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Impact of hedging on the cost of capital valuation for hybrid life insurance

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Abstract

In the Solvency II framework for insurance, the cost of capital rate is a critical metric that encapsulates the cost of holding capital to meet regulatory solvency requirements, while also reflecting the investor's opportunity cost of capital allocation. It is therefore essential for insurers to rigorously justify the magnitude of this rate, particularly from the perspective of investors who perceive it as a required rate of return on capital. Albrecher et al. (2022) investigated the magnitude of this rate in the economic triangle of the policyholder, the shareholder, and the regulator. This paper seeks to extend that analysis by incorporating access to the financial market and focusing on hybrid life liabilities, which combine financial and mortality risks, thereby affording an asset-liability management perspective that insurers can employ to optimize business run-off. Furthermore, by incorporating partial hedging strategies, we show how hedging can affect both the numerator (i.e. the risk margin) and the denominator (i.e. the solvency capital requirement) of the cost of capital ratio. First, we assess whether the hedging operation has improved the insurer's overall capital position. We then focus precisely on when the hedging operation is beneficial for both the policyholder and the shareholder, in the sense that it leads to a simultaneous reduction in their respective contributions.

Keywords:
Cost of capital rate, partial hedging strategies, the economic triangle, asset-liability, Solvency II.

1 Introduction

In recent years, the insurance sector has seen the emergence of hybrid life products that reflect the increasing entanglement between financial markets and biometric contingencies. More broadly, hybrid insurance products refer to payoffs whose cash flows are jointly driven by financial and biometric sources of risk. Notable examples of such general structures include securitization methods, such as a survival forward as examined in Zeddouk & Devolder (2024), and margrabe-style exchange options applied in mortality-linked contexts, see Devolder & Azizieh (2013). These products embody a dual nature by design, embedding both financial volatility and biometric uncertainty in their valuation and risk profile. Within this broad class, a notable subclass has received particular attention in the literature: hybrid life insurance products whose payoff structure can be formally modeled as the product of two risk components, one biometric and one financial. This specific structure allows for tractable modeling and clearer separation of risk drivers, which has proven beneficial in both pricing and risk management. These products have been examined in the literature, including studies by Dhaene et al. (2017) and Deelstra et al. (2020) in a static approach, as well as by Salahnejhad Ghalehjooghi & Pelsser (2021) and Belhouari, Devolder & Linders (2025) to adapt in a time-consistent multi-period setting. These studies encompass well-established insurance products such as return-guaranteed contracts, including guaranteed life insurance policies with a minimum interest return. At the other end of the spectrum, they also cover higher-risk products without guarantees, such as pure unit-linked contracts. Consequently, the regulator is conceived with prudence, adopting a meticulous approach to monitoring insurers and safeguarding their solvency, including a rigorous assessment of capital adequacy.

The market-consistent valuation is built on a first layer, namely the best estimate, reflecting the expected cost needed to cover the underlying risk. This first-tier valuation is typically sufficient only in the rare circumstances where the risk is fully hedgeable through the assets already held by the insurer. However, the nature of insurance risks is far from being entirely hedgeable. In practice, financial markets are incomplete and longevity risks remain unhedgeable. To address this, the risk margin (RM) forms a second layer in the construction of the market-consistent reserve. Outside of the scope of regulatory constraints, the risk margin is far from being uniquely defined. Numerous actuarial premium principles exist and can be used to determine the risk margin. Some of the more traditional approaches include the standard deviation premium principle, see Kaas et al. (2002), as well as its adaptation to settings with stochastic interest rates while ensuring actuarial-consistency, such as the generalized standard deviation premium principle introduced in Belhouari, Deelstra & Devolder (2025). In this paper, we posit

the assumption that the market-consistent reserve coincides precisely with the technical tariff premium provided by the policyholder.
Furthermore, Solvency 2 legislation mandates an additional capital buffer, known as the solvency capital requirement (SCR). For its computation, the literature includes alternative approaches beyond the regulatory guidelines. For example, Devolder & Lebègue (2016) employed various risk measures in a time-consistent framework to assess the final net asset–liability positions, diverging from the 99.5% value-at-risk model mandated by regulatory standards. Focusing on the legislation as it stands, we must clarify an important distinction; there are two distinct perspectives for determining the cost-of-capital rate:

- A financial vision — that of Solvency II — which fixes the remuneration rate of capital above the risk-free rate for an insurer and which is then applied uniformly to the SCR; the RM being then a consequence of this choice of cost-of-capital rate (for example 6% or 4.5% for Solvency II), see equation (2) in Möhr (2011). In this vision, the RM is linked to the risk only through the SCR. A more risky business will require more SCR, more capital, but that capital is then remunerated uniformly by the insurer through a uniform cost-of-capital.

- An actuarial vision which consists in calculating the risk margin product-by-product by a given risk measure and then the cost-of-capital rate for a one-year product is simply the ratio RM / SCR. In this perspective, the cost-of-capital rate is no longer a financial, exogenous remuneration parameter for the shareholder, but rather an output derived from two actuarial risk measures. Our paper adopts this second approach, as in Albrecher et al. (2022).

Due to the incompleteness of the insurance-finance market, the concept of partial hedging has been increasingly explored in recent academic work. The core idea is that when a product cannot be fully hedged using available asset portfolios, a portion of it can still be covered through a specific hedging strategy. A broad spectrum of hedging methodologies has been developed, ranging from quantile hedging introduced in Föllmer & Leukert (1999), to shortfall risk-minimizing strategies as scrutinized in Föllmer & Leukert (2000), and to the class of convex hedgers investigated in Dhaene et al. (2017). Recent literature has increasingly investigated the valuation of the cost-of-capital rate within incomplete markets. In particular, Albrecher et al. (2022) examined the theoretical underpinnings of this rate for a single-period liability by assuming that all risks are non-replicable, thereby excluding access to the financial market. In a more recent work, Albrecher et al. (2025) subsequently introduced financial market access by departing from the classical benchmark strategy of investing the total capital requirement solely in a risk-free bond. Instead, they proposed a pure asset allocation strategy in which a fraction of the capital is invested in a risky asset to back purely actuarial (non-life) liabilities that are independent of the financial market. Their findings indicate that while total capital and premiums can decrease if the risk allocation remains below a given threshold, the Solvency Capital Requirement (SCR) consistently increases. Consequently, this pure asset allocation approach systematically excludes any win–win agreement between shareholders and policyholders. In a parallel and continuous line of research, our paper introduces an integrated asset–liability management framework to manage hybrid life insurance products that depend on the financial market. While previous models either overlooked the financial market entirely or relied on pure asset allocation without addressing liability management, our approach proposes an integrated asset–liability management framework. By purchasing a replicating portfolio to partially hedge these hybrid liabilities, we actively address the liability side rather than relying solely on pure asset allocation, thereby managing both sides of the balance sheet. The replicating portfolio in our setting is explicitly financed by the policyholder's premium. Based on partial hedging, we split the risk into hedgeable and non-hedgeable components, thereby leading to a post-hedging balance sheet that departs from the standard unhedged. Our study will focus on hybrid life products with a one-year maturity and will adopt static hedging strategies that involve no intermediate re-balancing. We establish that an active hedging strategy can successfully reduce the overall capital requirements. More importantly, we demonstrate—both theoretically under a specific regime (see Lemma 1) and through numerical implementation with realistic inputs in the application section—that a genuine win–win configuration is achievable. Provided the insurer adopts a sufficiently cautious valuation approach towards market incompleteness, both the policyholder's premium and the shareholder's SCR can be simultaneously reduced.

The paper proceeds as follows. Section 2 introduces the asset-allocation management considered for the non-hedged benchmark case and asset–liability management for the hedged case, within a general structural framework of hybrid life insurance liabilities. Section 3 defines hedging operators and focuses especially on mean-variance hedging and quantile hedging under the tractable multiplicative risk structure for hybrid life products. We state a specific context in which hedged asset–liability management is profitable for the insurer, as well as for the shareholder and the policyholder. Furthermore, we point out that a substantial reduction in the solvency capital requirement may occur without a corresponding decrease in the total premium; this is due to the fact that, in our setting, the hedging portfolio is exclusively financed by the policyholder's contribution. In addition, we examine the implications for the resulting hierarchy of cost-of-capital rates. In Section 4, we examine a hybrid life product, which we will refer to as an inflation-indexed participation endowment product, where, under our asset–liability management framework, we were able to achieve a reduction in the total capital, along with a win–win agreement on the reduction of contributions between shareholders and policyholders, provided that the insurer adopts a sufficiently prudent valuation approach. We note that this win–win outcome is structurally asymmetric—yielding massive relief for the shareholder but only a marginal reduction for the policyholder; as a natural consequence of the hedge being exclusively financed by the policyholder's premium. In addition, we present numerical results that quantify the magnitude of the cost-of-capital rate in hedged versus unhedged cases. Section 5 provides concluding remarks and identifies future investigations. Detailed proofs are provided in the Appendix.

2 Cost-of-capital valuation

2.1 An overarching analytical perspective

In this section, we establish the probabilistic and market framework necessary for the valuation of the cost-of-capital. Our analysis adopts a single-period approach with two states, 0 and 1, where time 0 denotes the present and time 1 represents one year later, corresponding to the maturity of the liabilities.
We assume that the financial market is arbitrage-free, ensuring the existence of risk-neutral probability measures under which discounted asset prices evolve as martingales. The market is taken to be incomplete, meaning not all contingent claims can be perfectly replicated and prices may not be unique.

For simplicity, we focus on a single policyholder, aged x at time 0, and exclude diversifiable risk. The policyholder's remaining lifetime is denoted by τ. We further assume independence between the actuarial and financial risks.

Let $(\Omega^a \times \Omega^f, \mathcal{F}, \mathbb{P} = \mathbb{P}^a \times \mathbb{P}^f)$ be our probability space, where Ω^a and Ω^f denote the actuarial and financial sample spaces, respectively, and $\mathbb{P}^a$ and $\mathbb{P}^f$ are the corresponding real-world probability measures. $\mathcal{F}$ is the global filtration encompassing both insurance and financial information where we decompose it as follows:

- $\mathcal{F}^f$ is the financial filtration.
- $\mathcal{F}^a$ is the actuarial filtration, composed of:
  - $\mathfrak{G}$, the filtration generated by systematic mortality risk factors.
  - $\sigma(1_{\tau \leq u} : u \leq t)$, which captures information about unsystematic mortality risk.

Stochastic mortality rates $(\lambda_t)_{0\leq t\leq 1}$, adapted to $\mathfrak{G}$, are employed. Thus, the actuarial filtration is given by $\mathcal{F}^a = \mathfrak{G} \vee \sigma(1_{\tau\leq u} : u \leq t)$. [1]

We then define the global filtration as $\mathcal{F} \triangleq \mathcal{F}^a \vee \mathcal{F}^f$. Since our focus is on the construction of regulatory capital layers at time zero, we denote conditional expectations and variances by $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$, respectively, with the filtration omitted for brevity, as all information is implicitly taken at the initial instant.

The financial market comprises $d \geq 1$ stochastic risky assets and a risk-free bond, initially priced at one and accruing at a constant, continuously compounded rate $r > 0$, yielding a value of e^r at maturity. The value of the risk-free bond at time 1 is denoted by $S^{(0)}$, while the values of the risky assets are denoted by $\boldsymbol{s}^* = (s^{(1)}, \ldots, s^{(d)})$ at time 0 and by $\boldsymbol{S}^* = (S^{(1)}, \ldots, S^{(d)})$ at time 1. The full asset-price vectors including the risk-free bond are written as $\boldsymbol{s} = (s^{(0)}, \ldots, s^{(d)})$ at time 0 and $\boldsymbol{S} = (S^{(0)}, \ldots, S^{(d)})$ at time 1. We assume that assets are freely tradable in a market characterized by deep liquidity, transparency, efficiency, and free of transaction costs.

We consider an aggregate non-hedgeable insurance-contingent claim $Y \geq 0$, maturing at time 1, representing our liabilities. The payoff of Y is generally structured to jointly reflect financial and mortality risks, without imposing any specific functional form. The set of all liability configurations will be denoted by

$$\mathcal{Y} \triangleq \{Y\}. \tag{2}$$

We further assume that Y can be decomposed into a hedgeable part $Y_H \geq 0$ and a non-hedgeable part Y_{NH}[2], such that

$$Y = Y_H + Y_{NH}. \tag{3}$$

The claim Y_H reflects risks that can be mitigated at maturity through financial strategies constructed from linear combinations of our financial assets, i.e., $S^{(0)}, \ldots, S^{(d)}$, whereas Y_{NH} comprises components that fall outside this span. We denote by $\Theta^{(0)}, \ldots, \Theta^{(d)}$ the positions resulting from admissible hedging strategies on $S^{(0)}, \ldots, S^{(d)}$, respectively. Hence, the residual set of non-hedgeable liabilities after partial hedging is referred to as

$$\mathcal{Y}^{NH} \triangleq \left\{ Y - \sum_{j=0}^{d} \theta^{(j)} S^{(j)} \mid \quad \theta^{(0)} \in \Theta^{(0)}, \ldots, \theta^{(d)} \in \Theta^{(d)} \right\}. \tag{4}$$

[1]The one-year survival probability of an individual is

$$p_x \triangleq \mathbb{E}^{\mathbb{P}^a}\left[e^{-\int_0^1 \lambda_s \mathrm{d}s}\right]. \tag{1}$$

[2]Super-hedging is permitted; hence, the non-hedgeable residual component Y_{NH} may become negative in certain states of the world.

Insurance company management must make business decisions regarding capital allocation at time 0, prior to the commencement of the insurance period in which claim Y will manifest. We adopt the cost-of-capital approach proposed by Albrecher et al. (2022). Firstly, we introduce a risk assessment operator, denoted by ρ, presumed to be mandated by the regulator for the valuation of the liability side of the balance sheet. In the sequel, we shall proceed under the postulate that it is compelled to conform to a prescribed collection of axiomatic stipulations.

Definition 1 (Risk measure ρ)**.** *Let $\rho : \mathcal{Y} \cup \mathcal{Y}^{NH} \to \mathbb{R}^+$ be a function. It is called a regulator-approved risk measure if and only if it satisfies the following axioms:*

- *Normalization:* $\rho(0) = 0$.
- *Translation invariance: For all $x \in \mathbb{R}$ and $Y \in \mathcal{Y} \cup \mathcal{Y}^{NH}$,*
$$\rho(Y + x) = \rho(Y) + xe^{-r}.$$
- *Monotonicity: For all $Y^{(1)}, Y^{(2)} \in \mathcal{Y} \cup \mathcal{Y}^{NH}$, if $Y^{(1)} \leq Y^{(2)}$, then*
$$\rho(Y^{(1)}) \leq \rho(Y^{(2)}).$$
- *Positive homogeneity: For all $a > 0$ and $Y \in \mathcal{Y} \cup \mathcal{Y}^{NH}$,*
$$\rho(aY) = a\rho(Y).$$

We adopt well-known risk measures; for instance, Value at Risk,

$$\rho(Y) = e^{-r} \times \mathrm{VaR}_{\tilde{\alpha}}(Y) = e^{-r} \times \inf\{x \in \mathbb{R} \mid P^{\mathbb{P}^a \times \mathbb{P}^f}(Y \leq x) \geq \tilde{\alpha}\}, \quad Y \in \mathcal{Y} \cup \mathcal{Y}^{NH},$$

or Tail-Value-at-Risk, defined in full generality below, applies to risk distributions, whether continuous or mixed,

$$\rho(Y) = e^{-r} \times \mathrm{TVaR}_{\tilde{\alpha}}(Y) = e^{-r} \times \mathbb{E}^{\mathbb{P}^a \times \mathbb{P}^f}\left[Y \mid Y > \mathrm{VaR}_{\tilde{\alpha}}(Y)\right], \quad Y \in \mathcal{Y} \cup \mathcal{Y}^{NH},$$

to compute the regulatory capital requirement, that is,

$$C \triangleq \rho(Y) \quad \text{for } Y \in \mathcal{Y}$$

and

$$C_{NH} \triangleq \rho(Y_{NH}) \quad \text{for } Y_{NH} \in \mathcal{Y}^{NH},$$

where C denotes the regulatory capital requirement to cover the total payoff Y in the absence of partial hedging, and C_{NH} denotes the regulatory capital requirement to cover the non-hedgeable component Y_{NH} when a hedging strategy is adopted.

Accordingly, the monotonicity axiom ensures that $C_{NH} \leq C$ as a consequence of $Y_{NH} \leq Y$ almost surely.

The second step involves splitting the regulatory capital requirement into three components: the best estimate, risk margin, and the solvency capital requirement, thus highlighting the distinction between policyholder and investor contributions. Given access to a financial market, a second layer of business decisions concerns hedging strategies. We structure our analysis around two distinct cases: with and without hedging. We first analyze the standard framework without hedging, and then address the hedged scenario.

2.2 Asset-allocation management: risk-free benchmark

In this subsection, we assume that there is no risky asset to partially hedge the total liabilities Y and that the entire regulatory capital C, determined by applying the risk assessment operator ρ to Y, is invested in the risk-free asset $S^{(0)}$. Such an assumption corresponds to the situation studied in Albrecher et al. (2022). Consequently, at maturity the profit-and-loss ($P\&L$), is given by

$$P\&L = CS^{(0)} - Y.$$

Since in this case, no risky market-traded assets $\boldsymbol{S}^*$ are used to partially recover Y, the best estimate is the discounted expected value of Y under the real-world probability measure, thus,

$$\mu \triangleq e^{-r}\mathbb{E}^{\mathbb{P}^a \times \mathbb{P}^f}[Y] \geq 0. \tag{5}$$

The total premium paid by the policyholder is defined according to

$$P \triangleq \sup_{\mathbb{Q} \in \mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}}[\min(Ye^{-r}, C)] \geq 0, \tag{6}$$

where $\mathcal{M}^{a,f} = \{\mathbb{Q}^a \times \mathbb{Q}^f \mid \mathbb{Q}^a \in \mathcal{M}^a \text{ and } \mathbb{Q}^f \in \mathcal{M}^f\}$, assuming that $\mathcal{M}^a$ and $\mathcal{M}^f$ are convex and compact sets of probability measures uniformly equivalent to $\mathbb{P}^a$ and $\mathbb{P}^f$, respectively. The set $\mathcal{M}^a$ comprises actuarial probability measures, while $\mathcal{M}^f$ comprises equivalent financial martingale measures. A more precise definition of $\mathcal{M}^a$ and $\mathcal{M}^f$ will be provided later, depending on the assumed stochastic dynamics of the mortality rates $(\lambda_t)_{0 \leq t \leq 1}$ and the risky assets $S^{(1)}, \cdots, S^{(d)}$. Furthermore, we assume that the expectation mapping $\mathbb{Q} \mapsto \mathbb{E}^{\mathbb{Q}}[\cdot]$ is continuous on $\mathcal{M}^{a,f}$ for any bounded continuous function of the payoffs $Y \in \mathcal{Y}$ and $Y_{NH} \in \mathcal{Y}^{NH}$. The determination of the premium (6) assumes that the policyholder will not necessarily receive the payoff at maturity. If the regulatory capital proves inadequate to cover the full payoff, only the regulatory capital will be distributed; it is the insurance limited-liability option. Consequently, the risk margin, as will be defined here on an excluding own credit risk basis, may become negative; further reference to the work of Albrecher et al. (2022) is again pertinent. The risk margin (RM) is defined as the difference between the total premium P and the expected claim amount:

$$RM \triangleq P - \mu. \tag{7}$$

The solvency capital requirement (SCR), representing an additional buffer above the risk margin, is defined as

$$SCR \triangleq C - P \geq 0. \tag{8}$$

From the perspective at time $t = 0$, the equity remaining for distribution to the investor is $(C - Ye^{-r})^+$, indicating that the investor is not required to provide further capital should the insurer become insolvent. Using the infimum–supremum duality, a formula for the SCR can be derived as follows

$$SCR = \inf_{\mathbb{Q} \in \mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}} \left[(C - Ye^{-r})^+ \right]. \tag{9}$$

The regulator mandates the risk measure ρ. Meanwhile, the investor and policyholder jointly agree on a specific set $\mathcal{M}^{a,f}$—a family of probability measures reflecting the combined actuarial and financial risks—which governs the determination of both the premium and the solvency capital requirement. This interaction among the three parties forms the economic triangle at the core of insurance risk management: the policyholder, the shareholder, and the regulator. Ultimately, the cost of capital rate R^{cc} is delineated below

$$SCR \times R^{cc} \triangleq RM. \tag{10}$$

The definition of the cost of capital rate highlights the insurer's appeal to the investor: the more funds invested, the higher the charge to the policyholder, proportional to the invested amount. Understandably, this rate depends on both terms RM and SCR, previously determined. Consequently, the 6% Solvency II benchmark cost of capital rate does not apply consistently within our framework.

2.3 Partially hedged asset-liability management: replicating portfolio framework

In this section, we assume that there are risky assets to partially hedge the liabilities and consider the decomposition introduced in (3). Consequently, the asset-liability management strategy considered is to initially acquire the replicating portfolio $\sum_{j=0}^{d} \theta^{(j)} S^{(j)}$, priced at μ_H, which replicates the hedgeable component Y_H at maturity. Indeed, a detailed analysis of the profit and loss in this asset-liability management is essential to better understand the mechanism. At time 0, the total regulatory capital is equal to the capital C_{NH} of the non-hedged part plus the market-consistent price of the hedged part μ_H, i.e.

$$C_{NH} + \mu_H.$$

At time 1, given that μ_H is invested in a hedge portfolio producing Y_H, and C_{NH} is invested in the risk-free asset $S^{(0)}$, the available capital becomes $Y_H + C_{NH} S^{(0)}$. The profit-and-loss ($P\&L$) at maturity in the presence of hedging is given by

$$P\&L = Y_H + C_{NH} S^{(0)} - Y_H - Y_{NH} = C_{NH} S^{(0)} - Y_{NH}. \tag{11}$$

In the absence of hedging, the profit-and-loss $P\&L$ simply amounted to $CS^{(0)} - Y$, since the sole investment consists of allocating C to the risk-free asset. With hedging, only the capital C_{NH}, the reduced amount, will be used, but in return, the replicable component Y_H is eliminated, leaving only the non-hedgeable risk Y_{NH} to be managed.

The total best estimate is modified when hedging strategies are employed, and it is essential to distinguish explicitly between the hedged and unhedged best estimates. Due to market-consistency (see Dhaene et al. (2017)), the best estimate of the hedgeable component Y_H is evaluated as the discounted expectation under any financial equivalent martingale measure $\mathbb{Q}^f$, which may belong to the set of financial martingale measures $\mathcal{M}^f$, although it is not required to do so. The hedged best estimate is expressed as follows

$$\mu_H \triangleq e^{-r} \mathbb{E}^{\mathbb{Q}^f}[Y_H]. \tag{12}$$

It should be noted that μ_H represents the market-consistent price of the hedge portfolio that replicates Y_H.
Since the remaining component Y_{NH} is non-hedgeable, its best estimate is determined as the discounted expected value under the real-world probability measure, thus,

$$\mu_{NH} = e^{-r} \mathbb{E}^{\mathbb{P}^a \times \mathbb{P}^f}[Y_{NH}]. \tag{13}$$

In the presence of hedging, the risk margin, denoted RM_{NH}, is associated solely with the non-hedgeable component, as no additional risk loading is required for the hedgeable part. Specifically,

$$RM_{NH} = P_{NH} + \mu_H - (\mu_{NH} + \mu_H) = P_{NH} - \mu_{NH}, \tag{14}$$

where P_{NH} is the non-hedgeable premium. Based on the $P\&L$ deduced in (11), and discounting, the non-hedgeable premium formula naturally evolves into the following form

$$P_{NH} = \sup_{\mathbb{Q} \in \mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}} \Big[\min \Big(Y_{NH} e^{-r}, \, C_{NH} \Big) \Big]. \tag{15}$$

The solvency capital requirement, denoted by SCR_{NH} in the hedging case, is defined as the surplus above the non-hedgeable total premium

$$SCR_{NH} = C_{NH} - P_{NH}. \tag{16}$$

The SCR_{NH} formula is directly derived from the expression of P_{NH} combined with (16), through the limited-liability option equation for the unhedgeable component, namely: $C_{NH} = (C_{NH} - Y_{NH} e^{-r})^+ + \min(C_{NH}, Y_{NH} e^{-r})$ and the infimum–supremum duality, as follows

$$SCR_{NH} = \inf_{\mathbb{Q} \in \mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}} \left[\Big(C_{NH} - Y_{NH} e^{-r} \Big)^+ \right]. \tag{17}$$

The post partial hedging cost-of-capital rate, denoted R^{cc}_{NH}, is given by

$$R^{cc}_{NH} = \frac{RM_{NH}}{SCR_{NH}}. \tag{18}$$

2.4 Profitability of the hedged asset–liability management

In this paper, we will investigate three types of profitability: 1. the profitability of the regulatory capital requirement. 2. The profitability for the shareholders by reducing the solvency capital requirement. 3. The profitability for the policyholders by reducing the insurance premium. The first and foremost question to consider is whether the asset-liability management operation incorporating hedging achieves an initial reduction of the total regulatory capital requirement C, before addressing any further considerations.

Definition 2 (Profitability condition of hedging)**.**
The hedging operation is profitable if

$$C_{NH} + \mu_H < C. \tag{19}$$

Secondly, our investigation now focuses on the efficiency of hedging in reducing the solvency capital requirement.

Definition 3 (Hedging profitability for shareholder benefit)**.** *We say that a hedging operation is profitable for the shareholder if*

$$SCR_{NH} < SCR. \tag{20}$$

The condition (20) provides a straightforward assessment of whether a hedging strategy is beneficial to shareholders. The condition assesses that the solvency capital requirement decreases once the hedge is implemented, thus generating a saving, for the benefit of the shareholder, equal to $(SCR - SCR_{NH})$.

Subsequently, our investigation examines, from the policyholder's perspective, whether such an asset liability strategy reliably reduces the premium.

Definition 4 (Hedging-induced premium reduction). *A partial hedging strategy reduces the policyholder's premium if*

$$\mu_H + P_{NH} < P. \tag{21}$$

Policyholders benefit when this inequality holds, as the total premium borne under the hedging operation (i.e. $\mu_H + P_{NH}$) is lower than the unhedged premium P. Structurally, satisfying this condition is inherently more demanding than lowering the shareholder's capital requirement, since the policyholder explicitly finances the cost of the hedging portfolio.

The objective of Section 3 is to focus on a specific form of our hybrid liabilities (2), which will yield greater tractability of the derivations, explicitly specifying the hedging strategies that we will use for the partial coverage of Y, namely mean-variance hedging and quantile hedging. Furthermore, we investigate profitability in the sense of the definitions 2, 3, and 4.

3 Impact of Mean-Variance and Quantile hedging on the cost-of-capital valuation

3.1 Multiplicative risk structure in hybrid life products

Liabilities are modeled as hybrid life-contingent claims delivered by the insurer at time 1, which are henceforth assumed to take the following form

$$Y \triangleq g(\tau) \times h, \tag{22}$$

where $g : (0, \infty) \to [0, \infty)$ is a deterministic function of the remaining lifetime τ, and h is an $\mathcal{F}_1^f$-measurable financial payoff. The set (2) shall therefore reduce to

$$\mathcal{Y} \triangleq \{g(\tau) \times h\}. \tag{23}$$

This multiplicative structure reflects the standard formulation of hybrid life products, highlighting the separation of financial and actuarial risks. We assume the multiplicative form given in (22), as it offers greater tractability while still covering a broad class of hybrid life products. The subset of financial liability configurations (i.e. no actuarial payoff) can be separately defined by

$$\mathcal{Y}^f \triangleq \{h\} \subset \mathcal{Y}, \tag{24}$$

whereas the subset of actuarial liability configurations (i.e. no financial payoff) is indicated by

$$\mathcal{Y}^a \triangleq \{g(\tau)\} \subset \mathcal{Y}. \tag{25}$$

3.2 Hedging operators

So far we have only stated that the hedging portfolio for a claim $Y \in \mathcal{Y}$ is constructed at time 1 as $\sum_{j=0}^{d} \theta^{(j)} S^{(j)}$. This specification remains incomplete, as it does not indicate how the weights $\theta^{(0)}, \ldots, \theta^{(d)}$ are determined; without a penalization rule, a claim that depends only on $S^{(k)}$ could, in principle, be hedged using a portfolio constructed from assets $S^{(i)}$ with $i \neq k$, even if these assets are completely independent of $S^{(k)}$. We now formally specify the two hedging strategies that will be employed throughout the remainder of the paper. Prior to this, we rigorously introduce the operator linking hedging strategies to their underlying risks, namely the notion of a hedger as defined in Dhaene et al. (2017).

Definition 5 (Hedger). *A hedger $\boldsymbol{\theta}$ is a function $\boldsymbol{\theta} : \mathcal{Y} \to \Theta^{(0)} \times \cdots \times \Theta^{(d)}$ that assigns to each hybrid life payoff $Y \in \mathcal{Y}$ a corresponding hedging strategy $\boldsymbol{\theta}_Y = (\theta_Y^{(0)}, \cdots, \theta_Y^{(d)})$, satisfying the following properties:*

- *Normalization: $\boldsymbol{\theta}_0 = (0, \cdots, 0)$.*
- *Translation invariance: For any $a \in \mathbb{R}$,*

$$\boldsymbol{\theta}_{Y+a} = \boldsymbol{\theta}_Y + (ae^{-r}, 0, \cdots, 0).$$

The normalization axiom implies that if a product's payoff is identically zero, then the allocations to the assets must necessarily be zero in order to replicate it. The translation invariance axiom highlights that, if we add a constant payoff a at maturity, the hedging strategy consists of investing ae^{-r} in the risk-free bond and taking zero positions in the risky assets, i.e., $\boldsymbol{\theta}_a = (ae^{-r}, 0, \ldots, 0)$.
Since the payoff Y is not necessarily hedgeable, the hedger cannot generate perfect hedging strategies; hence, only partial hedging is considered. The value of the hedging portfolio of a hybrid life payoff Y at maturity, i.e. $\sum_{j=0}^{d} \theta_Y^{(j)} \times S^{(j)}$, is denoted by $\boldsymbol{\theta}_Y \cdot \boldsymbol{S}$, whereas its initial value, i.e. $\theta_Y^{(0)} + \sum_{j=1}^{d} \theta_Y^{(j)} \times s^{(j)}$, is denoted

by $\boldsymbol{\theta}_Y \cdot \boldsymbol{s}$.
Hedgers include, for instance, the mean-variance hedger, defined by

$$\boldsymbol{\theta}_Y^{MV} = \arg \min_{(v^{(0)},\cdots,v^{(d)})\in\Theta^{(0)}\times\cdots\times\Theta^{(d)}} \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}\left[\left(Y - \sum_{j=0}^{d} v^{(j)} S^{(j)}\right)^2\right], \tag{26}$$

where, the expectation of the hedging portfolio matches the expectation of the payoff Y, i.e.

$$\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{S}] = \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}[Y].^{3}$$

For a general class of convex hedgers, we refer to Dhaene et al. (2017), Chen et al. (2020), Barigou et al. (2022), Linders (2023).

A principal limitation of mean–variance hedging is its symmetry: the loss function penalizes shortfall and super-hedging equally across all states of the world. In practical applications, it is often preferable to use a convex, asymmetric loss function that penalizes under-hedging (shortfall) more heavily than super-hedging. To address this, we present the quantile regression hedging strategy (see e.g. in Barigou et al. (2022)). This approach is based on the following asymmetric loss function:

$$U_p(x)^{4} = \begin{cases} p \times x & \text{if } x \geq 0 \\ (1-p)\times(-x) & \text{if } x < 0 \end{cases}$$

for $p \in (0,1)$, where x represents the hedging error (positive for shortfall, negative for super-hedging).

The quantile hedger, denoted $\boldsymbol{\theta}^{qr}$, is then defined as

$$\boldsymbol{\theta}_Y^{qr} = \arg \min_{(v^{(0)},\cdots,v^{(d)})\in\Theta^{(0)}\times\cdots\times\Theta^{(d)}} \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}[U_p(Y - \sum_{j=0}^{d} v^{(j)} S^{(j)})]. \tag{27}$$

Here, instances of under-hedging (shortfalls) are weighted by p, while instances of super-hedging are weighted by $1-p$. A key consequence of this approach is that we ensure that the liability is fully covered with probability at least equal to the specified confidence level p; that is,

$$P^{\mathbb{P}^a\times\mathbb{P}^f}[Y \leq \boldsymbol{\theta}_Y^{qr} \cdot \boldsymbol{S}] \geq p.$$

It is important to note that, despite sharing the term "quantile hedging," this approach differs from the framework developed by Föllmer & Leukert (1999), where

[3] Hence, $\mu_{NH}^{MV} = 0$.

[4] In Barigou et al. (2022), this loss function is normalized by the factor $1-p$. While the minimum value differs, the optimal hedging vector remains unchanged.

the objective is to maximize the probability of super-hedging without imposing a fixed confidence level, but given an upper bound on the initial hedge cost. For the remainder of the paper, the hedger $\boldsymbol{\theta}$ is assumed to follow a quantile or a mean-variance hedging strategy.

3.3 Tractability under the multiplicative structure payoff

As hedging originates in financial theory rather than actuarial theory, it is valuable to identify a hedger capable of disentangling the hedging strategy for financial risk from the residual actuarial component. The mean-variance hedger exhibits this property in the specific form of hybrid liabilities (23), as established in the following proposition. A similar result appears as Theorem 6 in Dhaene et al. (2017).

Proposition 1. *Let Y be of the form $\mathcal{Y}$ as defined in (23). Then,*

$$\boldsymbol{\theta}_Y^{MV} = \boldsymbol{\theta}_h^{MV} \times \mathbb{E}^{\mathbb{P}^a}[g(\tau)]. \tag{28}$$

Proof. See the Appendix. $\square$

As it is customary in the literature to evaluate the hedge price in terms of a mathematical expectation, we consider the minimal variance martingale measure $\mathbb{Q}_{MV}^f$, previously presented in Černỳ & Kallsen (2009). In that work, the transition from the real-world financial probability to $\mathbb{Q}_{MV}^f$ was derived in a financial market comprising a single traded risky asset where the change of measure satisfies

$$\frac{d\mathbb{Q}_{MV}^f}{d\mathbb{P}^f} = \frac{\mathbb{E}^{\mathbb{P}^f}[S^{(1)^2}] - \mathbb{E}^{\mathbb{P}^f}[S^{(1)}]S^{(1)}}{\mathbb{V}^{\mathbb{P}^f}[S^{(1)}]} + \frac{S^{(1)} - \mathbb{E}^{\mathbb{P}^f}[S^{(1)}]}{\mathbb{V}^{\mathbb{P}^f}[S^{(1)}]} s^{(1)} S^{(0)}. \tag{29}$$

The corresponding Radon-Nikodym density is extended here to our setting with d risky assets as follows

$$\frac{d\mathbb{Q}_{MV}^f}{d\mathbb{P}^f} = 1 + (\boldsymbol{s}^* S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*]) \cdot M_{\boldsymbol{S}^*}^{-1} \cdot (\boldsymbol{S}^* - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])', \tag{30}$$

where $M_{\boldsymbol{S}^*}$ is the variance–covariance matrix of $(\mathrm{Cov}^{\mathbb{P}^f}[S^{(i)}, S^{(j)}])_{1\leq i,j\leq d}$ and $\cdot$ denotes the matrix product.

Remark 1.

Using the mean-variance hedging strategy, the best estimate for a purely financial payoff $h \in \mathcal{Y}^f$ under $\mathbb{Q}_{MV}^f$ equals the hedge price, i.e. $e^{-r}\mathbb{E}^{\mathbb{Q}_{MV}^f}[h] = \boldsymbol{\theta}_h^{MV} \cdot \boldsymbol{s}$. Consequently, Proposition 1 implies

$$e^{-r}\mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}_{MV}^f}[Y] = \boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{s}.$$

See the Appendix for further details.

It should be noted that when the mean-variance approach is used as a hedging strategy, the set of equivalent financial martingale measures $\mathcal{M}^f$, employed to determine the premium and capital buffer, need not necessarily contain $\mathbb{Q}^f_{MV}$.

Furthermore, the formulas for the components of the balance sheet, whether hedged or unhedged, simplify as follows

Non-hedged balance sheet:

$$\mu = e^{-r}\mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times \mathbb{E}^{\mathbb{P}^f}[h] \tag{31}$$

$$P = \sup_{\mathbb{Q}\in\mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}}\Big[\min(g(\tau)he^{-r}, C)\Big] \tag{32}$$

$$SCR = \inf_{\mathbb{Q}\in\mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}}\Big[(C - g(\tau)he^{-r})^+\Big] \tag{33}$$

Hedged balance sheet, with a hedging portfolio of value $\mu_H = \boldsymbol{\theta}_Y \cdot \boldsymbol{s}$:

$$\mu_{NH} = e^{-r}\mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times \mathbb{E}^{\mathbb{P}^f}[h] - e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}] \tag{34}$$

$$P_{NH} = \sup_{\mathbb{Q}\in\mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}}\Big[\min\Big((g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r},\, C_{NH}\Big)\Big] \tag{35}$$

$$SCR_{NH} = \inf_{\mathbb{Q}\in\mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}}\Big[\Big(C_{NH} - (g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}\Big)^+\Big] \tag{36}$$

The best estimates (31) and (34) become tractable due to the decoupling of the mean terms for financial and actuarial risks. For premium and solvency capital requirement formulas (formulas (32), (35), (33) and (36)), the simplifications are most apparent when the actuarial payoff follows a Bernoulli distribution; as this is the case that will be treated in the application section.

It is worth noting that the risk margin may reach a negative value, not only as a consequence of its construction on an excluding-own-credit-risk basis (as noted in Subsection 2.2), but can also be solely due to the premium being priced under a risk-neutral measure, whereas the best estimate is defined under the real-world probability measure. From a balance sheet perspective, this negative risk margin situation leads to a reduction in premiums and a mechanical increase in reported equity. See the example that follows.

Example 1 (Continuous capped call payoff)**.** As a simplified illustration, we assume that the financial market is complete, with a single risk-neutral probability measure $\mathbb{Q}^f$, and that mortality risk is ignored. We consider a capped call-type payoff defined by $Y = \min\big((S^{(1)} - \tilde{K})^+,\ \tilde{D}\big)$, where $\tilde{K} = 100$ is the strike price and $\tilde{D} = 20$ is the cap level. The underlying asset price process $(S_t)_{t\in[0,1]}$ is assumed to follow a geometric brownian motion under the physical measure $\mathbb{P}^f$

$$dS_t = \mu^s S_t\, dt + \sigma^s S_t\, d\widetilde{W}_t, \qquad s^{(1)} = 100,$$

where $\mu^s = 8\%$ represents the expected instantaneous return, exceeding the assumed risk-free rate $r = 2\%$, $\sigma^s = 0.2$ is the volatility, and $(\widetilde{W}_t)_{t\geq 0}$ is a standard brownian motion. The capital requirement C will be assessed by the risk operator $\rho(Y) = e^{-r} \times \sup_{\omega\in\Omega} Y(\omega)$, which is consistent with definition 1. We obtain that $C = e^{-r} \times \tilde{D}$. Hence, $Ye^{-r} \leq C$ almost surely, and the RM formula is then simplified to

$$RM = e^{-r}\mathbb{E}^{\mathbb{Q}^f}[Y] - e^{-r}\mathbb{E}^{\mathbb{P}^f}[Y].$$

Since $\tilde{D}$ is positive, Y can be rewritten in this way $Y = (S^{(1)} - \tilde{K})^+ - (S^{(1)} - (\tilde{K} + \tilde{D}))^+$. It is well known that for a call option with strike $\tilde{K}$, the price at time $t = 0$ admits the following expressions:
Under the real-world measure $\mathbb{P}^f$:

$$e^{-r}\mathbb{E}^{\mathbb{P}^f}[(S^{(1)} - \tilde{K})^+] = s^{(1)}\, e^{-r} e^{\mu^s}\, \Phi(d_1^p) - \tilde{K}e^{-r}\, \Phi(d_2^p),$$

and under the risk-neutral measure $\mathbb{Q}^f$:

$$e^{-r}\, \mathbb{E}^{\mathbb{Q}^f}\big[(S^{(1)} - \tilde{K})^+\big] = s^{(1)}\, \Phi(d_1^q) - \tilde{K}e^{-r}\, \Phi(d_2^q),$$

with

$$d_1^q = \frac{\ln(s^{(1)}/\tilde{K}) + (r + \frac{1}{2}{\sigma^s}^2)}{\sigma^s}, \quad d_2^q = d_1^q - \sigma^s, d_1^p = \frac{\ln(s^{(1)}/\tilde{K}) + (\mu^s + \frac{1}{2}{\sigma^s}^2)}{\sigma^s},\ d_2^p = d_1^p - \sigma^s.$$

After computing the calls with strikes $\tilde{K}$ and $\tilde{K} + \tilde{D}$, we can then conclude that $RM = 6.36 - 8.52 = -2.16 < 0$. Here, the negativity of the risk margin is attributable exclusively to the change of probability measure, since $\min(Ye^{-r}, C)$ is always equal to Ye^{-r}, for all $\omega \in \Omega$.

3.4 Profitability

We aim to derive a sufficient condition, expressed explicitly in terms of the full regulatory capital requirements C and C_{NH} and the hedging portfolio, that can serve as a preliminary filter before switching to the hedged balance sheet to identify cases where the hedged asset–liability management is profitable. Precisely, the following lemma articulates two separate situations: in one, the hedging operation is profitable in the sense of definition 2, the shareholder benefit is achieved in the sense of definition 3, and the policyholder benefit is achieved in the sense of definition 4; in the second situation, neither condition holds.

Lemma 1. *Let Y be of the form $\mathcal{Y}$ defined in (23). The following holds:*

$$SCR_{NH} \leq SCR, \quad C_{NH} + \boldsymbol{\theta}_Y \cdot \boldsymbol{s} \leq C, \quad P_{NH} + \boldsymbol{\theta}_Y \cdot \boldsymbol{s} \leq P,$$

if

$$(\boldsymbol{\theta}_Y \cdot \boldsymbol{S})\, e^{-r} \leq C - C_{NH} \quad \text{almost surely.} \tag{37}$$

Moreover, the inequalities are guaranteed in the strict sense of the profitability conditions (19), (20), and (21) as follows

$$\mathbb{P}\big[(\boldsymbol{\theta}_Y \cdot \boldsymbol{S})\, e^{-r} < C - C_{NH} \cap g(\tau) h e^{-r} < C\big] > 0 \implies \text{Conditions (19) and (20) hold.} \tag{38}$$

$$\mathbb{P}\big[(\boldsymbol{\theta}_Y \cdot \boldsymbol{S})\, e^{-r} < C - C_{NH} \cap (g(\tau) h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S}) e^{-r} > C_{NH}\big] \implies \text{Condition (21) holds.} \tag{39}$$

Conversely, if

$$(\boldsymbol{\theta}_Y \cdot \boldsymbol{S})\, e^{-r} \geq C - C_{NH} \quad \text{almost surely,} \tag{40}$$

then none of the profitability conditions (19), (20), and (21) hold.

Proof. See the Appendix. □

The inequality (37) in Lemma 1 asserts that if the hedge significantly reduces the regulatory capital requirement C_{NH}, then C_{NH} moves sufficiently below C and the gap $C - C_{NH}$ becomes an upper bound of the discounted hedging portfolio in all states of the world; the resulting setting would surely be favorable, since the hedge succeeds in lowering SCR, C and P. The inequality (40) in Lemma 1 formulates the corresponding converse case. Nevertheless, this lemma provides a useful pre-purchase filter for cases in which the discounted hedging portfolio appears either uniformly lower-bounded or uniformly upper-bounded by the capital saving $C - C_{NH}$, although these inequality conditions may fail to hold for highly risky hybrid life products.

Example 2 (Discrete probability space with hybrid life payoff)**.** Consider a probability space restricted to four states: two actuarial (ω_1^a, ω_2^a) and two financial (ω_1^f, ω_2^f), with $d = 1$. The actuarial states correspond to survival (ω_1^a) and death (ω_2^a) at maturity, with probabilities 0.6 and 0.4 respectively. Financial states have asset values $S^{(1)}(\omega_1^f) = 6$, $S^{(1)}(\omega_2^f) = 4$, initial value $s^{(1)} = 5$, and probabilities 0.2 and 0.8 respectively. We set $r = 0$, so $S^{(0)} = 1$.

The product probability $\mathbb{Q}^a_\zeta \times \mathbb{Q}^f_{MV}$ is used, where $\mathbb{Q}^f_{MV}$ is the minimal variance martingale measure and $\mathbb{Q}^a_\zeta$ is an Esscher transform with $\zeta = -2$. The hybrid life payoff is $Y = (S^{(1)} - 1)^2 \cdot \mathbf{1}_{\tau>1}$, paying out only if the policyholder survives.

Case (37) of Lemma 1

The Value at Risk at 99.5% is $C = 25$. Under the minimal variance and Esscher measures, we compute $\mathbb{Q}^f_{MV}(\omega^f_1) = \mathbb{Q}^f_{MV}(\omega^f_2) = 0.5$, and $\mathbb{Q}^a_\zeta(\omega^a_1) = 0.917$, $\mathbb{Q}^a_\zeta(\omega^a_2) = 0.083$. The solvency capital requirement is:

$$SCR = \mathbb{E}^{\mathbb{Q}^a_\zeta \times \mathbb{Q}^f_{MV}}[(25 - Y)^+] = 9.411.$$

Partial mean-variance hedging yields allocations $\theta_h^{(0),MV} = -23$ and $\theta_h^{(1),MV} = 8$. The hedge value is then:

$$\boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{S} = \begin{cases} 15 & \text{if } \omega^f = \omega^f_1, \\ 5.4 & \text{if } \omega^f = \omega^f_2. \end{cases}$$

After hedging, the unhedgeable capital is $C_{NH} = 10$. We verify condition (37) of Lemma 1:

$$\boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{S} = \{15, 5.4\} \leq C - C_{NH} = 15,$$

with $(\boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{S})(\omega^f_2) = 5.4 < C - C_{NH}$, and $Y(\omega^a_1, \omega^f_2) = 9 < C$, in accordance with assumption (38). Consequently, hedging should be strictly effective in satisfying profitability for the shareholder and the insurer. To verify this, we calculate SCR_{NH}, which yields

$$SCR_{NH} = \mathbb{E}^{\mathbb{Q}^a_\zeta \times \mathbb{Q}^f_{MV}}\left[(10 - Y_{NH})^+\right] = 4.611 < SCR,$$

with $C_{NH} + \boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{s} = 10 + 10.2 < C = 25$. Since $Y_{NH} \leq C_{NH}$ for every state of the world, Lemma 1 does not ensure a strict premium reduction; and the computation confirms that $P_{NH} + \boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{s} = 5.389 + 10.2 = P = 15.589$.

Case (40) of Lemma 1

Suppose we adjust $s^{(1)} = 4.5$, $\mathbb{P}^f(\omega^f_1) = 0.05$, and $\mathbb{P}^f(\omega^f_2) = 0.95$. Then $\mathbb{Q}^f_{MV}(\omega^f_1) = 0.25$ and $\mathbb{Q}^f_{MV}(\omega^f_2) = 0.75$. The confidence level for the Value at Risk is herein adjusted to 95%. The new capital is $C = 9$, and the unhedgeable capital is $C_{NH} = 3.6$. Now, $\boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{S} = \{15, 5.4\} \geq C - C_{NH} = 5.4$, verifying condition (40) of Lemma 1, where hedging would lead to

$$\begin{cases} SCR_{NH} = 0.9462 > SCR = 0.747, \\ C_{NH} + \boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{s} = 3.6 + 10.2 > C = 9, \\ P_{NH} + \boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{s} = 2.6538 + 10.2 > P = 8.253. \end{cases}$$

Since a strict inequality implies the corresponding non-strict inequality, (19), (20), and (21) do not hold. □

We now aim to establish an equivalent characterization of when partial hedging results in a direct reduction of the premium for the policyholder.

Proposition 2.

Let $\Delta RM \triangleq RM - RM_{NH}$. Then a reduction in the premium under partial hedging is achieved if, and only if, the difference between the cost of the hedge and the real-world expectation of the hedging portfolio does not exceed the risk margin differential, i.e.

$$\boldsymbol{\theta}_Y \cdot \boldsymbol{s} + P_{NH} < P \iff (\boldsymbol{\theta}_Y \cdot \boldsymbol{s} - e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}]) < \Delta RM.$$

Proof.
The proof follows directly from the definitions of P, P_{NH} and μ_{NH}. □

This equivalence consists of a comparison between the market risk premium of the hedging portfolio $(\boldsymbol{\theta}_Y \cdot \boldsymbol{s} - e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}])$ and the difference between RM and RM_{NH}. Moreover, this proposition indirectly highlights that a reduction in the total premium induced by hedging does not necessarily imply a reduction in the risk margin; see later the application in Section 4. Such a decrease is guaranteed when the hedge price exceeds the hedging portfolio's real-world expectation (i.e., $\boldsymbol{\theta}_Y \cdot \boldsymbol{s} \geq e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}]$).

Proposition 3.

Let $\Delta P \triangleq P - P_{NH}$, then the following statements hold:

- $\Delta P \geq 0$,
- *Although a permanent hierarchical ordering between RM and RM_{NH} cannot be established in general, the following bound holds:*

$$RM_{NH} \leq RM + e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}].$$

Proof. See the Appendix for further details. □

If the right-hand side of the equivalence in proposition 2 is not satisfied, it is theoretically conceivable that a hedge, despite generating SCR capital savings (i.e. (20)), does not translate into a reduced total premium. In such a situation, the investor thus benefits from massive capital relief, whereas the policyholder is confronted with a higher premium, since the hedge is exclusively financed by their contribution. This is illustrated in the following example.

Example 3. We consider a single risky asset evolving in a three-state setting $\omega_1^f, \omega_2^f, \omega_3^f$ with real world probabilities of 0.004, 0.596 and 0.4; leading to an incomplete financial market. The risky asset, with an initial value of $s^{(1)} = 10.2$, has three possible terminal values, $S^{(1)}(\omega_1^f) = 25$, $S^{(1)}(\omega_2^f) = 15$ and $S^{(1)}(\omega_3^f) = 6$. $\mathcal{M}^a$ is similar to Example 2 but with $\zeta = -0.5$, and the survival probability is now equal to 0.4. $\mathcal{M}^f$ includes two distinct martingale measures, $\mathbb{Q}_1^f$ and $\mathbb{Q}_2^f$, different from $\mathbb{Q}_{MV}^f$. Table 1 lists the set of probabilities $\mathcal{M}^f$.

States	$\mathbb{Q}_{MV}^f$	$\mathbb{Q}_1^f$	$\mathbb{Q}_2^f$
ω_1^f	0.0006582399	0.0005	0.05
ω_2^f	0.465277	$\frac{4.1905}{9}$	$\frac{3.25}{9}$
ω_3^f	0.5340647601	$\frac{4.805}{9}$	$\frac{5.3}{9}$

Table 1: The set $\mathcal{M}^f$.

The hybrid life payoff is a survival call option, $Y = (S^{(1)} - 7)^+ \mathbf{1}_{\tau > 1}$. Using the 99.5% Value at Risk as the risk measure operator and mean-variance hedging as the hedger ($C = 8$, $C_{NH} = 4.79308$), the subsequent results are obtained

$$\begin{cases} SCR = 6.046, \\ SCR_{NH} = 4.331, \\ RM = 0.018, \\ RM_{NH} = 0.46208, \\ \boldsymbol{\theta}_Y \cdot \boldsymbol{s} = 1.49332, \\ \mu = \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}] = 1.936. \end{cases}$$

Initially, partial hedging management is profitable since $C_{NH} + \boldsymbol{\theta}_Y \cdot \boldsymbol{s} < C$. Secondly, we can then observe that SCR decreased after hedging. Moreover, since $(\boldsymbol{\theta}_Y \cdot \boldsymbol{s} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}]) \geq (RM - RM_{NH})$, the total premium increased, as confirmed in the following:

$$\boldsymbol{\theta}_Y \cdot \boldsymbol{s} + RM_{NH} \geq \mu + RM.$$

This showed that hedging drastically lowered what the shareholder had to contribute but did not reduce the policyholder's contribution. Although this premium increase remains structurally small, in a competitive market, such an imbalance could still prove detrimental to the insurer's long-term business. This asymmetric distribution of the hedging benefits arises from our proposed asset–liability management framework, in which the hedging portfolio is explicitly and exclusively financed by the policyholder. □

Finally, we analyze the relative ordering of cost-of-capital rates between hedged and unhedged balance sheets. Unfortunately, a reduction in the total premium does not guarantee a definitive ordering between RM and RM_{NH}. Consequently, an effective P-SCR reducing hedge, does not produce a conclusive ordering between R^{cc}_{NH} and R^{cc}. Even when premium reduction establishes an order between RM and RM_{NH} (when $\boldsymbol{\theta}_Y \cdot \boldsymbol{s} \geq e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}]$), the resulting configuration

$$\begin{cases} SCR_{NH} < SCR, \\ RM_{NH} < RM, \end{cases} \tag{41}$$

yields no unambiguous relationship between the two rates.
Accordingly, we rely on the permanent ordering already identified for P and P_{NH} (see proposition (3)), in the context of SCR savings. More precisely, we examine a premise that links the interaction between $(P - P_{NH})$ and the reduction of the capital requirement of the solvency $(SCR - SCR_{NH})$. Thus, we present the following proposition

Proposition 4.

Let $\Delta SCR \triangleq SCR - SCR_{NH}$. Assume that the hedging is effective in reducing the solvency capital requirement, i.e., condition (20) holds. Then,

$$R^{cc}_{NH} \geq R^{cc} \Leftrightarrow e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}] \geq \Delta P - \Delta SCR \times R^{cc}. \tag{42}$$

Proof. See the Appendix. □

This proposition show that the relative ordering of the two cost-of-capital rates R^{cc}_{NH} and R^{cc} hinges on which component the hedging reduces more: the premium P, or the solvency capital requirement SCR. Indeed, if the SCR contracts more than the premium, the level $\Delta P - \Delta SCR \times R^{cc}$ falls and is more readily surpassed by the real-world expectation of the hedging portfolio, thereby inducing an upward adjustment of the cost-of-capital rate. Conversely, if the premium contracts more than the solvency capital requirement, the level $\Delta P - \Delta SCR \times R^{cc}$ increases and is more readily to surpass the real-world expectation of the hedging portfolio, thereby inducing a reduction of the cost-of-capital rate.

4 Inflation-indexed participation endowment

In this section, we examine an inflation-indexed participating endowment product in an incomplete market, in order to illustrate the three types of profitabilities introduced in Definitions 2, 3, and 4.

4.1 The ambiguity set $\mathcal{M}^{a,f}$

We confine our study, for the purposes of this application, to an incomplete financial market featuring a single risky traded asset $(S_t)_{0\leq t\leq 1}$. A standard approach to describe an incomplete financial market involves adopting a geometric dynamics for $(S_t)_{0\leq t\leq 1}$ with stochastic volatility (for instance, Heston model). We deliberately rejected this alternative, as its valuation relies on Fourier transforms and lengthy derivations that divert from the main objective of our study. Another option is to introduce ambiguity into the single constant volatility parameter of the usual Black–Scholes model; while this simplifies the computation, it does not yield a unique real-world probability measure. We model an incomplete financial market by augmenting the traded risky asset $(S_t)_{0\leq t\leq 1}$ with a non-traded macroeconomic indicator $(I_t)_{0\leq t\leq 1}$, representing the Consumer Price Index (CPI). The latter is not assumed to be a martingale under the equivalent risk-neutral measures and is imperfectly correlated with $(S_t)_{0\leq t\leq 1}$. Accordingly, the financial payoff will be jointly driven by these two risk factors. Thus, it provides an incomplete financial market, a scalable computational environment, and preserves a single real-world measure; this is a favorable compromise.
The stochastic dynamics of the risky traded asset and the indexation factor under the real-world financial probability measure $\mathbb{P}^f$ are specified as follows

$$dS_t = \mu_S S_t dt + \sigma_1 S_t d\widetilde{W}_{1,t} + \sigma_2 S_t d\widetilde{W}_{2,t}, \tag{43}$$

$$dI_t = \mu_I I_t dt + \alpha I_t d\widetilde{W}_{1,t}, \tag{44}$$

where μ_S is the expected return of $(S_t)_{0\leq t\leq 1}$, and $\sigma_1, \sigma_2 > 0$ are its volatility coefficients. $(\widetilde{W}_{1,t})_{0\leq t\leq 1}$ and $(\widetilde{W}_{2,t})_{0\leq t\leq 1}$ are two independent brownian motions. This multivariate geometric brownian motion model is fairly common in the literature; see, for example, Equation (2.2) of Barrera et al. (2022), for the particular choice $d = 1$ and $L = 2$. The drift parameter μ_I represents the expected instantaneous inflation rate, while the parameter $\alpha > 0$ governs the volatility of inflation shocks. The imperfect correlation structure indicates that the financial asset $(S_t)_{0\leq t\leq 1}$ offers only an imperfect hedge against inflation risk, reflecting the incompleteness of the market.

A measure change from $\mathbb{P}^f$ to an equivalent martingale measure $\mathbb{Q}^f$ is performed via the market price of risk vector $\big(\xi_1(t), \xi_2(t)\big)$, whose components follow the constraint equation below

$$\sigma_1\xi_1(t) + \sigma_2\xi_2(t) = \mu_S - r, \tag{45}$$

and $\big(\xi_1(t), \xi_2(t)\big)$ is a square-integrable stochastic process satisfying Novikov's condition. In this paper, we restrict ourselves to a constant market price of risk

$\big(\xi_1(t), \xi_2(t)\big) = (\xi_1, \xi_2)$. Under any such measure $\mathbb{Q}^f$, the stochastic risky asset dynamics becomes as follows

$$dS_t = rS_t dt + \sigma_1 S_t dW_{1,t} + \sigma_2 S_t dW_{2,t} \tag{46}$$

$$dI_t = \big(\mu_I - \alpha\xi_1\big) I_t dt + \alpha I_t dW_{1,t} \tag{47}$$

where $dW_{i,t} \triangleq d\widetilde{W}_{i,t} + \xi_i dt$ for $i \in \{1, 2\}$.
The set $\mathcal{A}$ of all admissible market prices of risk is the set of vectors $(\xi_1, \xi_2) \in \mathbb{R}^2$ satisfying the no-arbitrage condition (45)

$$\mathcal{A} \triangleq \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \sigma_1\xi_1 + \sigma_2\xi_2 = \mu_S - r\right\}.$$

To ensure a well-posed optimization problem in the computation of the premium and the solvency capital requirement, we directly constrain $\mathcal{A}$ by bounding the squared norm of the risk price vector, which is economically equivalent to imposing an upper limit on the squared maximal sharpe ratio attainable in the market. The operational set $\mathcal{A}^*$ is thus

$$\mathcal{A}^*_{\xi_0} \triangleq \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \sigma_1\xi_1 + \sigma_2\xi_2 = \mu_S - r \text{ and } \xi_1^2 + \xi_2^2 \leq \xi_0,\ \xi_0 > \frac{(\mu_S - r)^2}{\sigma_1^2 + \sigma_2^2}\right\}.$$

Under any $\mathbb{Q}^f$, the dynamics in (46) and (47) depend only on the first component of the vector belonging to $\mathcal{A}^*_{\xi_0}$. Thus, we introduce the following

$$\mathcal{B}_{\xi_0} \triangleq \left\{\xi_1 \in [\xi_1^{\min}, \xi_1^{\max}] \mid (\xi_1, \xi_2) \in \mathcal{A}^*_{\xi_0}\ \right\},^{5}$$

and, $\mathcal{M}^f \triangleq \left\{\mathbb{Q}^f_{\xi_1} \mid \xi_1 \in \mathcal{B}_{\xi_0}\right\}$.
To model stochastic mortality rates $(\lambda_t)_{t\geq 0}$, we employ an Ornstein–Uhlenbeck intensity process (see e.g. Luciano & Vigna (2008)). Accordingly, under the probability measure $\mathbb{P}^a$, it follows that

$$d\lambda_t = \mu_m \lambda_t dt + \sigma_m dW_m(t),$$

where $\mu_m > 0$ and $\sigma_m > 0$. The probability of survival to maturity can be expressed as

$$p_x = \mathbb{E}^{\mathbb{P}^a}\big[e^{-\int_0^1 \lambda_u du}\big] = e^{-m + \frac{1}{2}s^2}, \tag{48}$$

[5] The bounds $\xi_1^{\min}$ and $\xi_1^{\max}$ are determined as the two distinct roots of the following quadratic equation $(1 + \frac{\sigma_1^2}{\sigma_2^2})\xi_1^2 - 2(\frac{\mu_S - r}{\sigma_2})(\frac{\sigma_1}{\sigma_2})\xi_1 + ((\frac{\mu_S - r}{\sigma_2})^2 - \xi_0) = 0$, whose discriminant is strictly positive under the condition $\xi_0 > \frac{(\mu_S - r)^2}{\sigma_1^2 + \sigma_2^2}$.

where

$$m \triangleq \lambda_0 \frac{e^{\mu_m}-1}{\mu_m},$$
$$s^2 \triangleq \frac{\sigma_m^2}{2\mu_m^3}(1-e^{\mu_m})^2 + \frac{\sigma_m^2}{\mu_m^2}\left(1-\frac{e^{\mu_m}-1}{\mu_m}\right).$$

The mortality risk market is, by nature, assumed to remain incomplete. Under any probability measure $\mathbb{Q}^a_\zeta$, mortality loading is applied via an Esscher transform (see Belhouari, Deelstra & Devolder (2025)) with a negative parameter $\xi \leq 0$, thereby amplifying the survival probability by the factor $e^{-\zeta \times s^2}$, to ensure a prudential pricing:

$$p_x^\zeta = p_x \times e^{-\zeta \times s^2},$$

for $\zeta_0 \leq \zeta \leq 0$. Thus, $\mathcal{M}^a = \left\{\mathbb{Q}^a_\zeta \mid \zeta \in [\zeta_0, 0]\right\}$. Finally $\mathcal{M}^{a,f} = \mathcal{M}^a \times \mathcal{M}^f$. Since ξ_1 and ζ vary over closed and bounded intervals, the parameter set is compact. Moreover, the corresponding family of probability measures depends continuously on (ξ_1, ζ) in the weak topology, so that $\mathcal{M}^{a,f}$ is compact. Finally, the payoff in Equation (49) has at most linear growth in $(S^{(1)}, I^{(1)})$, while these variables admit uniformly bounded moments over the admissible parameter set; hence the associated family of payoffs is uniformly integrable. It follows that the relevant expectation functional is continuous, and therefore the assumptions of Section 2 are satisfied.

4.2 The hybrid life payoff specification

The focus of our application is a hybrid life insurance product, whose payoff at maturity is given by

$$Y_D = D \times Y \triangleq D \times 1_{\tau>1} \times \left[1 + \beta\left(S^{(1)} - I^{(1)}\right)^+\right], \tag{49}$$

with $I^{(1)} \triangleq I_1 = i^{(1)} e^{(\mu_I - \frac{1}{2}\alpha^2) + \alpha \widetilde{W}_{1,1}}$.

The payoff provides the surviving policyholder with a guaranteed minimum payment of $D > 0$, supplemented by a leveraged participation in the real performance of the risky asset. Specifically, the profit-sharing component $\beta(S^{(1)} - I^{(1)})^+$ ensures that a bonus is distributed only if the asset value $S^{(1)}$ outperforms the realized inflation index $I^{(1)}$ at maturity. In high-inflation regimes, the insurer is protected from paying large bonuses, while in low-inflation regimes, the policyholder benefits from making a large profit-sharing. Alternatively, one might examine a payoff structured as $Y_D = D \times 1_{\tau>1}\left[S^{(1)} + \beta(S^{(1)} - I^{(1)})^+\right]$, representing a pure unit-linked contract without downside protection, enhanced by a leveraged bonus whenever the asset outperforms inflation.

In both the hedged and unhedged cases, the derivation is confined to the valuation of the unit payoff (Y or Y_{NH}). Positive homogeneity of the risk measure ρ and of the mean-variance and quantile hedging operators entails that the aggregate valuation is the unit valuation multiplied by D.

4.3 Derivation of the unhedged balance sheet

The capital requirement C is ascertained by the Value at Risk and Tail Value at Risk [6] operators and quantified via Monte Carlo simulation, given there is no closed-form solution in the case of our payoff (49). To avoid a large number of formulas, the following proposition provides a closed form formula exclusively for the premium P and the best estimate μ; with RM and SCR implicitly inferred.

Proposition 5.

The best estimate μ in the unhedged case is given by

$$\mu = e^{-r} p_x \times \left[1 + \beta \left(s^{(1)} e^{\mu_S} \Phi(\tilde{d}) - i^{(1)} e^{\mu_I} \Phi(\tilde{d} - v) \right) \right].$$

The premium P in the unhedged case is given by

$$P = \sup_{\zeta \in [\zeta_0, 0[} p_x e^{-s^2 \zeta} \times \sup_{\xi_1 \in \mathcal{B}_{\xi_0}} JY_{\xi_1} = p_x e^{-s^2 \zeta_0} \times JY_{\xi_1^{\max}},$$

where a closed-form expression is obtained using Kirk's approximation, explained in Etesami (2020), to compute JY_{ξ_1}

$$JY_{\xi_1} \approx \Big[e^{-r} + \beta \Big(s^{(1)} (\Phi(d_{\xi_1}) - \Phi(d^*_{\xi_1})) + i^{(1)} e^{(\mu_I - \alpha \xi_1)} e^{-r} (\Phi(d^*_{\xi_1} - \sigma_{\xi_1}) - \Phi(d_{\xi_1} - v)) + K^+ e^{-r} \Phi(d^*_{\xi_1} - \sigma_{\xi_1}) \Big) \Big],$$

[6] Since the confidence level $\tilde{\alpha}$ typically surpasses q_x, C exceeds e^{-r} when $\mathrm{VaR}_{\tilde{\alpha}}$ is used and, consequently, when $\mathrm{TVaR}_{\tilde{\alpha}}$ is used as well.

where

$$
\begin{aligned}
\tilde{d} &= \frac{\ln(\frac{s^{(1)}}{i^{(1)}}) + (\frac{\alpha^2}{2} - \mu_I) + (\mu_S + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2) - \alpha\sigma_1}{v},\\
v &= \sqrt{(\sigma_1 - \alpha)^2 + \sigma_2^2},\\
d_{\xi_1} &= \frac{\ln(\frac{s^{(1)}}{i^{(1)}}) + (\frac{\alpha^2}{2} - \mu_I) + (r + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2) + \alpha\xi_1 - \sigma_1\alpha}{v},\\
d^*_{\xi_1} &= \frac{\ln\left(\frac{s^{(1)}}{i^{(1)}e^{-r}e^{(\mu_I - \alpha\xi_1)} + K^+ e^{-r}}\right) + \frac{1}{2}\sigma_{\xi_1}^2}{\sigma_{\xi_1}},\\
\sigma_{\xi_1}^2 &= (\sigma_1^2 + \sigma_2^2) - 2\sigma_1\alpha\left(\frac{i^{(1)}e^{(\mu_I - \alpha\xi_1)}}{i^{(1)}e^{(\mu_I - \alpha\xi_1)} + K^+}\right) + \alpha^2\left(\frac{i^{(1)}e^{(\mu_I - \alpha\xi_1)}}{i^{(1)}e^{(\mu_I - \alpha\xi_1)} + K^+}\right)^2,\\
K^+ &= \frac{Ce^r - 1}{\beta}.
\end{aligned}
$$

Proof. See the Appendix. □

Remark 2.

The Kirk approximation in Etesami (2020), used in proposition 5 to derive $\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}\left[(S^{(1)} - [I^{(1)} + \frac{Ce^r - 1}{\beta}])^+\right]$*, assumes that the discounted asset* $S^{(1)}$ *and the discounted inflation index* $I^{(1)}$ *are martingales; a premise that is not satisfied by* $I^{(1)}$ *in our setting. Consequently, the Kirk approximation employed in proposition 5 is more general than the one in Etesami (2020). Imposing the martingale condition* $\mu_I - \alpha\xi_1 = r$ *recovers their approximation as an exact subcase.*

4.4 Derivation of the hedged balance sheet

The hedged balance sheet of our application focuses on two hedging strategies: quantile hedging and mean-variance hedging. The closed-form expressions for the components $\theta_Y^{(1),MV}$ and $\theta_Y^{(0),MV}$ of the mean variance hedger, under our dynamics and for the payoff defined in (49) are as follows; a detailed proof is provided in the Appendix.

$$
\begin{aligned}
\theta_Y^{(1),MV} = p_x \times \beta \times \Bigg(&\frac{s^{(1)}e^{\mu_S}\left(e^{\sigma_1^2}e^{\sigma_2^2}\Phi(\tilde{d_{\sigma_S^2}}) - \Phi(\tilde{d})\right)}{s^{(1)}e^{\mu_S}\left(e^{\sigma_1^2}e^{\sigma_2^2} - 1\right)} + \\
&\frac{i^{(1)}e^{\mu_I}\left(\Phi(\tilde{d} - v) - e^{\alpha\sigma_1}\Phi(\tilde{d_{\sigma_S^2}} - v)\right)}{s^{(1)}e^{\mu_S}\left(e^{\sigma_1^2}e^{\sigma_2^2} - 1\right)}\Bigg),
\end{aligned}
\tag{50}
$$

and

$$\theta_Y^{(0),MV} = p_x \times e^{-r}\left[1 + \beta\left(s^{(1)}e^{\mu_S}\Phi(\tilde{d}) - i^{(1)}e^{\mu_I}\Phi(\tilde{d} - v)\right) - s^{(1)}e^{\mu_S}\theta_h^{(1)}\right], \quad (51)$$

with

$$\tilde{d}_{\sigma_S^2} = \frac{\ln(\frac{s^{(1)}}{i^{(1)}}) + (\frac{1}{2}\alpha^2 - \mu_I) + (\mu_S + \frac{3}{2}\sigma_1^2 + \frac{3}{2}\sigma_2^2) - 2\alpha\sigma_1}{v}.$$

Quantile hedging necessitates numerical methods. Calibration of the quantile hedge components is performed via numerical optimization, as set out below,

$$\boldsymbol{\theta}_Y^{qr} = \arg \min_{(v^{(0)},v^{(1)})\in\Theta^{(0)}\times\Theta^{(1)}} \sum_{i=1}^{N} U_p(Y_i - v^{(0)}S^{(0)} - v^{(1)}S_i^{(1)})$$

with N denoting the number of simulated trajectories, chosen sufficiently large. Once the hedge components are determined, the hedged best estimate, per (34), is

$$\mu_{NH} = \mu - \theta_Y^{(0)} - e^{-r}\theta_Y^{(1)}s^{(1)}e^{\mu_S}.$$

The capital requirement C_{NH} is determined with the aid of Monte Carlo simulation by the Value at Risk and Tail Value at Risk operators. We now provide a semi-closed-form expression for the premium, from which RM_{NH} and SCR_{NH} are once again inferred indirectly. In fact, under partial hedging, a closed-form expression for the premium P_{NH} becomes intractable:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\min(Y_{NH}e^{-r}, C_{NH})] = p_x e^{-s^2\zeta}\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min\left((h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}, C_{NH}\right)] - \\ (1 - p_x e^{-s^2\zeta}) \times \boldsymbol{\theta}_Y \cdot \boldsymbol{s}.\end{aligned}$$

In stark contrast to the unhedged setting, the presence of the variable $\theta_Y^{(1)}S^{(1)}$ obstructs any simplification of the term $\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min((h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}, C_{NH})]$, making the Kirk approximation unfeasible. By rewriting

$$\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min\left((h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}, C_{NH}\right)] = e^{-r}\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min\left(h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S}, C_{NH}S^{(0)}\right)],$$

and for a fixed value of $\xi_1 \in \mathcal{B}$, the problem reduces to the following Feynman–Kac partial differential equation (PDE)

$$\begin{aligned}\partial_t V_t + rS_t\,\partial_S V_t + \left(\mu_I - \alpha\xi_1\right)I_t\,\partial_I V_t + \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2)S_t^2\,\partial_S^2 V_t + \tfrac{\alpha^2}{2}I_t^2\,\partial_I^2 V_t \\ + \sigma_1\alpha S_t I_t\,\partial_{IS}^2 V_t = rV_t,\end{aligned} \quad (52)$$

where $V_0 = e^{-r}\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min\left(h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S}, C_{NH}S^{(0)}\right)]$. The PDE is solved numerically using the finite difference method, subject to the terminal condition

$$V_1 = \min\left(1 + \beta(S^{(1)} - I^{(1)})^+ - \boldsymbol{\theta}_Y \cdot \boldsymbol{S}, C_{NH}S^{(0)}\right).$$

As the finite-difference resolution of the usual Feynman–Kac PDE is well known, we omit its implementation details.

Proposition 6.

The premium P_{NH} in the hedged case is given by

$$P_{NH} = p_x e^{-s^2\zeta_0} V_0^{\xi_1^{\max}} - (1 - p_x e^{-s^2\zeta_0}) \times \boldsymbol{\theta}_Y \cdot \boldsymbol{s}.$$

Proof. See the Appendix for the monotonicity details leading to P_{NH}. □

4.5 Numerical results

We present the contract's specifications together with the parameters describing the underlying dynamics. We consider a 65-year-old female contracting an inflation-indexed participation endowment product. The mortality parameters and the value of μ_S follow those in Deelstra et al. (2020). Volatilities σ_1 and σ_2 are fixed uniformly so that $\sigma_S = \sqrt{\sigma_1^2 + \sigma_2^2} \approx 0.15$. The full set of parameters is summarized in the following table.

Parameters	Numerical Values
α	5%
μ_I	2%
D	100000
$i^{(1)}$	1
$s^{(1)}$	1
r	3%
σ_1	12%
σ_2	10%
μ_S	6%
β	80%
λ_0	0.015030
μ_m	0.113826
σ_m	0.002990

$\mathrm{corr}^{\mathbb{Q}^f}(ln(S^{(1)}), ln(I^{(1)})) = \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}} = 0.768$, hence $S^{(1)}$ and $I^{(1)}$ are positively correlated; accordingly, Kirk's approximation exhibits enhanced accuracy; see Lo

(2014), who generalized the approximation to multi-asset spread options using the Lie-Trotter operator.

The parameter ξ_0 bounds the squared maximal sharpe ratio and represents the insurer's degree of conservatism in the face of market incompleteness. Accordingly, our analysis considers three distinct calibrations

Cases	ξ_0
Low	$\xi_0 \in [0.3, 0.8[$
Medium	$\xi_0 \in [0.8, 1.3[$
High	$\xi_0 \in [1.3, 1.8]$

For the mortality parameter ζ_0, we adopt a conservative range, namely $\zeta_0 \in \left[2 \times \frac{-0.16}{s}, \frac{-0.16}{s}\right]$, following the Esscher-parameter calibration reported in Belhouari, Deelstra & Devolder (2025), i.e $\frac{-0.16}{s}$. The capital C, assessed using Value at Risk and Tail Value at Risk, equals $C^{\mathrm{VaR}_{0.995}} = 132029.5$ and $C^{\mathrm{TVaR}_{0.995}} = 137311.6$, respectively. The capital C_{NH} covering the non-hedgeable component and the hedge price are summarized in the table below for each hedging strategy [7].

	Mean-variance hedging	Quantile hedging
$C_{NH}^{\mathrm{VaR}_{0.995}}$	12183.9	4385.922
$C_{NH}^{\mathrm{TVaR}_{0.995}}$	13938.34	6007.132
$\boldsymbol{\theta}_Y \cdot \boldsymbol{s}$	99908.02	107582

First, observe that

$$\boldsymbol{\theta}_Y \cdot \boldsymbol{s} + C_{NH} < C,$$

which confirmed that the full hedging operation reduced total capital under both mean-variance and quantile hedging strategies. Secondly, we identify that quantile hedging is costlier than mean-variance hedging, but provides a better reduction in the capital of the non-hedgeable component C_{NH}. We now examine the interplay between the reductions in the policyholder's and the investor's contributions.

Figure 1, Figure 2, and Figure 3 show that, as the insurer's aversion to financial-market ambiguity ξ_0 increases, capital relief and premium reduction do not share the same monotonicity. The decline in ΔSCR indicates that, with greater market uncertainty, hedging becomes less effective at mitigating the solvency capital requirement. Conversely, the rise in ΔP suggests that hedging gains importance for pricing. Moreover, Figure 1 highlights that quantile hedging provides greater capital relief than mean-variance hedging, as it explicitly targets the tail of the loss distribution containing extreme risk values, unlike the symmetric error minimization inherent in the mean-variance approach. We observe in Figure 2 and Figure 3

[7] The quantile hedging confidence level is set to $p = 0.95$.

that a reduction in the total premium ($\Delta P > \boldsymbol{\theta}_Y \cdot \boldsymbol{s}$), both under quantile hedging and mean-variance hedging, is only achieved once the insurer's ambiguity aversion exceeds a certain threshold ($\xi_0 \approx 1.2$ for MVH and $\xi_0 \approx 1.3$ for QH). Intuitively, when the insurer's ambiguity aversion is low, the valuation of the unhedged risk remains relatively optimistic, making the objective market cost of the hedge an expensive addition that slightly increases the total premium. Conversely, under high ambiguity aversion, retaining the risk incurs a severe pricing penalty, making the market-priced hedge a cost-effective alternative that successfully reduces the overall premium. However, because the hedging portfolio is exclusively financed by the policyholder's premium, any variation in the premium: i.e. $\Delta P - \boldsymbol{\theta}_Y \cdot \boldsymbol{s}$, whether a slight increase or a reduction, remains marginal compared to the massive capital relief experienced by the shareholder. Hence, an insurer should avoid hedging if it does not view the risk as highly ambiguous, in order to prevent paying an unnecessarily high price for market protection, as even a small premium increase (case $\Delta P < \boldsymbol{\theta}_Y \cdot \boldsymbol{s}$) can be penalizing in a competitive market. Figure 4 shows that, for $\xi_0 > 1.3$, the post-hedging risk margin consistently exceeds the corresponding pre-hedging risk margin under both the mean–variance and quantile hedging strategies. Nevertheless, the total premium is reduced (Figure 2 and Figure 3) despite this increase. Numerical results then indicate that adopting a prudence level $\xi_0 > 1.3$ produces a win-win outcome for both strategies of hedging: the total capital requirement declines, and both the policyholder's and the shareholder's contributions are correspondingly reduced. This win-win configuration is inherently asymmetric: the shareholder's capital reduction is massive, while the policyholder's premium reduction is only marginal, reflecting the premise that the hedge is exclusively financed by the policyholder. Given that the insurer's prudence regarding actuarial market incompleteness increases for more negative values of ζ_0, a similar decreasing trend is observed for ΔSCR in Figure 5, where the plot is shown as a function of ζ_0. We further note that the impact of ζ_0 on ΔSCR is negligible: with a one-year maturity, the mortality loading at $\frac{-0.16}{s}$ is already near its maximum, so decreasing ζ_0 below $\frac{-0.16}{s}$ has little additional effect. The same conclusion applies to quantile hedging and to ΔP; the corresponding plots are omitted to avoid overcrowding the figures. To study the effect of the risk operator ρ on capital optimization, Figure 6 demonstrates that using TVaR yields a greater solvency capital relief (ΔSCR) than VaR. Because TVaR produced a significantly larger initial capital C, which inherently inflates the pre-hedging SCR; consequently, the subsequent hedging operation generates a much larger reduction.

To situate our findings with respect to proposition 4, Figure 7 plots $\Delta P - \Delta SCR \times R^{cc}$ against the insurer's caution parameter ξ_0, alongside the real-world expectation. Since the real-world expectation consistently exceeds this quantity, we therefore anticipate an increase in the cost-of-capital rate after partial hedging. Figure 8 confirms this result. The upward trend of the cost of capital rate with respect to ξ_0 is intuitive: as the insurer adopts a more conservative stance toward market ambiguity, it anticipates a wider range of adverse scenarios, which increases the assessed risk. This higher perceived risk requires the insurer to offer investors progressively higher remuneration for the capital they provide.

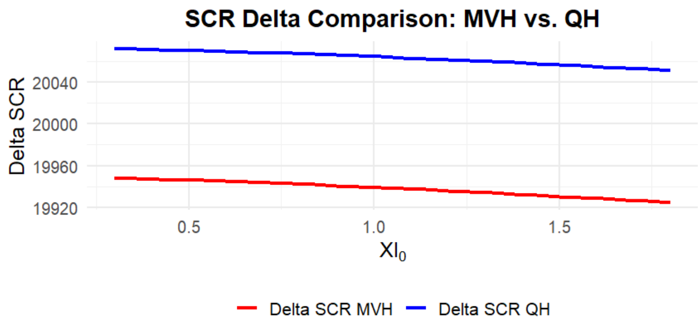

Figure 1: Impact of ξ_0 on ΔSCR: Quantile Hedging vs. Mean–Variance Hedging. ($\zeta_0 = \frac{-0.16}{s}$ and capital assessed using VaR)

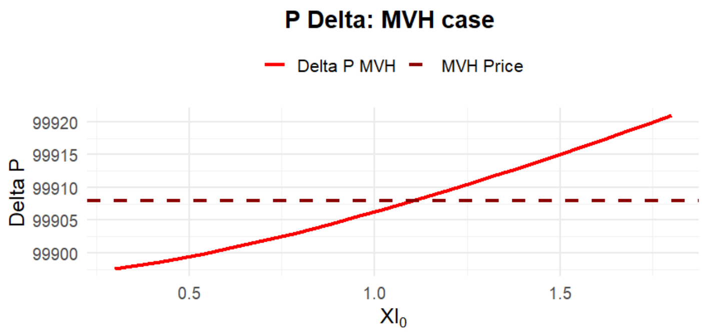

Figure 2: Impact of ξ_0 on ΔP in the Mean–Variance Hedging case. ($\zeta_0 = \frac{-0.16}{s}$ and capital assessed using VaR)

5 Conclusion

This paper extends the cost of capital valuation model presented in Albrecher et al. (2022) by adopting an asset–liability management perspective, relying on partial hedging. The asset side is represented by a hedging portfolio generated from a

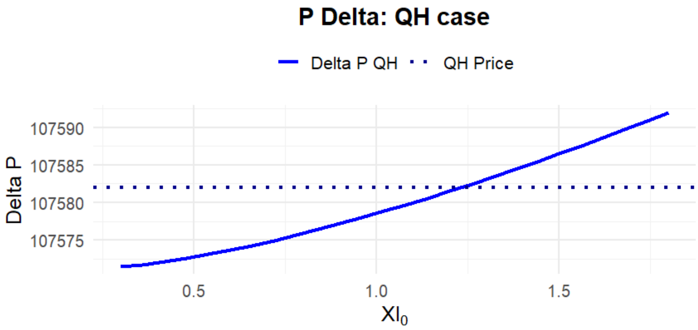

Figure 3: Impact of ξ_0 on ΔP in the Quantile Hedging case. ($\zeta_0 = \frac{-0.16}{s}$ and capital assessed using VaR)

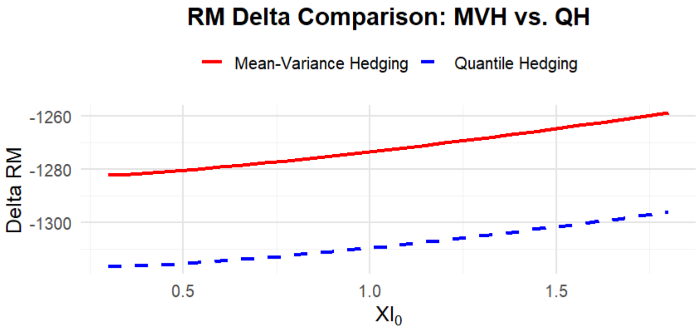

Figure 4: Impact of ξ_0 on ΔRM: Quantile Hedging vs. Mean–Variance Hedging. ($\zeta_0 = \frac{-0.16}{s}$ and capital assessed using VaR)

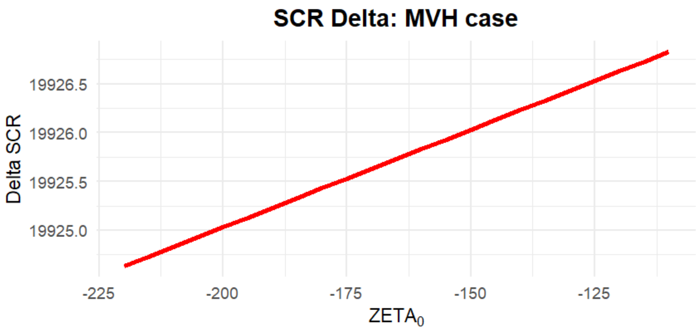

Figure 5: Impact of ζ_0 on ΔSCR in the Mean–Variance Hedging case. ($\xi_0 = 1.8$ and capital assessed using VaR)

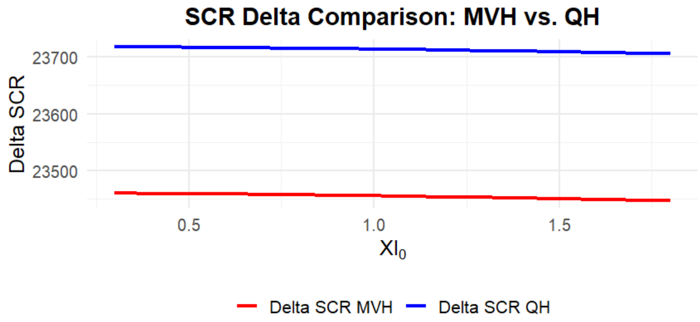

Figure 6: Impact of ξ_0 on ΔSCR: Quantile Hedging vs. Mean–Variance Hedging. ($\zeta_0 = \frac{-0.16}{s}$ and capital assessed using TVaR)

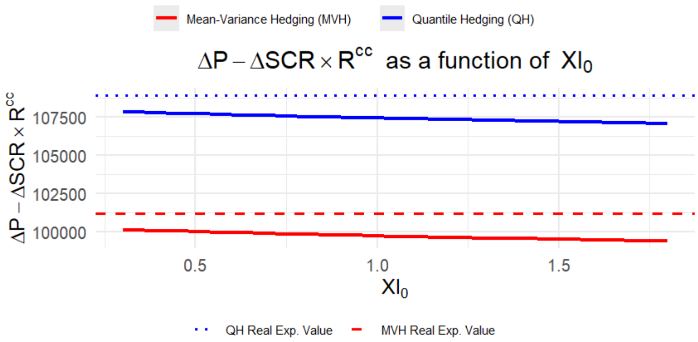

Figure 7: Impact of ξ_0 on $\Delta P - \Delta SCR \times R^{cc}$: Quantile Hedging and Mean–Variance Hedging. ($\zeta_0 = \frac{-0.16}{s}$, and capital assessed using VaR)

hedging strategy, and the liabilities comprise hybrid finance–actuarial risks; the remainder of the study focused on a multiplicative hybrid specification. Based on Dhaene et al. (2017)'s notion of the hedger, we derived post–asset–liability hedging formulas for the solvency capital requirement and the premium. We established three separate criteria for the effectiveness of hedging in reducing: the total capital requirement, the shareholder's capital, and the policyholder's contribution. We presented a lemma identifying two particular regimes: one in which hedging succeeds in being profitable for the interests of all contributors-the insurer, the shareholder, and the policyholder-and one in which, in adverse scenarios, it fails for all of them. Our exploratory results show that a hedging strategy may exist that distributes the total reduction of the capital requirement asymmetrically between the policyholder and the shareholder. Because the hedging portfolio is explicitly and exclusively financed by the policyholder's premium, the shareholder

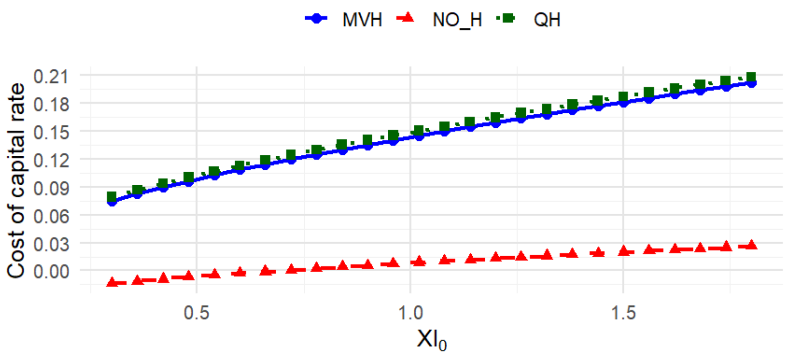

Figure 8: Impact of ξ_0 on the cost of capital rate: Without hedging vs. Quantile Hedging vs. Mean–Variance Hedging. ($\zeta_0 = \frac{-0.16}{s}$, and capital assessed using VaR)

benefits from a massive capital relief, whereas the policyholder sees only a marginal reduction, or even a slight increase in their contribution. We then investigated this interplay's impact on the relative ordering between the cost-of-capital rate with and without hedging. Situated in a suitably incomplete market and employing two hedging paradigms, namely quantile hedging and mean–variance hedging, we studied a hybrid life product denominated as inflation-indexed participation endowment. Numerical results indicated that both strategies reduced the insurer's total capital $C_{NH}+\mu_H$ relative to the unhedged baseline C, thereby evidencing the value of active asset–liability management for capital optimization. While quantile hedging incurred greater expense compared with mean–variance hedging, it nevertheless yielded a greater reduction in C_{NH} post-hedging. A sensitivity analysis was conducted to examine how the insurer's degree of ambiguity aversion influenced capital savings: ΔP and ΔSCR. Both hedging strategies consistently optimized the solvency capital requirement (SCR) across all degrees of ambiguity aversion. By contrast, a reduction in the premium could be achieved only when the insurer's ambiguity aversion was sufficiently high; hence, if the level of conservatism is not high enough, we may once again find ourselves in a situation where the shareholder contribution is massively reduced, but the policyholder's contribution has slightly increased. Consequently, insurers should acknowledge that, under standard levels of prudence, such asset–liability management acts primarily as a capital shield for the shareholder rather than a pricing optimizer for the policyholder, fundamentally because the cost of the market hedge is fully borne by the latter. To secure a win-win outcome between policyholders and shareholders, insurers should therefore deploy it primarily when adopting a sufficiently conservative stance toward market incompleteness, while recognizing that the policyholder's premium reduction will inherently remain marginal compared to the shareholder's capital relief. This highlights a structural asymmetry—derived from the financing of the hedge—between the policyholder and the shareholder, which insurers must manage in a competitive environment when applying partial hedging as part of asset–liability management. Operationally, capital optimization via hedging is not merely a risk management exercise but a strategic pricing decision: insurers must balance massive shareholder capital relief with maintaining competitive tariff structures for policyholders, as solvency efficiency does not necessarily imply commercial attractiveness, especially since even a marginal premium increase can be penalizing in a very competitive market. Moreover, we examined how the choice of risk operator could lead to further reductions in capital requirements. Finally, we compared the magnitude of the cost-of-capital rate before and after hedging in our model, where it is not a static parameter but a dynamic variable sensitive to market incompleteness and the specific hedging strategy employed.

Our study opens several avenues for future research. A natural extension would be to consider a dynamic hedging, which would allow for the re-balancing of hedging portfolios over time. Accordingly, hybrid life insurance products with maturities exceeding one year will be the subject of further study. Furthermore, while our framework assumes that the capital covering the unhedgeable component (C_{NH}) is invested in a risk-free bond, future research could examine risky allocations of this residual capital and their implications for cost-of-capital valuation.

Appendix

Proof of Proposition 1

Proof. We omit the MV notation; the proof implicitly concerns the mean-variance hedger $\boldsymbol{\theta}^{MV}$. Let $Y = g(\tau)h$ be in $\mathcal{Y}$. Let $M_{\boldsymbol{S}^*}$ denote the variance–covariance matrix of $(\mathrm{Cov}^{\mathbb{P}^f}[S^{(i)}, S^{(j)}])_{1\leq i,j\leq d}$, and set $I_Y = (\mathrm{Cov}^{\mathbb{P}^a\times\mathbb{P}^f}[S^{(1)}, Y], \cdots, \mathrm{Cov}^{\mathbb{P}^a\times\mathbb{P}^f}[S^{(d)}, Y])$, $\boldsymbol{\theta}^*_Y = (\theta_Y^{(1)}, \cdots, \theta_Y^{(d)})$, $\boldsymbol{\theta}^*_h = (\theta_h^{(1)}, \cdots, \theta_h^{(d)})$, and $I_h = (\mathrm{Cov}^{\mathbb{P}^f}[S^{(1)}, h], \cdots, \mathrm{Cov}^{\mathbb{P}^f}[S^{(d)}, h])$. Starting from equation (49) in Dhaene et al. (2017), $\boldsymbol{\theta}_Y$ and $\boldsymbol{\theta}_h$ are determined by solving the following system of equations

$$\begin{cases} \boldsymbol{\theta_Y}^{*'} = M_{\boldsymbol{S}^*}^{-1} \cdot I_Y^{'}, \\ \theta_Y^{(0)} = e^{-r}\Big(\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}[Y] - \sum_{j=1}^{d} \theta_Y^{(j)} \cdot \mathbb{E}^{\mathbb{P}^f}[S^{(j)}]\Big), \end{cases} \qquad \begin{cases} \boldsymbol{\theta}_h^{*'} = M_{\boldsymbol{S}^*}^{-1} \cdot I_h^{'}, \\ \theta_h^{(0)} = e^{-r}\Big(\mathbb{E}^{\mathbb{P}^f}[h] - \sum_{j=1}^{d} \theta_h^{(j)} \cdot \mathbb{E}^{\mathbb{P}^f}[S^{(j)}]\Big). \end{cases} \tag{53}$$

Then to seal the proof, it suffices to verify that $\boldsymbol{\theta}^*_Y = \mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times \boldsymbol{\theta}^*_h$ and $\theta_Y^{(0)} = \mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times \theta_h^{(0)}$. By the presumed independence between actuarial and financial markets and the covariance definition, one derives for $1 \leq i \leq d$

$$\begin{aligned}\mathrm{Cov}^{\mathbb{P}^a\times\mathbb{P}^f}[S^{(i)}, Y] &= \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}[S^{(i)}hg(\tau)] - \mathbb{E}^{\mathbb{P}^f}[S^{(i)}]\times\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}[hg(\tau)]\\ &= \mathbb{E}^{\mathbb{P}^a}[g(\tau)]\times(\mathbb{E}^{\mathbb{P}^f}[S^{(i)}h] - \mathbb{E}^{\mathbb{P}^f}[S^{(i)}]\times\mathbb{E}^{\mathbb{P}^f}[h])\\ &= \mathbb{E}^{\mathbb{P}^a}[g(\tau)]\times\mathrm{Cov}^{\mathbb{P}^f}[S^{(i)}, h].\end{aligned}$$

Hence, $I_Y = \mathbb{E}^{\mathbb{P}^a}[g(\tau)]\times I_h$, from which it follows that

$$\boldsymbol{\theta}_Y^{*'} = (M_{\boldsymbol{S}^*}^{-1}\cdot I_h^{'})\times\mathbb{E}^{\mathbb{P}^a}[g(\tau)] = \mathbb{E}^{\mathbb{P}^a}[g(\tau)]\times\boldsymbol{\theta}_h^{*'}.$$

Leveraging the previously established result for $\boldsymbol{\theta}_Y^*$, one obtains

$$\begin{aligned}\theta_Y^{(0)} &= e^{-r}\times(\ \mathbb{E}^{\mathbb{P}^a}[g(\tau)]\mathbb{E}^{\mathbb{P}^f}[h] - \sum_{j=1}^{d}\theta_h^{(j)}\times\mathbb{E}^{\mathbb{P}^a}[g(\tau)]\times\mathbb{E}^{\mathbb{P}^f}[S^{(j)}])\\ &= \mathbb{E}^{\mathbb{P}^a}[g(\tau)]\times e^{-r}\times(\ \mathbb{E}^{\mathbb{P}^f}[h] - \sum_{j=1}^{d}\theta_h^{(j)}\times\mathbb{E}^{\mathbb{P}^f}[S^{(j)}]).\end{aligned}$$

□

Proof of Remark 1

Proof. Initially, we must ensure that $\frac{d\mathbb{Q}^f_{MV}}{d\mathbb{P}^f}$ in (30) indeed constitutes a density, namely that $\mathbb{E}^{\mathbb{P}^f}[\frac{d\mathbb{Q}^f_{MV}}{d\mathbb{P}^f}] = 1$. We have that

$$\mathbb{E}^{\mathbb{P}^f}[\frac{d\mathbb{Q}^f_{MV}}{d\mathbb{P}^f}] = 1 + \mathbb{E}^{\mathbb{P}^f}[(\boldsymbol{s}^*S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])\cdot M_{\boldsymbol{S}^*}^{-1}\cdot(\boldsymbol{S}^* - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])^{'}].$$

We observe that the row vector $(\boldsymbol{s}^*S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])$ and the matrix $M_{\boldsymbol{S}^*}^{-1}$ are $\mathcal{F}_0-$measurable, hence they can therefore be taken outside the expectation operator

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^f}[\frac{d\mathbb{Q}^f_{MV}}{d\mathbb{P}^f}] &= 1 + (\boldsymbol{s}^*S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])\cdot M_{\boldsymbol{S}^*}^{-1}\cdot\mathbb{E}^{\mathbb{P}^f}[(\boldsymbol{S}^* - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])^{'}]\\ &= 1 + (\boldsymbol{s}^*S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])\cdot M_{\boldsymbol{S}^*}^{-1}\cdot(\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*] - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])^{'}\\ &= 1.\end{aligned}$$

We now proceed to confirm $e^{-r}\mathbb{E}^{\mathbb{Q}^f_{MV}}[h] = \boldsymbol{\theta}_h^{MV}\cdot\boldsymbol{s}$. Let h be in $\mathcal{Y}^f$. The density $\frac{d\mathbb{Q}^f_{MV}}{d\mathbb{P}^f}$ in (30) yields

$$\begin{aligned}e^{-r}\mathbb{E}^{\mathbb{Q}^f_{MV}}[h] &= e^{-r}\mathbb{E}^{\mathbb{P}^f}[\frac{d\mathbb{Q}^f_{MV}}{d\mathbb{P}^f}\times h]\\ &= e^{-r}\mathbb{E}^{\mathbb{P}^f}[h] + e^{-r}(\boldsymbol{s}^*S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*])\cdot M_{S^*}^{-1}\cdot\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*h - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*]h]^{'}.\end{aligned}$$

By $\mathcal{F}_0-$measurability of $\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*]$, it follows that

$$\begin{aligned}
e^{-r}\mathbb{E}^{\mathbb{Q}^f_{MV}}[h] &= e^{-r}\mathbb{E}^{\mathbb{P}^f}[h] + e^{-r}(\boldsymbol{s}^* S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*]) \cdot M_{\boldsymbol{S}^*}^{-1} \cdot (\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^* h] - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*]\mathbb{E}^{\mathbb{P}^f}[h])' \\
&= e^{-r}\mathbb{E}^{\mathbb{P}^f}[h] + e^{-r}(\boldsymbol{s}^* S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*]) \cdot M_{\boldsymbol{S}^*}^{-1} \cdot I_h',
\end{aligned}$$

where $I_h = (\mathrm{Cov}^{\mathbb{P}^f}[S^{(1)}, h], \cdots, \mathrm{Cov}^{\mathbb{P}^f}[S^{(d)}, h])$. Let us denote $\boldsymbol{\theta}_h^* = (\theta_h^{(1)}, \cdots, \theta_h^{(d)})$, then based on the system of equations established in the proof of proposition 1, we obtain that

$$\begin{aligned}
e^{-r}\mathbb{E}^{\mathbb{Q}^f_{MV}}[h] &= e^{-r}\mathbb{E}^{\mathbb{P}^f}[h] + e^{-r}(\boldsymbol{s}^* S^{(0)} - \mathbb{E}^{\mathbb{P}^f}[\boldsymbol{S}^*]) \cdot \boldsymbol{\theta}_h^{*,'}, \\
&= e^{-r} \times (\ \mathbb{E}^{\mathbb{P}^f}[h] - \sum_{j=1}^{d} \theta_h^{(j)} \times \mathbb{E}^{\mathbb{P}^f}[S^{(j)}]) + \sum_{j=1}^{d} s^{(j)}\theta_h^{(j)} \\
&= \theta_h^{(0)} + \sum_{j=1}^{d} s^{(j)}\theta_h^{(j)} \\
&= \boldsymbol{\theta}_h^{MV} \cdot \boldsymbol{s}.
\end{aligned}$$

Finally, assuming independence between actuarial and mortality risks, and by proposition 1, it follows that, $e^{-r}\mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f_{MV}}[Y] = \mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times \boldsymbol{\theta}_h^{MV} \cdot \boldsymbol{s} = \boldsymbol{\theta}_Y^{MV} \cdot \boldsymbol{s}$. $\square$

Proof of Lemma 1.

First case: profitability achieved.

Profitability in the sense of definition 2. Verification that the inequality $C_{NH} + \boldsymbol{\theta}_Y \cdot \boldsymbol{s} \leq C$ holds in the weak sense follows immediately from the fact that $e^{-r}\mathbb{E}^{\mathbb{Q}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}] = \boldsymbol{\theta}_Y \cdot \boldsymbol{s}$, for some $\mathbb{Q}^f \in \mathcal{M}^f$, under condition (37). To verify the strict inequality, we define the set $\mathcal{Z} = \{(\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} < C - C_{NH}\}$. By assumption (38), we have that $\mathbb{P}^f[\mathcal{Z}] > 0$. Since all financial risk-neutral measures are equivalent to the real-world measure, it follows that: $\mathbb{Q}^f(\mathcal{Z}) > 0, \quad \forall \mathbb{Q}^f \in \mathcal{M}^f$. Hence:

$$\begin{cases} (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} < C - C_{NH}, & \text{for } \omega \in \mathcal{Z}, \\ (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} = C - C_{NH}, & \text{for } \omega \in \mathcal{Z}^c. \end{cases}$$

Thus, for a $\mathbb{Q}^f \in \mathcal{M}^f$,

$$\int_{\mathcal{Z}} (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} d\mathbb{Q} + \int_{\mathcal{Z}^c} (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} d\mathbb{Q} < C - C_{NH},$$

and consequently $\boldsymbol{\theta}_Y \cdot \boldsymbol{s} < C - C_{NH}$.

Profitability in the sense of definition 3. We will prove only the strict inequality; the elements of the proof for the non-strict case are already contained in it. Let $\mathcal{H}$ denote the following subset:

$$\mathcal{H} = \mathcal{Z} \cap \{g(\tau)he^{-r} < C\}.$$

By assumption (38), we have that $(\mathbb{P}^a \times \mathbb{P}^f)[\mathcal{H}] > 0$. Since all risk-neutral measures are equivalent to the real-world measure, it follows that: $\mathbb{Q}(\mathcal{H}) > 0, \quad \forall \mathbb{Q} \in \mathcal{M}^{a,f}$. From this, and since the function $x \mapsto (x)^+$ is non-decreasing, we obtain that:

$$\begin{cases} (C_{NH} - g(\tau)he^{-r} + (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r})^+ < (C - g(\tau)he^{-r})^+, & \text{for } \omega \in \mathcal{H}, \\ (C_{NH} - g(\tau)he^{-r} + (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r})^+ = (C - g(\tau)he^{-r})^+, & \text{for } \omega \in \mathcal{H}^c. \end{cases}$$

Thus for all $\mathbb{Q} \in \mathcal{M}^{a,f}$,

$$\int_{\mathcal{H}} \left(C_{NH} - g(\tau)he^{-r} + (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}\right)^+ d\mathbb{Q} + \int_{\mathcal{H}^c} \left(C_{NH} - g(\tau)he^{-r} + (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}\right)^+ d\mathbb{Q}$$
$$< \int_{\mathcal{H}} \left(C - g(\tau)he^{-r}\right)^+ d\mathbb{Q} + \int_{\mathcal{H}^c} \left(C - g(\tau)he^{-r}\right)^+ d\mathbb{Q},$$

which is equivalent to

$$\mathbb{E}^{\mathbb{Q}}\left[\left(C_{NH} - g(\tau)he^{-r} + (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}\right)^+\right] < \mathbb{E}^{\mathbb{Q}}\left[\left(C - g(\tau)he^{-r}\right)^+\right].$$

By definition, the infimum is the greatest lower bound. Since $\mathcal{M}^{a,f}$ is a compact set and the expectation mapping is continuous, the infimum on the right-hand side is attained at some measure $\mathbb{Q}^* \in \mathcal{M}^{a,f}$. Evaluated at this specific measure, the established strict inequality yields:

$$SCR_{NH} \leq \mathbb{E}^{\mathbb{Q}^*}\left[\left(C_{NH} - g(\tau)he^{-r} + (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}\right)^+\right] < \mathbb{E}^{\mathbb{Q}^*}\left[\left(C - g(\tau)he^{-r}\right)^+\right] = SCR.$$

Thus, the strict inequality is preserved, yielding $SCR_{NH} < SCR$.

Profitability in the sense of definition 4. We will prove only the strict inequality; the elements of the proof for the non-strict case are already contained in it. Let $\mathcal{G}$ denote the following subset:

$$\mathcal{G} = \mathcal{Z} \cap \left\{(g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} > C_{NH}\right\}.$$

Since $(\mathbb{P}^a \times \mathbb{P}^f)[\mathcal{G}] > 0$ (by assumption (39)), then $\mathbb{Q}(\mathcal{G}) > 0, \quad \forall \mathbb{Q} \in \mathcal{M}^{a,f}$. Under condition (37), it can be verified that

$$\begin{cases} (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} + \min\left((g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}, C_{NH}\right) < \min(g(\tau)he^{-r}, C), & \text{for } \omega \in \mathcal{G}, \\ (\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} + \min\left((g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}, C_{NH}\right) = \min(g(\tau)he^{-r}, C), & \text{for } \omega \in \mathcal{G}^c. \end{cases}$$

Thus for all $\mathbb{Q} \in \mathcal{M}^{a,f}$,

$$\mathbb{E}^{\mathbb{Q}}\left[(\boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} + \min\Big((g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r},\, C_{NH}\Big)\right] < \mathbb{E}^{\mathbb{Q}}\left[\min\left(g(\tau)he^{-r}, C\right)\right],$$

i.e.

$$\min\Big((g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r},\, C_{NH}\Big) < \mathbb{E}^{\mathbb{Q}}\left[\min\left(g(\tau)he^{-r}, C\right)\right] - \boldsymbol{\theta}_Y \cdot \boldsymbol{s}.$$

Since $\mathcal{M}^{a,f}$ is a compact set and the expectation mapping is continuous, the supremum on the left-hand side is attained at some measure $\mathbb{Q}^{**} \in \mathcal{M}^{a,f}$. Evaluated at this specific measure, the established strict inequality implies that:

$$\begin{aligned} P_{NH} &= \mathbb{E}^{\mathbb{Q}^{**}}\left[\min\left((g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r}, C_{NH}\right)\right] \\ &< \mathbb{E}^{\mathbb{Q}^{**}}\left[\min(g(\tau)he^{-r}, C)\right] - \boldsymbol{\theta}_Y \cdot \boldsymbol{s} \leq P - \boldsymbol{\theta}_Y \cdot \boldsymbol{s}. \end{aligned}$$

Second case: profitability not achieved. By analogy, the proof can be completed by using the reversed inequality in the weak sense, namely $\geq$, and is therefore omitted.

□

Details about proposition 3.

Proof. Proof. Since $C_{NH} \leq C$ and the hedgeable component is almost surely positive, we can establish the inequality

$$\min\Big(\big(g(\tau)h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S}\big)e^{-r},\, C_{NH}\Big) \leq \min(\ g(\tau)he^{-r}, C).$$

Through the expectation operator, followed by taking the supremum (the least upper bound), with both operations preserving the order of inequalities, we establish that $P_{NH} \leq P$. Combining this result with the equality $\mu - \mu_{NH} = e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}]$, it follows that

$$RM_{NH} \leq RM + e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}].$$

□

Proof of proposition 4.

Proof. We have the following equivalence:

$$R^{cc}_{NH} \geq R^{cc} \iff RM_{NH} \times SCR \geq RM \times SCR_{NH}.$$

Cancelling the $RM \times SCR$ term and reversing the inequality accordingly, we obtain

$$(RM - RM_{NH}) \times SCR \leq (SCR - SCR_{NH}) \times RM.$$

Dividing both sides by the positive term SCR yields:

$$RM - RM_{NH} \leq (SCR - SCR_{NH}) \times \frac{RM}{SCR}.$$

We recall the identity derived from the definition of the risk margins: $RM - RM_{NH} = (P - P_{NH}) - e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}]$. Substituting this and the definition $R^{cc} = \frac{RM}{SCR}$ into the inequality gives:

$$e^{-r}\mathbb{E}^{\mathbb{P}^f}[\boldsymbol{\theta}_Y \cdot \boldsymbol{S}] \geq (P - P_{NH}) - (SCR - SCR_{NH}) \times R^{cc}.$$

The equivalence holds, since the corresponding operations can be carried out in the reverse direction. $\square$

Proof of proposition 5.

Proof. From (31), it follows that $\mu = e^{-r}p_x \times \Big(1 + \beta\mathbb{E}^{\mathbb{P}^f}[(S^{(1)} - I^{(1)})^+]\Big)$. The term $\mathbb{E}^{\mathbb{P}^f}[(S^{(1)} - I^{(1)})^+]$ is obtained through two measure changes with Radon–Nikodym densities $\frac{d\mathbb{P}^f_S}{d\mathbb{P}^f} = \frac{S^{(1)}}{\mathbb{E}^{\mathbb{P}^f}[(S^{(1)}]}$ and $\frac{d\mathbb{P}^f_I}{d\mathbb{P}^f} = \frac{I^{(1)}}{\mathbb{E}^{\mathbb{P}^f}[I^{(1)}]}$, as follows

$$\mathbb{E}^{\mathbb{P}^f}[(S^{(1)} - I^{(1)})^+] = s^{(1)}e^{\mu_S}\mathbb{P}^f_S(S^{(1)} > I^{(1)}) - i^{(1)}e^{\mu_I}\mathbb{P}^f_I(S^{(1)} > I^{(1)}).$$

The resulting relations $\mathbb{P}^f_S(S^{(1)} > I^{(1)}) = \Phi(\tilde{d})$ and $\mathbb{P}^f_I(S^{(1)} > I^{(1)}) = \Phi(\tilde{d} - v)$ yield the final formula for μ.
Since the financial and actuarial markets are independent, we have

$$\sup_{\mathbb{Q} \in \mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}}[\min(g(\tau)he^{-r}, C)] = \sup_{\zeta} p_x e^{-s^2\zeta} \times \sup_{\mathbb{Q}^f_{\xi_1} \in \mathcal{M}^f} \mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min\big(he^{-r}, C\big)].$$

The financial component is left to be determined. Since $C \geq e^{-r}$, we observe that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min\big(he^{-r}, C\big)] = e^{-r} + \beta e^{-r}\Big(&\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[(S^{(1)} - I^{(1)})^+] - \\ &\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[(S^{(1)} - (I^{(1)} + \frac{Ce^r - 1}{\beta}))^+]\Big).\end{aligned}$$

To derive the term $\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[(S^{(1)} - I^{(1)})^+]$, we mirror the computation of $\mathbb{E}^{\mathbb{P}^f}[(S^{(1)} - I^{(1)})^+]$ as done previously; the only change is replacing μ_S with r and and subtracting $\alpha\xi_1$ from μ_I.

The second term $\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[(S^{(1)} - (I^{(1)} + \frac{Ce^r-1}{\beta}))^+]$ defies closed-form evaluation because $\left(I^{(1)} + \frac{Ce^r-1}{\beta}\right)$ is not log-normally distributed and $\left(\frac{S^{(1)}}{I^{(1)}+\frac{Ce^r-1}{\beta}}\right)$ is therefore not log-normally distributed either; hence the Kirk approximation of Etesami (2020) is applied to approximate it under $\mathbb{Q}^f_S = \frac{S^{(1)}}{\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[(S^{(1)}]}$ by the following log-normal variable

$$\frac{S^{(1)}}{I^{(1)} + \frac{Ce^r-1}{\beta}} \sim N(\ln\left(\frac{s^{(1)}}{i^{(1)}e^{-r}e^{(\mu_I-\alpha\xi_1)} + \frac{C-e^{-r}}{\beta}}\right) + \frac{1}{2}\sigma^2_{\xi_1}, \sigma^2_{\xi_1}),$$

with $\sigma^2_{\xi_1} = (\sigma_1^2 + \sigma_2^2) - 2\sigma_1\alpha\left(\frac{i^{(1)}e^{(\mu_I-\alpha\xi_1)}}{i^{(1)}e^{(\mu_I-\alpha\xi_1)}+\frac{Ce^r-1}{\beta}}\right) + \alpha^2\left(\frac{i^{(1)}e^{(\mu_I-\alpha\xi_1)}}{i^{(1)}e^{(\mu_I-\alpha\xi_1)}+\frac{Ce^r-1}{\beta}}\right)^2$. The remainder of the derivation follows from computing $\mathbb{Q}^f_S(\frac{S^{(1)}}{I^{(1)}+\frac{Ce^r-1}{\beta}} > 1)$.
By combining the expressions obtained for both terms and performing the necessary simplifications the final formula is reached. The financial payoff component in (49) is increasing in ξ_1 (since under $\mathbb{Q}^f_{\xi_1}$, $I^{(1)} = i^{(1)}e^{(\mu_I-\alpha\xi_1-\frac{1}{2}\alpha^2)+\alpha W_{1,1}}$); accordingly, the monotonicity of both the minimum operator and the expectation implies that $JY_{\xi_1} = \mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min(he^{-r}, C)]$ likewise increases with ξ_1, reaching its supremum at $\xi_1^{\max}$. Regarding the mortality term $\left(p_x e^{-s^2\zeta}\right)$, it decreases in ζ and therefore achieves its supremum at ζ_0. Thus, $P = p_x e^{-s^2\zeta_0} \times JY_{\xi_1^{\max}}$.

□

Details about (50) and (51).

Proof. By adapting theorem 5 from Dhaene et al. (2017) to the case of a single risky asset, then the components of the mean-variance hedger for Y are as follows

$$\begin{cases} \theta_Y^{(1),MV} &= \mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times \frac{\mathrm{Cov}^{\mathbb{P}^f}[S^{(1)},h]}{\mathbb{V}^{\mathbb{P}^f}[S^{(1)}]}, \\ \theta_Y^{(0),MV} &= \mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times e^{-r} \times (\ \mathbb{E}^{\mathbb{P}^f}[h] - \mathbb{E}^{\mathbb{P}^f}[S^{(1)}] \times \theta_h^{(1)}\ ). \end{cases}$$

By differentiating, one obtains

$$\begin{aligned} \theta_Y^{(1),MV} &= \mathbb{E}^{\mathbb{P}^a}[g(\tau)] \times \theta_h^{(1),MV} \\ &= \frac{p_x\beta s^{(1)}e^{\mu_S}\left(\mathbb{E}^{\mathbb{P}^f}[\frac{S^{(1)}}{\mathbb{E}^{\mathbb{P}^f}[S^{(1)}]}(S^{(1)} - I^{(1)})^+] - \mathbb{E}^{\mathbb{P}^f}[(S^{(1)} - I^{(1)})^+]\right)}{s^{(1)2}e^{2\mu_S}\left(e^{(\sigma_1^2+\sigma_2^2)} - 1\right)}. \end{aligned}$$

The expectation $\mathbb{E}^{\mathbb{P}^f}[(S^{(1)} - I^{(1)})^+]$ was solved in proposition 5. For the remaining term $\mathbb{E}^{\mathbb{P}^f}[\frac{S^{(1)}}{\mathbb{E}^{\mathbb{P}^f}[S^{(1)}]}(S^{(1)} - I^{(1)})^+]$, the derivation is effected through the three Radon–Nikodym measure changes $\frac{d\mathbb{P}^f_S}{d\mathbb{P}^f} = \frac{S^{(1)}}{\mathbb{E}^{\mathbb{P}^f}[(S^{(1)}]}$, $\frac{d\mathbb{P}^f_{S2}}{d\mathbb{P}^f_S} = \frac{S^{(1)}}{\mathbb{E}^{\mathbb{P}^f_S}[(S^{(1)}]}$ and $\frac{d\mathbb{P}^f_{SI}}{d\mathbb{P}^f_S} = \frac{I^{(1)}}{\mathbb{E}^{\mathbb{P}^f_S}[I^{(1)}]}$, as follows

$$\mathbb{E}^{\mathbb{P}^f}[\frac{S^{(1)}}{\mathbb{E}^{\mathbb{P}^f}[S^{(1)}]}(S^{(1)} - I^{(1)})^+] = \mathbb{E}^{\mathbb{P}^f_S}[(S^{(1)} - I^{(1)})^+]$$
$$= s^{(1)}e^{\mu_S}e^{\sigma_1^2}e^{\sigma_2^2}\mathbb{P}^f_{S2}(S^{(1)} > I^{(1)}) -$$
$$i^{(1)}e^{\mu_I}e^{\alpha\sigma_1} \times \mathbb{P}^f_{SI}(S^{(1)} > I^{(1)}).$$

Using the following distribution under $\mathbb{P}^f_{S2}$,

$$\frac{S^{(1)}}{I^{(1)}} \sim N(\ln(\frac{s^{(1)}}{i^{(1)}}) + (\frac{1}{2}\alpha^2 - \mu_I) + (\mu_S + \frac{3}{2}\sigma_1^2 + \frac{3}{2}\sigma_2^2) - 2\alpha\sigma_1, v^2),$$

with $v = \sqrt{(\sigma_1 - \alpha)^2 + \sigma_2^2}$, the final formula of $\theta_Y^{(1),MV}$ is obtained.
Using the value obtained for $\theta_Y^{(1),MV}$, the formula for the component $\theta_Y^{(0),MV}$ can be easily inferred. $\square$

Proof of proposition 6.

Proof. $\mathbb{E}^{\mathbb{Q}}[\min(Y_{NH}e^{-r}, C_{NH})]$ is ultimately expressed by substituting $V_0^{\xi_1}$, obtained from the solution of the PDE. The final step consists of studying the monotonicity to determine the supremum. $V_0^{\xi_1}$ is increasing in ξ_1, as with $JY_{\xi_1} = \mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[\min(he^{-r}, C)]$ in the non-hedged case; subtracting $\boldsymbol{\theta}_Y \cdot \boldsymbol{S}$ has no effect. As for monotonicity in ζ, we have

$$\partial_\zeta \mathbb{E}^{\mathbb{Q}}[\min(Y_{NH}e^{-r}, C_{NH})] = \partial_\zeta \Big[p_x e^{-s^2\zeta} V_0^{\xi_1} - (1 - p_x e^{-s^2\zeta}) \times \boldsymbol{\theta}_Y \cdot \boldsymbol{s} \Big]$$
$$= -s^2 \times p_x e^{-s^2\zeta}[V_0^{\xi_1} + \boldsymbol{\theta}_Y \cdot \boldsymbol{s}].$$

Since

$$\boldsymbol{\theta}_Y \cdot \boldsymbol{s} + V_0^{\xi_1} = \begin{cases} e^{-r}\mathbb{E}^{\mathbb{Q}^f_{\xi_1}}[h] & \text{if } (h - \boldsymbol{\theta}_Y \cdot \boldsymbol{S})e^{-r} \le C_{NH} \\ C_{NH} + \boldsymbol{\theta}_Y \cdot \boldsymbol{s} & \text{else} \end{cases}$$

then $\boldsymbol{\theta}_Y \cdot \boldsymbol{s} + V_0^{\xi_1} \ge 0$ and hence

$$\partial_\zeta \mathbb{E}^{\mathbb{Q}}[\min(Y_{NH}e^{-r}, C_{NH})] \le 0,$$

yielding a decreasing pattern in ζ. Therefore:

$$\sup_{\mathbb{Q} \in \mathcal{M}^{a,f}} \mathbb{E}^{\mathbb{Q}}[\min(Y_{NH}e^{-r}, C_{NH})] = p_x e^{-s^2\zeta_0} V_0^{\xi_1^{\max}} - (1 - p_x e^{-s^2\zeta_0}) \times \boldsymbol{\theta}_Y \cdot \boldsymbol{s}.$$

$\square$

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