ON THE DISTRIBUTION OF THE INTEGRAL OF A FUNCTION WITH RESPECT TO A BROWNIAN BRIDGE

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LIDAM Discussion Paper LFIN
2025 / 01

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On the distribution of the integral of a function with respect to a Brownian Bridge

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March 19, 2025

Abstract

We derive a couple of results associated with the distribution of the integral J of a square-integrable function with respect to a Brownian bridge. These results are useful to design sampling algorithms and provide a tight upper bound for the distribution of the running supremum of J.

1 Introduction

An Itô integral $I = (I_t)_{t\geq 0}$ of a deterministic integrand is a stochastic integral of a function σ with respect to a Brownian motion W:

$$I_t := \int_0^t \sigma(u)dW_u \, . \tag{1}$$

It is well-known that this process is a continuous local martingale with zero expectation. Moreover, under suitable integrability conditions of σ, it is a genuine martingale whose variance is given by Itô isometry: $\mathbb{E}[I_t] = 0$ and $\mathbb{V}[I_t] = \int_0^t \sigma^2(u)du$; see, e.g., [6]. This type of integrals shows up in various applications related to stochastic calculus. For instance, this is the case when dealing with a geometric Brownian motion featuring a time-dependent deterministic volatility function, with the Ornstein-Uhlenbeck process or with Φ-martingales, to name but a few. More information about these processes can be found in, e.g., [8], [2] and [5], respectively.

Often, one is led to study the behavior of the stochastic integral I when conditioning with respect to the value of W at some given time $T > 0$. This is the case, for example, when exploiting conditional independence combined with the tower law as well as for variance reduction. In quantitative finance, for instance, this is particularly useful to price exotic options (see, e.g., [8, p.114]). This leads us to study the behavior of a stochastic integral I subject to $W_T = c$ or, equivalently, of a stochastic integral J of a deterministic function σ with respect to a Brownian bridge $B = (B_t)_{t\geq 0}$:

$$J_t := \int_0^t \sigma(u)dB_u \quad \text{subject to} \quad (B_0, B_T) = (0, c) \quad \text{where} \quad c \in \mathbb{R} \, , \quad T > 0 \, . \tag{2}$$

Indeed, by definition, the dynamics of the Brownian bridge B coincide with that of a Brownian motion W subject to the *terminal condition* $W_T = c$. Consequently, it holds that

$$J_t \sim I_t|_{W_T = c} \, . \tag{3}$$

Recall that B is a continuous Gaussian process with mean and covariance functions given, for any $s, t \leq T$, by (see, e.g., [6, Section 5.6.B] or [8, Section 4.7.2] for more details about the Brownian bridge)

$$\mathbb{E}[B_t] = c\frac{t}{T} \quad \text{and} \quad \mathbb{C}\text{ov}[B_s, B_t] = s \wedge t - \frac{st}{T} \, . \tag{4}$$

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These expressions can be easily extended to the case where s,t could be beyond T: Assuming a grid $\mathcal{T}^{(n)} = (t_0, t_1, \dots, t_n)$ of strictly increasing times, sampling the process B at times in $\mathcal{T}^{(n)}$ yields the vector $\mathbf{B}_0^{(n)} = (B_{t_0}, B_{t_1}, \dots, B_{t_n})$, which is multivariate normal with parameters given by the functions

$$\mu^B(t) := \mathbb{E}[B_t] = c\frac{t\wedge T}{T} \quad \text{and} \quad \Sigma^B(s,t) := \mathbb{C}\text{ov}[B_s, B_t] = s\wedge t - \frac{(s\wedge T)(t\wedge T)}{T} \ . \tag{5}$$

In what follows, we assume that σ is square-integrable up to any of the considered times and, in particular, up to T. Recalled that a function $\sigma : \mathbb{R}^+ \to \mathbb{R}$ is said *square-integrable up to t* if

$$\|\sigma\|_t < \infty \quad \text{where} \quad ||\sigma||_t^2 := \int_0^t \sigma^2(u)du \ .$$

We compute a couple of marginal and joint distributions related to J and $\mathbf{J}^{(n)}$, and illustrate the results associated with the supremum using numerical simulations.

2 Marginal and conditional distributions of J

We define the following functions for further references

$$\Sigma(t) := \int_0^t \sigma(u)du \ , \quad \Sigma(s,t) = \Sigma(t) - \Sigma(s) \ , \quad \mu_T(s,t) = \frac{\Sigma(s,t)}{T} \ , \quad \mu_T(t) = \mu_T(0,t)$$

and

$$\gamma_T^2(s,t,u) = \|\sigma\|_t^2 - \|\sigma\|_s^2 - T\mu_T^2(s,u) \ , \quad \gamma_T^2(t,u) = \gamma_T^2(0,t,u) \ .$$

The next lemma gives the expression of the distribution of J_t.

Lemma 1. *Let J be given by (2) where σ is square-integrable up to $t\vee T$. Then, for all $t,T\geq 0$,*

$$J_t \sim \mathcal{N}\left(c\mu_T(t\wedge T), \gamma_T^2(t, t\wedge T)\right) \ . \tag{6}$$

Proof: Let us first show that

$$J_t \sim \mathcal{N}\left(c\mu_T(t), \gamma_T^2(t,t)\right) \quad \text{when} \quad 0\leq t\leq T \ . \tag{7}$$

We start from the definition of the Itô integral I in (1) as the limit of an Itô sum:

$$I_t \overset{n\to\infty}{\longrightarrow} I_t^{(n)} := \sum_{i=1}^n \sigma(t_{i-1})\left(W_{t_i} - W_{t_{i-1}}\right) \quad \text{with} \quad t_i := i\delta_n \ , \quad \delta_n = t/n \ , \quad i\in\{1,2,\dots,n\} \ .$$

Now, define $Z_i := (W_{t_i} - W_{t_{i-1}})/\sqrt{\delta_n} \sim \mathcal{N}(0,1)$ and $\tilde{Z} := (W_T - W_t)/\sqrt{T-t} \sim \mathcal{N}(0,1)$, where

$$\sum_{i=1}^n \sqrt{\delta_n}Z_i + \sqrt{T-t}\tilde{Z} = (W_{t_n} - W_0) + (W_T - W_t) = W_T \ .$$

It is known from [9] that if $(Z_1,\dots,Z_{n+1})$ are $n+1$ i.i.d. standard Normal variables, then $(w_1Z_1,\dots,w_{n+1}Z_{n+1})$ subject to $\mathbf{w}\mathbf{Z}' = c$ is distributed as $(X_1,\dots,X_n, c-\sum_{i=1}^n X_i)$ where $\mathbf{X}^{(n)} = (X_1,\dots,X_n)$ is normally distributed with known mean vector $\boldsymbol{\mu}\in\mathbb{R}^n$ and covariance matrix $\boldsymbol{\Sigma}\in\mathbb{R}^{n\times n}$:

$$\mathbf{X}^{(n)}|_{W_T} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad \text{where} \quad \mu_i = c\frac{w_i^2}{||\mathbf{w}||^2} \quad \text{and} \quad \Sigma_{i,j} = \frac{w_i^2}{||\mathbf{w}||^2}\left(\delta_{ij}||\mathbf{w}||^2 - w_j^2\right) \ .$$

Exploiting this result with $(Z_1,\dots,Z_n,\tilde{Z})$, $\mathbf{w} := (\sqrt{\delta_n}\mathbf{e}_n, \sqrt{T-t})$ where $\mathbf{e}_n$ is a n-dimensional vector of ones and $c = W_T$, we have $w_{i,i\leq n}^2 = \delta_n$, $w_{n+1}^2 = T-t$ and $||\mathbf{w}||^2 = n\delta_n + (T-t)$, so that

$$\mu_i = \frac{W_T}{n\delta_n + (T-t)} = \frac{W_T}{T}\delta_n \quad \text{and} \quad \Sigma_{i,j} = \frac{\delta_n}{n}(\mathbb{1}_{\{i=j\}}\,n - \frac{s}{T}) = \mathbb{1}_{\{i=j\}}\,\delta_n - \frac{\delta_n^2}{T} \ .$$

Hence, noting Φ the cumulative distribution function of the standard normal variable,

$$\begin{aligned}
F_{I_t^{(n)}|W_T}(x) &= \mathbb{P}\left(\sum_{i=1}^{n}\sigma(t_{i-1})X_i \le x \,\middle|\, W_T\right) \\
&= \Phi\left(\frac{x-\sum_{i=1}^{n}\sigma(t_{i-1})\mu_i}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma(t_{i-1})\sigma(t_{j-1})\Sigma_{i,j}}}\right) \\
&= \Phi\left(\frac{x-\frac{W_T}{T}\sum_{i=1}^{n}\sigma(t_{i-1})\delta_n}{\sqrt{\sum_{i=1}^{n}\sigma^2(t_{i-1})\delta_n-\frac{1}{T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma(t_{i-1})\sigma(t_{j-1})\delta_n^2}}\right) .
\end{aligned}$$

Taking the limit as $n\to\infty$, the sums converge to Riemann integrals, and one finally gets

$$F_{I_t|W_T}(x)=\Phi\left(\frac{x-\frac{W_T}{T}\Sigma(t)}{\sqrt{||\sigma||_t^2-\frac{\Sigma^2(t)}{T}}}\right)=\Phi\left(\frac{x-W_T\mu_T(t)}{\sqrt{||\sigma||_t^2-T\mu_T^2(t)}}\right)=\Phi\left(\frac{x-W_T\mu_T(t)}{\gamma_T(t,t)}\right) .$$

From (3), replacing W_T by c in the above expression yields the distribution of J_t, and proves (7).

Let us now show that

$$J_t\sim\mathcal{N}\left(c\mu_T(T),\gamma_T^2(t,T)\right)\quad\text{when}\quad t\ge T\,. \tag{8}$$

We have from (7) that $J_T\sim\mathcal{N}\left(c\mu_T(T),\gamma_T^2(T,T)\right)$. Moreover, J_t-J_T is independent from B_T, hence $J_t-J_T\sim I_t-I_T|_{W_T=c}\sim I_t-I_T\sim\mathcal{N}\left(0,\int_T^t\sigma^2(u)du\right)$. Therefore, with $Z,Z^\perp$ two i.i.d. standard normal variables independent from W_T,

$$\begin{aligned}
F_{I_t|W_T}(x) &= \mathbb{P}\left(I_T|_{W_T}+(I_t-I_T)|_{W_T}\le x\right) \\
&= \mathbb{P}\left(W_T\mu_T(T)+\gamma_T(T,T)Z+\sqrt{\int_t^T\sigma^2(u)du}Z^\perp\le x\,\middle|\,W_T\right) \\
&= \mathbb{E}\left[\Phi\left(\frac{x-W_T\mu_T(T)-\sqrt{||\sigma||_t^2-||\sigma||_T^2}Z^\perp}{\gamma_T(T,T)}\right)\middle|\,W_T\right] \\
&= \Phi\left(\frac{x-\frac{W_T}{T}\Sigma(T)}{\sqrt{||\sigma||_t^2-T\mu_T^2(T)}}\right) \\
&= \Phi\left(\frac{x-W_T\mu_T(T)}{\gamma_T(t,T)}\right) .
\end{aligned}$$

where we have used that $\mathbb{E}[\Phi(\mu+\sigma Z)]=\Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$. Replacing W_T by c proves (8). □

The next lemma gives the distribution of $I_s^t:=I_t-I_s$ when conditioning with respect to the Brownian motion W at time T and possibly also at time s.

Lemma 2. *Define $J_s^t:=J_t-J_s$, and similarly for I, where σ is square-integrable up to $t\vee T$ and $0\le s\le t,T$. Then,*

$$J_s^t|_{B_s}\sim\mathcal{N}\left((c-B_s)\mu_{T-s}(s,t\wedge T),\gamma_{T-s}^2(s,t,t\wedge T)\right) . \tag{9}$$

Moreover,

$$J_s^t\sim\mathcal{N}\left(c\mu_T(s,t\wedge T),\gamma_T^2(s,t,t\wedge T)\right) . \tag{10}$$

Proof:
Observe that $J_s^t|_{B_s}\sim I_s^t|_{(W_s,W_T)=(B_s,c)}$ and let us first show that

$$I_s^t|_{W_s,W_T}\sim\mathcal{N}\left((c-B_s)\mu_{T-s}(s,t),\gamma_{T-s}^2(s,t,t)\right)\quad\text{when}\quad t\le T\,. \tag{11}$$

We have

$$I_s^t|_{W_s,W_T}=\int_0^{t-s}\sigma(u+s)dW_{u+s}|_{W_s,W_T}\sim\int_0^{t-s}\tilde{\sigma}(u)d\tilde{W}_u|_{\tilde{W}_{T-s}=W_T-W_s}$$

where $\tilde{\sigma}(u) = \sigma(u+s)$ and $\tilde{W}_u = W_{u+s}$. The last integral above is similar to J_t where $t \leftarrow t-s$, $T \leftarrow T-s$ and $\sigma \leftarrow \tilde{\sigma}$. Hence, it is normally distributed with mean $\tilde{\mu}_{T-s}(t-s) := \tilde{\Sigma}(t-s)/(T-s)$ and variance $\tilde{\gamma}^2_{T-s}(t-s) := \|\tilde{\sigma}\|^2_{t-s} - \frac{\tilde{\Sigma}^2(t-s)}{T-s}$. On the other hand,

$$\tilde{\Sigma}(t-s) = \int_0^{t-s} \tilde{\sigma}(u)du = \int_0^{t-s} \sigma(u+s)du = \int_s^t \sigma(u)du = \Sigma(s,t)$$

and

$$\tilde{\gamma}^2_{T-s}(t-s) = \int_0^{t-s} \tilde{\sigma}^2(u)du - \frac{\tilde{\Sigma}^2(t-s)}{T-s} = \int_s^t \sigma^2(u)du - \frac{\Sigma^2(s,t)}{T-s} .$$

This shows that $\tilde{\mu}_{T-s}(t-s) = \frac{\Sigma(s,t)}{T-s} = \mu_{T-s}(s,t)$ and $\tilde{\gamma}^2_{T-s}(t-s) = \gamma^2_{T-s}(s,t,t)$. Finally, for $t > T$,

$$I_s^t|_{W_s,W_T} \sim I_s^t|_{W_T-W_s} \sim I_s^T|_{W_T-W_s} + I_T^t$$

where the last two variables are independent normal variables with means $\mu_{T-s}(s,T)(W_T - W_s)$ and 0, and variances $\gamma^2_{T-s}(s,T,T) = \|\sigma^2\|_T - \|\sigma^2\|_s - (T-s)\mu^2_{T-s}(s,T)$ and $\int_T^t \sigma^2(u)du$, respectively. The sum of the variances coincides with $\|\sigma\|_t^2 - \|\sigma\|_s^2 - (T-s)\mu^2_{T-s}(s,T) = \gamma^2_{T-s}(s,t,T)$.

On the other hand, the tower law yields

$$\begin{aligned}
F_{J_s^t}(x) = \mathbb{E}[F_{J_s^t|B_s}(x)] &= \mathbb{E}\left[\Phi\left(\frac{x-(c-B_s)\mu_{T-s}(s,t\wedge T)}{\gamma_{T-s}(s,t,t\wedge T)}\right)\right] \\
&= \mathbb{E}\left[\Phi\left(\frac{x - c\mu_{T-s}(s,t\wedge T) + \mu_{T-s}(s,t\wedge T)\left(\frac{cs}{T} + \sqrt{s-\frac{s^2}{T}}Z\right)}{\gamma_{T-s}(s,t,t\wedge T)}\right)\right] \\
&= \mathbb{E}\left[\Phi\left(\frac{x - c\mu_{T-s}(s,t\wedge T)\frac{T-s}{T} - \mu_{T-s}(s,t\wedge T)\sqrt{s\frac{T-s}{T}}Z}{\gamma_{T-s}(s,t,t\wedge T)}\right)\right] \\
&= \Phi\left(\frac{x - c\mu_T(s,t\wedge T)}{\sqrt{\gamma^2_{T-s}(s,t,t\wedge T) + \mu^2_{T-s}(s,t\wedge T)s\frac{T-s}{T}}}\right) \\
&= \Phi\left(\frac{x - c\mu_T(s,t\wedge T)}{\sqrt{\|\sigma\|_t^2 - \|\sigma\|_s^2 - T\mu_T^2(s,t\wedge T)}}\right) \\
&= \Phi\left(\frac{x - c\mu_T(s,t\wedge T)}{\gamma_T(s,t,t\wedge T)}\right) .
\end{aligned}$$

□

3 Joint distribution of J

As expected from the multivariate normal property of B, the same applies to J when σ satisfies the squared-integrability condition.

Lemma 3. *Let σ be squared-integrable up to $s \vee t \vee T$. The stochastic integral J is a Gaussian process with mean and covariance functions*

$$\mu^J(t) := \mathbb{E}[J_t] = C\mu_T(t\wedge T) \quad \textit{and} \quad \Sigma^J(s,t) := \mathbb{C}ov[J_s, J_t] = \|\sigma\|^2_{s\wedge t} - \frac{1}{T}\Sigma(s\wedge T)\Sigma(t\wedge T) . \tag{12}$$

Proof. We prove that (J_s, J_t) is jointly normal with known $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. To this end, we start by considering that $s \leq t \leq T$, compute the marginal mean and variance, and then show that (J_s, J_s^t) is jointly normal. From Lemma 1, the elements of $\boldsymbol{\mu}$ and the diagonal elements of $\boldsymbol{\Sigma}$ are necessarily given by

$$\mu_1 = \mathbb{E}[J_s] = c\mu_T(s) , \quad \mu_2 = \mathbb{E}[J_t] = c\mu_T(t) , \quad \Sigma_{1,1} = \mathbb{V}[J_s] = \gamma_T^2(s,s) \quad \text{and} \quad \Sigma_{2,2} = \mathbb{V}[J_t] = \gamma_T^2(t,t) .$$

Let us now compute $\mathbb{C}\mathrm{ov}[J_s, J_s^t]$ when $s \leq t$. In this case, J_s is independent from J_s^t given B_s, and it holds from the tower law that

$$\mathbb{E}\left[J_s J_s^t\right] = \mathbb{E}\left[\mathbb{E}[J_s J_s^t | B_s]\right] = \mathbb{E}\left[\mathbb{E}[J_s|B_s]\,\mathbb{E}[J_s^t|B_s]\right] .$$

On the other hand, $J_s|B_s$ has the same distribution as J_T when replacing T by s and c by B_s by (6) in Lemma 1, and the distribution of $J_s^t|B_s$ is given by (9) in Lemma 2. Hence,

$$\mathbb{E}[J_s|B_s] = \mu_s(s)B_s = \frac{\Sigma(s)}{s}B_s\ , \quad \mathbb{E}[J_s^t|B_s] = \mu_{T-s}(s,t)(c-B_s) = \frac{\Sigma(s,t)}{T-s}(c-B_s) \tag{13}$$

such that

$$\begin{aligned}
\mathbb{E}\left[J_s J_s^t\right] = \mathbb{E}\left[\mathbb{E}[J_s|B_s]\,\mathbb{E}[J_s^t|B_s]\right] &= \frac{\Sigma(s)\Sigma(s,t)}{s(T-s)}(c\,\mathbb{E}[B_s] - \mathbb{V}[B_s^2] - \mathbb{E}^2[B_s]) \\
&= \frac{\Sigma(s)\Sigma(s,t)}{s(T-s)}(c^2\frac{s}{T} - s + \frac{s^2}{T} - \frac{c^2 s^2}{T^2}) \\
&= \frac{\Sigma(s)\Sigma(s,t)}{s(T-s)}(c^2\frac{s}{T}(1-\frac{s}{T}) - s(1-\frac{s}{T})) \\
&= \frac{c^2 - T}{T^2}\Sigma(s)\Sigma(s,t)\ .
\end{aligned}$$

Using the mean of J_s^t available from (10) in Lemma 2, one obtains

$$\mathbb{C}\mathrm{ov}\left[J_s, J_s^t\right] = \frac{c^2-T}{T^2}\Sigma(s)\Sigma(s,t) - c^2\mu_T(s)\mu_T(s,t) = \frac{-1}{T}\Sigma(s)\Sigma(s,t) \quad \text{when } s \leq t\ . \tag{14}$$

We conclude that the covariance between J_s, J_t when $s \leq t$ is given by

$$\Sigma_{1,2} = \mathbb{C}\mathrm{ov}[J_s, J_t] = \mathbb{V}[J_s] + \mathbb{C}\mathrm{ov}[J_s, J_s^t] = \gamma_T^2(s,s) - \frac{1}{T}\Sigma(s)\Sigma(s,t) = \|\sigma\|_s^2 - \frac{1}{T}\Sigma(s)\Sigma(t)\ .$$

Although intuitive, we show that (J_s, J_t) is bivariate normal. To this end, it is enough to prove that so is (J_s, J_s^t). We have

$$\begin{aligned}
\mathbb{P}(J_s \leq x, J_s^t \leq y) &= \mathbb{E}[\mathbb{P}(J_s \leq x, J_s^t \leq y | W_s)] \\
&= \mathbb{E}[\mathbb{P}(J_s \leq x|W_s)\mathbb{P}(J_s^t \leq y|W_s)] \\
&= \mathbb{E}[\mathbb{P}(J_s \leq x|W_s)\mathbb{P}(J_s^t \leq y|W_s)]
\end{aligned}$$

where, from Lemma 1,

$$\mathbb{P}(J_s \leq x|W_s) = \Phi\left(\frac{x - W_s\mu_s(s)}{\gamma_s(s,s)}\right)$$

and, from Lemma 2,

$$\mathbb{P}(J_s^t \leq y|W_s) = \Phi\left(\frac{y - (c-W_s)\mu_{T-s}(s,t)}{\gamma_{T-s}(s,t,t)}\right)\ .$$

Setting $Z = W_s/\sqrt{s} \sim \mathcal{N}(0,1)$, the joint probability becomes

$$\mathbb{P}(J_s \leq x, J_s^t \leq y) = \mathbb{E}\left[\Phi\left(\frac{x - (c\frac{s}{T} + \sqrt{s - \frac{s^2}{T}}Z)\mu_s(s)}{\gamma_s(s,s)}\right)\Phi\left(\frac{y - (c - c\frac{s}{T} - \sqrt{s - \frac{s^2}{T}}Z))\mu_{T-s}(s,t)}{\gamma_{T-s}(s,t,t)}\right)\right]$$

which can be written as

$$\mathbb{E}[\Phi(\mu + \sigma Z)\Phi(\tilde{\mu} + \tilde{\sigma}Z)] = \Phi_2\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\tilde{\mu}}{\sqrt{1+\tilde{\sigma}^2}}; \frac{\sigma\tilde{\sigma}}{\sqrt{1+\sigma^2}\sqrt{1+\tilde{\sigma}^2}}\right)\ .$$

The values $\mu, \tilde{\mu}, \sigma, \tilde{\sigma}$ can be found by identification, and coincide with the moments of J_s and J_s^t:

$$\mu = \mathbb{E}[J_s] = c\mu_T(s)\ , \quad \tilde{\mu} = \mathbb{E}[J_s^t] = c\mu_T(s,t)\ , \quad \sigma^2 = \mathbb{V}[J_s] = \gamma_T^2(s,s) \quad \text{and} \quad \tilde{\sigma}^2 = \mathbb{V}[J_s^t] = \gamma_T^2(s,t,t)$$

Having shown that (J_s, J_s^t) is bivariate normal, we conclude that so is $(J_s, J_t) = (J_s, J_s + J_s^t)$. As a check, one can verify that the mean vector $\boldsymbol{\mu}$ satisfies

$$\mu_1 = \mu = \mu^J(s) \ , \quad \mu_2 = \mathbb{E}[J_s + J_s^t] = \mu + \tilde{\mu} = c\mu_T(t) = \mu^J(t)$$

and the covariance matrix $\boldsymbol{\Sigma}$ indeed corresponds to

$$\Sigma_{1,1} = \sigma^2 = \mathbb{V}[J_s] = \Sigma^J(s,s) \ , \quad \Sigma_{2,2} = \mathbb{V}[J_s + J_s^t] = \sigma^2 + \tilde{\sigma}^2 + 2\,\mathbb{C}\mathrm{ov}[J_s, J_s^t] = \gamma_T^2(t,t) = \mathbb{V}[J_t] = \Sigma^J(t,t) \ ,$$

and

$$\Sigma_{1,2} = \mathbb{C}\mathrm{ov}[J_s, J_t] = \mathbb{V}[J_s] + \mathbb{C}\mathrm{ov}[J_s, J_s^t] = \gamma_T^2(s,s) - \frac{1}{T}\Sigma(s)\Sigma(s,t) = \|\sigma\|_s^2 - \frac{1}{T}\Sigma(s)\Sigma(t) = \Sigma^J(s,t) \ .$$

Hence, the case $s > t$ amounts to exchange the roles of s, t in $\Sigma_{1,2}$.

It remains to tackle the case where s, t can be larger than T. To this end, it suffices to adjust the expression of the moments. For the means (resp. variances), it suffices to cap the first argument of μ_T (resp. the second argument of γ_T^2) at T. For the covariance,

$$\begin{aligned}
\mathbb{C}\mathrm{ov}[J_s, J_T + J_T^t] &= \mathbb{C}\mathrm{ov}[J_s, J_T] = \|\sigma\|_s^2 - \frac{1}{T}\Sigma(s)\Sigma(T) && \text{if } s \le T \le t \\
\mathbb{C}\mathrm{ov}[J_t, J_T + J_T^s] &= \mathbb{C}\mathrm{ov}[J_t, J_T] = \|\sigma\|_t^2 - \frac{1}{T}\Sigma(t)\Sigma(T) && \text{if } t \le T \le s \\
\mathbb{C}\mathrm{ov}[J_s, J_s + J_s^t] &= \mathbb{V}[J_s] = \gamma_T^2(s,T) = \|\sigma\|_s^2 - \frac{1}{T}\Sigma(T)\Sigma(T) && \text{if } T \le s \le t \\
\mathbb{C}\mathrm{ov}[J_t, J_t + J_t^s] &= \mathbb{V}[J_t] = \gamma_T^2(t,T) = \|\sigma\|_t^2 - \frac{1}{T}\Sigma(T)\Sigma(T) && \text{if } T \le t \le s
\end{aligned}$$

where we have used that the increments of J after T are independent from J up to any time before T. All these expressions coincide with the definition of the covariance function $\Sigma^J(s,t)$. $\square$

Remark 1. *The special case where $J = B$ is recovered by setting $\sigma \equiv 1$, in which case $\Sigma(s,t) = t - s$ and $\|\sigma\|_t^2 = \Sigma(t) = t$.*

Lemma 4. *The distribution of $(J_{\underline{s}}^s, J_{\tilde{s}}^{\overline{s}})$ is jointly normal. The means and variances are given in (9) and the covariance is*

$$\mathbb{C}ov\left[J_{\underline{s}}^s, J_{\tilde{s}}^{\overline{s}}\right] = ||\sigma||_s^2 - ||\sigma||_{s\wedge\tilde{s}}^2 - \frac{1}{T}\left(\Sigma(s\wedge T)(\Sigma(\overline{s}\wedge T) - \Sigma(\tilde{s}\wedge T)) + \Sigma(\underline{s}\wedge T)(\Sigma(\tilde{s}\wedge T) - \Sigma(\overline{s}\wedge T))\right) \ .$$

Proof:
We restrict ourself to compute the covariance. Decomposing the increments of J,

$$\mathbb{C}\mathrm{ov}[J_s - J_{\underline{s}}, J_{\overline{s}} - J_{\tilde{s}}] = \mathbb{C}\mathrm{ov}[J_s, J_{\overline{s}}] + \mathbb{C}\mathrm{ov}[J_{\underline{s}}, J_{\tilde{s}}] - \mathbb{C}\mathrm{ov}[J_s, J_{\tilde{s}}] - \mathbb{C}\mathrm{ov}[J_{\underline{s}}, J_{\overline{s}}]$$

where we know from (14) that

$$\mathbb{C}\mathrm{ov}[J_s, J_t] = \|\sigma\|_{s\wedge t}^2 - \frac{1}{T}\Sigma(s\wedge T)\Sigma(t\wedge T) \ .$$

Hence,

$$\mathbb{C}\mathrm{ov}[J_s, J_{\overline{s}}] = \|\sigma\|_s^2 - \frac{1}{T}\Sigma(s\wedge T)\Sigma(\overline{s}\wedge T) \ , \quad \mathbb{C}\mathrm{ov}[J_{\underline{s}}, J_{\tilde{s}}] = \|\sigma\|_{\underline{s}}^2 - \frac{1}{T}\Sigma(s\wedge T)\Sigma(\overline{s}\wedge T)$$

$$\mathbb{C}\mathrm{ov}[J_s, J_{\tilde{s}}] = \|\sigma\|_{s\wedge\tilde{s}}^2 - \frac{1}{T}\Sigma(s\wedge T)\Sigma(\tilde{s}\wedge T) \ , \quad \mathbb{C}\mathrm{ov}[J_{\underline{s}}, J_{\overline{s}}] = \|\sigma\|_{\underline{s}}^2 - \frac{1}{T}\Sigma(\underline{s}\wedge T)\Sigma(\overline{s}\wedge T)$$

Subtracting the sum of the first two covariances with the the sum of the last two yields

$$\|\sigma\|_s^2 - \|\sigma\|_{s\wedge\tilde{s}}^2 - \frac{1}{T}\left(\Sigma(s\wedge T)(\Sigma(\overline{s}\wedge T) - \Sigma(\tilde{s}\wedge T)) + \Sigma(\underline{s}\wedge T)(\Sigma(\tilde{s}\wedge T) - \Sigma(\overline{s}\wedge T))\right) \ .$$

$\square$

4 Exact sampling algorithms

Sampling the vectors $\mathbf{B}_0^{(n)}$ and $\mathbf{J}_0^{(n)}$ of B and J on a discrete time grid $\mathcal{T}^{(n)}$ can be achieved by exploiting the corresponding distributions. Therefore, we restrict ourselves to provide a procedure to jointly sample the pair of random vectors $(\mathbf{B}^{(n)}, \mathbf{J}^{(n)})$.

The algorithm consists in first sampling B on $\mathcal{T}^{(n)}$ from its distribution, and then sampling J using the conditional distributions $J_{t_k}^{t_{k+1}}|_{B_{t_k},B_{t_{k+1}}}$. We assume that either $t < t_0$, $T \in \mathcal{T}^{(n)}$ or $T > t_n$ to ease the exposition. For any $s \leq t$ such that $t \leq T$ or $T \leq s$,

$$J_s^t|_{(B_s,B_t)} \sim I_s^t|_{(W_s,W_t)=(B_s,B_t)} \ ,$$

where the distribution of the random variable in the right-hand side can be recovered from (11) with $t = T$ and $c = B_t$. This leads to

$$J_s^t|_{(B_s,B_t)} \sim \mathcal{N}\left((B_t - B_s)\mu_{t-s}(s,t), \gamma_{t-s}^2(s,t,t)\right) \quad \forall 0 \leq s,t \quad \text{s.t.} \quad T \notin (s,t) \ .$$

Defining $\delta_k = t_{k+1} - t_k > 0$ and $\delta X_k = X_{t_{k+1}} - X_{t_k}$, the increments of J on the interval (t_k, t_{k+1}) are independent and distributed as

$$\delta J_k|_{B_{t_k},B_{t_{k+1}}} \sim \mathcal{N}\left(\mu_k \delta B_k, \gamma_k^2\right) \ , \quad k \in \{0,1,2,\dots,n-1\}$$

where

$$\mu_k := \mu_{\delta_k}(t_k,t_{k+1}) = \frac{\Sigma_k}{\delta_k} \ , \quad \gamma_k^2 := \gamma_{\delta_k}^2(t_k,t_{k+1},t_{k+1}) = \|\sigma^2\|_{t_{k+1}} - \|\sigma^2\|_{t_k} - \frac{\Sigma_k^2}{\delta_k} \quad \text{and} \quad \Sigma_k := \Sigma(t_k,t_{k+1}) \ .$$

5 Distribution of the supremum

We note $M_t^X := \sup_{s\in[0,t]} X_s$ the running supremum of a stochastic process X. We are interested in the distribution of M_t^J, where J is the integral of σ with respect to B. Although the analytical expression can be found for the distribution of the maximum of $\mathbf{J}_0^{(n)}$, this is not the case for M_t^J. Yet, we can provide an upper bound which converges to the true value as $n \to \infty$, as we now explain.

5.1 Existing results for the Brownian motion

The distribution of the running maximum of a Brownian motion is known analytically; see e.g. [7, p.630]:

$$F_{M_t^W}(x) := \mathbb{P}\left(M_t^W \leq x\right) = \mathbb{1}_{\{x\geq 0\}}\left[2\Phi\left(\frac{x}{\sqrt{t}}\right) - 1\right] \ .$$

Interestingly, every continuous local martingale M can be written as a time-changed Brownian motion where the clock is $t \to \langle M \rangle_t$, the quadratic variation of M [3]. Applying this to $M = I$, we conclude that I has the same dynamics as W time-changed with the clock $t \mapsto \|\sigma\|_t^2$. Consequently,

$$F_{M_t^I}(x) = F_{M^W_{\|\sigma\|_t^2}}(x) \ .$$

Unfortunately, this result is useless here because of the condition $W_T = c$ associated with J.

5.2 Existing results for the Brownian bridge

The joint density of of (M_t^W, W_t) can be found in [6, Section 4.3.C]. This coincides with the density of M_t^B when $T = t$ and $c = W_t$. Therefore, the distribution of M_T^B is:

$$F_{M_T^B}(x) = \mathbb{1}_{\{x>c^+\}}\left[1 - \frac{\phi\left(\frac{2x-c}{\sqrt{T}}\right)}{\phi\left(\frac{c}{\sqrt{T}}\right)}\right] \ .$$

The distribution of M_t^B is also known analytically when $t \neq T$. Noting $\Delta = T - t > 0$, we have (see [1, Theorem 2.1] and [1, Theorem 2.2]))

$$F_{M_t^B}(x) = \mathbb{1}_{\{x>0\}} \left[\Phi\left(\frac{xT - ct}{\sqrt{tT\Delta}}\right) - e^{-\frac{2x(x-c)}{T}} \Phi\left(\frac{2xt - xT - ct}{\sqrt{tT\Delta}}\right) \right] , \quad t < T , \tag{15}$$

and

$$F_{M_t^B}(x) = \mathbb{1}_{\{x>c^+\}} \left[1 - e^{-\frac{2x(x-c)}{T}} - 2\left(1 - e^{-\frac{2x(x-c)}{T}}\right) \Phi\left(\frac{c - x}{\sqrt{|\Delta|}}\right) \right] , \quad t > T . \tag{16}$$

5.3 Distribution of the maximum J_t on a set of discrete times

Recall that if $\mathbf{X}^{(n)} = (X_1, X_2, \ldots, X_n)$ is a vector of n normal variables with mean $\boldsymbol{\mu}^{(n)}$ and covariance matrix $\boldsymbol{\Sigma}^{(n)}$, its joint distribution is

$$F_n(\mathbf{x}; \boldsymbol{\mu}^{(n)}, \boldsymbol{\Sigma}^{(n)}) := \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = \Phi_n\left(\mathbf{x}^*; \mathbf{R}\right)$$

where $\Phi_n(;\mathbf{R})$ is the joint distribution of n standard normal variables with correlation matrix $\mathbf{R}$ and, for $1 \leq i, j \leq n$,

$$x_i^* := \frac{x_i - \mu_i^{(n)}}{\sigma_i} , \quad \sigma_i^2 = \Sigma_{i,i}^{(n)} \quad \text{and} \quad R_{i,j} = \frac{\Sigma_{i,j}^{(n)}}{\sigma_i \sigma_j} .$$

Applying this to $\mathbf{x} = x\mathbf{e}_n$ yields the probability of the maximum $M^{\mathbf{X}^{(n)}}$ of $\mathbf{X}^{(n)}$:

$$F_{M^{\mathbf{X}^{(n)}}}(x) = F_n\left(x\mathbf{e}_n; \boldsymbol{\mu}^{(n)}, \boldsymbol{\Sigma}^{(n)}\right) . \tag{17}$$

Corollary 1. *Let J be the integral of a deterministic function σ with respect to a Brownian bridge B satisfying $B_T = c$, $\mathbf{J}^{(n)} = (J_{t_1}, \ldots, J_{t_n})$ and $\mathbf{J}_0^{(n)} = (J_{t_0}, \mathbf{J}^{(n)})$ where $t_0 = 0$. Then,*

$$F_{M^{\mathbf{J}_0^{(n)}}}(x) = \mathbb{1}_{\{x\geq 0\}} F_{M^{\mathbf{J}^{(n)}}}(x) = F_n\left(x\mathbf{e}_n; \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$$

where, for any $i, j \in \{1, 2, \ldots, n\}$, $\mu_i = c\mu_T(t_i \wedge T)$ and $\Sigma_{i,j} = \|\sigma\|_{t_i \wedge t_j}^2 - \frac{1}{T}\Sigma(t_i \wedge T)\Sigma(t_j \wedge T)$.

The following corollary gives the corresponding result for the Brownian bridge.

Corollary 2. *Let B be a Brownian bridge satisfying $B_T = c$. Then,*

$$F_{M^{\mathbf{B}_0^{(n)}}}(x) = \mathbb{1}_{\{x\geq 0\}} F_{M^{\mathbf{B}^{(n)}}}(x) = \mathbb{1}_{\{x\geq 0\}} F_n\left(x\boldsymbol{e}_n; \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$$

where $\mu_i = \frac{c}{T}(t_i \wedge T)$ and $\Sigma_{i,j} = t_i \wedge t_j - \frac{1}{T}(t_i \wedge T)(t_j \wedge T)$ for all $1 \leq i, j \leq n$.

Observe that when there exists $i = i^\star$ such that $t_i = T$, $\mu_{i^\star}^{(n)} = c$ and $\Sigma_{i^\star,j}^{(n)} = \Sigma_{j,i^\star}^{(n)} = 0$, so that

$$F_{M^{\mathbf{B}_0^{(n)}}}(x) = \mathbb{1}_{\{x\geq c^+\}} F_{n-1}\left(x\boldsymbol{e}_{n-1}; \boldsymbol{\mu}^{\bar{i^\star}}, \boldsymbol{\Sigma}^{\bar{i^\star}}\right)$$

where $\boldsymbol{\mu}^{\bar{i^\star}}$ is the $(n-1)$-dimensional vector obtained by dropping row $i^\star$ of $\boldsymbol{\mu}$ in Corollary 2 and, similarly, $\boldsymbol{\Sigma}^{\bar{i^\star}}$ is the $(n-1)$-by-$(n-1)$ covariance matrix obtained by dropping the $i^\star$ row and column of $\boldsymbol{\Sigma}$.

5.4 Upper bound to the distribution of the running maximum M_T^J

It is easy to see that the supremum M_t^X of a continuous-time stochastic process X on $[0, t]$ is bounded below by $\max_{t\in\mathcal{T}^{(n)}} X_t$ for any grid $\mathcal{T}^{(n)}$ made of increasing times $0 \leq t_0 < t_1 < \ldots < t_n \leq t$ and $n \in \mathbb{N}$:

$$M_t^X = \sup_{s\in[0,t]} X_s \geq \max_{t\in\mathcal{T}^{(n)}} X_t = \max \mathbf{X}_0^{(n)} .$$

Assuming that $t_0 = 0$, $X_0 = x_0$ is known,

$$F_{M_t^X}(x) \leq \mathbb{1}_{\{x\geq x_0\}} F_{M^{\mathbf{X}^{(n)}}}(x)$$

with equality in the limit where $n \to \infty$ such that $\sup |t_i - t_{i-1}| \to 0$. This provides an upper bound for the distribution of the running maximum of a continuous-time stochastic process that can be infinitely tight. One can thus apply this result to J and B, setting $t_0 = B_0 = J_0 = 0$.

6 Example

We illustrate the above results for the supremum of the Brownian bridge B, of the integral J of the square-integrable function $\sigma(u) = e^{\kappa u}$ and of the Ornstein-Uhlenbeck process X driven by J. For the ease of exposition, we consider the running maxima up to $t < T$.

We can sample the Brownian bridge B on any grid $\mathcal{T}$ using the exact distribution (2). The analytical distribution of the maximum of B on $\mathcal{T}$ is given in Corollary 2, which is an upper bound for M_t^B. Note that the distribution of the running supremum of B is available from (15).

Similarly, we can sample the integral J on any grid $\mathcal{T}$ using the exact distribution provided in Lemma 3. Defining $g(x) = (e^x - 1)/x$, we get for the integrand $\sigma(u) = e^{\kappa u}$ that $\Sigma(t) = tg(\kappa t)$ and $\|\sigma\|_t^2 = tg(2\kappa t)$, so that the means and covariances are given by

$$\mu_i = c\frac{t_i g(\kappa t_i)}{T}\ , \quad \sigma_i^2 = t_i g(2\kappa t_i) - T\left(\frac{\mu_i}{c}\right)^2 \quad \text{and} \quad R_{i,j\neq i} = \frac{t_{i\wedge j} g(2\kappa t_{i\wedge j}) - t_i t_j g(\kappa t_i) g(\kappa t_j)/T}{\sigma_i \sigma_j}\ .$$

Injecting these expressions in Corollary 1 yields the distribution of the maximum of J on the grid $\mathcal{T}$, which is an upper bound for M_t^J.

Finally, the dynamics of the Ornstein-Uhlenbeck driven by the Brownian bridge B are given by

$$dX_t = \kappa(\theta - X_t)dt + \eta dB_t\ , \quad X_0 = x_0\ ,$$

whose solution, at any time $t \in \mathcal{T}$, can be written as

$$X_t = x_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \eta e^{-\kappa t} J_t\ , \quad \text{where} \quad J_t = \int_0^t e^{\kappa u} dB_u\ .$$

Observe that X_t is a linear combination of J_t, hence, the distribution of the maximum of the Ornstein-Uhlenbeck process X on $\mathcal{T}$ is known analytically from that of J. The mean and variance of X_{t_i} are given by $\mu_i = x_0 e^{-\kappa t_i} + \theta(1 - e^{-\kappa t_i}) + \eta e^{-\kappa t_i} c\frac{t_i g(\kappa t_i)}{T}$ and $\sigma_i^2 = \eta^2 e^{-2\kappa t_i}\left(t_i g(2\kappa t_i) - T\left(\frac{\mu_i}{c}\right)^2\right)$, while the correlations are the same as those of the J_{t_i}'s, i.e., are given by the matrix $\mathbf{R}$ above. The distribution of the maximum of X on $\mathcal{T}$ is given by injecting these expressions in (17).

The left panels of Figure 1 display the distribution of M_4^B, the running supremum of a Brownian bridge B on $[0, t]$ with $t = 4$, $T = 5$ and $c = 3$. The blue curve corresponds to the theoretical expression, given in (15). The black curve shows the empirical distribution of $\max \mathbf{B}_0^{(n)} = \max(0, M^{\mathbf{B}^{(n)}})$ based on $M = 5{,}000$ trajectories on the grid $\mathcal{T}^{(n)} = \{0 = t_0, t_1, \ldots, t_n\}$ where $n = 5$ (top), $n = 50$ (middle), $n = 100$ (bottom), $t_i = i\delta$ and $\delta = t/n$ generated by Monte Carlo simulations. The values of $\mathbf{B}^{(n)}$ on a same trajectory are sampled from the multivariate normal distribution with moments given in (4). The red curve shows the analytical expression of the distribution of $\max \mathbf{B}_0^{(n)}$ provided in Corollary 2. The gray curve is the same as the black curve but on a finer grid $\mathcal{T}^{(m)}$ with $m = 250$. It provides the Monte Carlo estimation of the distribution of the maximum of $\mathbf{B}^{(m)}$. For $m > n$, $F_{M^{\mathbf{B}^{(m)}}}$ is expected to be a tighter – but more costly– upper bound to $F_{M_t^B}$ compared to $F_{M^{\mathbf{B}^{(n)}}}$ and, for m sufficiently large, the former can be seen as a proxy for M_t^B. One can check that, indeed, the distribution of $M^{\mathbf{B}^{(m)}}$ (gray) is closer from M_t^B (blue) compared to $M^{\mathbf{B}^{(n)}}$ (black).

The center and right panels show similar results for M_4^J and M_4^X. We set the process parameters to $(x_0, \kappa, \theta, \eta) = (0, 0.2, 1, 0.2)$. As for B, the black curve shows the empirical distribution of $\max \mathbf{J}_0^{(n)}$ (resp. $\max \mathbf{X}_0^{(n)}$). Because we analyze both B and J in this exercise, we rely on the exact sampling scheme detailed in Section 4.[1] In this particular case,

$$\Sigma_k = (e^{\kappa t_{k+1}} - e^{\kappa t_k})/\kappa$$

The red curve corresponds to the distribution of $M^{\mathbf{J}^{(n)}}$ and $M^{\mathbf{X}^{(n)}}$. Recall that the theoretical expressions of the distributions of the running supremums M_t^J and M_t^X are not available but, as explained above for B, they can are approached by $F_{M^{\mathbf{J}^{(m)}}}$ and $F_{M^{\mathbf{X}^{(m)}}}$ for large m (gray curves correspond to $m = 250$).

[1] To double check the results, we provide in the on-line appendix Figure 2, which is the same as Figure 1 but when J is simulated using the Euler-Maruyama scheme $J_{t_i} - J_{t_{i-1}} = \mathrm{e}^{\kappa t_{i-1}}(B_{t_i} - B_{t_{i-1}})$. The results are essentially the same, although the Euler scheme is biased for finite n.

The upper bounds are evaluated using the function *pvnorm* available from the R package *mvtnorm* [4]. The fact that the red and the black curves coincides confirms that the upper bound actually corresponds to the true distribution of B and J sampled on the grid $\mathcal{T}^{(n)}$. The fact that the gray curves coincide with the blue curves on the left panels shows that the distribution of the supremum of B can be reasonably well estimated in Monte Carlo with $m = 250$, and hence can serve as a benchmark for J and X. The results confirm the validity of the upper bounds and the convergence toward the distribution of the running supremum. For the considered cases, the analytical approximations found using $n = 50$ look reasonable. The analytical estimation remains faster than Monte Carlo even in dimension $n = 100$.

7 Conclusion

We provide a couple of distributions related to the integral J of a deterministic, square-integrable function σ. We derive the analytical expression of the distribution of $M^{\mathbf{J}}$, the maximum of $\mathbf{J}$, defined as the process J sampled on a discrete grid $\mathcal{T}$. The distribution of $M^{\mathbf{J}}$ can serve as an upper bound to the distribution of M_t^J, the running supremum of J. Moreover, the distribution of $M^{\mathbf{J}}$ converges to that of M_t^J when $\min \mathcal{T} = 0$, $\max \mathcal{T} = t$ and the largest time-step in $\mathcal{T}$ tends to zero. These results are useful in various applications, for example, to fasten convergence via importance sampling.

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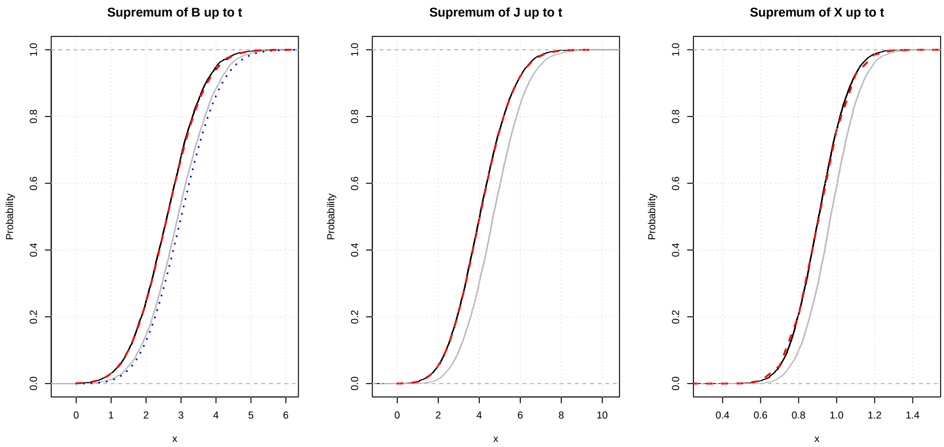
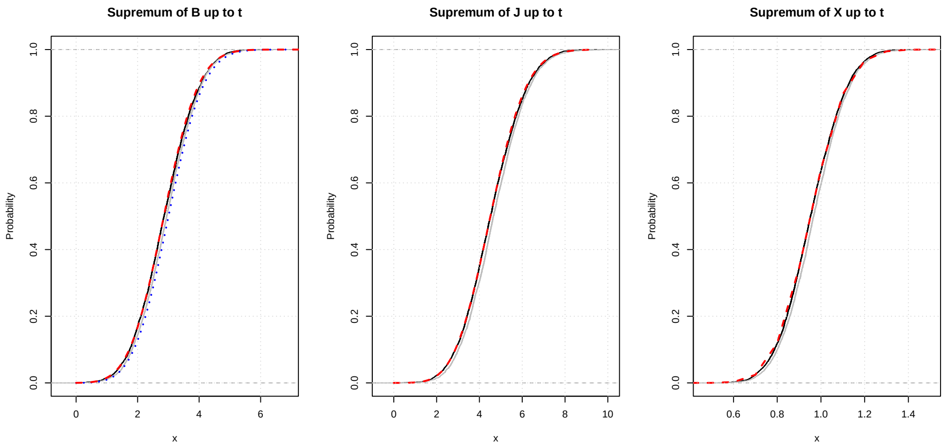
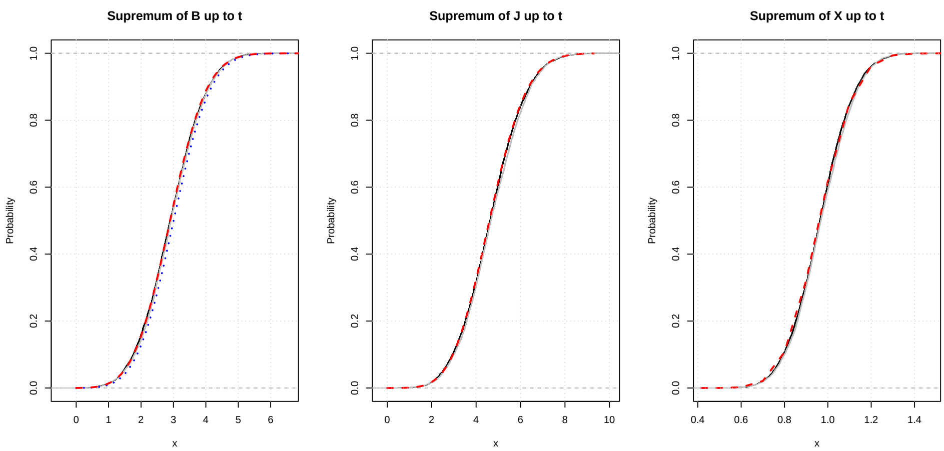


(a) $n = 5$



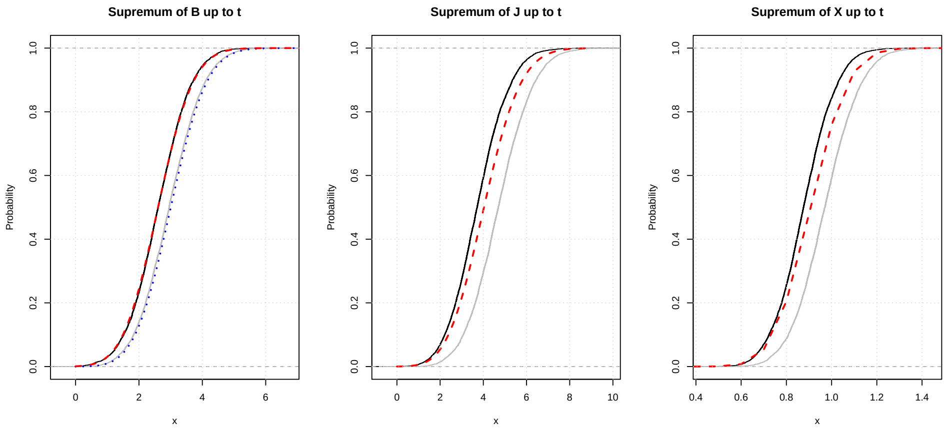
(b) $n = 50$



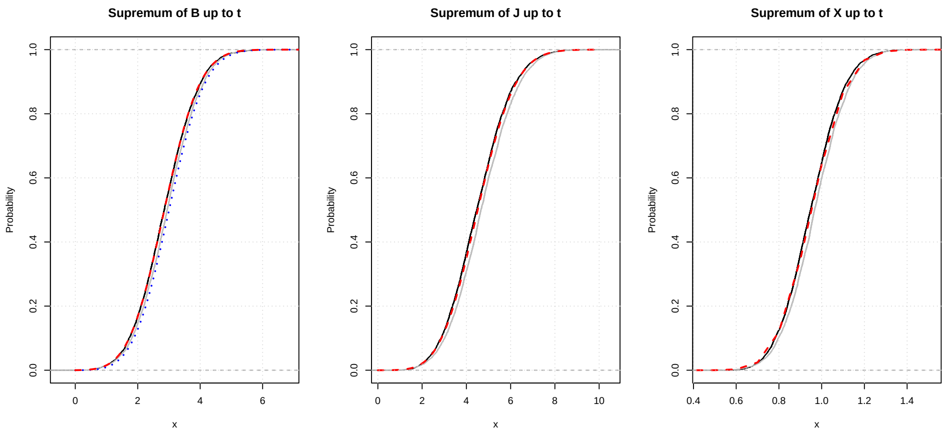
(c) $n = 100$

Figure 1: Running supremum M_t^B (left), M_t^J (center), and M_t^X (right). The black curve depicts the empirical distribution generated by Monte Carlo simulations where B is sampled from the multivariate normal distribution according to (4), and J is simulated using Section 4 with a time step of $\delta = t/n$ and $n = 5$ (top) $n = 50$ (middle) or $n = 100$ (bottom). The red curve shows the upper bound of the distribution of the supremum found using corollaries 1 and 2. The blue curve corresponds to the theoretical expression of M_t^B, given in (15).

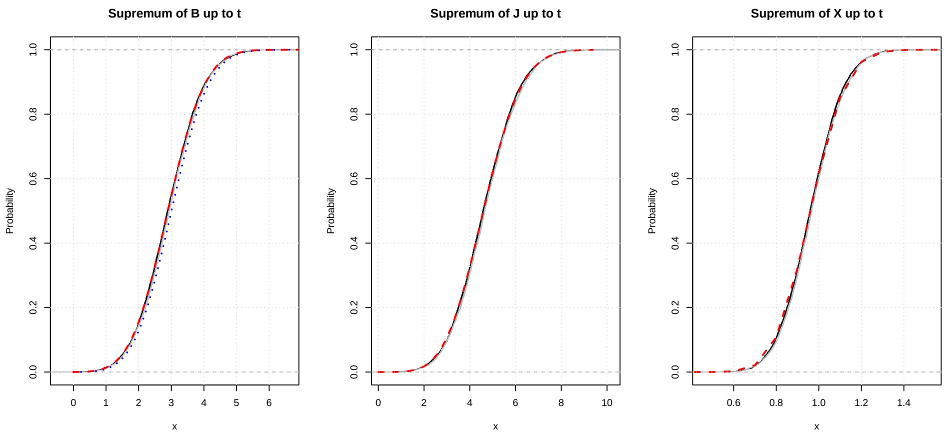
On-line appendix



(a) $n = 5$



(b) $n = 50$



(c) $n = 100$

Figure 2: Same as Figure 1 but J is simulated using the Euler scheme: $J_{t_i} - J_{t_{i-1}} = e^{\kappa t_{i-1}}(B_{t_i} - B_{t_{i-1}})$.