Paths to Stability for Overlapping Group Structures

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Abstract

We study the stability of overlapping group structures where each group possesses a constitution that contains the rules governing both the composition of the group and the conditions needed to leave the group and/or to become a new member of the group. We provide a necessary and sufficient condition on preferences that guarantees the existence and the emergence of constitutionally stable group structures. We show that although more blocking power for the individuals might enlarge the set of constitutionally stable group structures, it could happen that the society will never reach a stable group structure.

Key words: Overlapping group structures, Constitutions, Stability, Blocking power.

JEL classification: C72, C78, D85.

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1 Introduction

The understanding of how and why groups form and the precise way in which they affect outcomes of social and economic interactions has been apprehended assuming that each individual can only be member of one of these groups.[1] However, there are many situations in which individuals might be member of more than one group. Free trade agreements are signed among overlapping collections of countries. Joint ventures are formed among overlapping collections of firms. Overlapping groups of individuals may be involved in relationships involving public-goods provision, reciprocity or information-sharing.

Mauleon, Roehl and Vannetelbosch (2018) provide a theoretical framework that introduces the notion of constitution to model for each group the rules governing both the composition of the group and the conditions needed to leave the group and/or to become a new member of the group. They propose the concept of constitutional stability to predict the group structures that are going to emerge at equilibrium when the deviating coalition has to take into account the constitution of the group it wants to modify. Mauleon, Roehl and Vannetelbosch (2018) examine both the existence of constitutionally stable group structures as well as whether the society will reach one of these stable group structures. They provide requirements on constitutions and individuals' preferences guaranteeing that, from every initial group structure, there always exists a sequence of feasible deviations that are not blocked leading to a constitutionally stable group structure.

In the present paper, we investigate two fundamental questions that were not answered in Mauleon, Roehl and Vannetelbosch (2018). Does there exist an alternative way of guaranteeing stability without restricting both the constitutions and the individuals' preferences? We show that the existence of a common ranking in the society guarantees that from every initial group structure a constitutionally stable group structure will be reached. In the spirit of Banerjee, Konishi and Sönmez (2001) and Farrell and Scotchmer (1988) the common ranking property guarantees the consensus of all players in the society with respect to the ranking of two subsequent group structures that result from the deviation of a feasible deviating coalition that has not been blocked.

Notice that the constitutions grant the group members a certain level of blocking power. That is, the members of each group might have certain property rights or degree of authority allowing them to inhibit changes in the composition of the group that do not conform to their own preferences. What happens in terms of stability if more blocking power is given to the individuals? We show that the relationship between higher blocking power and stability is not monotone and depends on the

[1] Ray (2007) and Ray and Vohra (2015) provide surveys of models of coalition formation.

adopted perspective of stability. Although the set of constitutionally stable group structures might become larger, it could happen that the society will never reach one of these stable group structures because all sequences of feasible deviations that are not blocked do not necessarily lead to them.

Up to now, very little theoretical work exists on the stability for overlapping coalition formation. Myerson (1980) describes how the outcome of a cooperative game might depend on which groups of players hold cooperative planning conferences, and studies allocation rules, which are functions mapping conference structures to payoff allocations. Shenoy and Kraus (1996) and Dang, Dash, Rogers and Jennings (2006) propose heuristic algorithms for overlapping coalition formation without addressing the question of the stability of overlapping coalitions. Conconi and Perroni (2002) introduce a cooperative game of multi-dimensional agreement formation and define the notion of stable agreement structure, a core-like equilibrium concept where subsets of players can put forward objections to a certain proposed configuration of agreements. Albizuri, Aurrecoechea and Zarzuelo (2006) propose an extension of the Owen's coalitional value (1977) to an overlapping coalition formation setting. Allouch and Wooders (2008) analyze from a general equilibrium perspective club economies where clubs may overlap. Under the assumption that there is small communication cost of deviating from a given outcome, they prove that for all sufficiently large economies the core is nonempty and the set of price-taking equilibrium outcomes is equivalent to the core. Chalkiadakis, Elkind, Markakis, Polukarov and Jennings (2010) introduce a model for cooperative games with overlapping coalitions that is applicable in situations where agents need to allocate different parts of their resources to simultaneously serve different tasks as members of different coalitions. They explore the stability concept of the core. Our concept of constitutional stability generalizes previous stability concepts in the literature in which the rules governing the composition of each group as well as the exit of current members and/or the arrival of new members were exogenously given (see Jehiel and Scotchmer, 2001; Drèze and Greenberg, 1980; Caulier, Mauleon and Vannetelbosch, 2013; Caulier, Mauleon, Sempere-Monerris and Vannetelbosch, 2013) or not explicitly considered (like in the core where it is assumed that the deviators only form coalitions among themselves and, thus, no composition and/or admission rules are considered).

An important stream in the literature studies whether a decentralized process of successive blocking leads to a stable outcome in different models that can be seen as special cases of the present model where groups or coalitions do not overlap. See Roth and Vande Vate (1990), Klaus and Klijn (2007), Kojima and Ünver (2008) and Herings, Mauleon and Vannetelbosch (2017) for two-sided matching problems; Chung (2000) and Diamantoudi, Miyagawa and Xue (2004) for roommate problems; Koczy and Lauwers (2004), Koczy (2006), Yang (2010,2011, 2012) and Béal, Rémila and Solal (2013) for cooperative games, and Diamantoudi and Xue (2003) for hedonic games. We extend these results on paths to stability to the framework of overlapping group structures, a difficult framework for which there were no general results regarding necessary and sufficient conditions guaranteeing the convergence of the society to a constitutionally stable group structure.

The paper is organized as follows. Section 2 introduces the framework of overlapping groups and the notion of constitutions. Section 3 provides an alternative criterion for guaranteeing convergence to a constitutionally stable group structure and studies the relationship between blocking power and stability. Finally, Section 4 concludes.

2 The Model

We consider the model of Mauleon, Roehl and Vannetelbosch (2018). Let $N = \{i_1, \ldots, i_n\}$ be a finite set of players and let $M = \{c_1, \ldots, c_m\}$ be a finite set of groups (or its labels). A group structure h is a mapping $h : M \longrightarrow 2^N$ assigning to each group $c \in M$ a subset of players $h(c) \in 2^N$, with $h(c)$ representing the members of group c. That is, a group structure h indicates which players are members of which groups. The triple (N, M, h) is simply a mathematical hypergraph. Let $\mathcal{H}$ be the set of all group structures and let $h^\emptyset \in \mathcal{H}$, with $h^\emptyset(c) = \emptyset$ for all $c \in M$, be the empty group structure (i.e. no player is member of any group). The cardinality of $\mathcal{H}$ is $|\mathcal{H}| = 2^{mn}$. For instance, take a society consisting of three players, $N = \{i_1, i_2, i_3\}$ and four groups, $M = \{c_1, c_2, c_3, c_4\}$, where i_1, i_2 and i_3 are members of c_1, i_2 and i_3 are members of c_2 and c_3, and i_1 is the only member of c_4. Then, the group structure h is simply given by

$$h(c) = \begin{cases} \{i_1, i_2, i_3\}, & \text{if } c = c_1 \\ \{i_2, i_3\}, & \text{if } c \in \{c_2, c_3\} \\ \{i_1\}, & \text{if } c = c_4. \end{cases}$$

Each $i \in N$ has rational preferences $\succeq^i$over $\mathcal{H}$. The tuple $\succeq = (\succeq^i)_{i \in N}$ is called a preference profile. Given the preferences, players might have incentives to join or exit some group in a given group structure. Given a group $c \in M$ and a subset of players $D \subseteq N$, let $h \pm (c, D)$ be the group structure that is obtained from $h \in \mathcal{H}$ if the members of c, $h(c)$, change to $h(c) \pm D$ due to the arrival and/or departure of the players in the deviating coalition D. Players in $D \cap h(c)$ leave the group and players in $D \setminus h(c)$ join it. Formally:

$$(h \pm (c, D))(c') := \begin{cases} h(c) \pm D & \text{if } c = c' \\ h(c') & \text{if } c \neq c' \end{cases} \tag{1}$$

If $D \cap h(c) = \emptyset$, we write $h + (c, D)$ instead of $h \pm (c, D)$ meaning that no player leaves the group. If $D \subseteq h(c)$, we write $h - (c, D)$ instead of $h \pm (c, D)$ indicating that no player joins the group.

A constitution describes both the rules governing the composition of the group and the conditions to fulfill in order to be accepted into the group. Formally, the constitution $\mathcal{C}_h^c$ of group $c \in M$ in the group structure $h \in \mathcal{H}$, is a pair $\mathcal{C}_h^c = (\mathcal{D}_h^c, \mathcal{S}_h^c)$ where (i) $\mathcal{D}_h^c \subseteq 2^N \setminus \{\emptyset\}$ describes the set of feasible deviating coalitions,[2] and (ii) for each $D \in \mathcal{D}_h^c$, $\mathcal{S}_h^c(D) \subseteq 2^{h(c)}$ specifies a non-empty set of supporting coalitions containing only members of $h(c)$ that can grant the admission of the feasible deviating coalition into the group. Without the consent of at least one of these supporting coalitions the deviation of $D \in \mathcal{D}_h^c$ would be blocked. In addition, if $S \in \mathcal{S}_h^c(D) \setminus \{\emptyset\}$, we assume that $S' \in \mathcal{S}_h^c(D)$ for all $S' \supseteq S$. That is, if S is a supporting coalition for a certain deviating coalition, all coalitions containing S have also the power to support this deviation.

Let $\mathcal{C}^c = (\mathcal{C}_h^c)_{h \in \mathcal{H}}$ be the constitutions of group c in each possible group structure $h \in \mathcal{H}$. Let $\mathcal{C} := (\mathcal{C}^c)_{c \in M}$ be the constitutions of each group in each possible group structure. The quadruple $(N, M, \succeq, \mathcal{C})$ is called a society.

3 Constitutionally Stable Group Structures

3.1 Paths to stability

What about possible deviations? A group structure h' is obtainable from h via the deviating coalition D, $D \subseteq N$, if (i) there is a unique group c whose composition is changed: $h' = h \pm (c, D)$, and (ii) the deviating coalition is feasible according to the constitution of group c: $D \in \mathcal{D}_h^c$. The group structures that are obtainable from a given group structure h are such that they only differ from h in that a unique group has changed its composition via the deviation of a feasible deviating coalition according to the constitution of that group. Following Mauleon, Roehl and Vannetelbosch (2018), a group structure h is constitutionally stable if all coalitional deviations to some obtainable group structure are blocked.

Definition 1. Given the society $(N, M, \succeq, \mathcal{C})$, a group structure h is constitutionally stable with respect to the constitutions $\mathcal{C}$ if for any $D \subseteq N$, $h \pm (c, D)$ obtainable from h via D, we have either $h \succeq^i h \pm (c, D)$ for at least one $i \in D \setminus h(c)$, or in each supporting coalition $S \in \mathcal{S}_h^c(D)$ there exists $j \in S$ such that $h \succeq^j h \pm (c, D)$.

[2] If $\mathcal{D}_h^c = 2^N \setminus \{\emptyset\}$, then there are no restrictions on feasible deviations in group c. But, if $\mathcal{D}_h^c \subset 2^N \setminus \{\emptyset\}$, then some changes in the composition of the group are not possible.

Definition 1 tells us that, under the constitutions $\mathcal{C}$, a group structure $h \in \mathcal{H}$ is constitutionally stable if and only if any feasible deviation of some coalition $D \in \mathcal{D}_h^c$ to some obtainable group structure $h \pm (c, D)$ is deterred because at least one of the deviating players joining $h(c)$ does not strictly benefit from deviating or at least one of the members of every supporting coalition $S \in \mathcal{S}_h^c(D)$ is not strictly better off from the deviation. Some members of $h(c)$ may have the power to force other members to leave c even when the excluded players suffer from this exclusion. Hence, moving from $h \in \mathcal{H}$ to $h \pm (c, D)$ does not necessarily need the consent of players leaving $h(c)$. However, a player who is not in $h(c)$ cannot be forced to join $h(c)$. Only if she strictly benefits, she will join it.

Let $\Gamma(\mathcal{C})$ be the set of constitutionally stable group structures with respect to the constitutions $\mathcal{C}$. Generically, constitutionally stable group structures might fail to exist. Moreover, even if constitutionally stable group structures exist, there is no guarantee that the society will reach one of them and not be stuck in a (closed) cycle where a number of group structures are repeatedly visited as a result of a sequence of feasible deviations that are not blocked.

Given $h \in \mathcal{H}$ and $c \in M$, let $\mathcal{A}_h^c(\mathcal{C}) := \{D \in \mathcal{D}_h^c \mid \exists S \in \mathcal{S}_h^c(D)$ such that $h \pm (c, D) \succ^i h \ \forall\ i \in S \cup (D \setminus h(c))$ with $\emptyset \notin S \cup (D \setminus h(c))\}$ be the set of all accepted feasible deviations that are not blocked.[3] That is, the feasible deviations that strictly benefit all the deviating players joining the group and/or all members of at least one of the supporting coalitions.

An improving path is a sequence of group structures that can emerge when players join or leave some groups based on the improvement the resulting group structure offers them relative to the current one. Each group structure in the sequence differs from the previous one in that one group is modified by a feasible deviating coalition and every player joining the group strictly prefers the resulting group structure to the current one. Moreover, the deviation should not be blocked and, hence, there should be a supporting coalition that strictly benefits from the deviation. Formally, an improving path from $h_0 \in \mathcal{H}$ to $h_k \in \mathcal{H}$ is a sequence of group structures $(h_0, h_1, \ldots, h_k)$ such that for all $0 \leq l < k$ we have $h_{l+1} = h_l \pm (c_l, D_l)$ with $c_l \in M$ and $D_l \in \mathcal{A}_{h_l}^{c_l}(\mathcal{C})$. If there exists an improving path from $h \in \mathcal{H}$ to $h' \in \mathcal{H}$, we write $h \mapsto h'$. Moreover, let $I(h) = \{h' \in \mathcal{H} \mid h \mapsto h'\}$ be the set of group structures that can be reached by an improving path starting at h. Notice that h is constitutionally stable if and only if there is no improving path starting at h; that is, $I(h) = \emptyset$.

A set of group structures $H \subseteq \mathcal{H}$ is closed if there is no improving path leading out of it, i.e., $I(h) \subseteq H$ for all $h \in H$. Moreover, a set of group structures $H \subseteq \mathcal{H}$

[3] We restrict $\emptyset \notin S \cup (D \setminus h(c))$ to guarantee that at least one player strictly benefits from deviating.

with $|H| \geq 2$ is a cycle if for any pair $h, h' \in H$, there exists an improving path from h to h'.

Lemma 1 (Mauleon, Roehl and Vannetelbosch, 2018). *Let the society $(N, M, \succeq, \mathcal{C})$ be given. There exists no closed cycle if and only if, for each group structure $h \in \mathcal{H}$ that is not constitutionally stable, there is an improving path leading from this group structure to a constitutionally stable one.*

Thus, the non-existence of closed cycles not only implies existence of constitutionally stable group structures but it also guarantees that the society will reach one of these stable group structures. Mauleon, Roehl and Vannetelbosch (2018) provide reasonable restrictions on players' preferences and consistency conditions on the constitutions to guarantee the non-occurrence of closed cycles and the convergence to a constitutionally stable group structure. Does there exist an alternative way of guaranteeing stability without restricting both the constitutions and the individuals' preferences? We show here that an alternative way to exclude the occurrence of closed cycles is to impose the existence of a common ranking that guarantee the consensus of all players in the society with respect to the ranking of two subsequent group structures that result from the deviation of a feasible deviating coalition that has not been blocked.

Definition 2. Given the society $(N, M, \succeq, \mathcal{C})$, a common ranking $\trianglerighteq$ is a complete and transitive ordering over $\mathcal{H}$ such that for all $h \in \mathcal{H}$ and $c \in M$, $D \in \mathcal{A}_h^c(\mathcal{C})$ implies $h \pm (c, D) \trianglerighteq h$.

A common ranking $\trianglerighteq$ reflects a certain level of consensus between the players in the sense that, whenever a feasible deviation D from h to an obtainable group structure $h \pm (c, D)$ is not blocked, then all players in the society agree that the resulting group structure $h \pm (c, D)$ should be ranked above h. Indeed, our definition of common ranking implies that there do not exist null players in our society; that is, players who do not affect others and never have a payoff on their own. Here, all players agree that the group structure that results after an accepted feasible deviation should be ranked above the initial group structure. The main idea is that the set of group structures can be decomposed into several equivalence classes and once a higher class is reached, this will not be reversed afterwards. Indeed, a deviation takes place only if the joining and supporting players agree that the resulting group structure is not contained in a lower class than the current one. In this case, all players in the society rank the group structure h' that results after an accepted feasible deviation ($h' = h \pm (c, D)$) above the initial group structure h. Thus, if from h' an accepted feasible deviation takes place moving to h'', all players would rank h'' above h' and h, because the order is transitive. Note that a priori this is not a restriction at all because it would be possible, for instance, to choose $\trianglerighteq$ in such a way that all group structures are equivalent (i.e., $h \trianglerighteq h'$ as well as $h' \trianglerighteq h$ for all $h, h' \in \mathcal{H}$). This immediately implies that a (not necessarily unique) common ranking always exists. However, the more consensus about beneficial deviations between the players, the stronger the restrictions that can be imposed by a common ranking.

Farrell and Scotchmer (1988) introduce the common ranking property that requires the existence of a linear ordering over all coalitions which coincides with any player's preference ordering over coalitions to which she belongs. A relaxed version of the common ranking property, the top-coalition property, is introduced by Banerjee, Konishi and Sönmez (2001) to guarantee the existence of a core partition. The common ranking that we introduce here is similar to the one of Farrell and Scotchmer (1988) and orders the group structures differing only in that a unique group $h(c)$ of players has changed its composition according to the preferences of the joining and supporting players involved in the change of $h(c)$.

Proposition 1. *Let the society $(N, M, \succeq, \mathcal{C})$ be given.*

(i) *There are no cycles if and only if there exists a common ranking $\trianglerighteq$ such that for all $H \subseteq \mathcal{H}$ there is a unique $\trianglerighteq$–maximal group structure $h^* \in H$.*

(ii) *There are no closed cycles if and only if there exists a common ranking $\trianglerighteq$ such that for all $h \in \mathcal{H}$ there is a unique $\trianglerighteq$–maximal group structure $h^* \in I(h)$.*

Proof. **(i)** ($\Rightarrow$) In order to show that existence of $\trianglerighteq$ implies the non-existence of cycles, we will consider the counterposition of this statement. Therefore, assume there is a cycle $H \subseteq \mathcal{H}$. Since there exists a path from each group structure in H to every other group structure in H, if $\trianglerighteq$ is a common ranking, we must have $\overline{h} \trianglerighteq \widehat{h}$ as well as $\widehat{h} \trianglerighteq \overline{h}$ for all $\overline{h}, \widehat{h} \in H$. Thus, there is no unique $\trianglerighteq$-maximal element in H.

($\Leftarrow$) For the other direction suppose there exists no cycle. The following algorithm proceeds in a similar way as the one in the proof of Theorem 1 in Jackson and Watts (2001). We start with the binary relation $\trianglerighteq_1$where $h \trianglerighteq_1 \overline{h}$ if and only if there exists an improving path from $\overline{h}$ to h. Because there is no cycle, $\trianglerighteq_1$ is strict, with $\triangleright_1$ denoting the strict version of $\trianglerighteq_1$. Moreover, for all $h \in \mathcal{H}$, $c \in M$, and $D \in \mathcal{D}^c_h$, deviating from h to $h \pm (c, D)$ always implies $h \pm (c, D) \ \triangleright_1 \ h$ by construction. However, $\triangleright_1$ is not necessarily complete. Let $\widetilde{h}, \widehat{h} \in \mathcal{H}$ with neither $\widetilde{h} \ \triangleright_1 \ \widehat{h}$ nor $\widehat{h} \triangleright_1 \widetilde{h}$. We construct $\overline{\triangleright}_1$ by adding $\widetilde{h} \ \overline{\triangleright}_1 \ \widehat{h}$ to $\triangleright_1$, i.e., $h \ \overline{\triangleright}_1 \ \overline{h}$ if and only if $h \triangleright_1 \overline{h}$ or $h = \widetilde{h}$ and $\overline{h} = \widehat{h}$. Moreover, let $\triangleright_2$ be the transitive closure of $\overline{\triangleright}_1$. We will show that $\triangleright_2$ still represents the preference profile of the players, i.e., deviating from h

to $h \pm (c, D)$ always implies $h \pm (c, D) \ \triangleright_2 \ h$ for all $c \in M$ and $D \in \mathcal{D}_h^c$. Suppose this is not true, that is, suppose there exist $h' \in \mathcal{H}$, $c \in M$, $D \in \mathcal{D}_{h'}^c$, and $S \in \mathcal{S}_{h'}^c$ with $h' \pm (c, D) \succ^i h'$ for all $i \in (D \setminus h'(c)) \cup S$ but $h' \ \triangleright_2 \ h' \pm (c, D)$. Thus, there exists a sequence of group structures $(h_0, h_1, \ldots, h_k)$ with $h_0 = h'$, $h_k = h' \pm (c, D)$ and $h_0 \ \overline{\triangleright}_1 \ h_1 \ \overline{\triangleright}_1 \ \ldots \ \overline{\triangleright}_1 \ h_k$. Assume the sequence is of minimal length. This implies that $h_l = h_{l'}$ only if $l = l'$ for all $l, l' \in \{0, 1, \ldots, k\}$. Suppose there exists an $l \in \{1, \ldots, k\}$ with $\{h_{l-1}, h_l\} = \{\widehat{h}, \widetilde{h}\}$. Because $h_{l'} \neq \widehat{h}, \widetilde{h}$ for all $l' \notin \{l-1, l\}$ this yields

$$h_l \ \triangleright_1 \ h_{l+1} \ \triangleright_1 \ \ldots \ \triangleright_1 \ h_k = h' \pm (c, D) \ \triangleright_1 \ h' = h_0 \ \triangleright_1 \ \ldots \ \triangleright_1 h_{l-1}$$

and, thus, there exists an improving path from $\widetilde{h}$ to $\widehat{h}$ or vice versa. This contradicts the assumption that the two group structures are not comparable under $\triangleright_1$. Therefore, there exists no $l \in \{1, \ldots, k\}$ with $\{h_{l-1}, h_l\} = \{\widehat{h}, \widetilde{h}\}$. From this follows $h_0 \ \triangleright_1 \ h_1 \ \triangleright_1 \ \ldots \ \triangleright_1 \ h_k$ which contradicts the assumption that there is no cycle. Thus, $\triangleright_2$ still represents the preferences of the players and by construction it is also transitive and strict. If it is not complete, the previous steps can be iterated. Because the set of group structures is finite, the iteration will stop after finitely many steps and we obtain a common ranking $\triangleright$ which is strict. In particular, strictness implies that for each $H \subseteq \mathcal{H}$ there is a unique $\triangleright$-maximal group structure $h^* \in H$.

(ii) ($\Rightarrow$) The first direction proceeds analogously to the first direction of Part (i). Let a common ranking $\trianglerighteq$ and a set of group structures $H \subseteq \mathcal{H}$ be given. If H forms a closed cycle, we have $I(h) = I(h') = H$ and $h \trianglerighteq h'$ as well as $h' \trianglerighteq h$ for all $h, h' \in H$. But this would contradict that there is a unique $\trianglerighteq$-maximal group structure in H and, thus, there cannot exist a closed cycle.

($\Leftarrow$) For the other direction suppose there exist no closed cycles. The first step of the construction of the common ranking proceeds in the same way as the one of Part (i). That is, we start with $\trianglerighteq_1$ where $h \trianglerighteq_1 \overline{h}$ if and only if there exists an improving path from $\overline{h}$ to h. But note that here this binary relation is not necessarily strict. Since by assumption there are no closed cycles, there exists at least one constitutionally stable group structure $h' \in \mathcal{H}$. If this group structure is uniquely determined, according to Lemma 1 it is contained in every closed subset $H \subseteq \mathcal{H}$ and $\trianglerighteq_1$ can then obviously be extended to a complete ranking where h' is the unique maximal element. Therefore, in the following, suppose there exists a further constitutionally stable group structure $h'' \in \mathcal{H}$. In particular, this implies that neither $h' \trianglerighteq_1 h''$ nor $h'' \trianglerighteq_1 h'$. Let $\widetilde{h}, \widehat{h} \in \mathcal{H}$ be an arbitrary pair of group structures not comparable under $\trianglerighteq_1$. Analogously to above, $\overline{\trianglerighteq}_1$ is constructed by adding $\widetilde{h} \ \overline{\triangleright}_1 \ \widehat{h}$ to $\trianglerighteq_1$, i.e., $h \ \overline{\trianglerighteq}_1 \ \overline{h}$ if and only if $h \ \trianglerighteq_1 \ \overline{h}$ or $h = \widetilde{h}$ and $\overline{h} = \widehat{h}$. Again, let $\trianglerighteq_2$ be the transitive closure of $\overline{\trianglerighteq}_1$. Note that by construction $h' \ \trianglerighteq_2 \ h''$

would imply $h' \;\rhd_2\; h''$ and vice versa. If $\unrhd_2$ is not complete, because of finiteness of $\mathcal{H}$ we can iterate the previous steps until a complete ranking $\unrhd$ is reached. We will show that h' and h'' are still not equivalent under $\unrhd$. This, in fact, has the following implication: If h^* is $\unrhd$–maximal in a closed subset $H \subseteq \mathcal{H}$, it has to be constitutionally stable by construction and w.l.o.g. we may assume $h^* = h'$. Then, for any other stable group structure $h'' \in H$, we must have $h' \unrhd h''$ and, thus, h' is the unique $\unrhd$–maximal element in H.

In order to show h' and h'' are still not equivalent under $\unrhd$, let $\unrhd_k$ be the binary relation constructed in the k-th step of the algorithm described in the previous passage. For $k = 1, 2$ we already know that $h' \;\unrhd_k\; h''$ would imply $h' \;\rhd_k\; h''$ and vice versa. We will show inductively that this is also satisfied for all other k. Therefore, let $k \geq 3$ and suppose that h' and h'' are still not equivalent under $\unrhd_{k-1}$. Moreover, assume this is not satisfied under $\unrhd_k$, i.e., we have $h' \;\unrhd_k\; h''$ as well as $h'' \;\unrhd_k\; h'$. This assumption will lead to a contradiction. Let $\widetilde{h}^{(k-1)}, \widehat{h}^{(k-1)} \in \mathcal{H}$ be the corresponding pair of group structures not comparable under $\unrhd_{k-1}$ which is added in the next step. We will distinguish three cases:

Case 1: $h' \;\rhd_{k-1}\; h''$.

Because we assume h' and h'' are not equivalent under $\unrhd_{k-1}$, this implies that there exists a sequence of group structures $(h_1, \dots, h_l)$ with $h_1 = h''$, $h_l = h'$, and $h_1 \;\overline{\unrhd}_{k-1}\; \dots \;\overline{\unrhd}_{k-1}\; h_l$. Moreover, from this also follows that there exists $1 \leq l' \leq l-1$ with $\{h_{l'}, h_{l'+1}\} = \{\widetilde{h}^{(k-1)}, \widehat{h}^{(k-1)}\}$. But then

$$h_{l'+1} \;\unrhd_{k-1}\; \dots \;\unrhd_{k-1}\; h' \;\rhd_{k-1}\; h'' \;\unrhd_{k-1}\; \dots \;\rhd_{k-1}\; h_{l'},$$

which contradicts that $\widetilde{h}^{(k-1)}$ and $\widehat{h}^{(k-1)}$ are not comparable under $\rhd_{k-1}$.

Case 2: $h'' \;\rhd_{k-1}\; h'$.

This case proceeds analogously to the previous one by just reversing the roles of h' and h''.

Case 3: h' and h'' are not comparable under $\unrhd_{k-1}$.

If h' and h'' are equivalent under $\unrhd_k$ but not under $\unrhd_{k-1}$, there must be two sequences of group structures $(h_1, \dots, h_l)$ and $(\overline{h}_1, \dots, \overline{h}_{\overline{l}})$ with $h_1 = \overline{h}_{\overline{l}} = h'$, $h_l = \overline{h}_1 = h''$, and

$$h_1 \;{}_{k-1}\overline{\unrhd} \dots \;\overline{\unrhd}_{k-1}\; h_l = \overline{h}_1 \;\overline{\unrhd}_{k-1}\; \dots \;\overline{\unrhd}_{k-1}\; \overline{h}_{\overline{l}}.$$

Moreover, there exist $1 \leq l' \leq l-1$ and $1 \leq \overline{l}' \leq \overline{l}-1$ with $\{h_{l'}, h_{l'+1}\} = \{\overline{h}_{\overline{l}'}, \overline{h}_{\overline{l}'+1}\} = \{\widetilde{h}^{(k-1)}, \breve{h}^{(k-1)}\}$. In particular, this yields

$$h_{l'} \;\overline{\unrhd}_{k-1}\; h_{l'+1} \;\unrhd_{k-1}\; \dots \;\unrhd_{k-1}\; h'' \;\unrhd_{k-1}\; \dots \;\unrhd_{k-1}\; \overline{h}_{\overline{l}'} \;\overline{\unrhd}_{k-1}\; \overline{h}_{\overline{l}'+1}$$

which could only be satisfied if $\widetilde{h}^{(k-1)}$ and $\widehat{h}^{(k-1)}$ are comparable under $\unrhd_{k-1}$. $\square$

Proposition 1 provides an alternative criterion for guaranteeing convergence to a constitutionally stable group structure. Part (i) states that requiring non-existence of cycles is equivalent to requiring the existence of a special common ranking which identifies a unique maximal element in every subset of group structures. A common ranking meets this requirement if and only if it is strict. In this case, it is a variation of "Generalized Ordinal Potentials" introduced by Monderer and Shapley (1996). In particular, part (i) of Proposition 1 is closely related to their Lemma 2.5. Moreover, it also relates to Theorem 1 in Jackson and Watts (2001). According to part (ii), having this feature only in particular subsets of $\mathcal{H}$ is still strong enough for excluding closed cycles. Therefore, the society induces a constitutionally stable group structure for sure if and only if there exists a common ranking which is sufficiently restrictive and that guarantees the consensus of all players in the society with respect to the ranking of two subsequent group structures that result from the deviation of an accepted deviating coalition that has not been blocked.

3.2 Blocking power and stability

In our formulation, the constitutions grant the group members a certain level of blocking power allowing them to inhibit changes in the composition of the group that do not conform to their own preferences. Next proposition studies the consequences, in terms of stability, of enhancing the blocking power of the individuals. We show that the relationship between higher blocking power and stability depends on the adopted perspective of stability.

Proposition 2. *Let two societies $(N, M, \succeq, \mathcal{C})$ and $(N, M, \succeq, \bar{\mathcal{C}})$ be given and assume that the constitution $\mathcal{C}$ restricts more the set of feasible deviations and the set of supporting coalitions than the constitution $\overline{\mathcal{C}}$, $\mathcal{C} \subseteq \overline{\mathcal{C}}$, i.e., $\mathcal{D}_h^c \subseteq \overline{\mathcal{D}}_h^c$ and $\mathcal{S}_h^c(D) \subseteq \overline{\mathcal{S}}_h^c(D)$ for all $h \in \mathcal{H}$, $c \in M$, and $D \in \mathcal{D}_h^c$. Then, the non-existence of closed cycles under $\overline{\mathcal{C}}$ does not imply that there are no closed cycles under $\mathcal{C}$ even if $\Gamma(\overline{\mathcal{C}}) \subseteq \Gamma(\mathcal{C})$.*

Proof. It is sufficient to construct a suitable example. The one we consider here is a variation of an example from Bogomolnaia and Jackson (2002). There are three players $N = \{i_1.i_2, i_3\}$ and one group $M = \{c\}$. Thus, $|\mathcal{H}| = 8$. The set of all possible group structures are given by:

	$h_1(c)$	$h_2(c)$	$h_3(c)$	$h_4(c)$	$h_5(c)$	$h_6(c)$	$h_7(c)$	$h^{\emptyset}$
c	$\{i_1\}$	$\{i_2\}$	$\{i_3\}$	$\{i_1, i_2\}$	$\{i_1, i_3\}$	$\{i_2, i_3\}$	$\{i_1, i_2, i_3\}$	$\emptyset$

and the players' preferences are

$$h_4 \succ^{i_1} h_7 \succ^{i_1} h_5 \succ^{i_1} h_1 \succ^{i_1} h_2 \sim^{i_1} h_3 \sim^{i_1} h_6 \sim^{i_1} h^{\emptyset}$$
$$h_6 \succ^{i_2} h_7 \succ^{i_2} h_4 \succ^{i_2} h_2 \succ^{i_2} h_1 \sim^{i_2} h_3 \sim^{i_2} h_5 \sim^{i_2} h^{\emptyset}$$
$$h_5 \succ^{i_3} h_7 \succ^{i_3} h_6 \succ^{i_3} h_3 \succ^{i_1} h_1 \sim^{i_3} h_2 \sim^{i_3} h_4 \sim^{i_3} h^{\emptyset}.$$

Consider the following constitution $\mathcal{C}^c = (\mathcal{D}^c, \mathcal{S}^c)$:

$$\mathcal{D}^c_h = 2^N \setminus \{\emptyset\} \quad \text{and} \quad \mathcal{S}^c_h(D) = \{S \subseteq h(c) \,|\, (h(c) \setminus D) \subseteq S, S \neq \emptyset\} \tag{2}$$

for all $h \neq h^{\emptyset}$. Given $\mathcal{C}^c$, a priori all possible deviating coalitions are feasible and a deviation $D \neq h(c)$ takes place if and only if all members of the resulting group structure benefit from the deviation, i.e., $h \pm (c, D) \succ^i h$ for all $i \in h(c) \pm D$. This implies that players who are undesired can be dismissed if the other members of the group agree on this. For the (pathological) special case of $D = h(c)$, it is required that at least one player approves the deviation. Now, given the constitution as defined in (2), we have that h_7 is the unique constitutionally stable group structure and $H := \{h_4, h_6, h_5\}$ forms a closed cycle. In fact, once H is reached, there is no improving path leading to h_7 because the players act too myopically.

However, consider the following constitutions $\overline{\mathcal{C}}^c = (\overline{\mathcal{D}}^c, \overline{\mathcal{S}}^c)$. Let $\overline{\mathcal{D}}^c_h = 2^N \setminus \{\emptyset\}$ and

$$\overline{\mathcal{S}}^c_h(D) = \begin{cases} \{S \subseteq h(c) \,|\, (h(c) \setminus D) \subseteq S,\, S \neq \emptyset\} & \text{, if } D \cap h(c) \neq \emptyset \\ \{S \subseteq h(c) \,|\, S \neq \emptyset\} & \text{, if } D \cap h(c) = \emptyset \end{cases}$$

for all $h \neq h^{\emptyset}$. Here, granting access to c to a coalition of players that are not member of the group just needs the support of only one member of the group. This obviously implies $\mathcal{C}^c \subsetneq \overline{\mathcal{C}}^c$ and, thus, the players have less blocking power under $\overline{\mathcal{C}}^c$ than under $\mathcal{C}^c$ (but note that the sets of constitutionally stable group structures coincide). However, under $\overline{\mathcal{C}}^c$, $H = \{h_4, h_6, h_5\}$ does not form a closed cycle any more because for all $h \in H$ there is always one member of $h(c)$ who supports deviating from h to h_7. Therefore, given $\overline{\mathcal{C}}^c$, there exists no closed cycle.

Notice that the setting analyzed in this example is actually not completely the same as in Bogomolnaia and Jackson (2002), because in their paper they study coalition formation (i.e., the set of players is always decomposed into a partition) while we have just one group containing some of the players. However, "core stability" in their setting corresponds to constitutional stability with respect to the constitution defined in (2). □

From the definition of constitutional stability we have that if the sets of feasible deviations and supporting coalitions shrink, the blocking power of each individual increases and the set of constitutionally stable group structures might become larger but never smaller. However, the consequences on stability of enhancing the blocking power of individuals strongly depend on the adopted perspective of stability. Although the set of constitutionally stable group structures might become larger the greater the blocking power of the individuals, it could happen that the society will never reach one of these stable group structures because all improving paths leading to them could be destroyed and closed cycles could occur.

4 Conclusion

We have studied the stability of overlapping group structures where each group possesses a constitution that contains the rules governing both the composition of the group and the conditions needed to leave the group and/or to become a new member of the group. Mauleon, Roehl and Vannetelbosch (2018) have provided requirements on both constitutions and individuals' preferences guaranteeing that, from every initial group structure, there always exists an improving path leading to a constitutionally stable group structure. In this paper we have shown that the relationship between higher blocking power and stability is not monotone and depends on the adopted perspective of stability. Although the set of constitutionally stable group structures might become larger, it could happen that the society will never reach one of these stable group structures because all improving paths leading to them could be destroyed and closed cycles could occur. We have then provided a necessary and sufficient condition on preferences that guarantees the existence and the emergence of constitutionally stable group structures. The common ranking property is independent on constitutions and hence guarantees a monotone relationship between higher blocking power and stability.

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