2012/59

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in an overlapping generation model

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DISCUSSION PAPER

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Equity and efficiency in an overlapping generation model

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December 2012

Abstract

The paper addresses *inter*generational and *intra*generational equity in an overlapping generation economy. We aim at defining an egalitarian distribution of a constant stream of resources, when preferences are ordinal and non-comparable. We establish the impossibility of efficiently distributing resources while treating equally agents with same preferences that belong to possibly different generations. We thus propose an egalitarian criterion based on the equal-split guarantee: this requires all agents to find their assigned consumption bundle at least as desirable as the equal division of resources. Finally, we show how to construct a cardinalization of the preferences that enables well-being comparisons: this allows defining the family of critical-level utilitarian orderings that top-rank the egalitarian solution.

Keywords: intergenerational equity, intragenerational equity, overlapping generation model, no-envy, equal-split guarantee, allocation rules, utilitarian welfare function.

JEL classification: D61, D63, D91

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The authors wish to thank F. Maniquet and R. Boucekkine for guidance and discussions. The comments by J. Davila, C. d'Aspremont, M. Fleurbaey, E. Ramaekers and particularly W. Thomson, T. Sprumont, and G. Valletta as well as by seminar participants in Brussels, Louvain-la-Neuve, Paris, Vielsam are gratefully acknowledged. This research is part of the ARC project 09/14-018 on "sustainability" (French speaking community of Belgium.

1 Introduction

In the last decades, the scientific community of economists, environmentalists and moral philosophers has shown increasing interest in intergenerational equity. We contribute to this debate by addressing the moral obligations that present generations bear to future ones. In particular, our goal is to shed light on *intra*- and *inter*-generational justice by studying egalitarian criteria for a dynamic overlapping generation economy.

The main economic approach to intergenerational equity describes each generation by their utility level and studies how to rank infinite vectors of these utilities (utility streams) satisfying equity and efficiency conditions. The seminal contribution of Diamond (1965) showed a strong tension between efficiency and equity: a continuous and complete ranking of these utility streams cannot satisfy both Pareto efficiency and "finite anonymity". The anonymity concept expresses equal concern for all generations by requiring the ranking to be invariant to permutations of the utilities of a finite number of generations.

The building blocks of this approach are the cardinal measurability and the comparability assumptions. These provide the necessary support for making anonymity an appealing ethical concept and define the "better-off than" relation in terms of utility differences. Unfortunately, economic theory does not provide yet a clear solution about how to construct these utilities from agent's choices and how to guarantee that the welfare measure of individuals is comparable.

An alternative approach studies well-defined resource models and allows for ordinal and non comparable preferences. The present paper contributes to this literature: it examines egalitarian solutions to the problem of distributing a stream of resources in an overlapping generation model.

We establish a Diamond type result. At each Pareto efficient allocation, the dynamic structure of the economy precludes treating equally agents who have the same preferences (a weak counterpart of anonymity for the non comparable framework). As this clash arises already for finitely many periods, alternative concepts of equity should be considered to build appealing egalitarian criteria.

In the literature on resource allocation with ordinal and non comparable preferences, two key concept of equity have attracted the most attention: the "no-envy" criterion, introduced by Foley (1967) and Kolm (1972), requires

Belgium). This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

no agent to prefer the consumption bundle assigned to any other agent; the "equal-split guarantee", proposed by Steinhaus (1948), introduces an ordinal lower bound to well-being levels.[1] We show that, although in the static framework these equity requirements are both (and together) compatible with Pareto efficiency, this is not the case for the dynamic model and several difficulties arise.

The main result of the paper is to provide a class of egalitarian criteria based on efficiency and the "equal-split guarantee". Each ordering is constructed as a discounted utilitarian aggregation of a particular utility representation of individual preference profiles; when this measure of social welfare is maximized on a properly defined subset of allocations, the axioms of equity and efficiency hold. This corresponds to a critical-level discounted utilitarian ordering where the critical-level is defined with respect to the equal-split of resources.[2]

Our model is that of an infinite horizon overlapping generations in which agents live for two periods. In each period, one unit of each good has to be shared among young and old agents living during that period; population size is kept constant. Endowing each period with equal resources is necessary to avoid that the egalitarian treatment of agents impedes to distribute all the available resources, as required by efficiency.[3] Every generation consists of a finite number of agents, each described by ordinal and non comparable preferences defined on the two dimensional space of vectors of goods.

In this framework, the tensions between "no-envy" based equity conditions and efficiency can be explained by the possibly different conditions that agents face at different periods of time. In particular, it is not possible to treat in the same way two agents with identical preferences when they have to share resources with different types of agents.

An example from the static framework will illustrate the point. Take

[1]For a recent survey, see Thomson (2011).

[2]The concept of critical-level utilitarianism was first introduced by Blackorby and Donaldson (1984) for population ethics.

[3]When periods are differently endowed in terms of resources, treating agents equally might impede to distribute all the resources available during the resource-rich period. This negative conclusion is called "leveling down objection" and opposes equity with efficiency. Parfit (1997) discusses the differences between a *pure* egalitarian view, that cares only about equality, and a *pluralist* egalitarian view, that aims at combining equity with efficiency. For an illustration consider the two following alternatives: in A everyone gets 10 units; in B half of the agents get 10, half get 20. According to the first view A is better but incurs in the leveling down; according to the second view B is better, but needs to compromise equity. Our strategy is to consider a model with the most favorable conditions in terms of compatibility between efficiency and equity, i.e. with constant flow of resources; an extension to non constant resources is introduced in Section *5*.

three couples that have to share a roasted chicken for dinner. The first couple consists of two "chest-lovers": as both prefer the chest of the chicken over the legs, the only way to equally and efficiently share the chicken is to give each agent half chest and one leg. The second couple is formed by two "leg-lovers": again, as both prefer legs over chest, equal sharing is required. In the third couple, a chest-lover and a leg-lover are to share the chicken: when all the chest is given to the chest-lover and both legs are given to the leg-lover, both agents are better-off compared to the same-preference agents of the other couples. In this example, an egalitarian treatment of people with equal preferences (across couples) is not compatible with efficiency. In the overlapping generation framework —where generations replace couples— some compensation is possible, but not sufficient to overcome this conflict. Moreover, the assumption of non transferability of resources is only partially responsible for this clash: these difficulties vanish only when goods can be freely transferred over time (transforming the model in a static resource allocation problem).

In Dubey (2010), a recent contribution related to ours, the tension between Pareto efficiency and "no-envy" is of a different nature. He considers an overlapping generation framework with a single good and one agent per generation with time-invariant preferences. For this model, the impossibility of treating agents equally derives from the exogenous quantity that the first old generation —the generation that is old in the first period— is assigned to consume when young. When this consumption level is sufficiently high, equality prevents from distributing resources efficiently: all agents would benefit from an allocation with lower consumption when young and higher consumption when old; but this change would entail giving a larger bundle to the first old generation. What the author calls "fair" are cases in which there is a sufficiently low young-period consumption of the first old generation: by the impossibility of assigning a better (feasible) bundle to all future generations without reducing the welfare of the first old generation, Pareto efficiency and "no-envy" are compatible. In a similar framework, Shinotsuka et al. (2007) study the relation between this tension and the exogenous growth rate of the population; in particular, they discuss the appeal of different concepts of "no-envy". Compared to their works, we sterilize the role of the assignment that are not under the control of the social planner by limiting the equity conditions about the first generation. We nevertheless obtain negative results due to the more general framework with agents with different preferences and many goods.

From a welfarist perspective, Quiggin (2012) shows that the overlapping generation structure has far reaching consequences for intergenerational equity. In particular, assuming that agent's preferences are additively separable with respect to the consumption during the young and old period of life, he proves that efficiency and being utilitarian within periods implies being *undiscounted* utilitarian across generations. Despite also in our framework a (critical-level) undiscounted utilitarian ordering would satisfy our equity and efficiency conditions and be unweighted utilitarian within each period (given separability), we propose the discounted version and show that this does not bring to an unfavorable treatment of future living agents.

The paper is organized as follows. In Section *2*, we present the model. In Section *3*, we show the main result: we prove the compatibility between Pareto efficiency and the "equal-split guarantee" and propose a family of social orderings based on one particular cardinalization of individual well-beings. In Section *4*, we discuss the tension between Pareto efficiency and "no-envy" based equity concepts. In Section 5, we present an extension of the framework: non constant resource endowments do not affect the existence of the egalitarian criterion and strengthen the negative conclusions about envy related axioms. We present some concluding remarks in Section *6*.

2 Framework

2.1 The model

We consider a two-period overlapping generation model. Let $t \in \mathbb{N}$ be the time index; generation t is the cohort of agents that are young at period t. Each generation t consists of a constant (finite) number I of agents.[4]

The first period is $t = 1$; this brings forth the *Adam generation* (generation 0), consisting of the I "first-old agents": the model describes their preferences and allocations only at time 1. Therefore, in each period $t \in \mathbb{N}_+$, there are I young agents of generation t and I old agents of generation $t-1$.

In each period there are L infinitely divisible and privately appropriable goods indexed by $l \in \{1, ..., L\}$; for notational simplicity, we also identify by L the set of goods.

Each member $i \in I$ of generation $t \in \mathbb{N}_+$ is allocated a **consumption bundle** $a_i(t) = (c_i(t), d_i(t))$, where **consumption when young** is the vector $c_i(t) = \left(c_i^1(t), ..., c_i^l(t), ..., c_i^L(t)\right) \in \mathbb{R}_+^L$ and **consumption when old** is the vector $d_i(t) = \left(d_i^1(t), ..., d_i^l(t), ..., d_i^L(t)\right) \in \mathbb{R}_+^L$. The agents of the Adam generation are only assigned a consumption when old, i.e.

[4]With a slight abuse of notation, we denote by I both for the set of agents and its cardinality.

$a_i(0) = d_i(0) = \left(d_i^1(0), ..., d_i^l(0), ..., d_i^L(0)\right) \in \mathbb{R}_+^L$ for each $i \in I$. Thus, $(c(t), d(t)) \in \mathbb{R}_+^{2LI}$ specifies the consumption bundles of agents of generation $t \in \mathbb{N}_+$ and $d(0) \in \mathbb{R}_+^{LI}$ specifies the consumption bundles of agents of generation 0. We denote an **allocation** by a, i.e. a list of consumption bundles for each agent $i \in I$ of each generation $t \in \mathbb{N}$.

Agent's preferences are denoted by $\succsim_{i,t}$ and are assumed to be complete, continuous, strictly convex, and strongly monotone.[5]We indicate by $\succ_{i,t}$ and $\sim_{i,t}$ the asymmetric and symmetric counterpart of $\succsim_{i,t}$.

The available resources are a constant stream that, without loss of generality, is normalized to 1 for each good and is assumed to be nontransferable across time. An allocation a is **feasible** if $\sum_{i \in I} d_i^l(t-1) + \sum_{i \in I} c_i^l(t) \leq 1$ for each good $l \in L$ and for each period $t \in \mathbb{N}_+$. Let A denote the set of feasible allocations.

An **economy** is defined by the specification of each agent's preferences: $E = \left(\{\succsim_{i,t}\}_{\forall i \in I, \forall t \in \mathbb{N}}\right)$. Let $\mathcal{E}$ denote the set of economies that satisfy the above assumptions.

2.2 Normative Objective

The paper addresses *inter*generational and *intra*generational justice implications of an egalitarian distribution of resources for the described overlapping generations model.

Our first step, borrowed from fair allocation theory, is to identify selection functions from the set of feasible allocations.[6] Formally, an allocation rule (or simply **rule**) is a correspondence $\psi : \mathcal{E} \to A$ that chooses for each specification of the economy a subset of its feasible allocations. A rule satisfies some efficiency and equity requirements (**axioms**) if the selected allocations do for each economy in the domain. We thus aim at identifying rules that satisfy appealing axioms.

The second step is to construct a (family of) social orderings of feasible allocations that top-rank the solutions identified by a rule. The advantage of a complete ranking of feasible allocations is the possibility of studying second-best choices, when constraints of any nature are reducing the subset of attainable resource distributions.[7]

[5] The convexity requirement is not necessary for our existence result in Section 3. Nevertheless, we do assume this restriction to avoid that the difficulties with alternative equity conditions (see Section 4) originate from it.

[6] For a discussion of the axiomatic approach see Thomson (2001).

[7] On orderings for well-defined static frameworks (the "social ordering function" approach), see Fleurbaey and Maniquet (2011).

We start defining our efficiency axiom. An allocation $a \in A$ is **(Pareto) efficient** if there is no other allocation $a' \in A^f$ s.t. for each agent $i \in I$ of generation $t \in \mathbb{N}$, $a'_i(t) \succsim_{i,t} a_i(t)$ and for some agent $i \in I$ of generation $t \in \mathbb{N}$, $a'_i(t) \succ_{i,t} a_i(t)$.

In the static counterpart of our model, i.e. with only one period, this axiom is compatible with appealing equity conditions.[8] A well-known example of an equitable and efficient rule is the equal-division Walrasian. The idea is the following: each agent is assigned her favorite bundle from a budget set defined by the equal split of resources and the Walrasian prices. As the budget set is the same for all agents, no agent would be better-off with the bundle assigned to someone else. Moreover, each agent prefers the bundle she is assigned to the equal division of resources, guaranteeing an ordinal lower bound for welfare. Finally *Pareto efficiency* holds by the First Welfare Theorem.

Unfortunately, the appealing results for the static framework do not hold true for our dynamic extension. A similar equal-division Walrasian rule need not satisfy the same interesting properties because Walrasian prices might not be constant over time: a good l available at time t is demanded only by young and old agents living in that period; thus the same good might well have different prices in different periods and the budget sets are not the same for agents of different generations. As we will show in Section 4, several equity axioms related to the "no-envy" family either fail to be compatible with *efficiency* or fail to be ethically appealing.

The egalitarian solution we propose for this economy is presented next.

3 Pareto efficiency and equal-split guarantee: an egalitarian solution

Before defining our equity axiom, we identify the ways to equally divide resources: let the **equal-split bundle** be a consumption bundle $\left(\bar{c}, \bar{d}\right)$ such that $\bar{c}^l + \bar{d}^l = \frac{1}{I}$ for each good $l \in L$. An allocation $\bar{a}$ is an **equal-split allocation** if for each agent $i \in I$ of generation $t \in \mathbb{N}_+$, $\bar{a}_i(t) = \left(\bar{c}, \bar{d}\right)$ and for each agent $i \in I$ of generation 0, $\bar{a}_i(0) = \bar{d}$.

Clearly, for each equal-split bundle there is a corresponding equal-split allocation. This includes as a particular case the equal-split allocation that assigns equal goods to young and old agents; for the corresponding equal-split bundle it holds that $\bar{c}^l = \bar{d}^l = \frac{1}{2I}$ for each good $l \in L$ and, thus,

[8] See Moulin (1990). These are models of the "classical problem of fair division", surveyed in Thomson Thomson (2011).

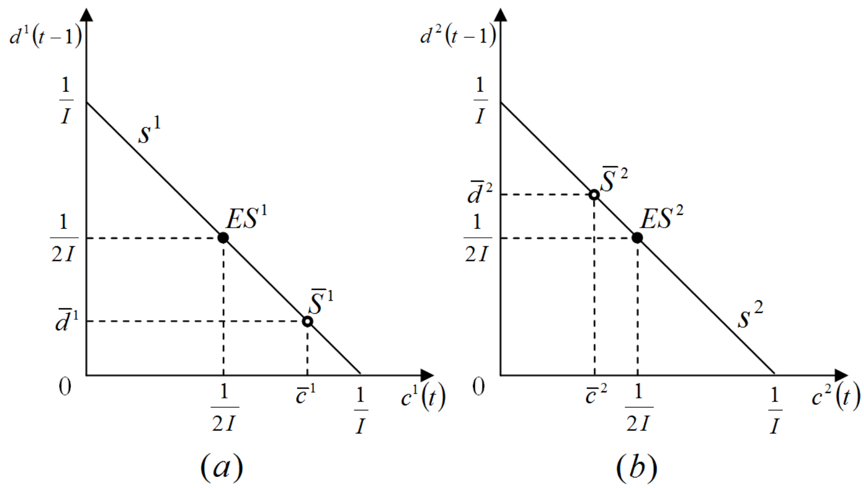


Figure 3.1: Equal-split allocations

$\bar{c} = \bar{d}$. Graphically, this case is represented by the equal-split allocation ES. Consider a generic period $t \in \mathbb{N}_+$. Fig. 1a shows the distribution of good 1 between young agents, $c^1(t)$, and old agents, $d^1(t-1)$; Fig. 1b shows the corresponding distribution for good 2.

This "age-independent" equal-split of the resources is not the unique way to divide resources equally among agents. In particular, each good can be unequally shared among the two living generations: as each agent of generation $t \in \mathbb{N}_+$ consumes both when young and when old, the (lifetime) consumption bundles will be equal. In the graph, the equal split $\bar{S}$ is an example of this latter case: at this allocation, each agent receives an amount $\bar{S}^1$ of good 1 ($\bar{c}^1$ when young and $\bar{d}^1$ when old) and an amount $\bar{S}^2$ of good 2 ($\bar{c}^2$ when young and $\bar{d}^2$ when old).[9] More generally, we above defined an *equal-split allocation* to be any time-invariant profile of bundles that divide each unit of good $l \in L$ among young and old agents in such a way that all young receive the same amount $\bar{c}^l$ and all old receive the same amount $\bar{d}^l$, i.e. $\bar{c}^l + \bar{d}^l = \frac{1}{I}$. Graphically, this corresponds to setting a young/old split for each good: a point on segment s^1 for good 1 (Fig. 1a) and a point on segment s^2 for good 2 (Fig. 1b).

Let A^s be the set of equal-split allocations. In the next, we show our result for any equal split allocation $\bar{a} \in A^s$. We will come back to the

[9] The agents of the first old generation are assigned the quantities $\bar{d}^1$ and $\bar{d}^2$ of the two goods.

particular role covered by the age-independent equal-split in Section 4, where we show that it implies some additional equity properties.

Our key axiom of equity requires that each agent is assigned a bundle that she finds more desirable than the equal-split bundle.

Given an equal-split allocation $\bar{a} \in A^s$, a feasible allocation a satisfies the **equal-split guarantee** if for each agent $i \in I$ of generation $t \in \mathbb{N}_+$, $a_i(t) \succsim_{i,t} \bar{a}_i(t) = (\bar{c}, \bar{d})$ and for each agent $i \in I$ of generation 0, $a_i(0) \succsim_{i,0} \bar{a}_i(0) = (\bar{d})$.

The next theorem shows that for each equal-split allocation $\bar{a} \in A^s$ there is a rule that satisfies this equity requirement together with *Pareto efficiency.*

Theorem 1. *On the domain $\mathcal{E}$, there exists a rule that satisfies Pareto efficiency and the equal-split guarantee.*

Proof. First notice that any equal-split allocation is feasible by definition, $\bar{a} \in A^s \subset A$, and that our assumption on resources guarantee that the set of feasible allocations is compact with respect to the product topology. Compactness is preserved by taking its intersection with the set of allocations that each agent of each generation finds at least as desirable as the equal-split bundle:

$$A^{s+} = \left\{a \in A \,\middle|\, a_i(t) \succsim_{i,t} (\bar{c}, \bar{d})\ \forall i \in I, \forall t \in \mathbb{N}_+ \text{ and } a_i(0) \succsim_{i,0} (\bar{d})\ \forall i \in I\right\}$$

is compact. For each $i \in I$ and each $t \in \mathbb{N}$, let $u_{i,t}(a_i(t))$ be a utility function representing agent's preferences. Let $\rho \in (0,1)$ and consider the following maximization program:

$$\max_{a \in A^{s+}} \sum_{t=0}^{\infty} \rho^t \left[\sum_{i \in I} u_{i,t}(a_i(t))\right].$$

By the Weierstrass Theorem (maximization of a continuous function on a compact set), it achieves a maximum. Let a^* be a solution to this program. Assume by contradiction that there is a $a' \in A$ that Pareto dominates a^*, i.e. for each $i \in I$ and each $t \in \mathbb{N}$, $u_{i,t}(a'_i(t)) \geq u_{i,t}(a^*_i(t))$ and for some $i \in I$ and $t \in \mathbb{N}$, $u_{i,t}(a'_i(t)) > u_{i,t}(a^*_i(t))$. By transitivity of the preferences, for each $i \in I$ and each $t \in \mathbb{N}$, $u_{i,t}(a'_i(t)) \geq u_{i,t}(a^*_i(t)) \geq u_{i,t}(\bar{c}, \bar{d})$ and for each $i \in I$, $u_{i,0}(a'_i(0)) \geq u_{i,0}(a^*_i(0)) \geq u_{i,0}(\bar{d})$, with at least a strict inequality; thus, $a' \in A^{s+}$. Moreover, $\sum_{t=0}^{\infty} \rho^t \left[\sum_{i\in I} u_{i,t}(a'_i(t))\right] > \sum_{t=0}^{\infty} \rho^t \left[\sum_{i\in I} u_{i,t}(a^*_i(t))\right]$, contradicting the maximality of $a^* \in A^{s+}$. The fact that a^* satisfies the *equal-split guarantee* with respect to allocation $\bar{a} \in A^s$ is a consequence of the definition of A^{s+}. $\square$

In proving the existence of an egalitarian solution satisfying *Pareto efficiency* and the *equal-split guarantee*, an arbitrary utility representation of the profile of preferences has been introduced. For each specification of the utility representations, a different allocation would arise from the maximization of the above program.

In the following, we propose a class of criteria to select among the allocations that satisfy *Pareto efficiency* and the *equal-split guarantee.* The first step is to construct a cardinalization of the preferences that introduces a well defined "better-off than" relation among the agents. This is equivalent to selecting a utility representation of the preferences for each agent.

Call **equal-split proportional** the utility function that evaluates the allocation agent $i \in I$ of generation $t \in \mathbb{N}$ is assigned, $a_i(t)$, by the proportion of the *equal-split bundle* the agent is indifferent with. Let $\bar{a}$ be an equal-split allocation. For each $i \in I$ and each $t \in \mathbb{N}$, let $u_{i,t}^{ESP}(a_i(t))$ be the equal-split proportional utility defined by :

$$u_{i,t}^{ESP}(a_i(t)) = \lambda \quad \Longleftrightarrow \quad a_i(t) \sim_{i,t} \lambda \bar{a}_i(t)$$

The appeal of this utility representation stems from the importance of the equal-split allocation: the cardinalized well-being of an agent is defined by how much the agent is better/worse-off compared to this reference allocation. This is selecting from the "egalitarian equivalent correspondence" (see Pazner and Schmeidler, 1978) obtained by requiring the reference bundle to be proportional to each generation's social endowment, i.e. the equal-split bundle multiplied by the number of agents in a generation.[10]

We can thus present a family of social orderings by aggregating such utilities. The literature on social welfare functionals is extensive; here, without discussing the differences among the criteria that were studied so far, we propose (as example) the well-known discounted utilitarian criterion.[11]

Let $\beta \in (0,1)$ is the social discount factor. The discounted utilitarian social ordering is given by the discounted sum of each generation's welfare $G_t(a_t)$:

[10] The egalitarian equivalent solution is a rule that selects in the Pareto set an allocation for which all agents are indifferent with respect to a reference bundle; a characterization of the egalitarian equivalence based on monotonicity (requiring no agent to be worse-off when the feasible set is enlarged) is proposed by Dutta and Vohra (1993). When the reference bundle is proportional to the social endowment, the Pazner-Schmeidler rule arises; for a characterization in large economies, see Sprumont and Zhou (1999); for a defense of the corresponding social ordering, see Fleurbaey and Maniquet (2005). For a general discussion, see Thomson (2011).

[11] A characterization of the most used social welfare functionals for the static framework is contained in d'Aspremont and Gevers (2002).

$$So\,(a) = \sum_{t=0}^{\infty} \beta^t G_t\,(a_t)\,. \tag{3.1}$$

For each generation $t \in \mathbb{N}$, its welfare $G_t\,(a_t)$ is defined as the unweighted sum of the individuals' equal-split proportional utilities:[12]

$$G_t\,(a_t) = \sum_{i=1}^{I} u_{i,t}^{ESP}\,(a_i\,(t)) \tag{3.2}$$

We can now state the related maximization program:

$$\begin{array}{ll} \max_{a\in A} & So\,(a) \\ \text{s.t.} & u_{i,t}^{ESP}\,(a_i\,(t)) \geq 1 \quad \forall i \in I, \forall t \in \mathbb{N}_+ \end{array} \tag{3.3}$$

The maximum of this program achieves the *efficient* and equitable allocations selected by the rule. Note that the social ordering So is used to rank only a subset of the feasible allocations, in particular those satisfying the *equal-split guarantee* requirement. An alternative formulation is thus $\max_{a\in A^{s+}} So\,(a)$. All the allocations that do not satisfy the equal-split requirement, thus $a \in A \backslash A^{s+}$, are considered socially inacceptable. In summary, the maximization program corresponds to a *critical-level discounted utilitarian* distribution of resources.

A remark about the discount rate of the social ordering is useful. For a generic period $t \in \mathbb{N}_+$, let us consider the distribution of a good $l \in L$ among the young and the old agents living in that period; call $c_t^l = \sum_{i\in I} c_i^l\,(t)$ and $d_{t-1}^l = \sum_{i\in I} d_i^l\,(t-1)$ the total amount of the good each group is allocated. The first-order condition for a maximum requires that the marginal contribution of c_t^l to generation's welfare $G_t\,(a_t)$ is a fraction β of the marginal contribution of d_{t-1}^l to the previous generation's welfare $G_{t-1}\,(a_{t-1})$; formally, $\frac{\partial G_{t-1}(a_{t-1})}{\partial d_{t-1}^l} = \beta \frac{\partial G_t(a_t)}{\partial c_t^l}$. Thus, the presence of discounting in the social ordering function amounts to giving higher weight to the utility contribution of consumption when old in each period. This does *not* imply however that future generations are treated worse than previous ones: each generation (except the Adam generation) is relatively favored by the discount $\beta < 1$ with respect to the next one, but unfavored with respect to the previous one.

[12] More general formulations can be applied without altering the results; for instance, by introducing the CES aggregator or the generalized Gini evaluation function.

4 The tension with no-envy related axioms

4.1 No-envy in dynamics

As anticipated above, the appealing results of the static framework do not extend to this dynamic overlapping generation model. In this section we provide a formal discussion of this negative conclusion and show how deep the conflict with *efficiency* is.

A strong axiom for intergenerational and intragenerational equity requires agents to find their assigned consumption bundle at least as desirable as the one assigned to other agents, independently of their generations. Formally:

An allocation a satisfies **no-envy** if:

i) for each agent $i \in I$ of generation $t \in \mathbb{N}_+$ and for each agent $\iota \in I$ of generation $\tau \in \mathbb{N}_+$, $(c_i(t), d_i(t)) \succsim_{i,t} (c_\iota(\tau), d_\iota(\tau))$;

ii) for each agent $i \in I$ of generation $t \in \mathbb{N}_+$and for each agent $\iota \in I$ of generation 0, $(c_i(t), d_i(t)) \succsim_{i,t} (0, d_\iota(0))$;

iii) for each pair of agents $i, \iota \in I$ of generation 0, $d_i\,(0) \succsim_{i,0} d_\iota\,(0)$.

The first condition states the *no-envy* comparisons among agents that live in both periods, i.e. those belonging to generations $t \in \mathbb{N}_+$. The second condition requires agents of these generations not to envy the arbitrary lifetime consumption bundles of agents of the Adam generation, obtained by setting their consumption when young to zero.[13] The third condition imposes *no-envy* among agents of generation 0.

This asymmetric treatment of agents is necessary as agents of generation 0 are old at time 1 and the model does not specify the consumption when young of these agents and their complete preferences (over consumption in both periods). If this information is added to the specification of the economy and taken into account for ethical considerations, the egalitarian treatment of agents would introduce a "distributional persistence" that proves to be incompatible with *efficiency.* In fact, this alternative "symmetric no-envy" formulation drives the following result of Dubey (2010): in a stationary preference framework (with one agent per period and one good), consumption when young of the first old agent generates a tension between equity and efficiency. A sufficiently high consumption when young causes an

[13]This is a natural way to introduce a partial comparison between the Adam generation and the others. The reverse statement obtained by requiring the agents of the first old generation not to envy the consumption when old of any other agents is not as appealing. Note that, as will become clear in the next subsection, condition *i)* is alone sufficient to derive the incompatibility with *efficiency.*

impossibility: a better treatment of all future generations requires the Adam generation to achieve a larger well-being. A sufficiently low consumption when young allows existence: a feasible allocation that increases the well-being of all future generations requires making the Adam generation worse off.

The above concept of *no-envy* is similar to the "no-envy in lifetime consumptions" introduced by Suzumura (2002) for the dynamic overlapping generation framework. The author also proposes two alternative *no-envy* concepts: "no-envy in overlapping consumptions", meaning that no agent should prefer the *current* consumption bundle of another agent living in the same period to his own; "no-envy in the lifetime rate of return",[14] which equalizes welfare measured according to this specific cardinalization of preferences.[15]

"No-envy in overlapping consumptions" applied to the present model would require each period to be treated as an autonomous static distribution problem among I young and I old agents. In the one-good case, the only allocation that satisfies this axiom divides equally the good among living agents: each agent $i \in I$ of generation $t \in \mathbb{N}_+$ would receive the same bundle $a_i(t) = \left(\frac{1}{2I}, \frac{1}{2I}\right)$. This allocation is in general not *efficient*: two agents of the same generations might achieve higher satisfaction by exchanging consumption when young for consumption when old.

The axiom of "no-envy in lifetime rate of return" is not immediately applicable to this multi-good model: this index is formally defined as $r_{i,t} = \frac{d_i(t)-(1-c_i(t))}{1-c_i(t)}$, where the amount of the unique good available (per agent) is normalized to unity. The equalization of the lifetime rate of return among agents and across generations defines a constant budget set (and relative prices) in which their allocation should lie, i.e. $1+r^* = d_i(t)+(1+r^*)\,c_i(t)$. Without generalizing the concept to this framework, we refer to Subsection 4.2, where we explore the weaker concept of *no-domination.*

With respect to the Suzumura formulation of "no-envy in lifetime consumptions", we weakened the role of the first old generation. Nevertheless, the tensions between *efficiency* and *no-envy* is even stronger as the next subsection shows.

[14]The concept of lifetime rate of return is due to Cass and Yaari (1966); it will be further discussed in the next paragraphs.

[15]Shinotsuka et al. (2007)have studied the logical implications between the three *no-envy* concepts in an overlapping generation (one-good) resource distribution problem with stationary preferences and exogenous population change.

4.2 Equal treatment of equals

The next axiom (ethically different, but logically weaker than *no-envy*) can be viewed as the counterpart of anonymity: it requires that two agents with the same preferences are treated in the same way independently of their identifiers, name and generation. In the utility stream literature, with cardinal measurability and comparability of preferences, anonymity requires the social ordering of alternatives to be independent of the names of the agents: a permutation of their utility levels should be rank invariant. In this literature, Diamond (1965) showed that the infinity of the utility streams determines the impossibility of providing a continuous and complete ranking that satisfies *Pareto efficiency* and "finite anonymity", for which the permutation are limited to a finite amount of generations.[16]

An allocation a satisfies **equal treatment of equals** if:

i) for each agent $i \in I$ of generation $t \in \mathbb{N}_+$ and for each agent $\iota \in I$ of generation $\tau \in \mathbb{N}_+$ such that $\succsim_{i,t} \equiv \succsim_{\iota,\tau}$, $a_i(t) \sim_{i,t} a_\iota(\tau)$;

ii) for each pair of agents $i, \iota \in I$ of generation 0 such that $\succsim_{i,0} \equiv \succsim_{\iota,0}$, $a_i(0) \sim_{i,0} a_\iota(0)$.

Theorem 2. *On the domain $\mathcal{E}$, no rule satisfies Pareto efficiency and equal treatment of equals.*

Proof. We construct an economy $E \in \mathcal{E}$ for which no allocation satisfies *efficiency* and *equal treatment of equals.*

Let $L = 2$ and set $\gamma > 15$. Agents are of two kinds, α and β, with preferences represented by the following functions:[17]

$$U_\alpha\left(c^1, c^2, d^1, d^2\right) = \gamma c^1 + c^2 + \gamma d^1 + d^2 \tag{4.1}$$
$$U_\beta\left(c^1, c^2, d^1, d^2\right) = c^1 + \gamma c^2 + d^1 + \gamma d^2 \tag{4.2}$$

Take generations $t \in [1, 9]$ such that: generations $t \in [1, 2]$ consist of two α agents; generations $t \in [3, 7]$ consist of an α and a β agent; generations $t \in [8, 9]$ consist of two type β agents.

By *equal treatment of equals,* the α agents of generations $t \in [1, 2]$ should achieve the same utility level. The same is true for the agents β of generations

[16] This negative conclusion can be overcome only at the cost of a non-constructive approach: the necessity of using the axiom of choice implies that a ranking that combines *Pareto efficiency* and *finite anonymity* exists but cannot be formally described nor used for computing the optimal alternative. See Svensson (1980), Zame (2007), and Lauwers (2010).

[17] In the argument the utility functions are not strictly convex, but linear for computational simplicity. The result holds true when adding a second order term that provides the convexity restriction.

$t \in [8,9]$. Easy computation shows that the highest utility they can reach (both agents α and β of generations $t \in \{1,2,8,9\}$) is $U^{max} = \frac{3(\gamma+1)}{4}$. The *Pareto efficient* distribution of the remaining resources implies that at least one agent (α or β) of generations $t \in [3,7]$ achieves a larger utility than $U^{min} = \frac{4}{5}\gamma$. As $U^{min} > U^{max}$ (since $\gamma > 15$), it is not possible to satisfy both *efficiency* and *equal treatment of equals*. □

This impossibility is stronger than Diamond's because it pertains to a finite time horizon. The incompatibility between *efficiency* and *equal treatment of equals* is explained by agents living in different contexts for which they cannot be fully compensated. In fact, the context of each agent is described not only by the amount of resources to share, but also by the preferences of agents of the same generation, by the preferences of agents of previous and next generations and, indirectly, by the preferences of all agents and their succession over time.

By assumption, we sterilized the differences of agents due to resources: each generation has the same opportunity to resources as any other generation. Nevertheless, circumstances are different and the transferability of resources among agents are limited by the overlapping structure of the economy. Interestingly, leaky transfers from a period to the previous or the next one do not solve the tension between efficiency and equity:[18] only when the resources become freely transferable, the model becomes "static" (with a total amount to share among all agents) and the clash is overcome.

Another way to identify the circumstances of the generations is by looking at the possible "gains from exchange" that can be achieved starting from an equal-split allocation, assigning all resources and such that each agent receives the same bundle of goods. These improvement possibilities are larger for more "different" preferences: when all agents have same preferences, no Pareto improvement is possible through exchange; as preferences differ, welfare improvement can be obtained. The clash between *efficiency* and *equal treatment of equals* then follows from the necessity of distributing all "gains from exchange" combined with the differences of these gains that might arise over time.

[18]We do not provide a formal proof of this result. Intuitively, the argument follows the previous proof by continuity: for any (costly) transfer function it is possible to construct a sequence of homogeneous (formed by agents with equal preferences) and cosmopolitan generations (agents with different preferences) such that those agents cannot achieve the same utility level at Pareto efficient distribution of resources.

4.3 No-domination

The intuition that the negative result is driven by differences in preferences is confirmed by the next result that combines *efficiency* with another well-known axiom of equity introduced by Thomson (1983). It requires that no agent is assigned a consumption bundle that is larger than that of someone else, in this dynamic version, independently of the generation she belongs to.

An allocation a satisfies **no-domination** if:

i) for each agent $i \in I$ of generation $t \in \mathbb{N}_+$ and for each agent $\iota \in I$ of generation $\tau \in \mathbb{N}_+$, $a_i(t) \not> a_\iota(\tau)$;

ii) for each pair of agents $i, \iota \in I$ of generation 0, $a_i(0) \not> a_\iota(0)$.

The interesting property of *no-domination* is the independence of the preferences of the agents: this concept excludes allocations that by being strictly greater than one other would imply envy for any preferences of the agents.

We can state the next result:

Theorem 3. *On the domain $\mathcal{E}$, there exists a rule that satisfies both Pareto efficiency and no-domination.*

Proof. The proof is constructive.Let $A^{old} \equiv \{a \in A \,|c_{i,t} = 0 \forall i \in I, \forall t \in \mathbb{N}_+\} \subset A$ be the set of allocations that assign goods exclusively to old agents. Let $a^* \in A^{old}$ be such that there exists no $a' \in A^{old}$ such that for each $i \in I, t \in \mathbb{N}$, $a'_i(t) \succsim_{i,t} a^*_i(t)$ and for some $i \in I, t \in \mathbb{N}$, $a'_i(t) \succ_{i,t} a^*_i(t)$. By contradiction, assume that a^* is not *efficient*. Then, there exists $\bar{a} \in A$, such that for each $i \in I, t \in \mathbb{N}$, $\bar{a}_i(t) \succsim_{i,t} a^*_i(t)$ and for some $i \in I, t \in \mathbb{N}$, $\bar{a}_i(t) \succ_{i,t} a^*_i(t)$. This implies that there exists a generation $t \in \mathbb{N}$ for which $\sum_{i \in I} \bar{c}_i\,(t) > 0$ (as in a^* agents are assigned only consumption when old). By the resource constraint, $\sum_{i \in I} \bar{d}_i\,(t-1) < 1_L$. As $\bar{a}$ Pareto dominates a^*, agents of generation $t-1$ need to be compensated so that $\sum_{i \in I} \bar{c}_i\,(t-1) > 0$. This "chain compensation" stops with generation 0, as agents of the Adam generation cannot be compensated with an increase in consumption when young: in $\bar{a}$, there is some $i \in I$ for which $a^*_i\,(0) \succ_{i,0} \bar{a}_i\,(0)$. The contradiction guarantees that any *efficient* allocation in A^{old} is also *efficient* in A.

It remains to prove that, for each economy, there exists an allocation in A^{old} that satisfies *efficiency* and *no-domination.* To do so, let us introduce the following definition and Lemma.

An allocation $x = (x_1, ..., x_I)$ satisfies the ***same size property*** *if for each pair $i, \iota \in I$, $\sum_{l \in L} x^l_i = \sum_{l \in L} x^l_\iota$.*

Lemma 1. *(Same size efficiency) For each vector of infinitely divisible and privately appropriable goods* $\Omega \in \mathbb{R}^L_{++}$ *to be distributed among I agents, there exists a rule that satisfies the same size property and Pareto efficiency.*

Proof. The result is presented in Moulin (1991).[19]

For each $t \in \mathbb{N}_+$, consider the problem of allocating the available resources $\Omega = \left(\{1\}_{l \in L}\right)$ to the I old agent: by Lemma 1, there exists an allocation that satisfies *efficiency* and the *same size property*; moreover, the latter implies that all agent's allocation belong to an hyperplane defined by unitary (and thus time-invariant) prices; thus, *no-domination* is as well satisfied and existence follows. □

4.4 Combining no-domination and equal-split guarantee

As both the *equal-split guarantee* and *no-domination* are compatible with *efficiency*, the next step is to understand to which extent these equity axioms can be combined.

Unfortunately *no-domination*, together with *efficiency*, does not allow choosing allocations such that each agent is better-off than she would be by consuming any *arbitrarily small* bundle of goods. This negative result is stronger than showing that there exists no egalitarian solution that combines *efficiency* with *no-domination* and the *equal-split guarantee*.

We introduce this minimal welfare guarantee through an axiom requiring that, given a scalar $\varepsilon \in (0, 1)$ and an *equal-split bundle* $\left(\bar{c}, \bar{d}\right)$, the allocation should assign to each agent a bundle that she finds at least as desirable as the ε-part of the *equal-split*. When $\varepsilon = 1$, the axiom of *equal-split guarantee* is obtained; as ε decreases the condition becomes weaker and weaker; for $\varepsilon = 0$ it is vacuous. This requirement is similar to the ε version of "individual rationality" introduced by Moulin and Thomson (1988): their axiom requires each agent to be at least as well-off as when consuming a bundle that is the ε-share of the aggregate available resources.[20]

Given $\bar{a} \in A^s$ and $\varepsilon \in (0, 1)$, an allocation a satisfies the **$\boldsymbol{\varepsilon}$ equal-split guarantee** if:

i) for each agent $i \in I$ of generation $t \in \mathbb{N}_+$, $a_i(t) \succsim_{i,t} \varepsilon \left(\bar{c}, \bar{d}\right)$;

ii) for each agent $i \in I$ of generation 0, $a_i(0) \succsim_{i,0} \varepsilon \left(\bar{d}\right)$.

[19] The author names it "budget constrained Pareto optimal method".

[20] The idea of introducing a parametrization in the distributional criteria has been further exploited in the literature of fair allocations: similar criteria are defined, among others, in Thomson (1987), Diamantaras and Thomson (1990), Berliant et al. (1992), Sprumont (1998).

Theorem 4. *Let $\varepsilon \in (0,1)$. On the domain $\mathcal{E}$, no rule satisfies Pareto efficiency, no-domination and the ε equal-split guarantee.*

Proof. The proof is again constructive. Take a two-goods economy with one agent per generation (each agent is thus identified by its generation). Let $\bar{a} \in A^s$ be an *equal-split allocation* $\bar{a} \in A^s$ such that $\bar{a}(t) = \left(\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2\right) \gg 0$.

STEP 1.

Consider three successive agents $t-1, t, t+1 \in \mathbb{N}_+$. Their preferences are represented by the following linear functions:[21]

$$\begin{aligned} U_{t-1}\left(c_{t-1}^1, c_{t-1}^2, d_{t-1}^1, d_{t-1}^2\right) &= \gamma\left(c_{t-1}^1 + c_{t-1}^2\right) + \zeta d_{t-1}^1 + d_{t-1}^2 \\ U_t\left(c_t^1, c_t^2, d_t^1, d_t^2\right) &= \delta c_t^1 + c_t^2 + d_t^1 + \delta d_t^2 \\ U_{t+1}\left(c_{t+1}^1, c_{t+1}^2, d_{t+1}^1, d_{t+1}^2\right) &= c_{t+1}^1 + \zeta c_{t+1}^2 + \gamma\left(d_{t+1}^1 + d_{t+1}^2\right) \end{aligned}$$

where $0 < \gamma < \delta < \zeta < \frac{\varepsilon}{3} \min\left[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2\right] < 1$. For each agent $i = t-1, t, t+1$ we denote by U_i^{ES} the utility level achieved at the *equal-split bundle*. The ε *equal-split guarantee* requires that $U_i(a_i) \geq \varepsilon U_i^{ES}$.

Part *a)*.

The distribution of goods available at time t is represented in the Edgeworth box of Figure 4.1a, where agent $t-1$'s origin is the bottom-left corner and agent t's origin is the top-right corner. The *equal-split bundle* is denoted ES; its ε component is εES^d for consumption when old and εES^c for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period t (among consumption when young of t and consumption when old of $t-1$) is such that either $d_{t-1}^1 = 1$ or $d_{t-1}^2 = 0$ (this corner solution follows from the linearity of preferences with $\delta < \zeta$).

When $d_{t-1}^2 = 0$, it is not possible to assign to agent $t-1$ a bundle that satisfies the ε *equal-split guarantee*: the maximum utility when $d_{t-1}^1 = 1, c_{t-1}^1 = c_{t-1}^2 = 1$ (represented in the graph by $U_{t-1}^{\max}$) is $U_{t-1}^{\max} = 2\gamma + 3\zeta < \varepsilon \min\left[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2\right] < \varepsilon \bar{d}^2 < \varepsilon U_{t-1}^{ES}$. In the graph, the indifference level satisfying the ε *equal-split guarantee* when consumption when young is maximum ($c_{t-1}^1 = c_{t-1}^2 = 1$) is $\bar{U}_{t-1}^d$; since this is higher than the $U_{t-1}^{\max}$, the *efficient* allocations that guarantee to agent $t-1$ the equity constraint lie on the segment $\overline{F0_t^c}$: $d_{t-1}^1 = 1$ and $d_{t-1}^2 > 0$.

Part *b)*.

[21] The linearity assumption is without loss of generality: the result holds true when a second order term is added.

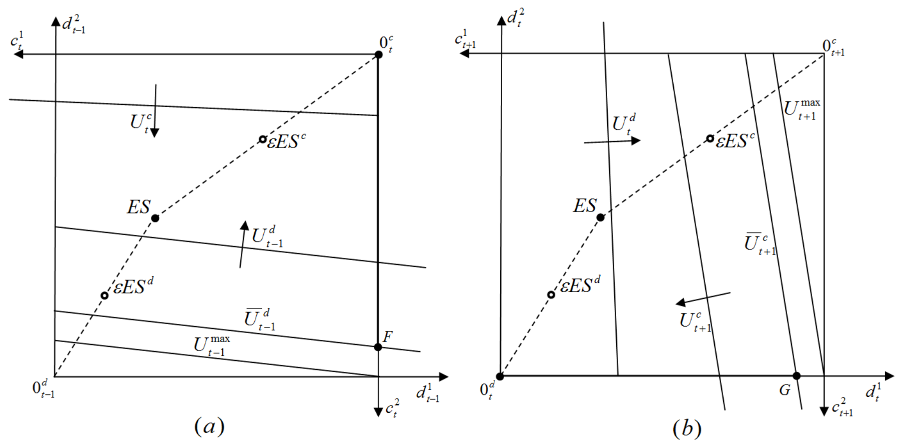


Figure 4.1: Determining the allocation of agent t.

The distribution of goods available at time $t+1$ is represented in the Edgeworth box of Figure 4.1b, where agent t's origin is the bottom-left corner and agent $t+1$'s origin is the top-right corner. The *equal-split bundle* is denoted ES; its ε component is εES^d for consumption when old and εES^c for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period $t+1$ (among consumption when young of $t+1$ and consumption when old of t) is such that either $c_{t+1}^1 = 0$ or $d_{t+1}^2 = 1$ (this corner solution follows from the linearity of preferences with $\delta < \zeta$).

When $c_{t+1}^1 = 0$, it is not possible to assign to agent $t+1$ a bundle that satisfies the ε *equal-split guarantee*: the maximum utility when $c_{t+1}^2 = 1, d_{t+1}^1 = d_{t+1}^2 = 1$ (represented in the graph by $U_{t+1}^{\max}$) is $U_{t+1}^{\max} = 2\gamma + 3\zeta < \varepsilon \min\left[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2\right] < \varepsilon \bar{d}^2 < \varepsilon U_{t+1}^{ES}$. In the graph, the indifference level satisfying the ε *equal-split guarantee*, given maximum consumption when old ($d_{t+1}^1 = d_{t+1}^2 = 1$), is $\bar{U}_{t+1}^c$; since this is higher than the $U_{t+1}^{\max}$, the *efficient* allocations that guarantee to agent $t+1$ the equity constraint lie on the segment $\overline{0_t^d G}$: $c_{t+1}^1 > 0$ and $c_{t+1}^2 = 1$.

Summing up, the lifetime consumption of agent t, $a(t)$, satisfies $c_t^1 = 0$, $c_t^2 < 1$, $d_t^1 < 1$, and $d_t^2 = 0$.

STEP 2.

Consider four successive agents $\tau-1, \tau, \tau+1, \tau+2 \in \mathbb{N}_+$ with the following utilities:

$$
\begin{aligned}
U_{\tau-1}\left(c_{\tau-1}^1, c_{\tau-1}^2, d_{\tau-1}^1, d_{\tau-1}^2\right) &= c_{\tau-1}^1 + c_{\tau-1}^2 + d_{\tau-1}^1 + \delta d_{\tau-1}^2 \\
U_{\tau}\left(c_{\tau}^1, c_{\tau}^2, d_{\tau}^1, d_{\tau}^2\right) &= c_{\tau}^1 + \zeta c_{\tau}^2 + \gamma\left(d_{\tau}^1 + \delta d_{\tau}^2\right) \\
U_{\tau+1}\left(c_{\tau+1}^1, c_{\tau+1}^2, d_{\tau+1}^1, d_{\tau+1}^2\right) &= \gamma\left(c_{\tau+1}^1 + \zeta c_{\tau+1}^2\right) + d_{\tau+1}^1 + \delta d_{\tau+1}^2 \\
U_{\tau+2}\left(c_{\tau+2}^1, c_{\tau+2}^2, d_{\tau+2}^1, d_{\tau+2}^2\right) &= c_{\tau+2}^1 + \zeta c_{\tau+2}^2 + d_{\tau+2}^1 + d_{\tau+2}^2
\end{aligned}
$$

where $0 < \gamma < \delta < \zeta < \frac{\varepsilon}{3}\min\left[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2\right] < 1$. For each agent $i = \tau-1, \tau, \tau+1, \tau+2$, U_i^{ES} denotes the utility level at the *equal-split bundle*; then, the ε equal-split condition requires that $U_i\left(a_i\right) \geq \varepsilon U_i^{ES}$.

Part *a)*.

The distribution of goods available at time τ is represented in the Edgeworth box of Figure 4.2a, where agent $\tau-1$'s origin is the bottom-left corner and agent τ's origin is the top-right corner. The *equal-split bundle* is denoted ES; its ε component is εES^d for consumption when old and εES^c for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period τ (among consumption when young of τ and consumption when old of $\tau-1$) is such that either $c_\tau^1 = 1$ or $c_\tau^2 = 0$ (this corner solution follows from the linearity of preferences with $\delta < \zeta$).

When $c_\tau^2 = 0$, it is not possible to assign to agent τ a bundle that satisfies the ε *equal-split guarantee*: the maximum utility when $c_\tau^1 = 1$ and $d_\tau^1 = d_\tau^2 = 1$ (represented in the graph by $U_\tau^{\max}$) is $U_\tau^{\max} = 2\gamma + 3\zeta < \varepsilon\min\left[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2\right] < \varepsilon\bar{d}^2 < \varepsilon U_\tau^{ES}$. In the graph, the indifference level satisfying the ε *equal-split guarantee*, given maximum consumption when old ($d_\tau^1 = d_\tau^2 = 1$), is $\bar{U}_\tau^c$; since this is higher than the $U_\tau^{\max}$, the *efficient* allocations that guarantee to agent τ the equity constraint lie on the segment $\overline{0_{\tau-1}^d G'}$: $c_\tau^1 = 1$ and $c_\tau^2 > 0$.

Part *b)*.

The distribution of goods available at time $\tau+2$ is represented in the Edgeworth box of Figure 4.2b, where agent $\tau+1$'s origin is the bottom-left corner and agent $\tau+2$'s origin is the top-right corner. The *equal-split bundle* is denoted ES; its ε component is εES^d for consumption when old and εES^c for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period $\tau+2$ (among consumption when young of $\tau+2$ and consumption when old of $\tau+1$) is such that either $d_{\tau+1}^1 = 1$ or $d_{\tau+1}^2 = 0$ (this corner solution follows from the linearity of preferences with $\delta < \zeta$).

When $d_{\tau+1}^2 = 0$, it is not possible to assign to agent τ a bundle that satisfies the ε *equal-split guarantee*: the maximum utility when $d_{\tau+1}^1 = 1$ and $c_\tau^1 = c_\tau^2 = 1$ (represented in the graph by $U_\tau^{\max}$) is $U_\tau^{\max} = 2\gamma + 3\zeta < \varepsilon\min\left[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2\right] < \varepsilon\bar{d}^2 < \varepsilon U_\tau^{ES}$. In the graph, the indifference

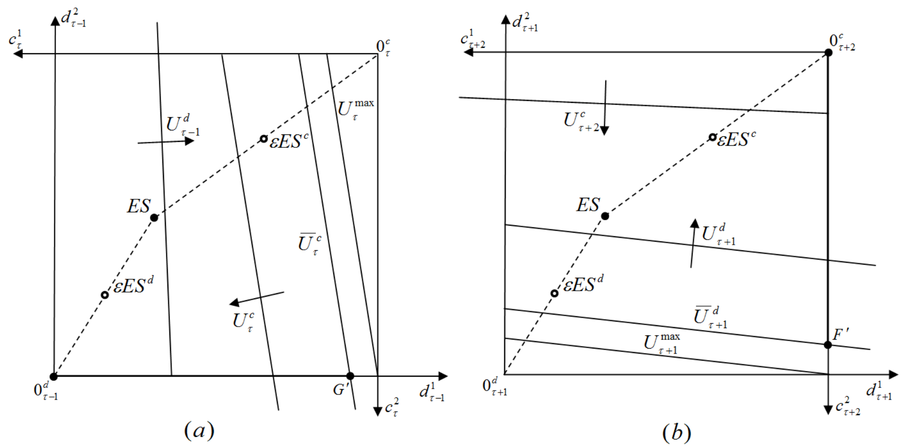


Figure 4.2: Determining the allocations of agents τ and $\tau+1$.

level satisfying the ε *equal-split guarantee* when consumption when young is maximum ($c^1_{\tau+1} = c^2_{\tau+1} = 1$) is $\bar{U}^d_{\tau+1}$; since this is higher than the $U^{\max}_{\tau+1}$, the *efficient* allocations that guarantee to agent $\tau+1$ the equity constraint lie on the segment $\overline{F'0^c_{\tau+2}}$: $d^1_{\tau+1} = 1$ and $d^2_\tau > 0$.

STEP 3.

The *efficient* distribution of resources available at $\tau+1$ requires that either:

i) $c^1_{\tau+1} = 0$ (and $d^1_\tau = 1$); or

ii) $c^2_{\tau+1} = 1$ (and $d^2_\tau = 0$).

From Step 1, $a(t)$ satisfies $c^1_t = 0$, $c^2_t < 1$, $d^1_t < 1$, and $d^2_t = 0$.

In case *i)*, $a(\tau)$ is such that $c^1_\tau > 0$, $c^2_\tau = 1$, $d^1_\tau = 1$ and $d^2_\tau \geq 0$: thus, $a(\tau)$ dominates $a(t)$.

In case *ii)*, $a(\tau+1)$ satisfies $c^1_{\tau+1} \geq 0$, $c^2_{\tau+1} = 1$, $d^1_{\tau+1} = 1$, $d^2_{\tau+1} > 0$ and thus dominates $a(t)$. $\square$

This further tension between *efficiency*, on the one side, and equity as *no-domination* and the ε *equal-split guarantee*, on the other side, is particularly strong as it arises already in a finite horizon economy. The result shows that *efficiency* and *no-domination* do not allow to guarantee that each agent is assigned an allocation that she finds at least as desirable as a strictly positive bundle (possibly infinitesimally small).

Interestingly, the reverse argument fails. Even if *efficiency* is imposed, the *equal-split guarantee* is compatible with (and actually implies) a certain degree of *no-domination.* The next axiom requires that no agent is assigned a bundle that is dominated by the ε-part of the bundle of any other agent.

Given an $\varepsilon \in [0,1]$, an allocation a satisfies **no-domination** if:

i) for each agent $i \in I$ of generation $t \in \mathbb{N}_+$ and each agent $\iota \in I$ of generation $\tau \in \mathbb{N}_+$, $\varepsilon a_i(t) \not\gg a_\iota(\tau)$;

ii) for each pair of agents $i, \iota \in I$ of generation 0, $\varepsilon a_i(0) \not\gg a_\iota(0)$.

Theorem 5. *If a rule satisfies Pareto efficiency and the equal-split guarantee with respect to the equal-split allocation $\bar{a} \in A^s$, then, for all $\varepsilon \leq \min\left[\min_l \bar{c}^l, \min_l \bar{d}^l\right]$, it satisfies ε no-domination.*

Proof. If the equal-split bundle has some zero components, $\varepsilon = 0$ and ε *no-domination* only requires non-negativity of the assignments. Thus, let the equal-split allocation be such that the equal split-bundle is strictly positive, i.e. $(\bar{c}, \bar{d}) \gg 0$, and let $\varepsilon \leq \min\left[\min_l \bar{c}^l, \min_l \bar{d}^l\right]$.

By contradiction, assume that ε *no-domination* is violated. Then there exist $i, \iota \in I$ and $t, \tau \in \mathbb{N}_+$ (or $t = \tau = 0$) such that $(c_i(t), d_i(t)) \ll \varepsilon (c_\iota(\tau), d_\iota(\tau))$ (resp. $d_i(0) \ll \varepsilon d_\iota(0)$).

By the *equal-split guarantee*, $(c_i(t), d_i(t)) \succsim_{i,t} (\bar{c}, \bar{d})$ (resp. $d_i(0) \succsim_{i,0} \bar{d}$); as $\varepsilon \leq \min\left[\min_l \bar{c}^l, \min_l \bar{d}^l\right]$, we have that for all bundles $(c_\iota(\tau), d_\iota(\tau)) \in [0,1]^{2L}$, $\varepsilon (c_\iota(\tau), d_\iota(\tau)) \ll (\bar{c}, \bar{d})$. By preference monotonicity, we derive a contradiction: $(c_i(t), d_i(t)) \succsim_{i,t} (\bar{c}, \bar{d}) \succ_{i,t} \varepsilon (c_\iota(\tau), d_\iota(\tau)) \succ_{i,t} (c_i(t), d_i(t))$ (resp. $d_i(0) \succsim_{i,0} \bar{d} \succ_{i,0} \varepsilon d_\iota(0) \succ_{i,0} d_i(0)$). $\square$

The asymmetry of the above results suggests that the *equal-split guarantee* implies (together with *efficiency*) a weak version of *no-domination*, while *no-domination* does not allow setting an arbitrarily small lower bound for well-being.

The last theorem also suggests how to select among the equal-split allocations for our equity requirement.[22] Since the upper bound of ε implied by *efficiency* and *equal-split guarantee* is determined by the equal-split bundle (or allocation), we can choose the latter to maximize ε. This shows the role of the above introduced "age-independent" equal-split of resources, assigning in each period the same amount of goods to young and old agents: when for each good $l \in L$, $\bar{c}^l = \bar{d}^l = \frac{1}{2I}$, ε is maximal.

5 Extensions

In this section we show that we can relax the assumption about constant stream of resources without affecting the appeal of the defined egalitarian

[22] We are indebted to Yves Sprumont for highlighting this point.

criterion. Conversely, this will further increase the tension between *efficiency* and *no-envy* related equity concepts.

The assumption that resources are constant over time goes in the direction of giving a chance for the existence of an egalitarian criterion in the ordinal and non-comparable framework. When the stream of resources is time varying, the impossibility of transferring resources over time would impede any compensation between generations of agents that are differently endowed. This directly entails the incompatibility of *equal treatment of equals* (or of *no-domination*) with *Pareto efficiency* even with stationary preferences.

Interestingly, an appealing extension of the *equal-split guarantee* can be introduced for non-constant streams of resources. For each $t \in \mathbb{N}_+$, let the vector of resources be $\omega_t = \left(\omega_t^1, ..., \omega_t^L\right) \in \mathbb{R}_{++}^L$ such that $\inf_{l \in L} \omega_t^l > \phi$ for some $\phi > 0$. This restriction guarantees that resources are bounded away from zero at any period $t \in \mathbb{N}_+ \bigcup \{\infty\}$.

Let an economy be a list $E = \left(\{\omega_t\}_{t \in \mathbb{N}_+}, \{\succsim_{i.t}\}_{i \in I, t \in \mathbb{N}}\right)$ and let $\bar{\mathcal{E}}$ be the domain of overlapping generation economies with non-constant resources.

An allocation a is **feasible for E** if $\sum_{i \in I} d_i^l(t-1) + \sum_{i \in I} c_i^l(t) \leq \omega_t^l$ for each good $l \in L$ and for each period $t \in \mathbb{N}_+$. Let $A(E)$ denote the set of feasible allocations.

For each $E \in \bar{\mathcal{E}}$, an allocation $\bar{a} \in A(E)$ is a **lower-bound division of resources** if there is a pair $\bar{c}, \bar{d} \in \mathbb{R}_{++}^L$ such that for each agent $i \in I$ of generation $t \in \mathbb{N}_+$, $\bar{a}_i(t) = \left(\bar{c}, \bar{d}\right)$ and for each agent $i \in I$ of generation 0, $\bar{a}_i(0) = \bar{d}$.

Let $A^{lb} \subset A(E)$ be the set of lower bound division of resources. As resources are bounded away from zero, A^{lb} is non-empty. It is thus possible to select one of these allocations as the lower-bound for the egalitarian distribution problem:

Given an $\bar{a} \in A^{lb}$, an allocation a satisfies the **lower-bound guarantee** if for each agent $i \in I$ of generation $t \in \mathbb{N}_+$, $a_i(t) \succsim_{i,t} \bar{a}_i(t) = \left(\bar{c}, \bar{d}\right)$ and for each agent $i \in I$ of generation 0, $a_i(0) \succsim_{i,0} \bar{a}_i(0) = \left(\bar{d}\right)$.

The possibility result of Theorem 1 thus extends: *efficiency* and the *lower-bound guarantee* are compatible.

As shown in Section 3, we can construct a similar cardinalization of the preferences and provide a family of egalitarian orderings that rank the feasible allocations.

For each agent $i \in I$ of generation $t \in \mathbb{N}$, let the **lower-bound proportional** utility function $u_{i,t}^{LBP}(a_i(t))$ satisfy the following condition:

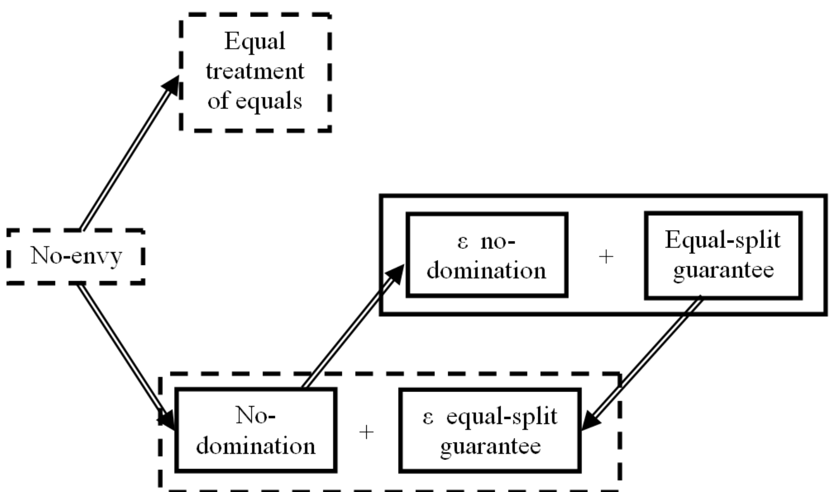
$$u_{i,t}^{LBP}\left(a_i\left(t\right)\right) = \lambda \quad \Longleftrightarrow \quad a_i\left(t\right) \sim_{i,t} \lambda \bar{a}_i\left(t\right)$$

The social ordering proposed extends when the equal-split proportional utility, $u_{i,t}^{ESP}\left(a_i\left(t\right)\right)$, is substituted by the lower-bound proportional utility, $u_{i,t}^{LBP}\left(a_i\left(t\right)\right)$.

6 Conclusions

We have studied the problem of equitably distributing a constant stream of goods in an overlapping generation framework with ordinal and non-comparable preferences.

The results can be summarized by the following graph, where the equity axioms are compared. *No-envy* is stronger than *equal-treatment of equals* and *no-domination*; *no-domination* implies ε *no-domination* while the *equal-split guarantee* implies ε *equal-split guarantee.* The tension between efficiency and equity is represented by the dashed boxes in which the axioms are, while the compatibility with *efficiency* is represented by continuous line boxes.



Equal-treatment of equals is not compatible with *efficiency* due to different conditions (co-living agents) that same-preference agents face when they live in different periods. This implies the clash between *efficiency* and *no-envy.*

It is possible to determine an egalitarian concept based on *no-domination*; this is however incompatible with guaranteeing each agent a bundle that she considers as desirable as an arbitrarily small share of the equal division of resources (*no-domination* and the ε *equal-split guarantee* are not compatible). This result also implies that *efficiency* is not together compatible with *no-domination* and the *equal-split guarantee*.

We suggest that an egalitarian criterion for this economy should be constructed starting from the *equal-split guarantee*. It has been shown that with *efficiency* it implies ε *no-domination* and that the equal-split allocation that entails the most concern for avoiding domination (thus maximizing ε) is the "age-independent" division of resources, which equally divides each good available in a period between the young and old agents alive.

Furthermore, we show how to formulate a cardinalization of the preferences that, providing an endogenous way to compare the well-being levels, allows applying welfarist aggregation functions to rank allocations; among these, we introduce a critical-level discounted utilitarian criterion and show that it does not discriminate future generations.

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