A RECURSIVE METHOD FOR COMPUTING MOMENTS OF HAWKES INTENSITIES: APPLICATION TO THE POTENTIAL APPROACH OF CREDIT RISK

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A recursive method for computing moments of Hawkes intensities: application to the potential approach of credit risk

John-John Ketelbuters* Donatien Hainaut†

Abstract

This paper explores an alternative to the structural models and reduced models in credit risk. The approach we use is called the potential approach. In the context of credit risk, it consists in assuming that the survival probability of a company is equal to the ratio of the expected value of a supermartingale divided by its initial value. This approach, that was previously used for modelling the term structure of interest rates, is extended by the use of a self-exciting processess that is time-changed by the inverse of an alpha-stable subordinator. We derive a new recursive method that allows to compute all the moments of a self-exciting process intensity. We show that this method can be used to approximate the survival probabilities in the potential aproach. More specifically, we prove that the approximation converges and we provide a bound on the numerical error. Finally, we calibrate the model and show that it allows to properly fit survival probability curves that are highly convex.

The most common paradigms in credit risk management are structural and reduced models. In the first approach, the balance sheet of a firm is explicitly modeled and the default occurs when the assets fall below the liabilities, as e.g. in Capinski (2007), He and Chen (2018), Jeon, Yoon, and Kang (2016) or Hainaut and Colwell (2014). In the reduced approach, the default is caused by the first jump of a point process, see e.g. Liang and Wang (2012), Hainaut and Le Courtois (2014) or Eckert, Jakob, and Fischer (2016). Chen and Panjer (2003) unify discrete structural and reduced models whereas Ballestra and Pacelli (2014) propose an hybrid approach. In this paper, we investigate a third alternative called the potential approach. This method has been successfully applied to interest rates by Rogers (1997), Flesaker and Hughston (1999) or Macrina (2014). Instead of modeling the short term rates, discount factors are the ratios of the expectation of a supermartingale on its initial value.

We apply and extend the potential approach to credit risk in two directions. Firstly, we work with a self-excited process that allows sudden and repeated jumps of default probabilities, as those caused by contagion between defaults of several firms. In this framework, the intensity of default arrival depends on recent past shocks. This approach finds its origin in the articles of Hawkes (1971a), Hawkes (1971b) and Hawkes and Oakes (1974). More recently, Errais, Giesecke, and Goldberg (2010) develop a self-excited approach for modelling defaults in a credit portfolio. Ait-Sahalia, Cacho-Diaz, and Laeven (2015) use a similar model to measure contagion between international markets. Hainaut (2016b) and Hainaut (2016a) propose a model explaining the dynamics of interest rates by self-excited jump-diffusions.

The second way in which we extend the potential approach is the use of the technique of time-change. This technique allows to obtain a better fit for survival probability curves that exhibit a high degree of convexity, as shown in Ketelbuters and Hainaut (2022). Such curves are observed for companies that face financial difficulties in the short-term, but has a good long-term viability provided that they overcome their difficulties. Airline companies during the Covid-19 crisis were given as an example above. Obtaining such convex shapes involves a non-Markov time-change built with the inverse of an α-stable subordinator. This approach is applied to diffusions in physics for describing the movements of heavy particles that can get immobilized (see e.g. Metzler and Klafter (2004) or Eliazar and Klafter (2004)). This type of time-changed Brownian motions, called subdiffusions, are also popular in econophysics (see Scalas (2006)) and applied

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to financial derivatives by Magdziarz (2009b) for illiquid markets. As shown e.g. in Magdziarz (2009a), the probability density function of sub-diffusions is solution of a Fractional Fokker-Planck equation. This equation is proposed in Barkai, Metzler, and Klafter (2000) and Metzler, Barkai, and Klafter (1999). Articles of Leonenko, Meerschaert, and Sikorskii (2013b) and Leonenko, Meerschaert, and Sikorskii (2013a) go a step further and explore fractional Pearson diffusions and their correlation.

Computing default probabilities in such a setting requires to be able to approximate the moment generating function of (fractional) Hawkes process intensities. To do this, the most common approach is to derive a (fractional partial) differential equation and to solve it numerically, as in e.g. Ketelbuters and Hainaut (2022). Since this approach is rather complicated both mathematically and numerically, we propose here to use an other approach. Our idea is to use the moments of the Hawkes process intensity to approximate the moment generating function via the Maclaurin series of the exponential function. In a recent article, Cui, Hawkes, and Yi (2020) describe a method to derive the moments of Hawkes processes and their intensities. In this paper, we go a step further and prove that the moments of the Hawkes process intensity always share the same form and can be computed recursively. This recursive procedure is new to the best of our knowledge and allows to efficiently compute default probabilities in our proposed model.

The structure of this paper is as follows. Sections 2 and 3 review the potential approach and the properties of a Hawkes jump-diffusion. In Section 4, we extend the Hawkes jump-diffusion to the subdiffusive setting by using the inverse of an α-stable subordinator as time-change. Section 5 introduces our new numerical method to compute all the moments of a Hawkes process intensity. We show that this method also works in the subdiffusive time-changed setting. We end Section 5 by showing how this result can be used to approximate default probabilities and provide upper bounds on the numerical error. Finally, Section 6 presents a numerical analysis and shows in particular that our model performs as well as in Ketelbuters and Hainaut (2022), while being much simpler to deal with numerically.

1 The potential approach for credit risk modelling

All stochastic processes considered in this paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where $\mathbb{Q}$ is the risk neutral measure. The default time of a firm is a random time denoted by τ. As mentioned in the introduction, the two most common approaches in credit risk management are the structural and reduced models. In the first one, the default is triggered when the market value of assets falls below debts. In the second approach, the default is caused by the first jump of a point process. In this paper, we investigate an alternative way, called the potential approach. Instead of modelling an event triggering the default, we propose a dynamics for the probability that the company survives until a certain maturity. This probability is of course in the interval $[0, 1]$ and is a decreasing function of the time horizon. The potential approach is based upon an $\mathbb{F}^\xi = (\mathcal{F}^\xi_t)_{t\geqslant 0}$-adapted stochastic process $(\xi_t)_{t\geqslant 0}$. The filtration $\mathbb{F}^\xi$ is defined precisely in the next section. If the company is still in activity at time s, the survival probability until time $t \geqslant s$ is denoted by $\mathbb{Q}(\tau > t|\tau > s)$ and defined as

$$\mathbb{Q}(\tau > t|\tau > s) = \frac{1}{\xi_s}\mathbb{E}_\mathbb{Q}[\xi_t|\mathcal{F}^\xi_s].$$

To ensure that $\frac{1}{\xi_s}\mathbb{E}_\mathbb{Q}[\xi_t|\mathcal{F}^\xi_s]$ takes its values in $[0, 1]$ and decreases with t, $(\xi_t)_{t\geqslant 0}$ needs no fulfill three conditions

(i) $\xi_t > 0$ for all t, $\mathbb{Q}$-a.s.;

(ii) $(\xi_t)_{t\geqslant 0}$ is a supermartingale, i.e. $\mathbb{E}_\mathbb{Q}[\xi_t|\mathcal{F}^\xi_s] \leqslant \xi_s$ for all $s \leqslant t$, $\mathbb{Q}$-a.s.;

(iii) the expectation satisfies $\lim_{t\to+\infty} \mathbb{E}_\mathbb{Q}[\xi_t] = 0$.

There exist various ways to build such a process in a Brownian setting. We refer to e.g. Rogers (1997) for examples. In this paper, we consider a process $(\xi_t)_{t\geqslant 0}$ that is a Hawkes jump-diffusion supermartingale. The construction and properties of this process are developed in the next sections.

2 Hawkes jump-diffusion supermatingales

This section is devoted to the construction of Hawkes jump-diffusion supermartingales. We consider a self-exciting counting process $(N_t)_{t\geqslant 0}$ whose intensity is given by

$$\mathrm{d}\lambda_t = \kappa(\theta - \lambda_t)\mathrm{d}t + \sigma\sqrt{\lambda_t}\mathrm{d}W_t + \eta\mathrm{d}P_t$$

where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion and

$$P_t = \sum_{k=1}^{N_t} J_k.$$

The jump sizes $J_1, J_2, \dots =^{\text{dist}} J$ are i.i.d. random variables that are assumed to be distributed according to a measure ν defined as

$$\nu(B) := \int_{B\cap\mathbb{R}^+} \rho e^{-\rho x}\mathrm{d}x, \quad B \in \mathcal{B}(\mathbb{R})$$

with $\rho > 0$. As a consequence, $J_1, J_2, \dots$ have an exponential distribution with parameter ρ (and thus expected value $1/\rho$). We denote by $\mathbb{F}^\xi = (\mathcal{F}_t^\xi)_{t\geqslant 0}$ the filtration generated by the process $(\lambda_t, P_t, N_t, W_t)_{t\geqslant 0}$. We assume that the random variables $J_1, J_2, \dots$ are in some sense independent of the process $(\lambda_t, P_t, N_t, W_t)_{t\geqslant 0}$. More precisely, if we denote by $T_1, T_2, \dots$ the stopping times that represent the jump times of $(\lambda_t, N_t, P_t)_{t\geqslant 0}$, that is

$$T_k = \inf\{\tau > 0 : N_\tau = k\}$$

then we assume that J_k is independent of $\mathcal{F}^\xi_{T_k-}$ for any $k = 1, 2 \dots$. In other words, nothing can be said about the jump size J_k until just before the associated jump time T_k. Of course, the jump sizes $J_1, J_2, \dots$ are not independent from the process $(N_t)_{t\geqslant 0}$, as the size of a jump influences the occurrence of subsequent jumps of $(N_t)_{t\geqslant 0}$. However, past jumps do not have any impact on subsequent jump sizes. Using our assumptions, we can obtain an expression for the expected value of the exponential of $(\lambda_t)_{t\geqslant 0}$.

Lemma 2.1. *For any $z < \rho/\eta$ and $t \geqslant u \geqslant 0$, it holds that*

$$\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t} - e^{z\lambda_u}] = \mathbb{E}_{\mathbb{Q}}\left[\int_u^t \left\{ z\kappa\theta e^{z\lambda_{s-}} + \left(-z\kappa + \frac{\sigma^2}{2}z^2 + \frac{\rho}{\rho - \eta z} - 1\right)\lambda_{s-}e^{z\lambda_{s-}} \right\}\mathrm{d}s\right].$$

Proof. Fix $z < \rho/\eta$. Ito's lemma applied on the process $(e^{z\lambda_t})_{t\geqslant 0}$ yields

$$\begin{aligned} e^{z\lambda_t} - e^{z\lambda_u} = &\int_u^t z e^{z\lambda_s}\kappa(\theta - \lambda_s)\mathrm{d}s + \int_u^t z e^{z\lambda_s}\sigma\sqrt{\lambda_s}\mathrm{d}W_s \\ &+ \frac{\sigma^2}{2}\int_u^t z^2 e^{z\lambda_s}\lambda_s\mathrm{d}s + \sum_{u<s\leqslant t}(e^{z\lambda_s} - e^{z\lambda_{s-}}). \end{aligned} \tag{1}$$

The jump part can be rewritten as

$$\sum_{u<s\leqslant t}(e^{z\lambda_s} - e^{z\lambda_{s-}}) = \sum_{k\geqslant 1}\mathbf{1}_{\{u<T_k\leqslant t\}}e^{z\lambda_{T_k-}}\left(e^{z\eta J_k} - 1\right).$$

From this equation, we compute that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\sum_{u<s\leqslant t}\left(e^{z\lambda_s}-e^{z\lambda_{s-}}\right)\right] &= \sum_{k\geqslant 1}\mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_{\{u<T_k\leqslant t\}}e^{z\lambda_{T_k-}}\left(e^{z\eta J_k}-1\right)\right] \\
&= \sum_{k\geqslant 1}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_{\{T_k\leqslant t\}}e^{z\lambda_{T_k-}}\left(e^{z\eta J_k}-1\right)\middle|\mathcal{F}^{\xi}_{T_k-}\right]\right] \\
&= \sum_{k\geqslant 1}\mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_{\{u<T_k\leqslant t\}}e^{z\lambda_{T_k-}}\mathbb{E}_{\mathbb{Q}}\left[\left(e^{z\eta J_k}-1\right)\middle|\mathcal{F}^{\xi}_{T_k-}\right]\right] \\
&= \left(\frac{\rho}{\rho-\eta z}-1\right)\sum_{k\geqslant 1}\mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_{\{u<T_k\leqslant t\}}e^{z\lambda_{T_k-}}\right] \\
&= \left(\frac{\rho}{\rho-\eta z}-1\right)\mathbb{E}_{\mathbb{Q}}\left[\int_u^t e^{z\lambda_{s-}}\mathrm{d}N_s\right] \\
&= \left(\frac{\rho}{\rho-\eta z}-1\right)\mathbb{E}_{\mathbb{Q}}\left[\int_u^t \lambda_{s-}e^{z\lambda_{s-}}\mathrm{d}s\right].
\end{aligned}
\tag{2}
$$

The first equality of Equation (2) follows from the monotone convergence theorem, as all the terms are non-negative. The second equality comes from the tower property. The third equality is a consequence of the $\mathcal{F}^{\xi}_{T_k-}$-measurability of the random variables T_k and λ_{T_k-}. The fourth equality is a consequence of our assumption of independence between $\mathcal{F}^{\xi}_{T_k-}$ and J_k, combined with the formula for the moment generating function of exponential random variables. The formula for this moment generating function is valid because of the assumption $\eta z<\rho$. Finally, the last two equalities are respectively consequences of the definition of stochastic integral and the fact that the process $(\int_0^t \lambda_{s-}\mathrm{d}s)_{t\geqslant 0}$ is the compensator of $(N_t)_{t\geqslant 0}$. By combining Equations (1) and (2), we find

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t}-e^{z\lambda_u}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\int_u^t ze^{z\lambda_s}\kappa(\theta-\lambda_s)\mathrm{d}s+\frac{\sigma^2}{2}\int_u^t z^2e^{z\lambda_s}\lambda_s\mathrm{d}s+\left(\frac{\rho}{\rho-\eta z}-1\right)\int_u^t\lambda_{s-}e^{z\lambda_{s-}}\mathrm{d}s\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\int_u^t\left\{z\kappa\theta e^{z\lambda_{s-}}+\left(-z\kappa+\frac{\sigma^2}{2}z^2+\frac{\rho}{\rho-\eta z}-1\right)\lambda_{s-}e^{z\lambda_{s-}}\right\}\mathrm{d}s\right],
\end{aligned}
$$

which is what we had to prove. $\square$

Lemma 2.1 can be used to prove that the random variables $(e^{z\lambda_t})_{t\geqslant 0}$ have finite expectation for some $z>0$. To this end, we study the function q

$$
(-\infty,\rho/\eta)\ni z\mapsto q(z):=-z\kappa+\frac{\sigma^2}{2}z^2+\frac{\rho}{\rho-\eta z}-1
$$

that appears in the expectation. Obviously we can restrict our attention to $z\in(0,\rho/\eta)$, as $e^{z\lambda_t}\leqslant 1$ implies that the expectation is finite when $z\leqslant 0$.

Lemma 2.2. *If $\sigma>0$ and if we define the constants*

$$
\zeta=\left(\frac{\sigma^2}{2}\rho+\eta\kappa\right)^2+2\eta\sigma^2(\eta-\kappa\rho) \tag{3}
$$

and

$$
\gamma_{\pm}=\frac{\left(\frac{\sigma^2}{2}\rho+\eta\kappa\right)\pm\sqrt{\zeta}}{\eta\sigma^2}, \tag{4}
$$

then $0<\gamma_-<\rho/\eta<\gamma_+$. Moreover, for $z\in(0,\rho/\eta)$,

$$
q(z)<0\Longleftrightarrow z\in(0,\gamma_-),
$$

$$q(z) = 0 \Longleftrightarrow z = \gamma_-,$$

and

$$q(z) > 0 \Longleftrightarrow z \in (\gamma_-, \rho/\eta).$$

If $\sigma = 0$ *then* $\frac{\kappa\rho-\eta}{\eta\kappa} > 0$ *and for* $z \in (0, \rho/\eta)$,

$$q(z) < 0 \Longleftrightarrow z \in (0, \frac{\kappa\rho - \eta}{\eta\kappa}),$$

$$q(z) = 0 \Longleftrightarrow z = \frac{\kappa\rho - \eta}{\eta\kappa},$$

and

$$q(z) > 0 \Longleftrightarrow z \in (\frac{\kappa\rho - \eta}{\eta\kappa}, \rho/\eta). \tag{5}$$

Proof. Since $\rho - \eta z > 0$ and $z > 0$, multiplying the inequality $q(z) \leqslant 0$ by $(\rho - \eta z)$ and dividing it by z yields

$$q(z) \leqslant 0 \Longleftrightarrow -\frac{\sigma^2}{2}\eta z^2 + \left(\frac{\sigma^2}{2}\rho + \kappa\eta\right) z + \eta - \kappa\rho \leqslant 0. \tag{6}$$

Assume that $\sigma > 0$. Then the determinant the second order equation in (6) is given by ζ in Equation (3). Some straightforward computations lead to

$$\zeta = \left(\frac{\sigma^2}{2}\rho - \eta\kappa\right)^2 + 2\eta^2\sigma^2, \tag{7}$$

so that ζ is strictly positive. As a consequence, $q(\gamma_\pm) = 0$, where $\gamma_\pm$ is given in Equation (4). Recall that the stability condition for the Hawkes process is the inequality $\eta - \kappa\rho < 0$. Therefore,

$$\zeta = \left(\frac{\sigma^2}{2}\rho + \eta\kappa\right)^2 + 2\eta\sigma^2(\eta - \kappa\rho) < \left(\frac{\sigma^2}{2}\rho + \eta\kappa\right)^2,$$

which implies that both quantities of Equation (4) are strictly positive. Moreover, Equation (7) implies that

$$\sqrt{\zeta} > \left|\frac{\sigma^2}{2}\rho - \eta\kappa\right| \geqslant \eta\kappa - \frac{\sigma^2}{2}\rho$$

and thus

$$\gamma_- = \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) - \sqrt{\zeta}}{\eta\sigma^2} < \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) - \left(\eta\kappa - \frac{\sigma^2}{2}\rho\right)}{\eta\sigma^2} = \frac{\rho}{\eta}. \tag{8}$$

In addition,

$$\begin{aligned}
\gamma_+ &= \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) + \sqrt{\left(\frac{\sigma^2}{2}\rho - \eta\kappa\right)^2 + 2\eta^2\sigma^2}}{\eta\sigma^2} \\
&> \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) + |\frac{\sigma^2}{2}\rho - \eta\kappa|}{\eta\sigma^2} \\
&\geqslant \frac{\sigma^2\rho}{\eta\sigma^2} = \frac{\rho}{\eta}.
\end{aligned}$$

All the claims of the $\sigma > 0$ case are proved. In the case $\sigma = 0$, the claims easily follow from Equation (6) and the stability condition $\eta - \kappa\rho < 0$. □

These lemmas allow to prove the integrability of $e^{z\lambda_t}$, as well as an upper bound for its expected value.

Corollary 2.1. *Define*

$$\gamma = \begin{cases} \gamma_- & \text{if } \sigma > 0, \\ \frac{\kappa\rho-\eta}{\eta\kappa} & \text{if } \sigma = 0. \end{cases}$$

Then for any $z \leqslant \gamma$ and any $t \geqslant 0$, the random variable $e^{z\lambda_t}$ is $\mathbb{Q}$-integrable. Moreover, we have the upper bound

$$\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}|\mathcal{F}_s^{\xi}] \leqslant 1 \vee e^{z\lambda_s + z\kappa\theta(t-s)}$$

whenever $t \geqslant s \geqslant 0$. Moreover,

$$\mathbb{E}_{\mathbb{Q}}[e^{\gamma\lambda_t}|\mathcal{F}_s^{\xi}] = e^{\gamma\lambda_s + \gamma\kappa\theta(t-s)}.$$

Proof. Since $e^{z\lambda_t} \leqslant 1$ when z is nonpositive, we only have to consider the case $z \in (0, \gamma)$. Fix any $t \geqslant u \geqslant s \geqslant 0$. From Lemmas 2.1 and 2.2, we find

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t} - e^{z\lambda_u}|\mathcal{F}_s^{\xi}\right] &= z\kappa\theta\mathbb{E}_{\mathbb{Q}}\left[\int_u^t e^{z\lambda_{\tau-}}\mathrm{d}\tau|\mathcal{F}_s^{\xi}\right] + q(z)\mathbb{E}_{\mathbb{Q}}\left[\int_u^t \lambda_{\tau-}e^{z\lambda_{\tau-}}\mathrm{d}\tau|\mathcal{F}_s^{\xi}\right] \\ &\leqslant z\kappa\theta\int_u^t \mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_{\tau-}}|\mathcal{F}_s^{\xi}\right]\mathrm{d}\tau, \end{aligned}$$

where we used Tonelli's theorem to swap the integral and the expectation. It implies that

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t}|\mathcal{F}_s^{\xi}\right] \leqslant z\kappa\theta\mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t}|\mathcal{F}_s^{\xi}\right],$$

$t \geqslant s$, and thus Grönwall's inequality yields

$$\mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t}|\mathcal{F}_s^{\xi}\right] \leqslant e^{z\lambda_s + z\kappa\theta(t-s)}.$$

In the case $z = \gamma$, Lemma 2.2 yields

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma\lambda_t}|\mathcal{F}_s^{\xi}\right] = z\kappa\theta\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma\lambda_t}|\mathcal{F}_s^{\xi}\right]$$

and hence

$$\mathbb{E}_{\mathbb{Q}}[e^{\gamma\lambda_t}|\mathcal{F}_s^{\xi}] = e^{\gamma\lambda_s + \gamma\kappa\theta(t-s)},$$

as announced. □

With the help of our previous results, we build a Hawkes jump-diffusion supermartingale. To this end, we rely on an eigenfunction of the infinitesimal generator of $(\lambda_t)_{t\geqslant 0}$. Let us consider a positive function $\psi \in C_+^2(\mathbb{R}^+)$ that satisfies the following differential equation

$$\mathcal{A}\psi(\lambda) = \beta\psi(\lambda),$$

where β is a strictly positive constant and $\mathcal{A}$ is the infinitesimal generator operator of $(\lambda_t)_{t\geqslant 0}$. By Ito's lemma, the infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}f(\lambda) = \kappa(\theta-\lambda)\frac{\mathrm{d}f}{\mathrm{d}\lambda} + \lambda\frac{\sigma^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} + \lambda\int_{\mathbb{R}^+}(f(\lambda+\eta z) - f(\lambda))\nu(\mathrm{d}z).$$

The function ψ is thus an eigenfunction of the infinitesimal generator $\mathcal{A}$, with associated eigenvalue β. By construction, the process $(M_t)_{t\geqslant 0}$ defined as $M_t = e^{-\beta t}\psi(\lambda_t)$ is a local-martingale. Indeed, according to Ito's lemma, we have for $t \geqslant u \geqslant 0$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[M_t - M_u|\mathcal{F}_u^{\xi}\right] &= \mathbb{E}_{\mathbb{Q}}\left[-\beta\int_u^t e^{-\beta s}\psi(\lambda_s)\mathrm{d}s + \int_u^t e^{-\beta s}\mathcal{A}\psi(\lambda_s)\mathrm{d}s\Big|\mathcal{F}_u^{\xi}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[-\beta\int_u^t e^{-\beta s}\psi(\lambda_s)\mathrm{d}s + \int_u^t e^{-\beta s}\beta\psi(\lambda_s)\mathrm{d}s\Big|\mathcal{F}_u^{\xi}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[-\beta\int_u^t M_s\mathrm{d}s + \beta\int_u^t M_s\mathrm{d}s\Big|\mathcal{F}_u^{\xi}\right] = 0. \end{aligned}$$

As a consequence, $\mathbb{E}_{\mathbb{Q}}[M_t|\mathcal{F}_u^{\xi}] = M_u$ whenever $u \leqslant t$ and thus

$$\mathbb{E}_{\mathbb{Q}}\left[\psi(\lambda_t)|\mathcal{F}_u^{\xi}\right] = e^{\beta(t-u)}\psi(\lambda_u).$$

It follows that $a > \beta$ implies that the process $(\xi_t)_{t\geqslant 0}$ that satisfies $\xi_t = e^{-at}\psi(\lambda_t)$ for all $t \geqslant 0$ is a supermartingale. As a matter of fact,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\xi_t|\mathcal{F}_u^{\xi}] &= \mathbb{E}_{\mathbb{Q}}\left[e^{-at}\psi(\lambda_t)|\mathcal{F}_u^{\xi}\right] \\ &= e^{-at+\beta(t-u)}\psi(\lambda_u) \\ &< e^{-au}\psi(\lambda_u) = \xi_u.\end{aligned} \tag{9}$$

Moreover, it is also clear from Equation (9) that $\lim_{t\to+\infty}\mathbb{E}_{\mathbb{Q}}[\xi_t] = 0$. We call the process $(\xi_t)_{t\geqslant 0}$ a *Hawkes diffusion supermartingale*. Thanks to its properties, this process is eligible for defining the term structure of default probabilities. The next step is thus to find such an eigenfunction ψ. The following result shows that ψ can be taken to be an exponential function.

Proposition 2.1. *Let γ be the constant defined in Corollary 2.1. Then the function ψ defined by*

$$\psi(\lambda) = e^{\gamma\lambda} \tag{10}$$

is an eigenfunction of the infinitesimal generator $\mathcal{A}$ with associated eigenvalue $\beta = \gamma\kappa\theta$.

Proof. Assuming that ψ satisfies Equation (10), Lemma 2.1 yields

$$\mathcal{A}\psi(\lambda) = \gamma\kappa\theta\psi(\lambda) + q(\gamma)\lambda\psi(\lambda),$$

where the function q is given at Equation (5). According to Lemma 2.2, $q(\gamma) = 0$, so that we further have

$$\mathcal{A}\psi(\lambda) = \gamma\kappa\theta\psi(\lambda).$$

We conclude that the function ψ is an eigenfunction of $\mathcal{A}$ with eigenvalue $\gamma\kappa\theta$. $\square$

Corollary 2.2. *For any $z \in (0, \gamma)$ and $w > \beta = \gamma\kappa\theta$, the process $(\xi_t)_{t\geqslant 0}$ defined by $e^{-wt+z\lambda_t}$ is a positive $(\mathbb{Q}, \mathbb{F}^{\xi})$-supermartingale whose expectation converges to 0.*

Proof. Corollary 2.1 implies that when $t \geqslant s \geqslant 0$, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\xi_t|\mathcal{F}_s^{\xi}] &= \mathbb{E}_{\mathbb{Q}}[e^{-wt+z\lambda_t}|\mathcal{F}_s^{\xi}] \\ &\leqslant e^{-wt+z\lambda_s+\beta(t-s)} \\ &= \xi_s e^{-(w-\beta)(t-s)} \\ &< \xi_s\end{aligned} \tag{11}$$

so that $(\xi_t)_{t\geqslant 0}$ is a supermartingale. The convergence of its expectation follows from Equation (11) and $e^{-(w-\beta)(t-s)} \to 0$ when $t \to +\infty$, since $w - \beta > 0$. $\square$

If we consider the supermartingale $(\xi_t)_{t\geqslant 0}$ and write τ for the time of default, the potential approach consists in assuming that the term structure of default probabilities is determined by

$$\mathbb{Q}(\tau > t|\tau > s) = \frac{e^{-wt}\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}|\mathcal{F}_s^{\xi}]}{e^{-ws+z\lambda_s}}. \tag{12}$$

The next section introduces the fractional version of the model.

3 Fractional Hawkes jump-diffusion supermatingales

In this section, we introduce a fractional Hawkes jump-diffusion supermartingale. This process is obtained by time-changing the supermartingale $(\xi_t)_{t\geqslant 0}$. We start by introducing the time-change process and the considered filtrations. Afterwards, we prove that the supermartingale $(\xi_t)_{t\geqslant 0}$ remains a supermartingale under the time-change and that it keeps all its desired properties.

Let $(U_t)_{t\geqslant 0}$ be an α-stable subordinator on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, with $\alpha \in (0,1)$. The parameter α will be referred to as the *fractional order* in the sequel. Define its inverse $(S_t)_{t\geqslant 0}$ as

$$S_t := \inf\{\tau \geqslant 0 : U_\tau \geqslant t\}.$$

We denote by $\mathbb{F}^U = (\mathcal{F}_t^U)_{t\geqslant 0}$ the natural filtration of $(U_t)_{t\geqslant 0}$, which we assume to be independent from $\mathbb{F}^\xi$. The supermartingale $(\xi_t)_{t\geqslant 0}$ will be time-changed by the inverse process $(S_t)_{t\geqslant 0}$. The inverse process $(S_t)_{t\geqslant 0}$ possesses the property of "randomly freezing time". This property can be observed on Figure 1. This figure displays corresponding sample paths of the process α-stable process $(U_t)_{t\geqslant 0}$ and its inverse $(S_t)_{t\geqslant 0}$ for some levels of the fractional order α. One can see that the frozen time periods of $(S_t)_{t\geqslant 0}$ (i.e. the periods of time for which the process remains constant) correspond to the jumps of the original process $(U_t)_{t\geqslant 0}$. Also note that by definition of $(S_t)_{t\geqslant 0}$, representing the path $[0, 10] \ni t \mapsto S_t(\omega)$ only requires the values of $t \mapsto U_t(\omega)$ until it reaches the value 10. This is why the graphs on the right-side are truncated at $y = 10$. Finally, note that the values of the fractional order close to 0 give the wildest behaviour for $(U_t)_{t\geqslant 0}$, i.e. paths with very large jumps. This translates into a lot of long motionless periods for the inverse $(S_t)_{t\geqslant t}$ when α is close to 0.

Let us now introduce precisely the filtration we consider for the time-changed supermartingale $(\xi_{S_t})_{t\geqslant 0}$. Define the filtration $\mathbb{F}^{\xi,U} = (\mathcal{F}_t^{\xi,U})_{t\geqslant 0}$ as $\mathcal{F}_t^{\xi,U} = \mathcal{F}_t^\xi \vee \mathcal{F}_t^U$. By definition of $(S_t)_{t\geqslant 0}$, for any $u, s \geqslant 0$ it holds that $\{S_u \leqslant s\} = \{U_s \geqslant u\}$, which clearly belongs to $\mathcal{F}_s^U$. As a consequence, for any fixed $u \geqslant 0$, S_u is a $\mathbb{F}^U$-stopping-time, and thus also a $\mathbb{F}^{\xi,U}$-stopping-time. We can therefore define the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geqslant 0}$ as the collection of stopped σ-algebras

$$\mathcal{G}_t := \mathcal{F}_{S_t}^{\xi,U} = \{A \in \mathcal{F}_\infty^{\xi,U} : A \cap \{S_t \leqslant s\} \in \mathcal{F}_s^{\xi,U} \text{ for all } s \geqslant 0\},$$

with $\mathcal{F}_\infty^{\xi,U} = \bigvee_{t\geqslant 0} \mathcal{F}_t^{\xi,U}$. We show now that the obtained process $(\xi_{S_t})_{t\geqslant 0}$ remains a supermartingale and can therefore be used as in the previous section to define a term structure of default probabilities.

Proposition 3.1. *The process $(\xi_{S_t})_{t\geqslant 0}$ is a positive $(\mathbb{Q}, \mathbb{G})$-supermartingale that satisfies $\lim_{t\to+\infty} \mathbb{E}_\mathbb{Q}[\xi_{S_t}] = 0$.*

Proof. By Corollary 2.2, $(\xi_t)_{t\geqslant 0}$ is a $(\mathbb{Q}, \mathbb{F}^\xi)$-supermartingale. Since the filtrations $\mathbb{F}^\xi$ and $\mathbb{F}^U$ are independent for any $s \geqslant 0$, it holds that

$$\mathbb{E}_\mathbb{Q}[\xi_t | \mathcal{F}_s^\xi \vee \mathcal{F}_s^U] = \mathbb{E}_\mathbb{Q}[\xi_t | \mathcal{F}_s^\xi]$$

for any $s, t \geqslant 0$. The property we use in the previous equation can be found in the chapter 9.7 of Williams (1991), at property (k). As a consequence, $(\xi_t)_{t\geqslant 0}$ is also a $(\mathbb{Q}, \mathbb{F}^{\xi,U})$-supermartingale. Then, since for any $s \leqslant t$, we have $S_s \leqslant S_t$ $\mathbb{Q}$-a.s., it follows from the Optional Stopping Theorem for supermartingales (see e.g. Theorem 3.3 in the chapter 2 of Revuz and Yor (2004)) that

$$\xi_{S_s} \geqslant \mathbb{E}_\mathbb{Q}[\xi_{S_t} | \mathcal{F}_{S_s}^{U,\xi}] = \mathbb{E}_\mathbb{Q}[\xi_{S_t} | \mathcal{G}_s].$$

This proves that $(\xi_{S_t})_{t\geqslant 0}$ is a $(\mathbb{Q}, \mathbb{G})$-supermartingale. Recall that from Piryatinska, Saichev, and Woyczynski (2004),

$$\mathbb{E}_\mathbb{Q}[e^{-\omega S_t}] = \mathrm{E}_\alpha(-\omega t^\alpha),$$

where the function E_α is the Mittag-Leffler function

$$\mathrm{E}_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + 1)}.$$

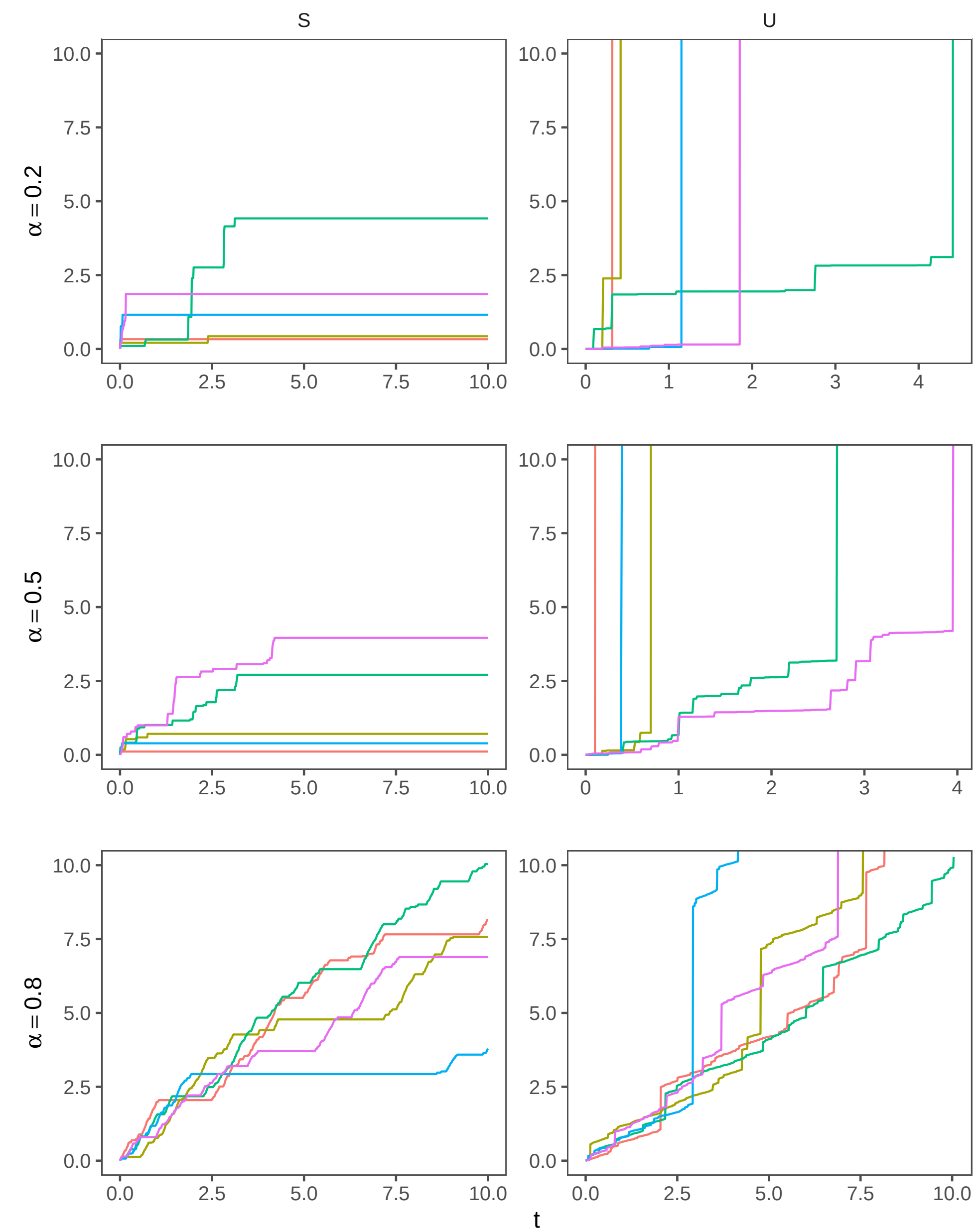

Figure 1: Five simulated paths of $(U_t)_{t\geqslant 0}$ and corresponding paths of $(U_t)_{t\geqslant 0}$ for fractional orders $\alpha \in \{0.2, 0.5, 0.8\}$.

As a consequence, from Corollary 2.1 and Tonelli's theorem,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\xi_{S_t}] &= \int_{\mathbb{R}^+} \mathbb{E}_{\mathbb{Q}}[\xi_u] g(t,u)\mathrm{d}u \\ &\leqslant \int_{\mathbb{R}^+} e^{(\beta-w)u} g(t,u)\mathrm{d}u \\ &= \mathrm{E}_\alpha((\beta-w)t^\alpha),\end{aligned}$$

where $u \mapsto g(t,u)$ denotes the probability density function of S_t, $t>0$. This proves the convergence of the expectation to 0 and ends the proof. $\square$

The supermartingale $(\xi_{S_t})_{t\geqslant 0}$ inherits from $(S_t)_{t\geqslant 0}$ the property of having motionless periods, during which the time appears to be frozen. The term structure of default probabilities is defined as follows: let $\tau^{(\alpha)}$ be the time of default in the fractional model. Then we assume that the term structure of default probabilities satisfies

$$\mathbb{Q}(\tau^{(\alpha)} > t | \tau^{(\alpha)} > s) = \frac{\mathbb{E}_{\mathbb{Q}}[e^{-wS_t + z\lambda_{S_t}} | \mathcal{G}_s]}{e^{-wS_s + z\lambda_{S_s}}}, \tag{13}$$

for $t \geqslant s \geqslant 0$.

The next section addresses the problem of computing the default probabilities, in both the non-fractional and fractional settings.

4 Computation of the default probabilities

The main challenge in computing the default probabilities of Equations (12) and (13) is the computation of $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}|\mathcal{F}_s]$ and $\mathbb{E}_{\mathbb{Q}}[e^{-wS_t+z\lambda_{S_t}}|\mathcal{F}_{S_s}]$, $t \geqslant s \geqslant 0$. To this end, several approaches can be thought of. A first possible approach is the one of Duffie, Pan, and Singleton (2000). It consists in showing that the expected value $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}|\mathcal{F}_s]$ can be written as $\exp\{a(t-s)+\lambda_s b(t-s)\}$ for some functions a, b and that these functions satisfy a system of differential equations. The expected value can then be found by solving this system numerically. In our context, the main drawback of this approach is that it does not transpose very well to the fractional model. As a matter of fact, obtaining the default probabilities in the fractional setting would require to compute numerically the integral

$$\int_0^{+\infty} \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_\tau}] g(t,\tau)\mathrm{d}\tau = \int_0^{+\infty} \exp\{a(\tau) + \lambda_0 b(\tau)\} g(t,\tau)\mathrm{d}\tau,$$

where $\tau \mapsto g(t,\tau)$ is the probability density function of S_t. Then one could use Proposition 3.2 in Gupta and Kumar (2022) that says that

$$g(t,\tau) = \frac{1}{\pi}\sum_{k=0}^{+\infty} \frac{(-\tau)^k}{k!}\Gamma((1+k)\alpha)t^{-(k+1)\alpha}\sin(\pi\alpha(k+1)), \quad \tau > 0$$

to evaluate the integral. This first approach seems to be computationally heavy, as it requires several layers of numerical approximations. A second approach is to derive a partial differential equation for the transform $(t,z) \mapsto \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}]$. This approach is used in e.g. Ketelbuters and Hainaut (2022) and has the benefit of having a natural extension to the fractional case. As a matter of fact, the transform $(t,z) \mapsto \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_{S_t}}]$ satisfies a fractional partial differential equation. Compared to the non-fractional case, the only difference is that the partial derivative with respect to time is a Caputo fractional derivative of order α. In both cases, the (fractional) partial differential equation can be solved numerically with a finite difference method. However, this second approach suffers from two drawbacks. The first drawback is that we have to rely on fractional calculus, which is rather complicated from a mathematical standpoint. The second drawback is that fractional partial differential equations are far from being the easiest equations to solve numerically. Indeed, performing finite difference approximations for fractional derivatives requires, at each iteration, to use the computed values for all the previous iterations. This substantially increases the computation time. This is why we propose here a third approach that rely on a recursive method to compute all the moments of λ_t. To the best of our knowledge, this recursive method is new. Once the moments $\mathbb{E}_{\mathbb{Q}}[\lambda_t^k]$ are computed for all nonnegative integers $k \leqslant n$, we can approximate the expectation $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}]$ with the help of the MacLaurin series expansion of the exponential function, that is

$$\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}] \approx \sum_{k=0}^{n} \frac{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^k]}{k!}.$$

This third approach avoids the shortcomings of the two other ones. Firstly, it is much faster computationally. Secondly, it has an easy extension to the fractional case, without relying on the complicated machinery of fractional calculus. That is, once the moments of λ_t are obtained, the moments of the time-changed process λ_{S_t} can be found very easily. Thirdly, the approximation can be shown to be convergent and a bound on the numerical error can be provided in both the non-fractional and fractional cases.

This section is organized as follows: we start by stating the recursive method that allows to compute the moments of λ_t and give a proof that this method works. Then, as a corollary to this method, we show that the moments of the time-changed process λ_{S_t} can be obtained in a similar manner. Finally, after introducing the approximations for the survival probabilities of Equations (12) and (13), we show that these approximations are convergent and give an upper bound on the associated numerical error.

The next result gives the recursive method for the moments of λ_t.

Proposition 4.1. *Let $z > 0$. For each $n \geqslant 1$, the moment $\mathbb{R}^+ \ni t \mapsto \mathbb{E}[(z\lambda_t)^n]$ satisfy*

$$\mathbb{E}[(z\lambda_t^n)] = \left(\sum_{j=1}^{n} d_{j,n} e^{-jat}\right) + c_n$$

with $a = \kappa - (\eta/\rho)$, for all $n \in \mathbb{N}$. Moreover, the constants $d_{k,n}, c_n$ can be computed recursively with the relations

$$d_{1,1} = z\lambda_0 - \frac{\kappa z\theta}{a}, \qquad c_1 = \frac{\kappa z\theta}{a},$$

$$d_{j,n} = \begin{cases} -\left(\frac{b_n d_{j,n-1}}{(j-n)a} + \sum_{p=2}^{n-j+1} z^{-1} m_p \binom{n}{p} \frac{d_{j,n-p+1}}{(j-n)a}\right) & j < n, \\ (z\lambda_0)^n - \left(\sum_{p=2}^{n} z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na}\right) - \frac{b_n c_{n-1}}{na} & \\ \qquad + \sum_{k=1}^{n-1} \left(\frac{b_n d_{k,n-1}}{(k-n)a} + \sum_{p=2}^{n-k+1} z^{-1} m_p \binom{n}{p} \frac{d_{k,n-p+1}}{(k-n)a}\right) & j = n, \end{cases}$$

$$c_n = \frac{b_n c_{n-1}}{na} + \sum_{p=2}^{n} z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na}.$$

The symbol b_n is used as a shorthand for

$$b_n = n\kappa z\theta + \frac{n(n-1)z\sigma^2}{2}$$

and $m_p = \mathbb{E}_{\mathbb{Q}}[(z\eta J)^p] = \left(\frac{z\eta}{\rho}\right)^p p!$.

Proof. Define the process $(\tilde{\lambda}_t)_{t \geqslant 0}$ as $\tilde{\lambda}_t := z\lambda_t$. This process satisfies the SDE

$$\mathrm{d}\tilde{\lambda}_t = \kappa(\tilde{\theta} - \tilde{\lambda}_t)\mathrm{d}t + \tilde{\sigma}\sqrt{\tilde{\lambda}_t}\mathrm{d}W_t + \tilde{\eta}\mathrm{d}P_t$$

where $\tilde{\theta} = z\theta$, $\tilde{\sigma} = \sqrt{z}\sigma$ and $\tilde{\eta} = z\eta$. Note that the jump intensity of $(P_t)_{t\geqslant 0}$ remains $(\lambda_t)_{t\geqslant 0} = (z^{-1}\tilde{\lambda}_t)_{t\geqslant 0}$. As a consequence, Ito's lemma applied on the process $(\tilde{\lambda}_t^n)_{t\geqslant 0}$ yields

$$\begin{aligned}\mathbb{E}\left[\tilde{\lambda}_t^n\right] - \tilde{\lambda}_0^n &= \mathbb{E}\Bigg[\int_0^t n\tilde{\lambda}_{s-}^{n-1}\left(\kappa(\tilde{\theta}-\tilde{\lambda}_{s-})\mathrm{d}s + \tilde{\sigma}\sqrt{\tilde{\lambda}_{s-}}\mathrm{d}W_s\right) \\ &\qquad + \frac{n(n-1)}{2}\int_0^t \tilde{\lambda}_{s-}^{n-2}\tilde{\lambda}_{s-}\tilde{\sigma}^2\mathrm{d}s + \int_0^t \left(\tilde{\lambda}_s^n - \tilde{\lambda}_{s-}^n\right)\mathrm{d}N_s\Bigg] \\ &= \mathbb{E}\left[n\int_0^t \left(\kappa\tilde{\theta}\tilde{\lambda}_{s-}^{n-1} - \kappa\tilde{\lambda}_{s-}^n\right)\mathrm{d}s + \frac{n(n-1)}{2}\int_0^t \tilde{\lambda}_{s-}^{n-1}\tilde{\sigma}^2\mathrm{d}s + \int_0^t \left(\tilde{\lambda}_s^n - \tilde{\lambda}_{s-}^n\right) z^{-1}\tilde{\lambda}_{s-}\mathrm{d}s\right]\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\partial\mathbb{E}\left[\tilde{\lambda}_t^n\right]}{\partial t} &= \mathbb{E}[\tilde{\lambda}_t^{n-1}]\left(n\kappa\tilde{\theta} + \frac{n(n-1)\tilde{\sigma}^2}{2}\right) - n\kappa\mathbb{E}[\tilde{\lambda}_s^n] + \mathbb{E}\left[\left((\tilde{\lambda}_t + \tilde{\eta}J)^n - \tilde{\lambda}_t^n\right)z^{-1}\tilde{\lambda}_t\right] \\ &= \mathbb{E}[\tilde{\lambda}_t^{n-1}]\left(n\kappa\tilde{\theta} + \frac{n(n-1)\tilde{\sigma}^2}{2}\right) - n\kappa\mathbb{E}[\tilde{\lambda}_s^n] + \mathbb{E}\left[\left(\sum_{p=0}^n\binom{n}{p}(\tilde{\eta}J)^p\tilde{\lambda}_t^{n-p} - \tilde{\lambda}_t^n\right)z^{-1}\tilde{\lambda}_t\right] \\ &= \mathbb{E}[\tilde{\lambda}_t^{n-1}]\left(n\kappa\tilde{\theta} + \frac{n(n-1)\tilde{\sigma}^2}{2}\right) - n\kappa\mathbb{E}[\tilde{\lambda}_s^n] + \sum_{p=1}^n\binom{n}{p}z^{-1}m_p\mathbb{E}\left[\tilde{\lambda}_t^{n-p+1}\right].\end{aligned}$$

Therefore, if we define the functions $(f_n)_{n\in\mathbb{N}}$, $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ as $f_n(t) = \mathbb{E}\left[\tilde{\lambda}_t^n\right]$, then each of these functions satisfies the first order linear differential equation

$$f_n'(t) = -naf_n(t) + h_{n-1}(t), \tag{14}$$

where

$$h_{n-1}(t) = \begin{cases} b_nf_{n-1}(t) & \text{if } n = 1, \\ b_nf_{n-1}(t) + \sum_{p=2}^n\binom{n}{p}z^{-1}m_pf_{n-p+1}(t) & \text{if } n \geqslant 2.\end{cases}$$

Equation (14) is easy to solve. Multiplying both sides by e^{nat} leads to

$$e^{nat}f_n'(t) + nae^{nat}f_n(t) = e^{nat}h_{n-1}(t)$$

and since the left-hand side is $(e^{nat}f_n(t))'$, an integration yields

$$f_n(t) = e^{-nat}\tilde{\lambda}_0^n + e^{-nat}\int_0^t e^{nas}h_{n-1}(s)\mathrm{d}s. \tag{15}$$

We prove now by induction that the functions $(f_n)_{n\in\mathbb{N}}$ are of the form

$$f_n(t) = \left(\sum_{j=1}^n d_{j,n}e^{-jat}\right) + c_n \tag{16}$$

and that the numbers $d_{j,n}, c_n$ can be computed recursively, as in the statement. From Equation (15), we find that

$$f_1(t) = \left(\tilde{\lambda}_0 - \frac{\kappa\tilde{\theta}}{a}\right)e^{-at} + \frac{\kappa\tilde{\theta}}{a}$$

so that Equation (16) holds for $n = 1$ with $d_{1,1} = \tilde{\lambda}_0 - \kappa\tilde{\theta}/a$ and $c_1 = \kappa\tilde{\theta}/a$. Assume that Equation (16) holds for $1, \dots, n-1$ and that $n \geqslant 2$. Then Equation (15) yields

$$f_n(t) = e^{-nat}\tilde{\lambda}_0^n + e^{-nat}\int_0^t e^{nas}h_{n-1}(s)\mathrm{d}s.$$

Using Equation (16), we can compute

$$\begin{aligned}\int_0^t e^{nas}h_{n-1}(s)\mathrm{d}s &= b_n\int_0^t e^{nas}\left[\left(\sum_{j=1}^{n-1} d_{j,n-1}e^{-jas}\right)+c_{n-1}\right]\mathrm{d}s\\ &\quad+\sum_{p=2}^{n} z^{-1}m_p\binom{n}{p}\int_0^t e^{nas}\left[\left(\sum_{j=1}^{n-p+1} d_{j,n-p+1}e^{-jas}\right)+c_{n-p+1}\right]\mathrm{d}s\\ &= -\sum_{j=1}^{n-1} e^{-(j-n)at}\left(\frac{b_n d_{j,n-1}}{(j-n)a}+\sum_{p=2}^{n-j+1} z^{-1}m_p\binom{n}{p}\frac{d_{j,n-p+1}}{(j-n)a}\right)\\ &\quad+e^{nat}\left[\frac{b_n c_{n-1}}{na}+\sum_{p=2}^{n} z^{-1}m_p\binom{n}{p}\frac{c_{n-p+1}}{na}\right]-\sum_{p=2}^{n} z^{-1}m_p\binom{n}{p}\frac{c_{n-p+1}}{na}\\ &\quad+\sum_{j=1}^{n-1}\left[\frac{b_n d_{j,n-1}}{(j-n)a}+\sum_{p=2}^{n-j+1} z^{-1}m_p\binom{n}{p}\frac{d_{j,n-p+1}}{(j-n)a}\right]-\frac{b_n c_{n-1}}{na}.\end{aligned}$$

As a consequence, we find

$$\begin{aligned}f_n(t) &= -\sum_{j=1}^{n-1} e^{-jat}\left(\frac{b_n d_{j,n-1}}{(j-n)a}+\sum_{p=2}^{n-j+1} z^{-1}m_p\binom{n}{p}\frac{d_{j,n-p+1}}{(j-n)a}\right)\\ &+e^{-nat}\left[\tilde{\lambda}_0^n-\left(\sum_{p=2}^{n} z^{-1}m_p\binom{n}{p}\frac{c_{n-p+1}}{na}\right)-\frac{b_n c_{n-1}}{na}+\sum_{j=1}^{n-1}\left(\frac{b_n d_{j,n-1}}{(j-n)a}+\sum_{p=2}^{n-j+1} z^{-1}m_p\binom{n}{p}\frac{d_{j,n-p+1}}{(j-n)a}\right)\right]\\ &+\left[\frac{b_n c_{n-1}}{na}+\sum_{p=2}^{n} z^{-1}m_p\binom{n}{p}\frac{c_{n-p+1}}{na}\right]\\ &=\left(\sum_{j=1}^{n} d_{j,n}e^{-jat}\right)+c_n,\end{aligned}$$

where

$$d_{j,n}=\begin{cases}-\left(\frac{b_n d_{j,n-1}}{(j-n)a}+\sum_{p=2}^{n-j+1} z^{-1}m_p\binom{n}{p}\frac{d_{j,n-p+1}}{(j-n)a}\right) & \text{if } j<n\\ \tilde{\lambda}_0^n-\left(\sum_{p=2}^{n} z^{-1}m_p\binom{n}{p}\frac{c_{n-p+1}}{na}\right)-\frac{b_n c_{n-1}}{na}+\sum_{k=1}^{n-1}\left(\frac{b_n d_{k,n-1}}{(k-n)a}+\sum_{p=2}^{n-k+1} z^{-1}m_p\binom{n}{p}\frac{d_{k,n-p+1}}{(k-n)a}\right) & \text{if } j=n\end{cases}$$

and

$$c_n=\frac{b_n c_{n-1}}{na}+\sum_{p=2}^{n} z^{-1}m_p\binom{n}{p}\frac{c_{n-p+1}}{na}.$$

This concludes the proof. □

Note that in the proof of the previous result, the method of using differential equations to find the moments of a Hawkes process intensity is not new, as this approach is used in e.g. Cui, Hawkes, and Yi (2020). The originality of the result lies in the observation that all the moments share the same form, i.e. a linear combination of exponential functions plus a constant, and that the associated coefficients can be computed recursively. According to the next corollary, it turns out that this result extends particularly well to the fractional case.

Corollary 4.1. *The moment* $\mathbb{R}^+\ni t\mapsto\mathbb{E}_{\mathbb{Q}}[(z\lambda_{S_t})^n]$ *satisfy*

$$\mathbb{E}[(z\lambda_{S_t})^n]=\left(\sum_{j=1}^{n} d_{j,n}\mathrm{E}_\alpha(-jat^\alpha)\right)+c_n,$$

for all $n \in \mathbb{N}$. The function E_α is the Mittag-Leffler function

$$\mathrm{E}_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+1)}.$$

The constants $a, c_n, d_{j,n}$ are given in Proposition 4.1.

Proof. The main tool of this proof is a result from Piryatinska, Saichev, and Woyczynski (2004), which states that

$$\mathbb{E}_{\mathbb{Q}}[e^{-\omega S_t}] = \mathrm{E}_\alpha(-\omega t^\alpha).$$

As a consequence, by Proposition 4.1 and the law of iterated expectations, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[(z\lambda_{S_t})^n] &= \left(\sum_{j=1}^{n} d_{j,n}\mathbb{E}_{\mathbb{Q}}\left[e^{-jaS_t}\right]\right) + c_n \\ &= \left(\sum_{j=1}^{n} d_{j,n}\mathrm{E}_\alpha(-jat^\alpha)\right) + c_n,\end{aligned}$$

as announced. □

We now have the tools to approximate the survival probabilities. In the non-fractional case, we approximate the survival probabilities $t \mapsto q(t) := \mathbb{Q}(\tau > t)$ by the sequence of functions $(q_n)_{n\in\mathbb{N}}$ defined as

$$q_n(t) := \sum_{k=0}^{n} \frac{e^{-wt}\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^k]}{e^{z\lambda_0}k!}. \tag{17}$$

We prove now the convergence of this approximation and give a bound for the numrical error.

Proposition 4.2. *Let $z \in [0, \gamma]$. For each $t \geqslant 0$, the approximation $q_n(t)$ converge towards $q(t)$ as $n \to +\infty$. Moreover, if $z \leqslant \gamma/2$, the error $q(t) - q_n(t)$ satisfies the following bounds*

$$0 \leqslant q(t) - q_n(t) \leqslant \frac{\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_t)^{n+1}\right] e^{\gamma(\lambda_0+\kappa\theta t)} + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2}\sqrt{e^{\gamma(\lambda_0+\kappa\theta t)}}}{e^{wt+z\lambda_0}(n+1)!}.$$

Proof. Note that by positivity of $(\tilde{\lambda}_t)_{t\geqslant 0}$, the sequence of random variables $(\sum_{j=0}^{n} \tilde{\lambda}_t^j)_{n\in\mathbb{N}}$ is increasing. As a consequence, the monotone convergence theorem states that

$$\mathbb{E}_{\mathbb{Q}}[e^{\tilde{\lambda}_t}] = \mathbb{E}_{\mathbb{Q}}\left[\lim_{n\to+\infty}\sum_{j=0}^{n}\frac{\tilde{\lambda}_t^j}{j!}\right] = \lim_{n\to+\infty}\sum_{j=0}^{n}\frac{\mathbb{E}_{\mathbb{Q}}\left[\tilde{\lambda}_t^j\right]}{j!},$$

which already proves the convergence of the approximation. We now derive a bound on the error we commit by keeping the $n+1$ first terms in the series. This error is the expected value of $\sum_{k\geqslant n+1}((z\lambda_t)^k/k!)$, which satisfies

$$\begin{aligned}\sum_{k=n+1}^{+\infty}\frac{(z\lambda_t)^k}{k!} &= \sum_{k=0}^{+\infty}\frac{(z\lambda_t)^{k+n+1}}{(k+n+1)!} \\ &\leqslant \frac{(z\lambda_t)^{n+1}}{(n+1)!}\sum_{k=0}^{+\infty}\frac{(z\lambda_t)^k}{k!} \\ &= \frac{(z\lambda_t)^{n+1}e^{z\lambda_t}}{(n+1)!}\end{aligned}$$

where the inequality is a consequence of $(k+n+1)! \geqslant k!(n+1)!$. The proof is completeted by observing that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_t)^{n+1}e^{z\lambda_t}\right] = \mathbb{E}_{\mathbb{Q}}\left[(z\lambda_t)^{n+1}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t}\right] + \mathbb{C}\mathrm{ov}_{\mathbb{Q}}((z\lambda_t)^{n+1}, e^{z\lambda_t}) \\
&\quad \leqslant \mathbb{E}_{\mathbb{Q}}\left[(z\lambda_t)^{n+1}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t}\right] + \sqrt{\mathbb{V}\mathrm{ar}_{\mathbb{Q}}((z\lambda_t)^{n+1})\mathbb{V}\mathrm{ar}_{\mathbb{Q}}(e^{z\lambda_t})} \\
&\quad \leqslant \mathbb{E}_{\mathbb{Q}}\left[(z\lambda_t)^{n+1}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{z\lambda_t}\right] + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2}\sqrt{\mathbb{E}_{\mathbb{Q}}[e^{2z\lambda_t}]} \\
&\quad \leqslant \mathbb{E}_{\mathbb{Q}}\left[(z\lambda_t)^{n+1}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma\lambda_t}\right] + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2}\sqrt{\mathbb{E}_{\mathbb{Q}}[e^{\gamma\lambda_t}]} \\
&\quad = \mathbb{E}_{\mathbb{Q}}\left[(z\lambda_t)^{n+1}\right]e^{\gamma(\lambda_0+\kappa\theta t)} + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2}\sqrt{e^{\gamma(\lambda_0+\kappa\theta t)}},
\end{aligned} \tag{18}$$

where the last equality follows from Corollary 2.1. □

Note that the moments $\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}]$ and $\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}]$ that appear in the upper bound of Proposition 4.2 can be computed with the help of the recursive method of Proposition 4.1.

In the fractional case, we approximate the survival probabilities $t \mapsto \mathbb{Q}(\tau^{(\alpha)} > t)$ by the functions $(q_n^{(\alpha)})_{n\in\mathbb{N}}$ as

$$\begin{aligned}
q_n^{(\alpha)}(t) &:= \sum_{k=0}^{n} \frac{\mathbb{E}_{\mathbb{Q}}\left[e^{-wS_t}(z\lambda_{S_t})^k\right]}{e^{z\lambda_0}k!} \\
&= \sum_{k=0}^{n} \frac{1}{e^{z\lambda_0}k!}\mathbb{E}_{\mathbb{Q}}\left[e^{-wS_t}\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_{S_t})^k|\mathcal{F}_t^S\right]\right] \\
&= \sum_{k=0}^{n} \frac{1}{e^{z\lambda_0}k!}\mathbb{E}_{\mathbb{Q}}\left[e^{-wS_t}\left(c_k + \sum_{j=1}^{k} d_{j,k}e^{-jaS_t}\right)\right] \\
&= \sum_{k=0}^{n} \frac{1}{e^{z\lambda_0}k!}\mathbb{E}_{\mathbb{Q}}\left[c_k e^{-wS_t} + \sum_{j=1}^{k} d_{j,k}e^{-(w+ja)S_t}\right] \\
&= \sum_{k=0}^{n} \frac{1}{e^{z\lambda_0}k!}\left(c_k \mathrm{E}_\alpha\left(-wt^\alpha\right) + \sum_{j=1}^{k} d_{j,k}\mathrm{E}_\alpha\left(-(w+ja)t^\alpha\right)\right).
\end{aligned} \tag{19}$$

In a similar way, we can also prove the convergence of the approximation in the fractional case, and give a bound on the error.

Proposition 4.3. *Let $z \in [0,\gamma]$. For each $t \geqslant 0$, the approximation $q_n^{(\alpha)}(t)$ converge towards $q^{(\alpha)}(t)$ as $n \to +\infty$. Moreover, if $z \leqslant \gamma/2$, the error $q^{(\alpha)}(t) - q_n^{(\alpha)}(t)$ satisfies the following bounds*

$$\begin{aligned}
&0 \leqslant q^{(\alpha)}(t) - q_n^{(\alpha)}(t) \\
&\leqslant \frac{e^{-z\lambda_0}}{(n+1)!}\left(\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_{S_t})^{n+1}\right]e^{\gamma\lambda_0}\mathrm{E}_\alpha((\gamma\kappa\theta - w)t^\alpha) + \sqrt{\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left((z\lambda_{S_t})^{n+1}\right)e^{\gamma\lambda_0}\mathrm{E}_\alpha((\gamma\kappa\theta - 2w)t^\alpha)}\right).
\end{aligned}$$

Proof. The convergence of $q_n^{(\alpha)}(t)$ towards $q^{(\alpha)}(t)$ follows from the monotone convergence theorem. As in the proof of Proposition 4.2,

$$\sum_{k=n+1}^{+\infty} \frac{(z\lambda_{S_t})^k}{k!} \leqslant \frac{(z\lambda_{S_t})^{n+1}e^{z\lambda_{S_t}}}{(n+1)!}.$$

As a consequence,

$$\begin{aligned}
q^{(\alpha)}(t) - q_n^{(\alpha)}(t) &\leqslant \frac{e^{-z\lambda_0}}{(n+1)!}\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_{S_t})^{n+1}e^{-wS_t+z\lambda_{S_t}}\right] \\
&\leqslant \frac{e^{-z\lambda_0}}{(n+1)!}\left(\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_{S_t})^{n+1}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{-wS_t+z\lambda_{S_t}}\right] + \sqrt{\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left((z\lambda_{S_t})^{n+1}\right)\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left(e^{-wS_t+z\lambda_{S_t}}\right)}\right) \\
&\leqslant \frac{e^{-z\lambda_0}}{(n+1)!}\left(\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_{S_t})^{n+1}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{-wS_t+z\lambda_{S_t}}\right] + \sqrt{\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left((z\lambda_{S_t})^{n+1}\right)\mathbb{E}_{\mathbb{Q}}\left(e^{-2wS_t+2z\lambda_{S_t}}\right)}\right) \\
&\leqslant \frac{e^{-z\lambda_0}}{(n+1)!}\left(\mathbb{E}_{\mathbb{Q}}\left[(z\lambda_{S_t})^{n+1}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{-wS_t+\gamma\lambda_{S_t}}\right] + \sqrt{\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left((z\lambda_{S_t})^{n+1}\right)\mathbb{E}_{\mathbb{Q}}\left[e^{-2wS_t+\gamma\lambda_{S_t}}\right]}\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[e^{-wS_t+\gamma\lambda_{S_t}}\right] &= \int_0^{+\infty} e^{-w\tau}\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma\lambda_\tau}\right]g(t,\tau)\mathrm{d}\tau \\
&= e^{\gamma\lambda_0}\int_0^{+\infty} e^{(\gamma\kappa\theta-w)\tau}g(t,\tau)\mathrm{d}\tau \\
&= e^{\gamma\lambda_0}\mathrm{E}_\alpha((\gamma\kappa\theta-w)t^\alpha)
\end{aligned}$$

and similarly

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-2wS_t+\gamma\lambda_{S_t}}\right] = e^{\gamma\lambda_0}\mathrm{E}_\alpha((\gamma\kappa\theta-2w)t^\alpha).$$

The result follows. □

Again, the moments that appear in the upper bound of Proposition 4.3 can be computed with the recursive method of Corollary 4.1.

5 Numerical experiments

This section presents the numerical results we obtained with the methods presented above. The model is calibrated on real data taken from Bloomberg on 25 January 2021. These data consist of survival probabilities for American Airlines. Note that these data are the same as in Ketelbuters and Hainaut (2022), so that we can compare the fit for both models. The model is calibrated in both settings, i.e. non-fractional and fractional. The parameters are found by minimizing the sum of squared errors, i.e.

$$\sum_{T=1}^{10}(Prob_T^{\mathrm{Model}} - Prob_T^{\mathrm{Market}})^2$$

where $Prob_T^{\mathrm{Market}}$ denotes the market survival up to time T probability from Bloomberg and $Prob_T^{\mathrm{Model}}$ denotes the corresponding survival probability implied by our model. The optimal parameters obtained for both models are given in Table 1 whereas Figure 2 shows the graphical result of the calibration. In particular, Figure 2 shows that the fit is as good as in Ketelbuters and Hainaut (2022). As explained above, this is achieved with a model that is easier to deal with both mathematically and numerically. Table 2 shows the numerical convergence of our method and Table 3 shows the values of the upper bounds that were given at Propositions 4.2 and 4.3. Finally, Figures 3, 4 and 5 show the impact of the different parameters on the implied survival probability curves, when all the other parameters are kept unchanged and equal to their value in Table 1.

Table 1: Parameters used for the computations.

	θ	σ	η	ρ	κ	λ_0	γ	β	z	w	α
Non-fractional	0.0551	0.1051	9.5599	3.1870	7.7522	0.0551	0.2044	0.0873	0.1022	0.2434	1.0000
Fractional	0.6751	0.0999	9.9308	2.3752	6.1281	0.6751	0.0760	0.3144	0.0380	0.3144	0.9083

Table 2: Convergence of the computed probabilities and market survival probabilities.

	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
Market data	0.795400	0.567300	0.457600	0.363300	0.301100	0.249200	0.203200	0.166100	0.133900	0.106100
Non-fractional										
$n=2$	0.787450	0.617377	0.484019	0.379467	0.297500	0.233237	0.182856	0.143358	0.112392	0.088114
$n=5$	0.787809	0.617661	0.484242	0.379642	0.297636	0.233345	0.182941	0.143424	0.112443	0.088155
$n=10$	0.787835	0.617681	0.484258	0.379654	0.297646	0.233352	0.182947	0.143429	0.112447	0.088158
$n=15$	0.787835	0.617682	0.484258	0.379655	0.297646	0.233353	0.182947	0.143429	0.112447	0.088158
$n=20$	0.787835	0.617682	0.484258	0.379655	0.297646	0.233353	0.182947	0.143429	0.112447	0.088158
Fractional										
$n=2$	0.766549	0.588248	0.458572	0.362939	0.291225	0.236678	0.194678	0.161986	0.136285	0.115891
$n=5$	0.771007	0.592472	0.462013	0.365698	0.293448	0.238485	0.196162	0.163218	0.137318	0.116765
$n=10$	0.771297	0.592780	0.462269	0.365904	0.293615	0.238621	0.196273	0.163310	0.137395	0.116830
$n=15$	0.771301	0.592785	0.462274	0.365908	0.293618	0.238623	0.196276	0.163312	0.137396	0.116832
$n=20$	0.771301	0.592785	0.462274	0.365908	0.293618	0.238623	0.196276	0.163312	0.137396	0.116832

Table 3: Bounds on the numerical error.

	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
Non-fractional										
$n=5$	0.0096	0.0079	0.0065	0.0053	0.0043	0.0036	0.0029	0.0024	0.0020	0.0016
$n=10$	0.0061	0.0050	0.0041	0.0034	0.0028	0.0023	0.0019	0.0015	0.0012	0.0010
$n=20$	0.0038	0.0031	0.0026	0.0021	0.0017	0.0014	0.0011	0.0009	0.0008	0.0006
$n=30$	0.0028	0.0023	0.0019	0.0016	0.0013	0.0011	0.0009	0.0007	0.0006	0.0005
$n=40$	0.0023	0.0019	0.0016	0.0013	0.0010	0.0009	0.0007	0.0006	0.0005	0.0004
Fractional										
$n=5$	0.0338	0.0377	0.0350	0.0317	0.0286	0.0259	0.0236	0.0216	0.0199	0.0184
$n=10$	0.0192	0.0239	0.0226	0.0205	0.0185	0.0168	0.0153	0.0140	0.0128	0.0119
$n=20$	0.0098	0.0146	0.0142	0.0130	0.0118	0.0107	0.0097	0.0089	0.0082	0.0076
$n=30$	0.0062	0.0108	0.0107	0.0099	0.0090	0.0081	0.0074	0.0068	0.0062	0.0058
$n=40$	0.0064	0.0087	0.0078	0.0080	0.0073	0.0072	0.0060	0.0055	0.0049	0.0045

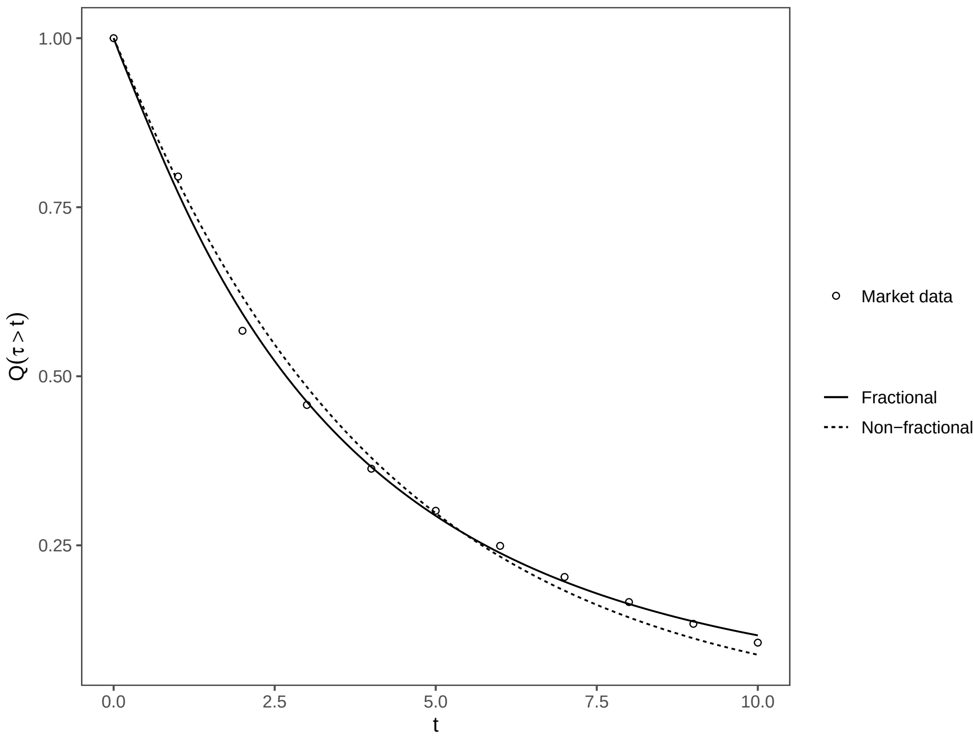

Figure 2: Results of the calibration.

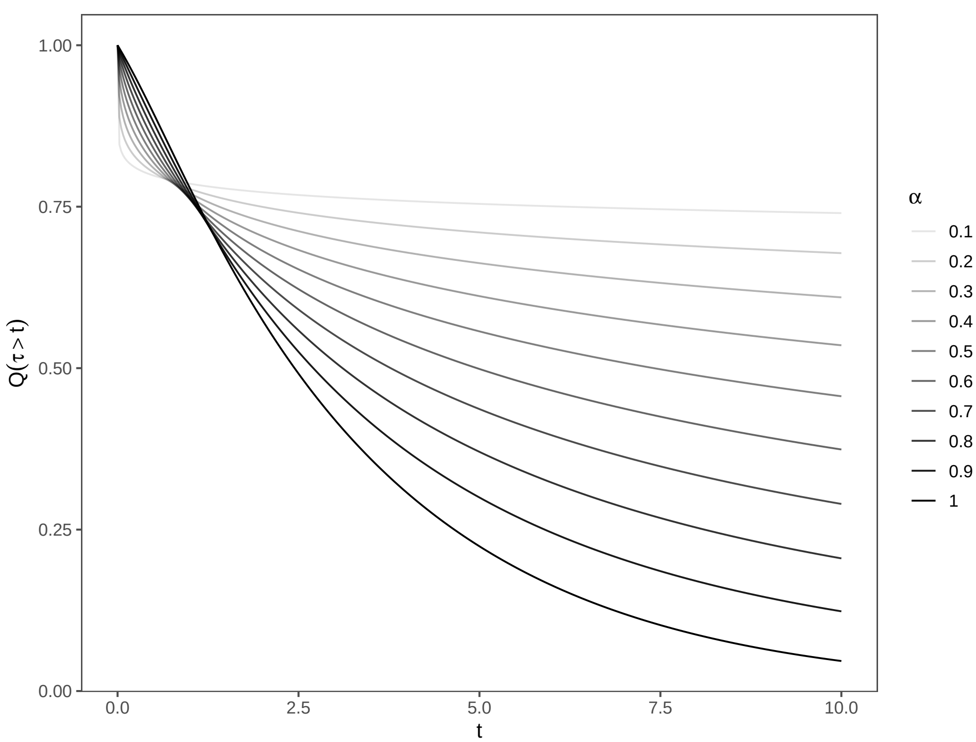

Figure 3: Survival probabilities for various fractional orders α.

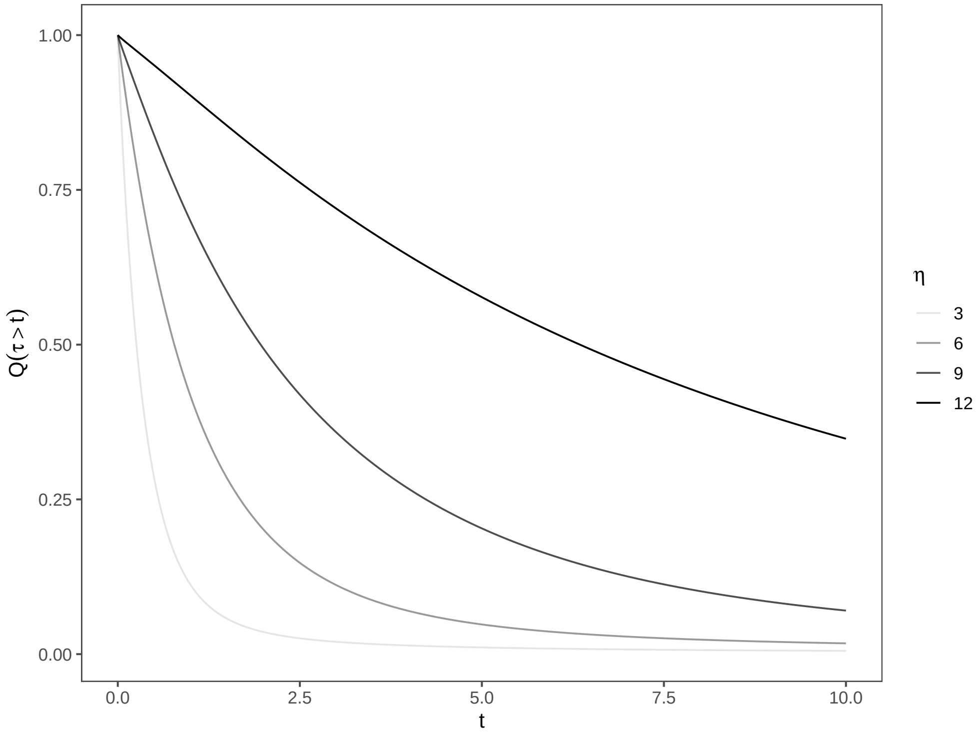

Figure 4: Survival probabilities for various values of η.

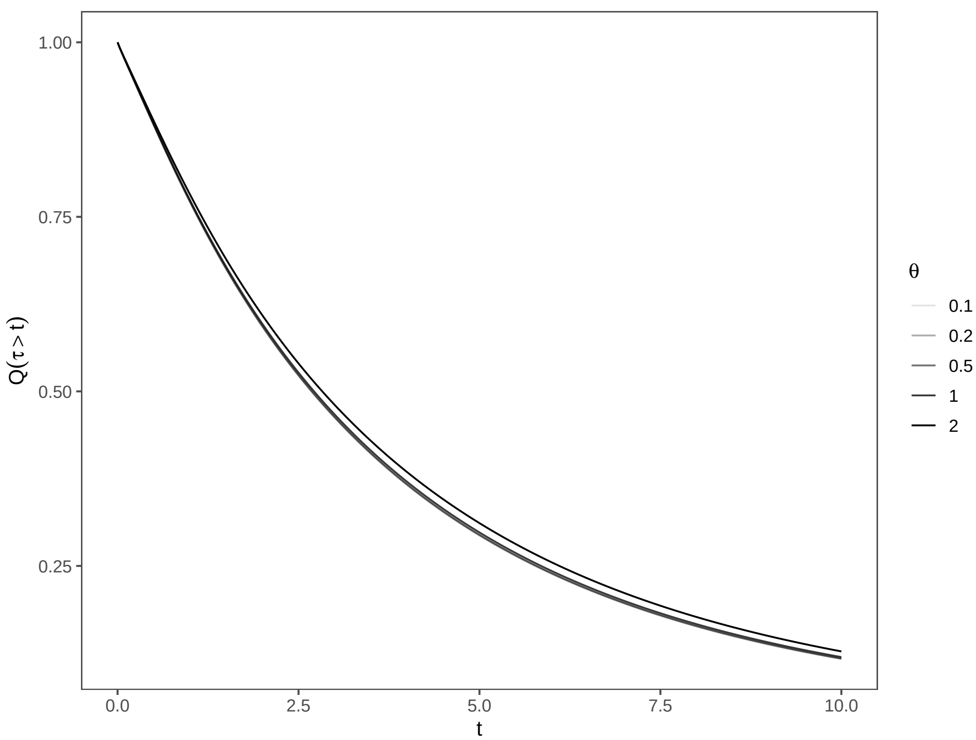

Figure 5: Survival probabilities for various values of θ.

Conclusions

In this paper, we proposed to adapt the potential approach initiated by Rogers (1997) for interest rate modelling, to credit risk. This has been done with a fractional Hawkes jump diffusion supermartingale. In order to compute the survival probabilities implied by the model, we proposed a new recursive method to compute all the moments of a (fractional) Hawkes process intensity. We provided a proof that our numerical method converges and gave an upper bound on the numerical error. Finally, our numerical experiments showed that our model gives a good fit compared to similar models, but with a better numerical tractability.

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