Comparison of Path-Complete Stability Criteria via Quantifier Elimination

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1 Introduction

In this work, we are interested in the stability analysis of discrete-time switching dynamical systems, i.e.

$$x(k+1) \;=\; f_{\sigma(k)}(x(k)), \qquad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state and $\sigma(k) \in \{1,\ldots,M\} := \langle M\rangle$ for an integer M is the mode of the system. We study especially the *multiple path-complete Lyapunov function* framework introduced in [1] that provides sufficient stability criteria for switching systems by solving Lyapunov inequalities encoded by a directed and labeled graph. Path-complete Lyapunov functions are defined by two components: a *template*, namely a set $\mathscr{V}$ in which the candidate Lyapunov functions are selected, and a *path-complete* graph $\mathscr{G} = (S,E)$ with $E \subseteq S \times S \times \langle M\rangle$. That is, for every finite sequence of modes, there exists a path in the graph whose sequence of labels is exactly the same. The stability criterion consists in finding a pair $(\mathscr{G} = (S,E), V = \{V_a \mid a \in S\})$ with $V \subseteq \mathscr{V}$ such that the following *Lyapunov inequalities* are satisfied:

$$\forall (a,b,i) \in E,\ \forall x \in \mathbb{R}^n:\ V_b(f_i(x)) \;\leq\; V_a(x). \qquad (2)$$

We refer to this notion as a *path-complete Lyapunov function* and we denote it by $V \in PCLF(F,\mathscr{G})$ where $F = \{f_i \mid i \in \langle M\rangle\}$. It turns out that in general, several PCLF, with different templates and different graphs, may be found to establish the stability of system (1) and therefore this brings up the question: are there path-complete graphs "better" than other ones? This has led to the introduction [2] of order relations between graphs. In our case, we will say that a graph $\mathscr{G}_2$ is better than a graph $\mathscr{G}_1$ with respect to a given template $\mathscr{V}$ and a family of switching systems $\mathscr{F}$ if

$$\begin{aligned}&\forall F \in \mathscr{F},\ [\exists V_1 \subseteq \mathscr{V}\ s.t.\ V_1 \in PCLF(F,\mathscr{G}_1)\\ &\quad \Rightarrow\ \exists V_2 \subseteq \mathscr{V}\ s.t.\ V_2 \in PCLF(F,\mathscr{G}_2)]\end{aligned} \qquad (3)$$

If (3) holds, we note $\mathscr{G}_1 \;\leq_{\mathscr{V},\mathscr{F}}\; \mathscr{G}_2$.

2 Results

In this work, we consider positive linear discrete-time switching systems, that are systems (1) where $f_i \in \mathbb{R}^{n\times n}_{\geq 0}$, and the template $\mathscr{C}$ of *linear copositive norms*. In this context, the Lyapunov inequalities (2) described by a path-complete graph $\mathscr{G} = (S,E)$ are given by

$$\forall (a,b,i) \in E, \forall x \in \mathbb{R}^n:\ {A_i}^\top v_b \;\leq\; v_a, \qquad (4)$$

componentwise. It follows that the comparison of two path-complete graphs can be achieved using quantifier elimination on the logical formula (3). By using the *Cylindrical Algebraic Decompositon* (CAD) algorithm, we can establish whether property (3) holds or not for a given pair of graphs. Let us apply this method on a trivial case to underline the output of the algorithm.

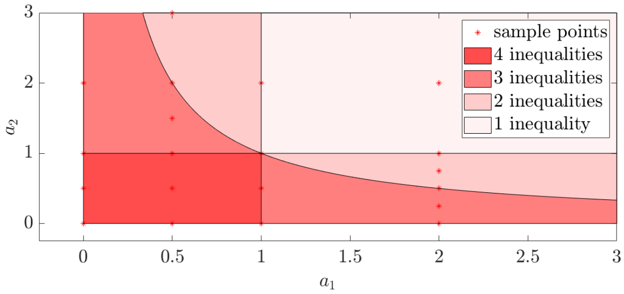

Figure 1: Visualisation of the output of our algorithm on the simplest case of a pair of 1-dimensional matrices $\{a_1,a_2\}$.

Example 1. Consider a positive linear switching system of dimension one with two modes a_1 and $a_2 \in \mathbb{R}_{\geq 0}$. Assume that we use the path-complete graph $\mathscr{G} = (\{a,b\},\{e_1 = (a,a,1), e_2 = (a,b,1), e_3 = (b,b,2), e_4 = (b,a,2)\})$ to establish the stability of the system. We denote by $\mathscr{G}_{e_i}$ the graph (which is no longer path-complete) that we obtain when we remove the edge e_i of $\mathscr{G}$. It is easy to see that for any $i \in \langle 4\rangle$, $\mathscr{G} \;\leq_{\mathscr{C},\mathscr{F}}\; \mathscr{G}_{e_i}$ and that there exists at least one system $F = \{a_1,a_2\}$ for which there is a solution $V \in \mathscr{C}$ such that $V \in PCLF(\mathscr{G}_{e_i})$ and none for $\mathscr{G}$. This result can be found by applying the CAD algorithm on the set (4) of four inequalities encoded by the graph $\mathscr{G}$. Indeed, we find the partition of the positive orthant in Figure 1 where the colour depicts the maximal number of inequalities that can be satisfied.

One can see on this trivial example that this algorithm can be used to compare any pair of path-complete graphs for the template of copositive norms and linear switching systems.

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References

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