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EXPONENTIAL GROWTH OF LIE ALGEBRAS OF FINITE GLOBAL DIMENSION

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ABSTRACT. Let L be a connected finite type graded Lie algebra. If $\dim L = \infty$ and $\mathrm{gldim}\, L < \infty$, then $\log \text{ index } L = \alpha > 0$. If, moreover, $\alpha < \infty$, then for some d, $\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k\alpha_k}$, where $\alpha_k \to \log \text{ index } L$ as $k \to \infty$.

We work with graded vector spaces V over a field $\Bbbk$ of characteristic $\neq 2$ and denote by $V_{(p,q)}$ the subspace $\{V_i \,|\, p < i < q\}$. The *logarithmic index* of any graded vector space V is defined by

$$\log \text{ index } V = \limsup_k \frac{\log \dim V_k}{k},$$

and an infinite sequence (q_i) is a *quasi-geometric growth sequence* for V if for some fixed n, $q_i < q_{i+1} \leq nq_i$, for all i, and if

$$\frac{\log \dim V_{q_i}}{q_i} \longrightarrow \log \text{index } V\,.$$

Now consider graded Lie algebras as defined in [4]; in particular we suppose $[x,[x,x]] = 0$, $x \in L_{\mathrm{odd}}$, if char $\Bbbk = 3$. L is connected finite type (a cft graded Lie algebra) if $L = \{L_i\}_{i\geq 1}$ and each L_i is finite dimensional. The *global dimension* (gldim L) and *depth* of a cft graded Lie algebra, L, are defined, respectively, by

$$\text{gldim } L = \max\{\, k \,|\mathrm{Ext}^k_{UL}(\Bbbk,\Bbbk) \neq 0\,\}$$

and

$$\text{depth } L = \min\,\{\, k \,|\, \mathrm{Ext}^k_{UL}(\Bbbk, UL) \neq 0\,\}\,.$$

It is easy to see that depth $L \leq$ gl dim L.

Our main result reads:

Theorem. *Suppose L is a cft graded Lie algebra and* $\dim L = \infty$. *If* $\mathrm{gldim}\, L < \infty$, *then* $\log \text{index}\, L > 0$. *If, moreover,* $\log \text{index}\, L < \infty$, *then for some d,*

$$\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k\alpha_k}\,,\ \textit{where } \alpha_k \to \log \text{ index } L\,,\ \textit{as } k \to \infty\,.$$

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Remark. Theorem 3 of [5] establishes the same conclusion when the hypothesis $\operatorname{gldim} L < \infty$ is weakened to depth $L < \infty$, but certain additional growth conditions on L are assumed.

Now suppose X is a simply connected topological space with each $H_i(X;\mathbb{Q})$ finite dimensional. Then the loop space homology, $H_*(\Omega X;\mathbb{Q})$, is the universal enveloping algebra of a cft graded Lie algebra L_X, isomorphic to $\pi_*(\Omega X)\otimes\mathbb{Q}$.

Corollary. *If* $\dim L_X = \infty$, $\operatorname{gldim} L_X < \infty$ *and* $\log\operatorname{index} L_X < \infty$*, then for some* d*,*

$$\sum_{i=1}^{d-1} \dim \pi_{k+i}(X)\otimes\mathbb{Q} = e^{k\alpha_k}\,,$$

where $\alpha_k \to \log\operatorname{index}\, L_X$ *as* $k\to\infty$*. In particular* $\sum_{i=1}^{d-1} \dim\pi_{k+i}(X)\otimes\mathbb{Q}$ *grows exponentially in* k*.*

Proof of the Theorem. First we establish

Lemma 1. *An infinite-dimensional cft graded Lie algebra* L *of finite global dimension has a quasi-geometric growth sequence.*

Proof. We use the same argument as in the proof of Theorem 2 in [5]: Put $m = \operatorname{gldim}\, L$, $a = \left(\frac{1}{2(m+1)}\right)^{m+1}$ and $\alpha = \log\operatorname{index}\, L$.

The Cartan-Eilenberg-Serre cochain complex $\mathcal{C}^*(L)$ is in fact a Sullivan algebra ([3]) of the form $\bigwedge(sL)^{\#}$, $(sL)^{\#}$ denoting the dual of the suspension of L and $\bigwedge V$ denoting the free graded commutative algebra on V. The differential in $\bigwedge(sL)^{\#}$ increases the length of word gradation by 1 and so gives a second gradation $H^{(p)}$ in $H(\bigwedge(sL)^{\#})$. As shown in [1], $\operatorname{Ext}^p_{UL}(\Bbbk,\Bbbk)\cong H^{(p)}$, and so our hypothesis implies $H^{(p)}=0$, $p>m$.

Note that for each k, $\mathcal{C}^*(L_{\geq k})$ is obtained from $\mathcal{C}^*(L)$ by dividing by the ideal generated by elements in $(sL)^{\#}$ of degree $\leq k$. Since $\operatorname{gldim}\, L_{\geq k}\leq m$ it follows that these quotient cochain complexes also satisfy $H^{(p)}=0$, $p>m$. The argument of [2], section 4, can therefore be applied verbatim to $\bigwedge(sL)^{\#}$ (with $\operatorname{cat}\bigwedge X\leq m$ replaced by $\operatorname{gldim}\, L\leq m$) to conclude that $\alpha>0$.

The same argument in the proof of Theorem 2 in [5] now shows that each L has a quasi-geometric growth sequence. Let n_i be an increasing sequence such that $(\dim L_{n_i})^{\frac{1}{n_i}}$ converges to e^{α}. By starting the sequence at some n_j we may assume $\dim{(L)}_{n_i} > \frac{1}{a}$, for all i. Thus the formula in ([2], top of page 189) gives a sequence $n_i = q_0 < q_1 < \cdots < q_k = n_{i+1}$ such that $q_{j+1}\leq(m+1)q_j$ and such that

$$\left(\dim L_{q_j}\right)^{\frac{1}{q_j+1}} \geq \left(a\dim L_{n_i}\right)^{\frac{1}{n_i+1}}\;,\quad j<k\,.$$

Hence interpolating the sequences n_i with the sequences q_j gives a quasi-geometric growth sequence (r_j). □

We now revert to the proof of the Theorem. Since $\operatorname{gldim} L<\infty$ we may choose a non-zero element $x\in L$ of even degree d. Let N be the sub-Lie algebra of elements of degree $>d$ that commute with x. Then

$$\operatorname{gldim} L \geq \operatorname{gldim}(\Bbbk x + N) = 1+\operatorname{gldim} N\,.$$

If $\log\operatorname{index} N = \log\operatorname{index} L = \alpha$, then certainly $\dim\, N=\infty$, and so N satisfies the hypotheses of the Theorem. By induction on global dimension it satisfies the conclusion. In particular, if $\alpha < \infty$, then for some d,

$$\sum_{j=1}^{d-1} \dim N_{k+j} = e^{k\beta_k}$$

with $\beta_k \to \alpha$. Write

$$\sum_{j=1}^{d-1} \dim L_{k+j} = e^{k\alpha_k}\,.$$

Then $\alpha_k \geq \beta_k$ and $\limsup \alpha_k = \alpha$ because $\alpha = \log\operatorname{index} L$. Thus $\alpha_k \to \alpha$, and the Theorem holds in this case.

Lemma 2. *There is a sequence of finitely generated sub-Lie algebras $E(i) \subset L$ such that* log index $E(i) \to \alpha$.

Proof. Otherwise for some $\varepsilon > 0$ we have $\log\operatorname{index} E \leq \alpha - \varepsilon$ for every finitely generated sub-Lie algebra $E \subset L$. Construct an increasing sequence of finitely generated sub-Lie algebras, $F(i)$, and increasing sequences (k_i) and (ℓ_i), as follows. Set $F(0) = 0$, and if $F(i)$ is constructed choose k_i and ℓ_i so that

(i) $\dim F(i)_k < e^{k(\alpha-\varepsilon/2)}\,, k \geq k_i\,,$

(ii) $\dim L_{\ell_i} \geq e^{\ell_i(\alpha-1/i)}$,

(iii) $\ell_i > (m+1)k_i$.

Then let $F(i+1)$ be the sub-Lie algebra generated by $F(i)$ and L_{ℓ_i}.

Now let $F = \bigcup_i F(i)$. Since $\dim F_{\ell_i} \geq e^{\ell_i(\alpha-\frac{1}{i})}$ it follows that $\log\operatorname{index} F = \alpha$. Moreover, because $F \subset L$, $\operatorname{gldim} F \leq m$. Thus by Lemma 1 there is an infinite sequence q_j such that for all j, $q_j < q_{j+1} \leq (m+1)q_j$ and $\dim F_{q_j} \geq e^{q_j(\alpha-\varepsilon/2)}$. In particular we may choose i and j so that $q_j \leq k_i < q_{j+1}$. But then $q_{j+1} \leq (m+1)q_j \leq (m+1)k_i < \ell_i$, and it follows that $F_{q_{j+1}} = F(i)_{q_{j+1}}$. This implies that $\dim F_{q_{j+1}} < e^{q_{j+1}(\alpha-\varepsilon/2)}$, a contradiction. □

Finally, we complete the proof of the theorem. It remains to consider the case $\log\operatorname{index} N < \log\operatorname{index} L$. Let $E(i) \subset L$ be finitely generated sub-Lie algebras such that $\log\operatorname{index} E(i) \to \log\operatorname{index} L$. Moreover, $\operatorname{gldim} E(i) \leq m$ and, according to Lemma 1, each $E(i)$ has a quasi-geometric growth sequence. Since $E(i)$ is finitely generated, Theorem 3 of [5] applies and states that for some d_i, $\frac{\log\dim E(i)_{(k,k+d_i)}}{k}$ converges to log index $E(i)$.

Fix $\varepsilon > 0$ and choose i so that log index $E(i) \geq \alpha - \varepsilon/4$. Then choose k_0 so that

$$\frac{\log\dim E(i)_{(k,k+d_i)}}{k} \geq \alpha - \varepsilon/3\,, \qquad k \geq k_0\,.$$

This implies that k_0 extends to an infinite sequence (k_ℓ) such that $k_\ell < k_{\ell+1} < k_\ell + d_i$ and such that

$$\frac{\log\dim L_{k_\ell}}{k_\ell} \geq \alpha - \varepsilon/2, \quad \ell \geq 0\,.$$

On the other hand, since $\log\operatorname{index} N < \log\operatorname{index} L$ we may assume (for k_0 sufficiently large and ε sufficiently small) that

$$\sum_{j\leq d_i/d} \dim N_{k_\ell+jd} \leq \frac{1}{2}\dim L_{k_\ell}\,, \quad \text{for all}\, \ell\,.$$

Since $N = (\ker \mathrm{ad} x)_{>d}$ we have

$$\dim L_{k_\ell + pd} \geq \dim L_{k_\ell} - \sum_{j=0}^{p-1} \dim N_{k_\ell + jd} \geq \frac{1}{2} \dim L_{k_\ell}\,, \quad p \leq d_i/d\,.$$

It follows that for $p \leq d_i/d$ and k_ℓ sufficiently large

$$\frac{\log \dim L_{k_\ell + pd}}{k_\ell + pd} \geq \frac{\log \frac{1}{2} + \log \dim L_{k_\ell}}{k_\ell} \cdot \frac{k_\ell}{k_\ell + pd} \geq \alpha - \varepsilon\,.$$

This establishes the Theorem. □

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