nodes with depth ≤ 5).

(v) h-DR inequalities generated by Separation Algorithm 1 (only at nodes with depth ≤ 5).

(vi) MIP separation of h-DR Inequalities (Separation Algorithm 2) with $h \le 8$ if the number of clients does not exceed 30 (only at nodes with depth ≤ 5).

Each procedure is run repeatedly until the violation falls below a given tolerance. The algorithm then moves on to the next procedure. When all six procedures have terminated, one restarts with procedure (i). However, in the heuristic/upper bounding phase separation procedures (iii)–(vi) are only performed at the root node.

6.4. Branching Strategy

In the upper bounding phase, the node selection strategy is best-first for the first 100 nodes of the search tree, then depth-first. In the optimality phase, the strategy is best-first. The variable selection strategy is the Xpress 7.6 default strategy.

7. Computational Results

We report detailed computational results on a large set of IRP instances, comparing the results of our Branch-and-Cut algorithm (denoted ABW), with the reported results of the Branch-and-Cut algorithm of Coelho and Laporte (2014) (denoted CL), based on a 3-index (arc ij on vehicle k) formulation; and of the Branch-and-Cut-and-Price algorithm of Desaulniers, Rakke, and Coelho (2016) (denoted DRC), in which the columns correspond to route and delivery patterns for a given period.

Our computation was carried out on an Intel Core i7-2620 processor clocked at 2.7 GHz with 8 GB RAM. The reported results for the CL and DRC algorithms were both obtained using an Intel Core i7-2600 processor clocked at 3.4 GHz with 8 cores and 16 GB RAM. Only a single core was used for both algorithms.

The test bed consists of the largest and most challenging instances in the data set introduced in Archetti et al. (2007), i.e., those with 50 clients and 3 periods, and those with 15, 20, 25, and 30 clients and 6 periods. All of the instances and the computational results for the CL and DRC algorithms are available on L. Coelho's webpage: http://www.leandro-coelho.com/instances/. The instances are partitioned into two groups: those with "low" storage costs, i.e., $h_i \in [0.01, 0.05]$ and $h_0 = 0.03$, and those with "high" storage costs, i.e., $h_i \in [0.1, 0.5]$ and $h_0 = 0.3$. As usual in testing exact algorithms for IRP, travel costs are given by the Euclidean distance between each pair of nodes, rounded to the nearest integer; the vehicle capacity is obtained by dividing the original capacity by the number of vehicles, rounded to the nearest integer. In our experiments $x.5$ was rounded to $x + 1$; the rounding policy adopted in the DRC and CL experiments is not detailed.

Each instance is labeled as nXX-TY-{low,high}-f, where XX is the number of clients, Y is the number of periods, the attribute {low, high} denotes the magnitude of the storage costs, and f is an integer identifying the instance.

We first report briefly on the behavior of our cutting plane procedures. In Table 1, we present lower bounds at the root node for four variants of two instances with 20 clients and 6 periods. The initial formulation consists of (1)–(16) plus the single-item lot-sizing reformulations presented in Avella, Boccia, and Wolsey (2015). Column 2 gives the number of vehicles K and column 3 the best upper bound obtained by our Branch-and-Cut algorithm. Columns 4–6 present the results obtained using just the subtour separation routines (i) and (ii), columns 7–9 using routines (i)–(iii) (multiperiod cuts), columns 10–12 using (i)–(iv), and finally columns 13–15 using (i)–(vi). In each case, we run the separation routines repeatedly until no more cuts are generated. For each set of three columns we give the lower bound obtained, the time for separation and re-optimization, and finally the %gap relative to the best solution given in Column 3.

In Tables 2 and 3, we report results with the ABW algorithm for the instances with 50 clients and 3 periods and with 30 clients and 6 periods along with the results of CL and DRC. Detailed results for the instances with 15, 20, and 25 clients and 6 periods are reported in the online appendix. Each table is organized as follows. Column *Name* shows the name of the instance and K is the number of vehicles. For each algorithm, *LB*, *BLB*, and *BUB* denote the lower bound at the root node, the best lower bound, and the best upper bound found after enumeration, respectively. Column *Time* show the time in CPU seconds spent to produce the final *BLB* and *BUB* values.

Table 1. Lower Bounds from Cuts at the Root Node for Two $n20$-$T6$ Instances

			(i)–(ii)			(i)–(iii)			(i)–(iv)			(i)–(vi)		
Name	*K*	*BUB*	*LB*	Time	%gap	*LB*	Time	%gap	*LB*	Time	%gap	*LB*	Time	%gap
Low-1	2	7,752.98	6,902.48	3	10.97	7,106.96	8	8.33	7,112.12	23	8.27	7,177.75	236	7.42
	3	9,073.71	8,099.41	6	10.74	8,348.60	19	7.99	8,395.32	90	7.48	8,486.83	305	6.47
	4	10,499.07	9,546.50	9	9.07	10,009.91	24	4.66	10,105.56	178	3.75	10,145.41	382	3.37
	5	12,603.24	11,049.65	10	12.33	11,486.53	37	8.86	11,570.32	265	8.20	11,602.3	216	7.94
High-1	2	15,677.13	15,058.76	3	3.94	15,214.86	7	2.95	15,236.52	21	2.81	15,325.86	156	2.24
	3	17,107.01	16,251.64	5	5.00	16,476.50	15	3.69	16,554.75	96	3.23	16,669.22	285	2.56
	4	18,558.81	17,690.91	6	4.68	18,105.31	18	2.44	18,244.36	136	1.69	18,289.07	200	1.45
	5	20,715.52	19,182.12	7	7.40	19,580.9	24	5.48	19,709.05	115	4.86	19,757.54	211	4.62

Table 2. Computational Results for $n50$-$T3$ Instances

				ABW					DRC					CL				
Name	K	UB_{heu}	Time	LB	BLB	BUB	%gap	Time	LB	BLB	BUB	%gap	Time	LB	BLB	BUB	%gap	Time
Low-1	2	4,737.86	1,800	4,529.66	4,554.42	4,737.86	3.87	3,600	4,457.2	4,457.2	7,673.91	41.92	7,200	4,163.4	4,567.16	4,567.16	0.00	1,536
Low-2	2	4,861.92	509	4,741.71	4,842.92	4,842.92	0.00	382	4,768.72	4,770.25	4,943.3	3.50	7,200	4,465.64	4,842.92	4,842.92	0.00	2,707
Low-3	2	4,749.44	571	4,629.22	4,705.74	4,705.74	0.00	1,261	4,636.45	4,645.51	8,054.85	42.33	7,200	4,463.88	4,652.31	4,730.31	1.65	7,200
Low-4	2	4,695.84	122	4,631.84	4,662.84	4,662.84	0.00	154	4,636.68	4,636.68	6,287.6	26.26	7,200	4,480.91	4,662.84	4,662.84	0.00	118
Low-5	2	4,901.08	1,800	4,501.37	4,541.65	4,901.08	7.33	3,600	4,531.34	4,531.34	7,392.95	38.71	7,200	4,241.23	4,507.91	4,567.43	1.30	7,200
High-1	2	15,155.32	1,029	15,053.33	15,097.10	15,097.10	0.00	1,246	14,999.18	14,999.18	18,161.08	17.41	7,200	14,697.6	15,097.1	15,097.1	0.00	1,763
High-2	2	15,350.38	97	15,239.22	15,342.94	15,342.94	0.00	641	15,273.48	15,273.48	18,778.43	18.66	7,200	14,997.6	15,186.5	15,356.4	1.11	7,200
High-3	2	15,609.98	563	15,447.61	15,520.90	15,520.90	0.00	1,245	15,455.64	15,468.69	18,388.44	15.88	7,200	15,294	15,520.9	15,520.9	0.00	3,197
High-4	2	16,775.12	215	16,734.58	16,775.12	16,775.12	0.00	151	16,747.42	16,747.42	19,641.16	14.73	7,200	16,593.7	16,775.1	16,775.1	0.00	801
High-5	2	16,078.42	921	15,950.13	16,003.45	16,067.87	0.40	3,600	15,978.09	15,978.09	18,790	14.96	7,200	15,689.6	15,968.2	16,027.7	0.37	7,200
Low-1	3	5,202.18	1,334	5,186.06	5,202.18	5,202.18	0.00	236	5,189.62	5,189.62	5,829.45	10.98	7,200	4,397.71	4,668.74	5,829.45	19.91	7,200
Low-2	3	5,957.09	1,800	5,284.68	5,385.02	5,957.09	9.60	3,600	5,332.17	5,351.07	6,188.80	13.54	7,200	4,848.35	5,015.45	6,188.80	18.96	7,200
Low-3	3	5,321.45	1,109	4,996.67	5,047.37	5,321.45	5.15	3,600	5,022.02	5,041.01	5,506.95	8.46	7,200	4,603.85	4,744.26	5,506.95	13.85	7,200
Low-4	3	5,386.50	1,800	5,155.01	5,216	5,386.50	3.17	3,600	5,189.8	5,213.18	5,292.2	1.49	7,200	4,627.81	5,053.54	5,292.20	4.51	7,200
Low-5	3	5,641.79	1,800	5,008.09	5,041.32	5,641.79	10.64	3,600	5,045.13	5,054.91	5,790.36	12.70	7,200	4,360.01	4,537.52	5,790.36	21.64	7,200
High-1	3	15,732.60	351	15,709.43	15,732.60	15,732.60	0.00	293	15,721.25	15,721.25	15,732.60	0.07	7,200	14,926.50	15,069.50	16,390.80	8.06	7,200
High-2	3	16,348.80	1,800	15,739.86	15,858.75	16,348.80	3.00	3,600	15,837.78	15,854.55	21,275.27	25.48	7,200	15,239.60	15,299.70	16,452.40	7.01	7,200
High-3	3	16,314.26	1,262	15,783.67	15,866.64	16,314.26	2.74	3,600	15,842.89	15,854.46	19,378.35	18.18	7,200	15,433.00	15,569.70	16,278.60	4.35	7,200
High-4	3	17,469.32	1,093	17,264.9	17,364.51	17,469.32	0.60	3,600	17,314.28	17,326.98	19,692.49	12.01	7,200	16,737.50	16,968.80	17,560.40	3.37	7,200
High-5	3	16,662.36	1,174	16,436.84	16,512.84	16,662.36	0.90	3,600	16,500.84	16,510.20	20,355.68	18.89	7,200	15,817.30	15,998.30	17,366.10	7.88	7,200
Low-1	4	6,689.22	1,800	5,708.49	5,754.65	6,689.22	13.97	3,600	5,734.78	5,760.4	8,904.76	35.31	7,200	4,643.52	4,750.38	7,533.48	36.94	7,200
Low-2	4	7,105.28	1,800	5,907.19	5,968.52	7,105.28	16.00	3,600	6,002.11	6,035.23	10,482.05	42.42	7,200	4,986.74	4,989.10	—	—	7,200
Low-3	4	5,589.17	1,100	5,398.38	5,428.63	5,589.17	2.87	3,600	5,419.76	5,446.15	8,693.82	37.36	7,200	4,824.93	4,863.11	6,516.87	25.38	7,200
Low-4	4	6,986	1,800	5,790.53	5,846.59	6,986	16.31	3,600	5,902.73	5,924.86	8,725.63	32.10	7,200	4,784.96	4,821.80	7,119.92	32.28	7,200
Low-5	4	6,456.78	1,800	5,641.01	5,680.33	6,456.78	12.03	3,600	5,684.83	5,700.53	9,251.72	38.38	7,200	4,526.44	4,553.35	—	—	7,200
High-1	4	16,508.64	1,561	16,234.48	16,302.13	16,508.64	1.25	3,600	16,270.65	16,281.23	19,385.03	16.01	7,200	15,165.00	15,190.70	17,738.80	14.36	7,200
High-2	4	17,733.72	1,799	16,346.88	16,460.04	17,733.72	7.18	3,600	16,512.64	16,538.71	20,987.64	21.20	7,200	15,503.90	15,540.90	17,674.80	12.07	7,200
High-3	4	16,412.80	1,018	16,195.60	16,265.77	16,412.8	0.90	3,600	16,244.83	16,266.05	19,451.19	16.38	7,200	15,587.70	15,682.10	17,536.60	10.58	7,200
High-4	4	18,618.03	1,800	17,891.05	17,994.82	18,618.03	3.35	3,600	18,024.35	18,043.89	20,817.08	13.32	7,200	16,923.90	16,975.00	19,111.70	11.18	7200
High-5	4	17,777.38	1,800	17,041.69	17,135.24	17,777.38	3.61	3,600	17,144.01	17,156.75	20,645.10	16.90	7,200	15,982.00	15,992.50	18,790.70	14.89	7,200
Low-1	5	7,167.86	1,800	6,419.22	6,446.11	7,167.86	10.07	3,600	6,439.11	6,452.71	10,429.19	38.13	7,200	4,864.57	4,881.49	—	—	7,200
Low-2	5	9,134.85	1,800	6,518.19	6,564.10	9,134.85	28.14	3,600	6,577.23	6,612.29	11,967.18	44.75	7,200	5,246.68	—	—	—	7,200
Low-3	5	6,889.06	1,800	5,823.12	5,881.77	6,889.06	14.62	3,600	5,913.07	5,935.12	10,045.05	40.91	7,200	4,930.35		—	—	7,200
Low-4	5	7,594.70	1,800	6,390.44	6,437.08	7,594.7	15.24	3,600	6,570.07	6,601.09	9,741.39	32.24	7,200	4,993.16	—	—	—	7,200
Low-5	5	7,035.78	1,800	6,197.02	6,228.27	7,035.78	11.48	3,600	6,225.53	6,233.77	6,233.77	0.00	7,098	4,746.76	—	—	—	7,200
High-1	5	17,954.77	1,800	16,944.35	16,989.26	17,954.77	5.38	3,600	16,972.92	16,986.23	20,925.28	18.82	7,200	15,387.40	—	—	—	7,200
High-2	5	18,233.75	1,800	16,986.48	17,067.47	18,233.75	6.40	3,600	17,093.89	17,129.53	22,496.79	23.86	7,200	15,769.70	15,775.50	19,517.90	19.17	7,200
High-3	5	17,391.73	1,800	16,608.09	16,675.93	17,391.73	4.12	3,600	16,739.11	16,757.53	20,840.11	19.59	7,200	15,752.10	—	—	—	7,200
High-4	5	20,111.17	1,800	18,524.5	18,598.81	20,111.17	7.52	3,600	18,689.82	18,728.47	21,847.94	14.28	7,200	17,140.90	—	—	—	7,200
High-5	5	18,688.59	1,800	17,589.85	17,659.84	18,688.59	5.50	3,600	17,678.44	17,695.79	17,695.79	0.00	5,741	16,207.30	16,265.80	19,438.00	16.32	7,200

Table 3. Computational Results for $n30$-$T6$ Instances

		ABW							DRC					CL				
Name	K	UB_{heu}	Time	LB	BLB	BUB	%gap	Time	LB	BLB	BUB	%gap	Time	LB	BLB	BUB	%gap	Time
Low-1	2	9,385	1,800	8,792.70	8,888.78	9,385.00	5.29	3,600	8,820.3	8,820.3	16,251.45	45.73	7,200	8,008.29	8,839.02	9,123.99	3.12	7,200
Low-2	2	8,763.46	1,800	8,074.90	8,236.6	8,763.46	6.01	3,600	8,193.31	8,193.31	15,729.43	47.91	7,200	7,783.33	8,197.54	8,565.34	4.29	7,200
Low-3	2	8,478.43	1,800	8,419.55	8,442.43	8,442.43	0.00	487	8,370.79	8,378.61	14,278.23	41.32	7,200	8,095.36	8,442.43	8,442.43	0.00	1,888
Low-4	2	8,610.78	1,800	7,863.04	7,947.95	8,610.78	7.70	3,600	8,008.77	8,039.12	15,738.27	48.92	7,200	7,538.53	7,735.56	8,313.72	6.95	7,200
Low-5	2	8,863.01	1,800	7,689.86	7,770.25	8,863.01	12.33	3,600	7,859.64	7,859.64	14,227.85	44.76	7,200	7,163.14	7,664.56	8,041.11	4.68	7,200
High-1	2	24,133.36	1,800	23,518.29	23,659.08	24,133.36	1.97	3,600	23,621.13	23,621.13	31,131.5	24.12	7,200	22,692.7	23,441.7	23,859.7	1.75	7,200
High-2	2	20,944.45	1,800	20,305.63	20,500.1	20,944.45	2.12	3,600	20,453.51	20,504.19	28,951.02	29.18	7,200	22,997.5	23,416.8	23,416.8	0.00	1,340
High-3	2	23,532.17	1,800	23,343.09	23,416.8	23,416.8	0.00	1,280	23,334.1	23,395.26	29,281.46	20.10	7,200	23,008.5	23,416.8	23,416.8	0.00	1,373
High-4	2	18,824.72	1,800	17,912.84	17,979.61	18,824.72	4.49	3,600	18,070.05	18,094.02	25,851.23	30.01	7,200	18,617.2	19,155.3	19,686.8	2.70	7,200
High-5	2	20,148.35	1,800	19,170.39	19,274.1	20,148.35	4.34	3,600	19,375.43	19,396.77	25,732.45	24.62	7,200	18,615.3	19,092.5	19,560.2	2.39	7,200
Low-1	3	11,367.08	1,800	9,981.67	10,020.79	11,367.08	11.84	3,600	10,187.07	10,201.41	19,097.65	46.58	7,200	8,263.12	8,607.01	11,522.8	25.30	7,200
Low-2	3	9,964.96	1,800	9,089.76	9,178.73	9,964.96	7.89	3,600	9,265.48	9,291.15	17,277.25	46.22	7,200	8,381.59	8,654.56	10,091.1	14.24	7,200
Low-3	3	9,590.32	1,800	9,102.72	9,151.21	9,590.32	4.58	3,600	9,176.51	9,198.47	16,127.15	42.96	7,200	8,241.71	8,410.07	9,941.07	15.40	7,200
Low-4	3	10,683.88	1,800	8,861.66	8,898.2	10,683.88	16.71	3,600	9,046.99	9,069.01	17,240.23	47.40	7,200	7,835.98	7,898.9	10,342.3	23.63	7,200
Low-5	3	10,128.46	1,800	8,689.75	8,732.31	10,128.46	13.78	3,600	8,909.75	8,928.70	16,066.91	44.43	7,200	7,252.9	7,397.05	10,161.4	27.20	7,200
High-1	3	25,844.98	1,800	24,733.90	24,777.11	25,844.98	4.13	3,600	24,988.49	25,011.69	34,010.58	26.46	7,200	23,012	23,452.6	26,360.3	11.03	7,200
High-2	3	22,028.40	1,800	21,353.87	21,458.7	22,028.4	2.59	3,600	21,532.91	21,588.27	30,179.24	28.47	7,200	20,642.5	20,994	21,931.9	4.28	7,200
High-3	3	24,567.56	1,800	24,075.41	24,131.96	24,567.56	1.77	3,600	24,175.33	24,201.97	31,128.5	22.25	7,200	23,158.2	23,428.4	24,602.6	4.77	7,200
High-4	3	20,456.06	1,800	18,866.46	18,923.09	20,456.06	7.49	3,600	19,083.34	19,124.46	27,368.71	30.12	7,200	17,854.2	17,976.8	19,832.5	9.36	7,200
High-5	3	21,700.98	1,800	20,129.20	20,199.77	21,700.98	6.92	3,600	20,434.55	20,462.29	27,545.45	25.71	7,200	18,738.5	18,985.2	21,965.7	13.57	7,200
Low-1	4	13,579.54	1,800	11,428.89	11,455.41	13,579.54	15.64	3,600	11,686.08	11,697.35	18,869.49	38.01	7,200	8,620.03	9,016.31	14,169.4	36.37	7,200
Low-2	4	11,108.81	1,800	10,251.51	10,311.45	11,108.81	7.18	3,600	10,422.53	10,436.33	18,873.86	44.70	7,200	9,017.71	9,143.11	12,336.5	25.89	7,200
Low-3	4	11,189.66	1,800	9,941.28	9,965.94	11,189.66	10.94	3,600	10,073.39	10,108.28	20,263.87	50.12	7,200	8,464.27	8,591.04	11,597	25.92	7,200
Low-4	4	11,370.01	1,800	9,946.88	9,983.02	11,370.01	12.20	3,600	10,174.94	10,238.87	17,246.3	40.63	7,200	8,146.38	8,149.34	—	—	7,200
Low-5	4	12,295.51	1,800	9,715.36	9,742.62	12,295.51	20.76	3,600	10,009.37	10,030.97	19,655.39	48.97	7,200	7,512.67	7,580.06	12,085	37.28	7,200
High-1	4	28,602.54	1,800	26,167.46	26,201.91	28,602.54	8.39	3,600	26,440.68	26,481.6	33,783.58	21.61	7,200	23,392.9	23,771.1	29,016.7	18.08	7,200
High-2	4	23,500.25	1,800	22,428.43	22,515.72	23,500.25	4.19	3,600	22,681.27	22,725.35	31,166.8	27.08	7,200	21,269.4	21,539.4	24,880.5	13.43	7,200
High-3	4	25,988.86	1,800	24,912.45	24,968.57	25,988.86	3.93	3,600	25,080.1	25,121.56	35,239.26	28.71	7,200	23,417.9	23,581.7	—	—	7,200
High-4	4	22,283.15	1,800	19,973.40	20,035.34	22,283.15	10.09	3,600	20,229.99	20,302.14	27,392.85	25.89	7,200	18,170.8	18,182.9	—	—	7,200
High-5	4	22,829.25	1,800	21,147.16	21,201.06	22,829.25	7.13	3,600	21,533.92	21,566.65	31,105.4	30.67	7,200	19,013.3	19,103.9	22,787.6	16.17	7,200
Low-1	5	14,824.81	1,800	12,878.96	12,896.62	14,824.81	13.01	3,600	13,194.27	13,210.61	19,001.15	30.47	7,200	9,110.97	—	—	—	7,200
Low-2	5	12,614.88	1,800	11,326.29	11,387.74	12,614.88	9.73	3,600	11,559.06	11,598.46	18,565.88	37.53	7,200	9,595.19	9,881.87	13,206.4	25.17	7,200
Low-3	5	13,150.13	1,800	10,878.07	10,891	13,150.13	17.18	3,600	11,066.22	11,088.49	18,458.13	39.93	7,200	8,725.88	8,835.89	17,774.5	50.29	7,200
Low-4	5	13,673.77	1,800	11,134.64	11,173.46	13,673.77	18.29	3,600	11,418.24	11,470.19	18,885.37	39.26	7,200	8,459.84	—	—	—	7,200
Low-5	5	13,261.28	1,800	10,813.85	10,814.65	13,261.28	18.45	3,600	11,141.07	11,174.33	17,450.56	35.97	7,200		—	—	—	7,200
High-1	5	31,219.77	1,800	27,610.72	27,661.09	31,219.77	11.40	3,600	27,981.24	28,007.94	33,935.15	17.47	7,200	23,931	—	—	—	7,200
High-2	5	25,076.12	1,800	23,573.71	23,658.8	25,076.12	5.65	3,600	23,833.74	23,881.11	30,878.57	22.66	7,200	21,874.7	22,086.3	26,185.2	15.65	7,200
High-3	5	27,420.74	1,800	25,854.63	25,903.65	27,420.74	5.53	3,600	26,026.74	26,101.72	33,466.92	22.01	7,200	23,724.5	—	—	—	7,200
High-4	5	23,624.53	1,800	21,161.20	21,209.33	23,624.53	10.22	3,600	21,476.72	21,515.16	30,100.03	28.52	7,200	18,494.3	—	—	—	7,200
High-5	5	24,281.36	1,800	22,260.78	22,324.9	24,281.36	8.06	3,600	22,670.68	22,703.38	28,942.78	21.56	7,200	19,321.4	—	—	—	7,200

For the instances $n50$-$T3$ we observe that:

—our lower bounds at the root node are generally slightly lower (less than 1% of difference) than those of DRC and always much higher than those of CL.

—our lower bounds after enumeration are generally slightly higher than those of DRC for $K = 2, 3$ and slightly worse for $K = 4, 5$, always much higher than those of CL.

—the best upper bounds are significantly lower than those of the other two algorithms for $K = 4, 5$.

—the overall effect is that our algorithm returns gaps that are significantly smaller than those of the other two algorithms, particularly for $K = 4, 5$.

—we solved to optimality nine instances that the DRC algorithm did not solve. Four of these instances were not solved by the CL algorithm either.

For the instances $n30$-$T6$ we observe that:

—our lower bounds at the root node as well as the lower bounds after enumeration are generally slightly lower (less than 2% of difference) than those of DRC and always much higher than those of CL.

—the best upper bounds are significantly lower than those of the other two algorithms, particularly for $K = 3, 4, 5$.

—the overall effect is that our algorithm returns gaps that are significantly smaller than those of the other two algorithms, particularly for $K = 3, 4, 5$.

—we solved to optimality the instances n30-T6-low-3 and n30-T6-high-3, $K = 2$, which were solved by CL, but not by DRC. For n30-T6-high-2 and n30-T6-high-4, $K = 2$, we get an optimal value less than the lower bound reported for the CL algorithm on L. Coelho's webpage.

Furthermore we observe (see the online appendix) that we solve to optimality four n15-T6, one n20-T6, and four n25-T6, $K = 2$, instances which the DRC algorithm did not solve. Instances n25-T6-high-1 and n25-T6-high-4 were not solved by the CL algorithm either.

Figure 2. (Color online) Average Gaps for ABW, CL, and DRC on n50-T3 Instances

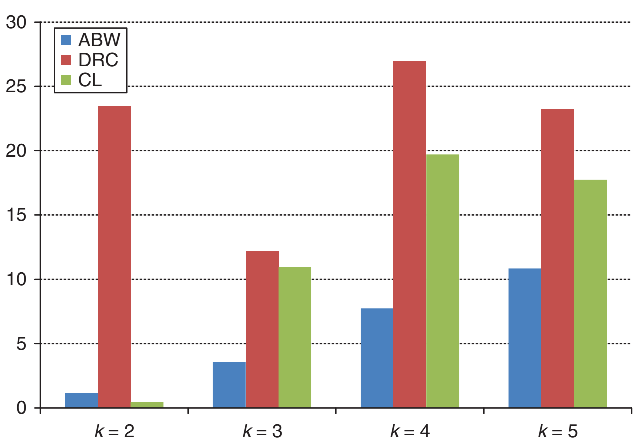
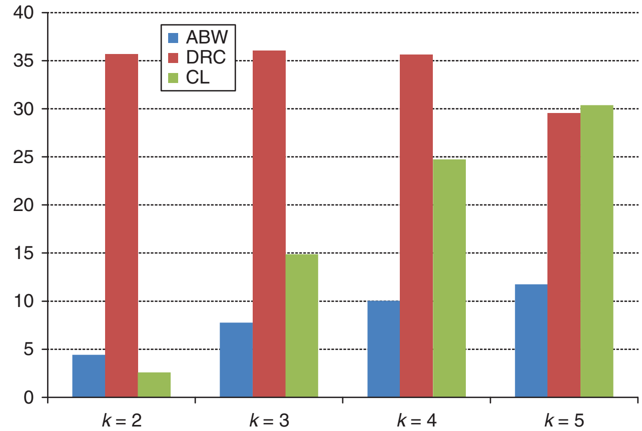


Figure 3. (Color online) Average Gaps for ABW, CL, and DRC on n30-T6 Instances



For the instances n25-T6-high-1, n25-T6-high-2, n25-T6-high-4, and n25-T6-high-5, $K = 2$, we get a feasible solution whose value is smaller than the lower bound reported for the CL algorithm on L. Coelho's webpage.

Finally, in Figures 2 and 3, for each set of 10 instances, the average gap of the ABW algorithm is compared with that of CL and DRC. The value of the average gap for each algorithm is computed without considering the instances for which the algorithm does not provide an upper bound. So the value of the average gap of the CL algorithm for the n50-T3 instances with $K = 5$ is not significant because the algorithm can only find upper bounds on 2 of the 10 instances.

8. Final Remarks

Though the test sets for IRP consist of instances with time-invariant demands and time-invariant stock upper bounds, the DR inequalities described here allow for time-varying demands. With restrictions to the set S_4, it is also straightforward to treat non-constant delivery or storage bounds. However, in any period the vehicle capacities must all be the same.

As noted earlier, restrictions of the SIRP arise in the CVRP and other single period variants such as the prize-collecting traveling salesman problem. It would be interesting to see if and when the DR inequalities are useful for these problems and also to test whether the cutting planes developed here complement the Branch-and-Price approach to IRP of Desaulniers, Rakke, and Coelho (2016).

Acknowledgments

The scientific responsibility is assumed by the authors. The authors thank Leandro Coelho for kindly providing the detailed results of the benchmark instances.

References

Adulyasak Y, Cordeau J-F, Jans R (2014) Formulations and branch-and-cut algorithms for multi-vehicle production and inventory routing problems. *INFORMS J. Comput.* 26(1):103–120.

Archetti C, Bianchessi N, Speranza MG (2013) Optimal solutions for routing problems with profits. *Discrete Appl. Math.* 161(4):547–557.

Archetti C, Bertazzi L, Laporte G, Speranza MG (2007) A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Sci.* 41(3):382–391.

Avella P, Boccia M, Wolsey LA (2015) Single-item reformulations for a vendor managed inventory routing problem: Computational experience with benchmark instances. *Networks* 65(2):129–138.

Coelho LC, Laporte G (2013) The exact solution of several classes of inventory-routing problems. *Comput. Oper. Res.* 40(2):558–565.

Coelho LC, Laporte G (2014) Improved solutions for inventory-routing problems through valid inequalities and input ordering. *Internat. J. Production Econom.* 155:391–397.

Coelho LC, Cordeau J-F, Laporte G (2014) Thirty years of inventory routing. *Transportation Sci.* 48(1):1–19.

Desaulniers G, Rakke JG, Coelho LC (2016) A branch-price-and-cut algorithm for the inventory-routing problem. *Transportation Sci.* 50(3):1060–1076.

Naddef D, Rinaldi G (2001) Branch-and-cut algorithms for the capacitated VRP. Toth P, Vigo D, eds. *The Vehicle Routing Problem* (SIAM, Philadelphia), 53–84.

Picard JC, Ratliff HD (1973) A graph-theoretic equivalence for integer programs. *Oper. Res.* 21(1):261–269.

Solyali O, Süral H (2011) A branch-and-cut algorithm using a strong formulation and an a priori tour-based heuristic for an inventory-routing problem. *Transportation Sci.* 45(3):335–345.