2-VERMA MODULES AND KHOVANOV-ROZANSKY LINK HOMOLOGIES

GRÉGOIRE NAISSE AND PEDRO VAZ

ABSTRACT. For a particular choice of parabolic subalgebra of $\mathfrak{gl}_{2n}$ we construct a parabolic 2-Verma module and use it to give a construction of Khovanov-Rozansky's HOMFLY-PT and $\mathfrak{gl}_N$ link homologies. We use our version of these homologies to prove that Rasmussen's spectral sequence from the HOMFLY-PT-link homology to the $\mathfrak{gl}_N$-link homology converges at the second page, proving a conjecture of Dunfield, Gukov and Rasmussen.

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CONTENTS

1. INTRODUCTION

One of the most celebrated homology theories of knots and links in 3-space is Khovanov and Rozansky's $\mathfrak{sl}_N$-link homology [11] categorifying the $\mathfrak{sl}_N$-link invariant. Soon after the appearance of [11] the existence of a triply-graded link homology categorifying the HOMFLY-PT polynomial was predicted in [4] by Dunfield, Gukov and Rasmussen, who made various conjectures about the structure of such homology theory. One of these conjectures was the existence of a family of differentials d_N, for N an integer, acting on the triply-graded homology and whose homology for each link L coincides with Khovanov and Rozansky's $\mathfrak{sl}_N$-link homology when $N > 0$.

A rigorous construction of a triply-graded link homology categorifying the HOMFLY-PT polynomial was given by Khovanov and Rozansky in [12] (see also [8]). The structure of this link homology was studied by Rasmussen in [23]. Rasmussen defined a family of differentials on the KR HOMFLY-PT link homology and showed that, for each $N > 0$, these differentials give rise to a spectral sequence starting at the KR HOMFLY-PT homology and converging to the $\mathfrak{sl}_N$ homology.

In this paper we give a new interpretation of the HOMFLY-PT polynomial in terms of parabolic Verma modules and a new construction of KR HOMFLY-PT link homology by introducing certain 2-Verma modules, the latter categorifying the parabolic Verma modules involved in the HOMFLY-PT polynomial. Using our version of HOMFLY-PT homology we prove that Rasmussen's spectral sequence converges at the second page, which implies the Dunfield-Gukov-Rasmussen conjecture for $N > 0$.

Summary of the paper and description of the main results. In the search for a construction of the HOMFLY-PT polynomial based on representation theory, Queffelec and Sartori [22] provided an algebraic gadget, called the double Schur algebra, which accommodates the Hecke algebra and the Ocneanu trace under the same roof. The double Schur algebra $\dot{S}_{q,\beta}(\ell, k)$ contains two copies of the Schur algebra (the cases $\ell = 0$ and $k = 0$), and is defined as a quotient of idempotented quantum $\mathfrak{gl}_{\ell+k}$, whose weight lattice has been shifted by a formal parameter. For link L presented as the closure of a braid b with k strands, Queffelec and Sartori constructed a certain element in the double Schur algebra, which is a multiple of a certain idempotent, coinciding with the HOMFLY-PT polynomial of L.

The construction in [22] finds a natural place in the terms of representations of the double Schur algebra. In this paper we extend the notion of Weyl modules to double Schur algebras and translate Queffelec and Sartori's results to this context. Concretely we show that, with the choice of highest weight $\beta = (\beta, \dots, \beta, 0, \dots, 0)$ (n β's and n 0's), the HOMFLY-PT polynomial of L can be obtained from the Weyl module $W(\beta)$ as a map which is a multiple of the identity, the coefficient being the HOMFLY-PT polynomial of L.

As representations of $U_q(\mathfrak{gl}_{\ell+k})$, Weyl modules $W(\mu)$ are isomorphic to parabolic Verma modules $M^{\mathfrak{p}}(\mu)$ for a certain parabolic subalgebra $\mathfrak{p}$ and we can rephrase the previous paragraph entirely in terms of parabolic Verma modules.

Theorem A (Theorem 2.6)**.** For a link presented as the closure of a braid b with k strands, the construction above defines an element $P^{\mathfrak{p}}(b) \in \mathrm{End}_{U_q(\mathfrak{gl}_{2k})}\big(M_{\mathfrak{p}}(\beta)\big)$ which is a link invariant. It is a multiple of the identity whose coefficient equals the HOMFLY-PT polynomial of the closure of b.

We construct a version of KLR algebra associated with the pair $(\mathfrak{p}, \mathfrak{gl}_{2k})$ that combines the original KLR algebra for $\mathfrak{gl}_{2k}$ with the superalgebra A_m introduced in [18] (see also [19] for a further study of A_m). This new algebra, denoted $R_{\mathfrak{p}}$, is a particular case of a general family of multigraded superalgebras introduced in [20] associated to symmetrizable Kac-Moody algebras. We pass to a suitable cyclotomic quotient $R_{\mathfrak{p}}^{\beta}$ and make use of the usual procedure to construct functors E_i^{β}, F_i^{β} (for i a simple root of $\mathfrak{gl}_{2k}$) from functors of restriction and extension for maps adding strands to diagrams. We prove that $U_q(\mathfrak{gl}_{2k})$ acts on the category of (cone bounded, locally finite dimensional) supermodules for $R_{\mathfrak{p}}^{\beta}$, whose (topological) Grothendieck group is therefore isomorphic to the parabolic Verma module $M^{\mathfrak{p}}(\mu)$. Since we work with a bigraded supercategory the Grothendieck group is a module over $\mathbb{Z}(\!(q,\lambda)\!)[\pi]/(\pi^2-1)$.

Theorem B (Proposition 3.25 and Theorem 3.26)**.** Functors F_i^{β} and E_i^{β} are exact and induce a $U_q(\mathfrak{gl}_{2n})$-action on the Grothendieck group of $R_{\mathfrak{p}}^{\beta}$ which becomes isomorphic to $M^{p}(\beta)$ after specializing $\pi = -1$.

We upgrade this data into a 2-category $\mathfrak{M}^{\mathfrak{p}}(\beta)$, which we call a *2-Verma module*.

The Rickard complex associated to a braid acts on the homotopy category of complexes of the Hom-categories of $\mathfrak{M}^{\mathfrak{p}}(\lambda)$ (this is a lift of the usual braiding induced by the embedding of the Hecke algebra into a Schur algebra that in turn embeds canonically in a double Schur algebra). For a closure of a braid b this procedure gives a chain complex $C(\mathrm{cl}(b)$.

Theorem C (Proposition 4.1, Corollary 4.2, Theorem 4.3)**.** The homotopy type of $C(\mathrm{cl}(b))$ is invariant under the Markov moves. Its homology groups are triply-graded link invariants and its bigraded Euler characteristic is the HOMFLY-PT polynomial of the closure of b. Moreover, $H(b)$ is isomorphic to Khovanov and Rozansky HOMFLY-PT homology $\mathrm{HKR}(b)$.

Introducing a differential d_N on $R_{\mathfrak{p}}^{\beta}$, turning it into a dg-algebra, one can recover $R_{\mathfrak{gl}_{2k}}^{\Lambda}$, the usual cyclotomic quotient of the KLR algebra associated with $\mathfrak{gl}_{2k}$, in the sense that the former is formal and quasi-isomorphic to the latter. The work of Mackaay and Yonezawa in [16] implies that replacing $R_{\mathfrak{p}}^{\beta}$ by $R_{\mathfrak{gl}_{2k}}^{\Lambda}$ in the construction above produces Khovanov and Rozansky's $\mathfrak{sl}_N$-link homology $\mathrm{HKR}_N(\mathrm{cl}(b))$ for the closure of b.

The differential d_N descends to $\mathrm{HKR}(b)$ and engenders a spectral sequence starting at $\mathrm{HKR}(b)$ and converging to $\mathrm{HKR}_N(b)$, which coincides with Rasmussen's spectral sequence in [23]. As a matter of fact, both $R_{\mathfrak{p}}^{\beta}$ and $R_{\mathfrak{gl}_{2k}}^{\Lambda}$ admit similar dg-enhancements, namely $(R_{\mathfrak{b}}, d_{\beta})$ and $(R_{\mathfrak{b}}, d_{\Lambda})$ for some KLR-like algebra $R_{\mathfrak{b}}$ that can be thought as associated to the Borel subalgebra of $\mathfrak{gl}_{2n}$. The Rickard complex can be lifted to this setting, and using this extra information, one can prove the Dunfield-Gukov-Rasmussen conjecture for $N > 0$:

Theorem D (Theorem 4.7)**.** For each link L and for each $N > 0$,

$$\mathrm{HKR}_N(L) \cong H(\mathrm{HKR}(L), d_N)$$

And in particular this implies

Corollary A (Corollary 4.16)**.** Rasmussen's spectral sequence converges at the second page.

All the results above have an analogue for the case of reduced homologies.

Plan of the paper. In Section 2 we describe the double Schur algebra following Queffelec and Sartori and how to use it to define the HOMFLY-PT polynomial. We interpret their construction in the context of parabolic Verma modules. Section 3 consists of higher representation theory: we introduce parabolic 2-Verma modules. We define the superalgebras $R_{\mathfrak{b}}$ and $R_{\mathfrak{p}}^{\beta}$ and develop a categorification of the parabolic Verma modules that are used to construct link homology theories categorifying the results in §2. We introduce a differential d_N on $R_{\mathfrak{p}}^{\beta}$ yielding a dg-algebra quasi-isomorphic to the cyclotomic KLR algebra $R^{\Lambda}_{\mathfrak{gl}_{2n}}$. Finally, Section 4 contains the topology. We introduce the braiding on the homotopy category of complexes of the Hom-categories of the 2-Verma module $\mathfrak{M}^{\mathfrak{p}}(\beta)$ through Rickard complexes and prove invariance under the Markov moves. We also prove our link homology groups are isomorphic to Khovanov and Rozansky's. We use the differential d_N to re-construct Rasmussen's spectral sequence starting at $\mathrm{HKR}(L)$ and converging to $\mathrm{HKR}_N(L)$. We prove its second page is already $\mathrm{HKR}_N(L)$, which in turn proves the Dunfield-Gukov-Rasmussen conjecture for $N > 0$.

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2. Parabolic Verma modules and link invariants

2.1. **Parabolic Verma modules.** Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra. Let also $\mathfrak{l}$ and $\mathfrak{n}$ be respectively the Levi factor and the nilpotent radical of $\mathfrak{p}$. Let V be an irreducible representation of $U_q(\mathfrak{l})$. It can be extended to $U_q(\mathfrak{p})$ by setting $U_q(\mathfrak{n})V = 0$. The *generalized Verma module* associated to $\mathfrak{p}$ and V is the induced module

$$M^{\mathfrak{p}}(V) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V.$$

Taking V finite dimensional gives a *parabolic Verma module*. When V is one-dimensional we say $M_{\mathfrak{p}}(V)$ is of *scalar type*, and when $\mathfrak{p}$ coincides with the Borel subalgebra $\mathfrak{b}$ we recover the usual Verma modules. Parabolic Verma modules are highest weight modules and therefore, can be expressed as quotients of usual (universal) Verma modules. See [6, Chapter 9] for further details on parabolic Verma modules and the manuscript [17] and references therein for a detailed study of generalized (and therefore of parabolic) Verma modules.

Let $\{\alpha_i : i \in I\}$ be the set of simple roots. For each partition $I = I_f \sqcup I_m$ we consider the parabolic subalgebra $\mathfrak{p}_I = \mathfrak{b} \oplus_{(i \in I_f)} g_{-\alpha_i}$, the Lie subalgebras $\mathfrak{g}_f$ and $\mathfrak{g}_m$ and the corresponding Cartan subalgebras $\mathfrak{h}_f$ and $\mathfrak{h}_m$. Then $\mathfrak{l} = \mathfrak{g}_f \oplus \mathfrak{h}_m$.

In this paper we consider $\mathfrak{g} = \mathfrak{gl}_k$ and concentrate on a certain class of parabolic Verma modules we now describe. Let $\Lambda_f = \{n_i\}_{i \in I_f}$ be an integral dominant highest weight for $\mathfrak{g}_f$, and let $\Lambda_m = (\lambda_i)_{i \in I_m}$ with the λ_i's formal parameters that we think as q_i^{β} for some generic β_i's. The irreducible $L(\Lambda_f)$ for $U_q(\mathfrak{g}_f)$ can be extended uniquely to an irreducible for $U_q(\mathfrak{l})$ by declaring that $U_q(\mathfrak{h}_m)$ acts on the highest weight vector as $K_i v_0 = \lambda_i v_0$. We denote this parabolic Verma module by $M^{\mathfrak{p}}(\Lambda)$, where $\Lambda_i = n_i$ if $i \in I_f$ or $\Lambda_i = \lambda_i$ if $i \in I_m$.

2.2. **Link invariants.** In [22] Queffelec and Sartori proposed an algebraic method to construct the HOMFLY-PT and the Alexander polynomials of links in 3-space. They defined a generalization of the idempotented q-Schur algebra they called the *doubled Schur algebra*. In this section we will briefly recall the basics of the construction in [22] and explain how it fits with parabolic Verma modules.

2.2.1. *The doubled Schur algebra.* In the following we let β be a formal parameter, we write λ for q^{β}, and work over the ring of Laurent polynomials in λ with coefficients in $\mathbb{Q}(q)$. We denote by $\Lambda^{\beta}_{\ell,k}$ the set of sequences $(\mu_{-\ell+1},\dots,\mu_0,\dots,\mu_k) \in (\beta - \mathbb{N}_0)^{\ell} \times \mathbb{N}_0^k$ for $\ell \geqslant 0$ and $k \geqslant 0$.

Remark 2.1. We follow this convention, slightly different from [22], because we want to relate it later with highest weight, rather than lowest weight, parabolic Verma modules, and moreover it will simplify the notation used in the following sections.

Let $I = \{-\ell+1,\dots,0,\dots,k-1\}$. Let $\alpha_i = (0,\dots,0,1,-1,0,\dots,0) \in \mathbb{Z}^{\ell+k}$, the entry 1 being at position $i \in I$. For $\mu_i \in \mathbb{Z} \sqcup \mathbb{Z}+\beta$ let

$$[\mu_i]_q = \begin{cases} \frac{q^{\mu_i}-q^{-\mu_i}}{q-q^{-1}}, & \text{if } \mu_i \in \mathbb{Z},\\ \frac{\lambda q^{\mu_i-\beta}-\lambda^{-1}q^{\beta-\mu_i}}{q-q^{-1}}, & \text{if } \mu_i \in \mathbb{Z}+\beta, \end{cases}$$

be the (shifted) quantum number.

Definition 2.2. The doubled Schur algebra $\dot{S}_{q,\beta}(\ell,k)$ is the $\mathbb{Q}(q)[\lambda^{\pm 1}]$-linear category defined by the following data:

- Objects: 1_μ for $\mu \in \Lambda^{\beta}_{\ell,k}$, together with a zero object.
- Morphisms: generated by morphisms

  $$E_i 1_\mu = 1_{\mu+\alpha_i}E_i \in \mathrm{Hom}(1_\mu, 1_{\mu+\alpha_i}), \qquad F_i 1_\mu = 1_{\mu-\alpha_i}F_i \in \mathrm{Hom}(1_\mu, 1_{\mu-\alpha_i}),$$

  for $i \in I$, together with the identiy morphism of 1_μ (denoted by the same symbol). The morphisms are subject to the relation below:
  (1) $1_\mu 1_\nu = \delta_{\mu,\nu}1_\nu$,
  (2) $(E_iF_j - F_jE_i)1_\mu = \delta_{i,j}[\mu_i - \mu_{i+1}]_q 1_\mu$,
  (3) $E_iE_j1_\mu = E_jE_i1_\mu$ and $F_iF_j1_\mu = F_jF_i1_\mu$ if $|i-j| > 1$, and

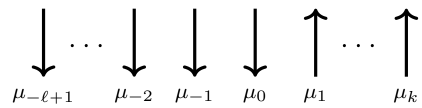
  $$E_i^2E_{i\pm1}1_\mu - [2]_qE_iE_{i\pm1}E_i1_\mu + E_{i\pm1}E_i1_\mu = 0,$$
  $$F_i^2F_{i\pm1}1_\mu - [2]_qF_iF_{i\pm1}F_i1_\mu + F_{i\pm1}F_i1_\mu = 0.$$

  (4) $1_\nu \in \mathrm{End}(1_\nu,1_\nu)$ is zero if $\nu \notin \Lambda^{\beta}_{\ell,k}$.

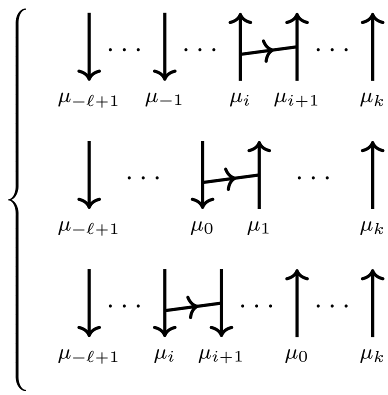
For $\ell = 0$ we recover the idempotented q-Schur algebra $\dot{S}_q(k)_d$, with $d = \mu_1 + \cdots + \mu_k$, and for $k = 0$ we recover $\dot{S}_q(\ell)_d$, with $d = \ell\beta - (\mu_{-\ell+1} + \cdots + \mu_0)$.

2.2.2. *Link invariants from the doubled Schur algebra.* In [22] it was given a diagrammatic presentation for the doubled Schur algebra in terms of webs. These are generated by the ladder operators below.

$1_\mu \mapsto$



$1_{\mu-\alpha_i}F_i1_\mu \mapsto$



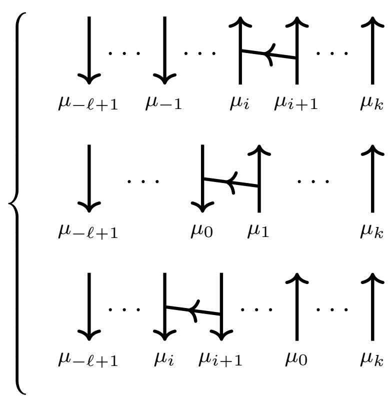
if $i > 0$,

if $i = 0$,

if $i < 0$,

and

$1_{\mu+\alpha_i}E_i1_\mu \mapsto$



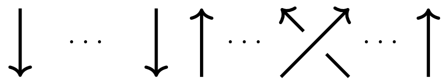
if $i > 0$,

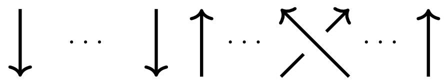
if $i = 0$,

if $i < 0$.

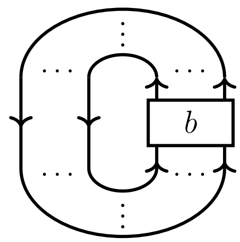
Notice that edges labelled with $\mathbb{N}_0$ are oriented upwards while edges labelled with $\beta - \mathbb{N}_0$ are oriented downwards. Multiplication corresponds to concatenation of diagrams, and in our conventions ab consists of placing the diagram of a on the top of the one for b if the labels match, being zero otherwise. On the $\mathbb{Q}(q)[\lambda^{\pm 1}]$-vector space spanned by all such webs we impose the relations of the doubled Schur algebra from Definition 2.2.

We consider links presented in the form of closures of braids. To start with, let b be a braid diagram in n strands. For such a diagram we assign an element of $\dot{S}_{q,\beta}(n)_n = \dot{S}_{q,\beta}(0,n)$ using a well known rule originally due to Lusztig [13, Definition 5.2.1]. This extends immediately to an element of $\dot{S}_{q,\beta}(n,n)$ from the embedding $\dot{S}_{q,\beta}(n)_n \hookrightarrow \dot{S}_{q,\beta}(n,n)$. Denote by $((\beta-1)^n,(1)^n)$ the label consisting of n entries equal to $\beta - 1$ and n entries equal to 1. For σ_i (resp. σ_i^{-1}) a positive (resp. negative) crossing between the ith and the $(i+1)$th strands counting from the left we have

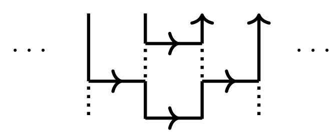
$\sigma_i \mapsto$  $\mapsto q^{-1}1_{((\beta-1)^n,(1)^n)} - F_iE_i1_{((\beta-1)^n,(1)^n)},$

$\sigma_i^{-1} \mapsto$  $\mapsto -F_iE_i1_{((\beta-1)^n,(1)^n)} + q1_{((\beta-1)^n,(1)^n)}.$

We consider closing braids on the left as in the diagram below.

$\mathrm{cl}(b) =$ 

To $\mathrm{cl}(b)$ we assing an element of $\dot{S}_{q,\beta}(n,n)$ obtained by adding cups and the bottom and caps at the top using the following pattern (dotted edges correspond to labels 0 and $\beta - 0$ and solid lines to labels 1 and $\beta - 1$):

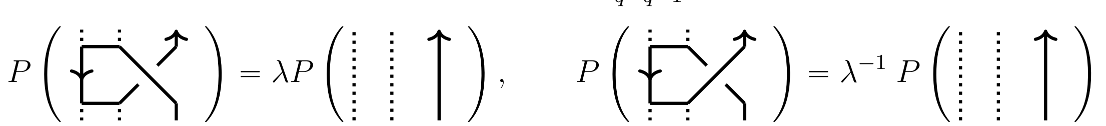
(1) 

and similar for the top of $\mathrm{cl}(b)$.

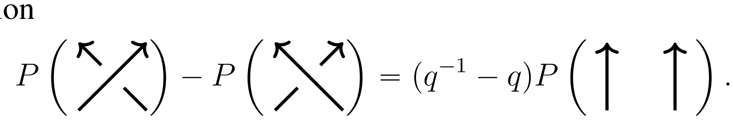
As explained in [22] this procedure gives an element $P(b) \in S_{q,\beta}(n,n)$ which is an endomorphism of $1_{((\beta)^n,(0)^n)}$, which in turn implies that $P(b) \in \mathbb{Q}(q)[\lambda^{\pm 1}]$. One of the main results in [22] is the following.

Theorem 2.3 (Theorems 3.1 and 3.8 in [22])**.** *For a braid b the element $P(b)$ is a framed link invariant which equals the HOMFLY-PT polynomial of the closure of b.*

The proof of Theorem 2.3 goes by first verifying that $P(b)$ is a braid invariant and then checking invariance under the Markov moves. In the process of showing that it equals the HOMFLY-PT polynomials it was shown that it gives the value $\frac{\lambda-\lambda^{-1}}{q-q^{-1}}$ for the unknot, and it satisfies



and the skein relation



Multiplying $P(b)$ by $\lambda^{w(\mathrm{cl}\, b)}$, where $w(\mathrm{cl}\, b)$ is the writhe of $\mathrm{cl}(b)$, results in the usual, framing independent, HOMFLY-PT polynomial.

By the usual specializations of λ we recover the $\mathfrak{sl}_N$ ($\lambda = q^N$) and the Alexander polynomials ($\lambda = 1$) of the closure of b. Note that for the latter one needs to cut open one of the strands to avoid getting the value zero associated to the unknot, and therefore to any link, as explained in [22, §4] (see the discussion on normalized invariants in §2.2.4 below for further details).

2.2.3. *Parabolic Verma modules and link invariants.* Let $\Lambda_+ \subset \Lambda^\beta_{\ell,k}$ be the subset consisting of sequences satisfying

$$\begin{cases} \mu_i - \mu_{i+1} \geqslant 0 & \text{for } i \neq 0, \\ \beta - (\mu_0 - \mu_1) \geqslant 0 & \text{for } i = 0, \end{cases}$$

Introduce a total order on $\Lambda^\beta_{\ell,k}$ by declaring $\mu > \nu$ if $\mu - \nu \in \Lambda_+$.

Definition 2.4. For any $\mu \in \Lambda_+$ we define

$$W(\mu) = \dot{S}_{q,\beta}(\ell,k)1_\mu/[\nu > \mu].$$

Here $[\nu > \mu]$ is the ideal generated by all elements of the form $1_\nu x 1_\mu$, for some $x \in \dot{S}_{q,\beta}(\ell,k)$ and $\nu > \mu$.

For $\ell = 0$ we recover the well-known Weyl modules for the q-Schur algebra $\dot{S}_q(k)_{\mu_1+\cdots+\mu_k}$. As in the case of $\ell = 0$, it is also true that $U_q(\mathfrak{gl}_{k+\ell})$ acts on $W(\mu)$: for $F_i \in U_q(\mathfrak{gl}_{k+\ell})$ and $1_\nu x 1_\mu \in W(\mu)$ we put $F_i.1_\nu x 1_\mu = 1_{\nu-\alpha_i} F_i x 1_\mu$ and similarly for $E_i \in U_q(\mathfrak{gl}_{k+\ell})$. Note that the Chevalley generators of $U_q(\mathfrak{gl}_{k+\ell})$ are indexed from $\{-\ell+1,\dots,k-1\}$, which is the set I introduced in the definition of $\dot{S}_{q,\beta}(\ell,k)$.

Let $\mathfrak{p}$ be the parabolic subalgebra obtained by adjoining the Chevalley generators F_i for $i \in I\backslash\{0\}$ to the Borel of $\mathfrak{gl}_{k+\ell}$.

Proposition 2.5. *For $\mu \in \Lambda_+$, the module $W(\mu)$ is isomorphic with the parabolic Verma module $M^{\mathfrak{p}}(\mu)$ as modules over $U_q(\mathfrak{gl}_{k+\ell})$.*

Proof. This follows at once from the fact that both $U_q(\mathfrak{gl}_{2k})$-modules are generated by a single vector of weight $((\beta)^\ell,(0)^k)$ and the PBW theorem, which is also true for $\dot{S}_{q,\beta}(\ell,k)$, and shows that the weights occuring in both modules are the same. ☐

From the previous section it is clear that all weights occurring in $P(b) \in \dot{S}_{q,\beta}(n,n)$ are smaller w.r.t. $>$ than $((\beta)^n,(0)^n)$, and thus $P(b)$ is sent to the same word in E's and F's under the quotient map $\dot{S}_{q,\beta}(n,n) \to W((\beta)^n,(0)^n)$. This means $P(b)$ acts on $W((\beta)^n,(0)^n)$ as an endomorphism of the highest weight object which equals the HOMFLY-PT poynomial of the closure of b.

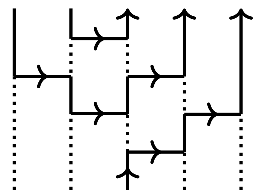
Notation. From now on, for the sake of keeping the notation simple we denote the highest weight modules $W((\beta)^n,(0)^n)$ and $M^{\mathfrak{p}}((\beta)^n,(0)^n)$ by $W(\beta)$ and $M^{\mathfrak{p}}(\beta)$ respectively.

Under the isomorphism in Proposition 2.5, this defines an endomorphism $P^{\mathfrak{p}}(b)$ of the highest weight object of the scalar Verma module $M^{\mathfrak{p}}(\beta)$ (seen as a linear category with objects indexed by the weights, in the obvious way) consisting of the same word in E's and F's as in $P(b)$, which yields the same element in $\mathbb{Q}(q)[\lambda^{\pm 1}]$.

Theorem 2.6. *For a braid b the element $\lambda^{w(\mathrm{cl}(b))}P^{\mathfrak{p}}(b) \in \mathrm{End}_{U_q(\mathfrak{gl}_{2n})}\big(M^{\mathfrak{p}}(\beta)\big)$ is a link invariant which equals the HOMFLY-PT polynomial of the closure of b.*

2.2.4. *Normalized link invariants.* In order to be able to compute the normalized HOMFLY-PT and $\mathfrak{sl}_N$-link invariants we follow the procedure described in [22, §4]. It consists in cut opening the braid closure diagram $\mathrm{cl}(b)$ into a special type of $(1,1)$-tangle diagram.

We open the diagram of $\mathrm{cl}(b)$ by cutting the outermost strand, following a pattern as shown in the example below for a braid with three strands,

(2) 

and the similar for the top part. We denote $\mathrm{cl}_o(b)$ the diagram obtained with this procedure.

The procedure described in §2.2.2 gives an element $\overline{P}(b) \in S_{q,\beta}(n-1,n)$ which is an endomorphism of $1_{((\beta)^{n-1},1,(0)^{n-1})}$, which in turn implies that $\overline{P}(b) \in \mathbb{Q}(q)[\lambda^{\pm 1}]$ (see [22, §4] for details).

Theorem 2.7 (Proposition 4.6 in [22])**.** *For a braid b the element $\overline{P}(b)$ is a framed link invariant which equals the reduced HOMFLY-PT polynomial of the closure of b.*

Notice we could have opened the diagram in a different way, by choosing a different strand to cut it open. We could have equally opened the diagram by cutting it using one of the inner strands at the expense of adding crossings to the original diagram. In [22] the authors proved that the link invariants obtained do not depend on this choice.

In order to parallel the construction of §2.2.3 using a parabolic Verma module, we consider $\mathfrak{gl}_{2n-1}$ with simple roots $\{2-n,\dots,n-1\}$. We form the parabolic subalgebra $\bar{\mathfrak{p}}$ given by adjoining $\{F_{\pm 1},\dots,F_{\pm(n-2)},F_{n-1}\}$ to the Borel of $\mathfrak{gl}_{2n-1}$ (that is all Chevalley generators of $\mathfrak{gl}_{2n-1}$ except F_0). Then we look at the parabolic Verma module $M^{\bar{\mathfrak{p}}}(\overline{\beta}) := M^{\bar{\mathfrak{p}}}((\beta)^{n-1},1,(0)^{n-1})$. Note that the irreducible module V used to construct this parabolic Verma module is two-dimensional with F_1 acting non-trivially on the highest weight vector v_0 (and thus the module is no longer of scalar type), and K_0 acts on v_0 as $q^{-1}\lambda$ (instead of λ in $\mathfrak{p}$).

The method described in §2.2.3 defines an endomorphism $\overline{P}^{\bar{\mathfrak{p}}}(b)$ of the highest weight object of $M^{\bar{\mathfrak{p}}}(\overline{\beta})$. The following is an easy consequence of the paragraphs above.

Theorem 2.8. *For a braid b the element $\lambda^{w(\mathrm{cl}_o(b))}\overline{P}^{\bar{\mathfrak{p}}}(b) \in \mathrm{End}_{U_q(\mathfrak{gl}_{2n-1})}\big(M^{\bar{\mathfrak{p}}}(\overline{\beta})\big)$ is a link invariant which equals the reduced HOMFLY-PT polynomial of the closure of b.*

Remark 2.9. From now on, for the means of higher representation theory, we will consider the parabolic Verma modules $M^{\mathfrak{p}}$ and $M^{\bar{\mathfrak{p}}}(\overline{\beta})$ as living in $\mathbb{Q}(\!(q,\lambda)\!) \supset \mathbb{Q}(q)[\lambda^{\pm 1}]$ with the polynomial fractions viewed as formal power series. See [18, §5.3] for more about rings of formal Laurent series in the context of categorification, see also [1] for a more general discussion about these rings.

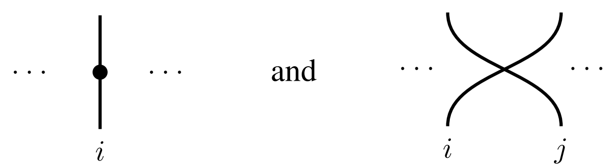
3. Parabolic 2-Verma modules

In this section we describe superalgebras which can be thought of as a mild mix of the Khovanov–Lauda–Rouquier algebras from [9, 24] and the (super)algebra A_m defined by the authors in [18]. We will follow the diagrammatic presentation for A_m given in [19]. These superalgebras contain KLR algebras as subalgebras concentrated in even parity. A general construction, giving all parabolic 2-Verma modules for Kac–Moody algebras, is described in [20]. Here we focus on the cases of the universal 2-Verma module, and the parabolic 2-Verma modules associated to the parabolic subalgebras $\mathfrak{p} \subseteq U_q(\mathfrak{gl}_{2n})$ and $\bar{\mathfrak{p}} \subseteq U_q(\mathfrak{gl}_{2n-1})$ from §2.2.3 and §2.2.4, respectively.

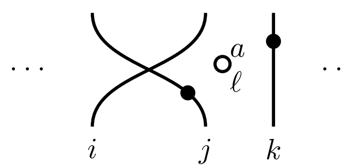
3.1. **Universal 2-Verma module of $U_q(\mathfrak{gl}_{2n})$.** Despite the fact the $\mathfrak{p}$-KLR and $\bar{\mathfrak{p}}$-KLR algebras can be defined in an ad-hoc way, we prefer to obtain them through the $\mathfrak{b}$-KLR algebra. This extra structure will be particularly precious for the proof of Theorem 4.7 in §4.6.

Recall that we identify the set of simple roots I with $\{0, \pm 1, \dots, \pm(n-1)\}$.

3.1.1. *The $\mathfrak{b}$-KLR algebra.* For each $\nu = \sum_{i \in I} \nu_i . i \in \mathbb{N}_0[I]$, we consider the collection of KLR diagrams $R(\nu)$, for the quiver A_{2n-1}, with bit of additional structure. Besides the KLR generators

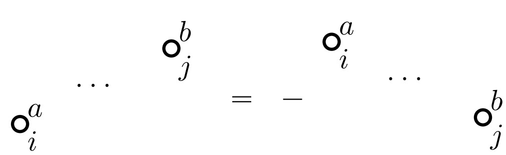


for all $i, j \in I$, regions can be decorated with *floating dots*, as in the example below.



Floating dots are labeled by a simple roots as subscript (e.g. $\ell \in I$ in the diagram above), and by a non-negative integer as superscript ($a \in \mathbb{N}_0$ in the example). By convention, we don't write a superscript when it is 0 and refer to the respective floating dot as *unlabeled*.

Floating dots are *odd* while all other KLR generators are *even*. This means we endow diagrams with a *parity* and declare floating dots to have parity one and (KLR) dots and crossings to have parity zero. The diagrams are equiped with a Morse height function that keeps track of the relative height of floating dots and forbids two floating dots to be at the same height. We allow all isotopies with respect to this Morse function without creating any critical point. We let floating dots anticommute. In terms of diagrams,

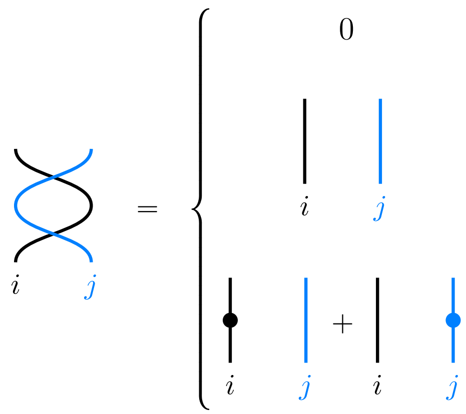


and it means a region cannot carry more than one floating dot (otherwise it is zero).

Let $\Bbbk$ denote the field of rationnals. Denote by $R_{\mathfrak{b}}(\nu)$ the $\Bbbk$-algebra obtained by linear combinations of these diagrams, together with multiplication given by gluing diagrams on top of each other whenever the labels agree, and zero otherwise. In our conventions ab means stacking the diagram for a atop the one for b, whenever they are composable (this is a consequence from the fact that we read diagrams from bottom to top). The diagrams are subject to the local relations (3) to (10) below.

• *The KLR relations:* For all $i, j, k \in I$ we have

(3)



if $i = j$,

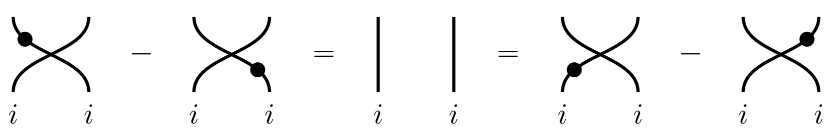
if $|i - j| > 1$,

if $|i - j| = 1$,

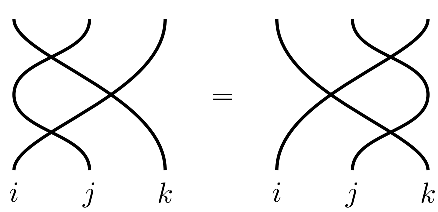
(4)

for $i \neq j$,

(5)

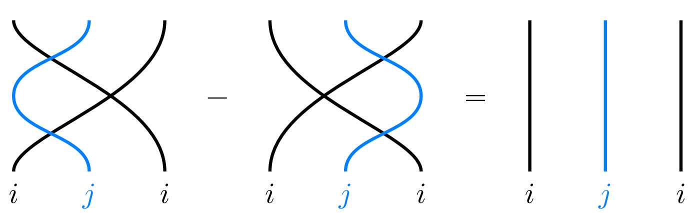


(6)



unless $i = k$ and $|i - j| = 1$,

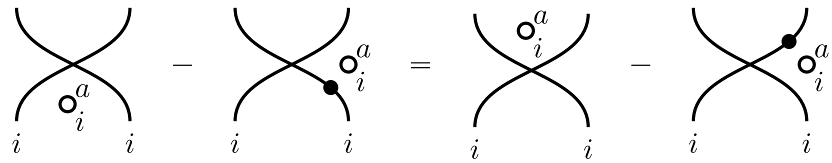
(7)



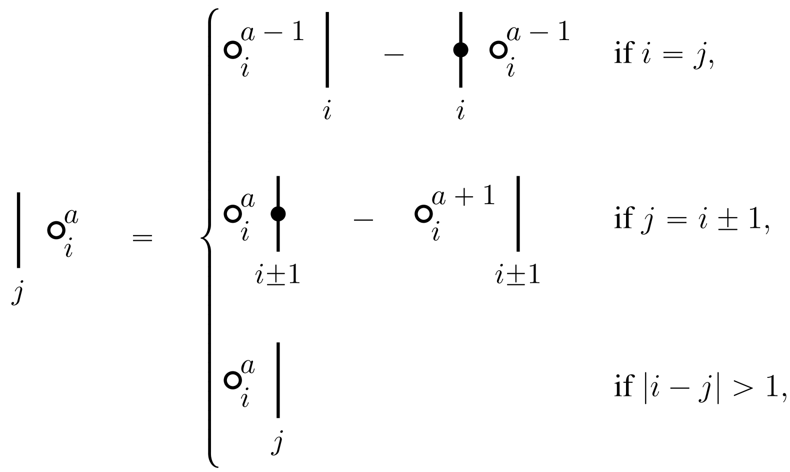
if $|i - j| = 1$,

• *The new relations:* For all $i, j \in I$ and $a \in \mathbb{N}_0$, we put

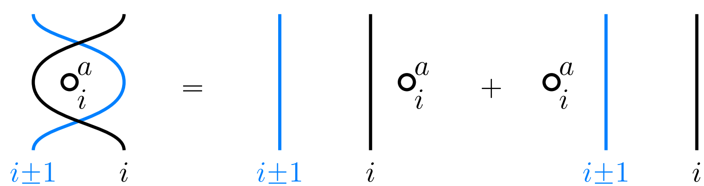
(8)



(9)



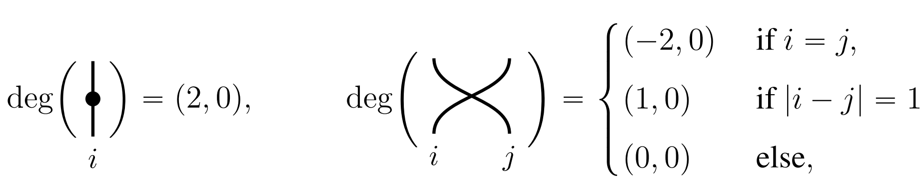
(10)



We also impose a floating dot in the left-most region of a diagram to be zero:

(11)

We turn $R_{\mathfrak{b}}(\nu)$ into a $\mathbb{Z} \times \mathbb{Z}^{2n-1}$-graded superalgebra. We refer to the first grading as the q-degree and the second as Λ-grading, where $\Lambda = \{\lambda_i\}_{i \in I}$. For $u \in R_{\mathfrak{b}}(\nu)$ we write $\deg(u) = (q(u), \Lambda(u))$, with $q(u)$ being the q-degree of u and $\Lambda(u) = \sum_{i \in I} \lambda_i(u) \cdot i$. We will also write $\lambda(u)$ for $\lambda_0(u)$ and call it the λ-grading. Finally, $p(u)$ is the parity of u. For the KLR generators we extend the KLR grading trivially to a multigrading by setting



and they have parity zero. In order to define the degree of a floating dot, let ℓ_i be the number of strands labeled i at its left for any $i \in I$. We declare that

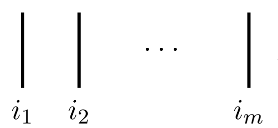
$$q\big(\circ^{a}_{i} \big) = 2(1 + a - \ell_i + \ell_{i\pm 1}), \qquad \lambda_j\big(\circ^{a}_{i} \big) = 2\delta_{ij}, \qquad p\big(\circ^{a}_{i} \big) = 1.$$

It is immediate that relations (3) to (10) above preserve the multigrading and the homological degree. We denote by a monomial $q^r \lambda_{1-n}^{s_{1-n}} \dots \lambda_{n-1}^{s_{n-1}}$ a shift by r units up in the q-grading and s_i units up in the λ_i-grading. We write a shift in parity by Π.

Definition 3.1. We define the $\mathfrak{b}$-KLR superalgebra as

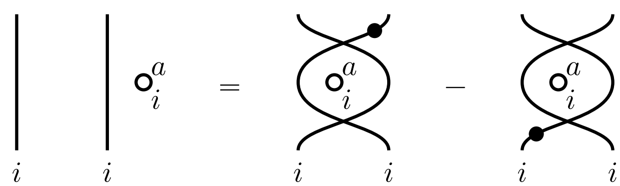
$$R_{\mathfrak{b}} = \bigoplus_{\nu \in \mathbb{N}_0[I]} R_{\mathfrak{b}}(\nu).$$

3.1.2. *A basis for $R_{\mathfrak{b}}$.* We will now construct a basis for $R_{\mathfrak{b}}$, viewed as a $\Bbbk$-module. Let us first introduce some notations borrowed from [9]. We denote by $\mathrm{Seq}(\nu)$ the set of all ordered sequences $\boldsymbol{i} = i_1 i_2 \dots i_m$ with each $i_k \in I$ and i appearing ν_i times in the sequence. We let $e_{\boldsymbol{i}} \in R_{\mathfrak{b}}(\nu)$ be the idempotent given by m vertical strands with labels in the order induced by $\boldsymbol{i}$,

$e_{\boldsymbol{i}} =$ 

We say a floating dot is in *tight position*, or that is is tight, if it is unlabeled and located in the region immediately at the right of the leftmost strand. By (11) and (9), we can also assume that this leftmost strand is labeled by the same label as the subscript of the floating dot since it would give zero otherwise.

Proposition 3.2. *The following equality holds in $R_{\mathfrak{b}}$*

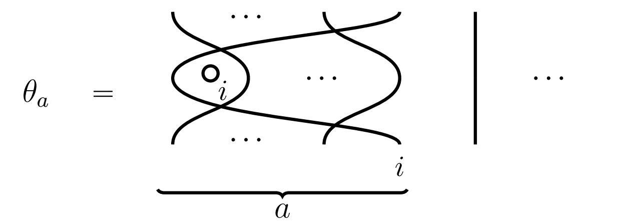


Proof. It is an immediate consequence of (3), (5) and (8). $\square$

Relations (9) and (10), and Proposition 3.2 imply that any diagram in $R_{\mathfrak{b}}$ can be written as a linear combination of diagrams involving KLR generators and only tight floating dots.

Lemma 3.3. *As a $\Bbbk$-module, $R_{\mathfrak{b}}$ is generated by diagrams involving only KLR generators and tight floating dots.*

Let ω denote a tight floating dot in a diagram given by m vertical strands with labels supposed to be given by the context. Let τ_a be a crossing between the a-th and the $(a+1)$-th strands. We also write $\tau_w = \tau_{\ell_r} \dots \tau_{\ell_1}$ for a reduced expression $w = s_{\ell_r} \dots s_{\ell_1} \in S_m$, and we let x_i denote a dot on the ith strand. Define the *tightened floating dot* $\theta_a = \tau_{a-1} \dots \tau_1 \omega \tau_1 \dots \tau_{a-1}$,



where the strand pulled to the left is the ath one and its label matches the subscript of the floating dot.

Fix $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)$. Since they are both sequences of the same elements, there is a subset ${}_{\boldsymbol{j}}S_{\boldsymbol{i}} \subset S_m$ of permutations $\phi \in S_m$ such that $i_k = j_{\phi(k)}$ for all $k \in \{1, \dots, m\}$. Suppose we have chosen a reduced expression for a fixed permutation $\phi = s_{\ell_r} \dots s_{\ell_1} \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$. Up to choosing another equivalent reduced expression, we can also suppose it is *left-adjusted*. By this we mean $\ell_r + \cdots + \ell_1$ is minimal. If we write $O_{s_{\ell_r} \dots s_{\ell_1}}(k) = \{s_{\ell_t} \dots s_{\ell_1}(k) | 0 \leqslant t \leqslant r\}$, then $s_{\ell_r} \dots s_{\ell_1}$ is left-adjusted if and only if

$$\min O_{s_{\ell_r} \dots s_{\ell_1}}(k) \leqslant \min O_{s_{\ell'_r} \dots s_{\ell'_1}}(k)$$

for all k and reduced expressions $s_{\ell'_r} \dots s_{\ell'_1} = \phi$.

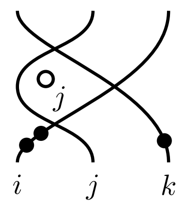
If we view $s_{\ell_r} \dots s_{\ell_1}$ as a diagram where each s_k corresponds to a crossing τ_k, it is given by the diagram where each strand is "pulled as far as possible to the left without creating double intersections". For example, the following permutation has two reduced expressions and we take the one on the left:

In general one can transform a reduced expression into a left-adjusted one by applying a sequence of braid moves, $s_{k+1}s_k s_{k+1} \to s_k s_{k+1} s_k$, and distant crossing permutations, transforming each triplet of intersections as in the example above from right to left. The left-adjusted reduced expression is unique up to distant crossings permutations.

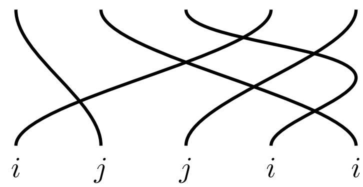
Now fix a left-adjusted reduced expression $\phi = s_{\ell_r} \dots s_{\ell_1} \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$, and choose m-uples $u = (u_1, \dots, u_m) \in \mathbb{N}_0^m$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \{0,1\}^m$. To such a triplet $(u, \alpha, s_{\ell_r} \dots s_{\ell_1})$ we associate an element of $e_{\boldsymbol{j}} R_{\mathfrak{p}}(\nu) e_{\boldsymbol{i}}$ given by:

- we take the braid-like diagram in which crossings correspond with the simple transpositions in the chosen left-adjusted reduced expression $s_{\ell_r} \dots s_{\ell_1}$,
- for each $\alpha_k = 1$, we look where the kth strand (counting from bottom left) attains its left-most position (in the sense that there is a minimal number of other strands at its left) and we insert a thightened floating dot at this position: that is we pull the strand to the far left at this height, and add a floating dot with matching label immediately at its right,
- we add u_k dots at the bottom of the kth strand for all k.

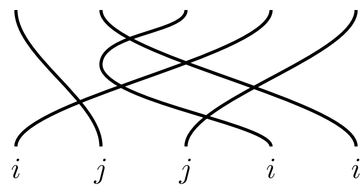
Example 3.4. *For example, with $\boldsymbol{i} = ijk$, $\boldsymbol{j} = kji$, $u = (2,0,1)$ and $\alpha = (0,1,0)$ we could get*



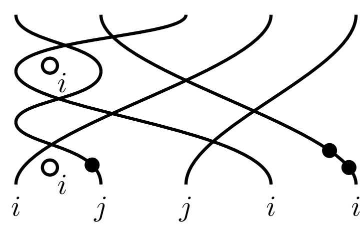
Example 3.5. *We take $\boldsymbol{i} = ijjii$ and $\boldsymbol{j} = jiiij$. We choose the following permutation*



It is not left-adjusted as we can pull the 4th strand to the left, so by applying two braid moves we get the following left-adjusted reduced expression



Now if we choose $u = (0,1,0,0,2)$ and $\alpha = (1,0,0,1,0)$, then applying the construction as above yields the following element of $e_{\boldsymbol{j}} R_{\mathfrak{p}}(\nu) e_{\boldsymbol{i}}$

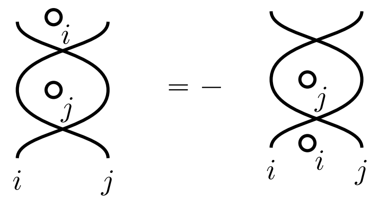


where we can see that the fourth strand has been pulled to the left to add a tight floating dot.

Let ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ denote the set of diagrams from $\boldsymbol{i}$ to $\boldsymbol{j}$ given by all choices of x, α and permutations ϕ, with a fixed choice of presentation $\phi = s_{l_r} \dots s_{l_1}$ for each of them.

Lemma 3.6. *Elements in ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ generate the $\mathbb{k}$-module $e_{\boldsymbol{j}} R_{\mathfrak{b}}(\nu) e_{\boldsymbol{i}}$.*

Proof. By Lemma 3.3, we can assume all floating dots are tight. Then the proof follows by an inductive argument on the number of strands, using the fact we can apply all braid relations and (KLR) dots slide at the cost of adding diagrams with fewer crossings, thanks to (3) and (6-7). Indeed, it means we can work up to braid isotopy and assume all dots are concentrated in the bottom of the diagram, allowing us to isolate each (tight) floating dot with only left-adjusted reduced type diagrams in-between. Then we observe that a strand can have up to only one tight floating at its immediate right, otherwise it would be zero because of the relation



that can be deduced from (3-10). Finally, for the same reasons, one can show that two tightened floating dots θ_a and θ_b anti-commute (up to adding diagrams with fewer crossings), so that we can bring any diagram to a linear combination of diagrams in ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$. See [20, Lemma 3.13] for more details. ☐

To prove ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ gives a basis for $e_{\boldsymbol{j}}R_{\mathfrak{b}}e_{\boldsymbol{i}}$, we first need to construct an action of $R_{\mathfrak{b}}$ on some polynomial-type ring.

3.1.3. *Polynomial action.* We fix $\nu \in \mathbb{N}_0[I]$ with $|\nu| = m$. For each simple root $i \in I$ we define

$$P_i = \Bbbk[x_{1,i}, \dots, x_{\nu_i,i}] \otimes_{\Bbbk} \textstyle\bigwedge^\bullet \langle \omega_{1,i}, \dots, \omega_{\nu_i,i} \rangle,$$

where $\bigwedge^\bullet \langle \omega_{1,i}, \dots, \omega_{\nu_i,i} \rangle$ is the exterior algebra of the $\Bbbk$-vector space generated by $\omega_{1,i}, \dots, \omega_{\nu_i,i}$. We write $P_I = \bigotimes_{i \in I} P_i$ where $\otimes$ means the supertensor product, so that P_I is a multigraded superring with $\deg(x_{r,i}) = (2,0)$ and $q(\omega_{r,i}) = 2(1-r)$, $\lambda_j(\omega_{r,i}) = 2\delta_{ij}$. Then we define

$$P_\nu = \bigoplus_{\boldsymbol{i} \in \mathrm{Seq}(\nu)} P_I 1_{\boldsymbol{i}},$$

where the $1_{\boldsymbol{i}}$'s are central idempotents.

There is an action of S_m on P_ν given by

$$s_k : P_I 1_{\boldsymbol{i}} \to P_I 1_{s_k \boldsymbol{i}},$$

sending

$$x_{p,j} 1_{\boldsymbol{i}} \mapsto \begin{cases} x_{p+1,j} 1_{s_k \boldsymbol{i}}, & \text{if } i_k = i_{k+1} = j \text{ and } p = \#\{s \leqslant k | i_s = j\}, \\ x_{p-1,j} 1_{s_k \boldsymbol{i}}, & \text{if } i_k = i_{k+1} = j \text{ and } p = 1 + \#\{s \leqslant k | i_s = j\}, \\ x_{p,j} 1_{s_k \boldsymbol{i}}, & \text{otherwise,} \end{cases}$$

and

$$\omega_{p,j} 1_{\boldsymbol{i}} \mapsto \begin{cases} \left(\omega_{p,j} + (x_{p,j} - x_{p+1,j})\omega_{p+1,j}\right) 1_{s_k \boldsymbol{i}}, & \text{if } i_k = i_{k+1} = j \text{ and } p = \#\{s \leqslant k | i_s = j\}, \\ \omega_{p,j} 1_{s_k \boldsymbol{i}}, & \text{otherwise.} \end{cases}$$

Note that the condition $p = \#\{s \leqslant k | i_s = j\}$ means i_k is the pth element in $\boldsymbol{i}$ with value $i_k = j$.

Proposition 3.7. *The operations above describe an action of the symmetric group S_m on P_ν.*

Proof. It is a straightforward generalization of [19, §2.1]. ☐

Then we define

$$\omega_{r,i}^{a} = \sum_{\ell=1}^{r} (-1)^{r+a+\ell} h_{a+\ell-r}(x_{\ell,i}, \dots, x_{r,i}) \omega_{\ell,i},$$

where $h_n(x_{\ell,i}, \dots, x_{r,i})$ is the nth complete homogeneous symmetric polynomial in variables $(x_{\ell,i}, \dots, x_{r,i})$. It is defined so that

$$\omega_{r,i}^{0} = \omega_{r,i}, \qquad \omega_{r,i}^{a} = \omega_{r-1,i}^{a-1} - x_{r,i}\omega_{r,i}^{a-1}.$$

We now have all the tools we need to define an action of $R_{\mathfrak{p}}(\nu)$ on P_ν. First we let $a \in R_{\mathfrak{b}}(\nu)e_{\boldsymbol{i}}$ act as zero on $P_I 1_{\boldsymbol{j}}$ whenever $\boldsymbol{j} \neq \boldsymbol{i}$. Otherwise, we declare that

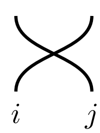
- the dot

$$\longmapsto x_{\ell_i+1,i} 1_{\boldsymbol{i}},$$

  acts as multiplication by $x_{\ell_i+1,i}1_{\boldsymbol{i}}$ with ℓ_i being the number of strands labeled i at the left,
- the floating dot acts as multiplication

$$\circ_j^a \longmapsto \sum_{r=0}^{\ell_{j\pm1}} (-1)^r \omega_{\ell_j}^{a+r}(\boldsymbol{i}) e_{\ell_{j\pm1}-r}(x_{1,j-1}, \dots, x_{\ell_{j-1},j-1}, x_{1,j+1}, \dots, x_{\ell_{j+1},j+1}),$$

  where $\ell_{j\pm1}$ is the number of strands labeled $j+1$ or $j-1$ at the left, and $e_n(\dots)$ is the nth elementary symmetric polynomial in variables $x_{k,j\pm1}$ for $k \leqslant \ell_{j\pm1}$,
- the crossing between the k-th and $(k+1)$-th strands,



  acts by

$$\begin{aligned}
f1_{\boldsymbol{i}} &\mapsto s_k(f_{\boldsymbol{i}}) && \text{if } |i-j| > 1,\\
f1_{\boldsymbol{i}} &\mapsto \frac{f1_{\boldsymbol{i}} - s_k(f1_{\boldsymbol{i}})}{x_{\ell_i+1,i} - x_{\ell_i+2,i}} && \text{if } i = j,\\
f1_{\boldsymbol{i}} &\mapsto (x_{\ell_i+1,i} + x_{\ell_j+1,j}) s_k(f) && \text{if } i = j+1,\\
f1_{\boldsymbol{i}} &\mapsto s_k(f) && \text{if } i = j-1.
\end{aligned}$$

Proposition 3.8. *The rules above define an action of $R_{\mathfrak{b}}(\nu)$ on P_ν.*

Proof. The proof is a computation, checking that the construction respects all the relations from the definition of $R_{\mathfrak{b}}$. This is done in detail in [20] **(Maybe we give them here as well)**. □

Theorem 3.9. *The family ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ is a basis for $e_{\boldsymbol{j}}R_{\mathfrak{b}}e_{\boldsymbol{i}}$ seen as a $\Bbbk$-module.*

Proof. The elements of ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ act as linearly independent endomorphisms on P_ν. The proof is similar to [25, Proposition 3.8] and the details are given in [20] **(Maybe we give them here as well)**. □

3.1.4. *Decomposition of* $R_{\mathfrak{b}}$. Our goal now is to describe how $R_{\mathfrak{b}}(m)$ can be decomposed as a module over $R_{\mathfrak{b}}(m-1)$, where $R_{\mathfrak{b}}(m)$ is the subalgebra of $R_{\mathfrak{b}}$ given by diagrams with m strands. We then use it to state a categorical version of the $\mathfrak{sl}_2$-commutator relations for each simple root. The following subsection is inspired by the work of Kang–Kashiwara in [7], and is a specialization of [20, §4] to the $\mathfrak{sl}_n$ case.

Denote by $R_{\mathfrak{b}}(m,i)$ the subalgebra given by diagrams with m strands together with a vertical strand with label i at the right. Similarly, denote by $R_{\mathfrak{b}}(\nu,i)$ the collection of diagrams $D \in R_{\mathfrak{b}}(m,i)$ such that $\mathrm{bot}(D) = \boldsymbol{j}i$ with $\boldsymbol{j} \in \mathrm{Seq}(\nu)$. The idempotents $e_{(m,i)}$ and $e_{(\nu,i)}$ are given respectively by the sum of all diagrams with $m+1$ vertical strands in $R_{\mathfrak{b}}(m,i)$ and in $R_{\mathfrak{b}}(\nu,i)$. These definitions extend in general for $R_{\mathfrak{b}}(m,\boldsymbol{i}) \subset R_{\mathfrak{b}}(m+|\boldsymbol{i}|)$ with $\boldsymbol{i}$ some sequence in I, and for $R_{\mathfrak{b}}(\boldsymbol{i},m) \subset R_{\mathfrak{b}}(m+|\boldsymbol{i}|)$. We will write $\otimes$ for $\otimes_{\Bbbk}$ and $\otimes_m$ for $\otimes_{R_{\mathfrak{b}}(m)}$ throughout this section.

For the sake of simplicity we now fix notation that we will use throughout this section and after. For a left, right or bimodule M and a free multigraded abelian (super)group $A \cong \bigoplus_{x\in X} x\mathbb{Z}$, we will write

$$M \otimes A = \bigoplus_{x\in X} q^{q(x)}\lambda_{1-n}^{\lambda_{1-n}(x)} \dots \lambda_{n-1}^{\lambda_{n-1}(x)} \Pi^{p(x)} M.$$

When putting an element x of $R_{\mathfrak{b}}$ next to a group A, it means we take the shift

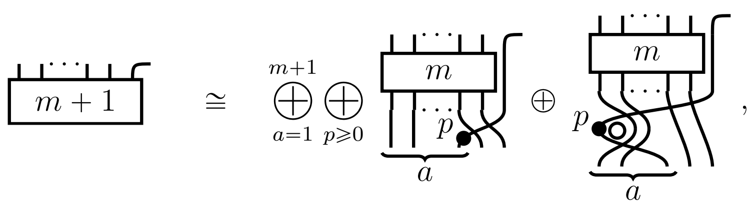
$$xA = q^{q(x)}\lambda_{1-n}^{\lambda_{1-n}(x)} \dots \lambda_{n-1}^{\lambda_{n-1}(x)} \Pi^{p(x)} A.$$

Proposition 3.10. *As a left* $R_{\mathfrak{b}}(m)$*-module,* $R_{\mathfrak{b}}(m+1)$ *decomposes as*

$$\bigoplus_{a=1}^{m+1} R_{\mathfrak{p}}(m) \otimes \tau_m \dots \tau_a(\mathbb{Z}[x_a] \oplus \theta_a[x_1]),$$

where $\theta_a[x_1] = \tau_{a-1}\dots\tau_1\omega\mathbb{Z}[x_1]\tau_1\dots\tau_{a-1}$.

The statement can be informally written in terms of diagrams as



where the strand pulled to the upper right is the ath one, counting from left, and the subscript of the floating dot is determined by the label of this strand.

Proof. The right coset decomposition of S_{m+1},

$$S_{m+1} = \bigsqcup_{a=1}^{m+1} S_m s_m \dots s_a,$$

gives a choice of minimal reduced presentations to construct ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$. By Theorem 3.9, this is a basis and so we get the decomposition as claimed, up to sliding some dots over crossings (all dots are concentrated in the bottom in Theorem 3.9). An easy exercise shows it still gives a basis (see [20, Corollary 3.16]). ∎

Lemma 3.11. *As a right $R_{\mathfrak{b}}(m)$-module, $R_{\mathfrak{b}}(m+1)$ admits a decomposition*

$$\bigoplus_{a=1}^{m+1} (\mathbb{Z}[x_a] \oplus \theta_a[x_1])\tau_a \dots \tau_m \otimes R_{\mathfrak{b}}(m)$$

and if $y \in \theta_{m+1}[x_1] \otimes R_{\mathfrak{b}}(m)$ then $y - w \in R_{\mathfrak{b}}(m) \otimes \theta_{m+1}[x_1]$ for some $w \notin R_{\mathfrak{b}}(m) \otimes \theta_{m+1}[x_1]$.

Proof. The first result is the symmetric of Proposition 3.10. The second follows from the same arguments as in [19, Lemma 3.5], which basically consist in showing one can slide elements from $R_{\mathfrak{b}}(m)$ over $\theta_{m+1}[x_1]$ at the cost of adding terms not contained in $R_{\mathfrak{b}}(m) \otimes \theta_{m+1}[x_1]$. ☐

The second statement of Lemma 3.11 means the projections of an element in $R_{\mathfrak{b}}(m+1)$ onto the summands $R_{\mathfrak{b}}(m)\theta_{m+1}[x_1]$ or $\theta_{m+1}[x_1]R_{\mathfrak{b}}(m)$ when viewed as left or right $R_{\mathfrak{b}}(m)$-module give the same result. Hence the projection $R_{\mathfrak{b}}(m+1) \to R_{\mathfrak{b}}(m)\theta_{m+1}[x_1]$ is a map of bimodules.

We are now able to construct short exact sequences that will be used to construct the categorical $\mathfrak{sl}_2$ commutator relations.

Proposition 3.12. *For all $i \in I$ there is a short exact sequence of $(R_{\mathfrak{b}}(m), R_{\mathfrak{b}}(m))$-bimodules*

$$\begin{aligned} 0 \to q^{-2} R_{\mathfrak{b}}(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)} R_{\mathfrak{b}}(m) \to e_{(m,i)} R_{\mathfrak{b}}(m+1) e_{(m,i)} \\ \to R_{\mathfrak{b}}(m) \otimes \mathbb{Z}[\xi] \oplus R_{\mathfrak{b}}(m) \otimes \theta_{m+1}\mathbb{Z}[\xi] \to 0, \end{aligned}$$

where $\theta_{m+1}e_{(\nu)}$ means a shift $q^{2\nu_{i\pm 1} - 4\nu_i}\lambda_i^2\Pi$ and $\deg(\xi) = (2,0)$. There are also isomorphisms of $(R_{\mathfrak{b}}(m), R_{\mathfrak{b}}(m))$-bimodules

$$e_{(m,i)} R_{\mathfrak{b}}(m+1) e_{(m,j)} \simeq q^{-i\cdot j} R_{\mathfrak{b}}(m) e_{(m-1,j)} \otimes_{m-1} e_{(m-1,i)} R_{\mathfrak{b}}(m),$$

for $i \neq j \in I$.

Proof. There is a $(R_{\mathfrak{b}}(m), R_{\mathfrak{b}}(m))$-bimodule morphism $s : R_{\mathfrak{b}}(m) \otimes_{m-1} R_{\mathfrak{b}}(m) \to R_{\mathfrak{b}}(m+1)$ given by

$$x \otimes y \mapsto x\tau_m y.$$

For the short exact sequence, we define the right arrow as the projection on the summands $R_{\mathfrak{b}}(m) \otimes \mathbb{Z}[x_{m+1}]$ and $R_{\mathfrak{b}}(m) \otimes \theta_{m+1}[x_1]$ in the decomposition from Proposition 3.10. By Lemma 3.11, this yields a bimodule morphism. The exactness follows also from Proposition 3.10 with the observation

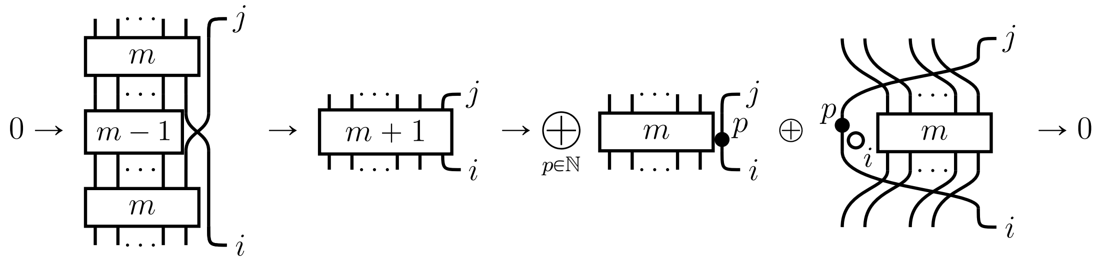
$$R_{\mathfrak{b}}(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)} R_{\mathfrak{b}}(m) \cong \bigoplus_{a=1}^{m} R_{\mathfrak{b}}(m)e_{(m-1,i)} \otimes \tau_{m-1} \dots \tau_a (\mathbb{Z}[x_a] \oplus \theta_a[x_1]),$$

thus

$$s\left(R_{\mathfrak{b}}(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)} R_{\mathfrak{b}}(m)\right) \cong \bigoplus_{a=1}^{m} R_{\mathfrak{b}}(m)e_{(m-1,i)} \otimes \tau_m \tau_{m-1} \dots \tau_a (\mathbb{Z}[x_a] \oplus \theta_a[x_1]).$$

Therefore, each direct summand of $R_{\mathfrak{b}}(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)} R_{\mathfrak{b}}(m)$ is identified with a direct summand of $e_{(m,i)} R_{\mathfrak{b}}(m+1) e_{(m,i)}$, except for $R_{\mathfrak{b}}(m) \otimes \mathbb{Z}[x_{m+1}]$ and $R_{\mathfrak{b}}(m) \otimes \theta_{m+1}[x_1]$. Since they form the right part of the sequence, we conclude it is a short exact sequence. ☐

We can informally view the short exact sequences and the isomorphisms of Proposition 3.12 in terms of diagrams as



with the cokernel vanishing whenever $i \neq j$. *Nota Bene*: we think of the rightmost picture not as the bimodule generated by those diagrams but as the quotient of $e_{(\nu,i)}R_{\mathfrak{b}}(m+1)e_{(\nu,i)}$ corresponding to such diagrams.

The categorical Serre relations are interpreted in $R_{\mathfrak{b}}$ as the following.

Proposition 3.13. *We have isomorphisms of* $(R_{\mathfrak{b}}(m+2), R_{\mathfrak{b}}(m))$*-bimodules*

$$e_{(\nu,ij)}R_{\mathfrak{b}}(m+2)e_{(\nu)} \simeq e_{(\nu,ji)}R_{\mathfrak{b}}(m+2)e_{(\nu)} \qquad \text{for } |i-j|>1,$$

and of $(R_{\mathfrak{b}}(m+3), R_{\mathfrak{b}}(m))$*-bimodules*

$$\begin{aligned} q^{-1}e_{(\nu,iji)}R_{\mathfrak{b}}(m+2)e_{(\nu)} &\oplus qe_{(\nu,iji)}R_{\mathfrak{b}}(m+2)e_{(\nu)} \\ &\simeq e_{(\nu,iij)}R_{\mathfrak{b}}(m+2)e_{(\nu)} \oplus e_{(\nu,jii)}R_{\mathfrak{b}}(m+2)e_{(\nu)} \qquad \text{for } i=j\pm 1. \end{aligned}$$

Proof. The first isomorphism follows from relation (3) and the second from the relations (3) and (6) as in [9, Proposition 2.13]. □

3.1.5. *Categorical* $U_q(\mathfrak{gl}_{2n})$*-action.* Let $R_{\mathfrak{b}}(\nu)\text{-mod}$ be the category of (graded) $R_{\mathfrak{b}}(\nu)$-supermodules with morphisms given by degree preserving supermodule maps. Let $\mathsf{F}_i : R_{\mathfrak{b}}(\nu)\text{-mod} \to R_{\mathfrak{b}}(\nu+i)\text{-mod}$ be the induction functor given by adding a strand of color i to the right of a diagram and $\mathsf{E}_i : R_{\mathfrak{b}}(\nu+i)\text{-mod} \to R_{\mathfrak{b}}(\nu)\text{-mod}$ be a shift of its right adjoint:

$$\begin{aligned} \mathsf{F}_i(M) &= R_{\mathfrak{b}}(\nu+i)e(\nu,i)\otimes_{R_{\mathfrak{b}}(\nu)} M, \\ \mathsf{E}_i(N) &= q^{2\nu_i-\nu_{i\pm1}+1}\lambda_i^{-1}e(\nu,i)N, \end{aligned}$$

for $M \in R_{\mathfrak{b}}(\nu)\text{-mod}$ and $N \in R_{\mathfrak{b}}(\nu+i)\text{-mod}$. We also define

$$\mathsf{Q} = \bigoplus_{k\geqslant 0} \Pi q^{2k+1}\,\mathrm{Id}\,.$$

We introduce the notations $\oplus_{[a]_q}M = \oplus_{\ell=0}^{a-1}q^{-k+1+2\ell}M$ for $a \in \mathbb{N}$ and $\oplus_{[\beta_i+k]_q}M = \mathsf{Q}q^k\lambda_i M \oplus \Pi\mathsf{Q}q^{-k}\lambda_i^{-1}M$ for $k \in \mathbb{Z}$.

A direct consequence of Theorem 3.22 and Proposition 3.23 ahead is the following, which describe how F_i and E_i give a categorical action of $U_q(\mathfrak{gl}_{2n})$ on $\bigoplus_\nu R_{\mathfrak{b}}(\nu)\text{-mod}$.

Theorem 3.14. *There is a natural short exact sequence of functors*

$$0 \to \mathsf{F}_i\mathsf{E}_i 1_\nu \to \mathsf{E}_i\mathsf{F}_i 1_\nu \to \oplus_{[\beta_i+\nu_{i\pm1}-2\nu_i]_q}1_\nu \to 0, \tag{12}$$

for all $i \in I$, and isomorphisms

$$\mathsf{E}_i\mathsf{F}_j 1_\nu \simeq \mathsf{F}_j\mathsf{E}_i 1_\nu, \tag{13}$$

for $i \neq j \in I$. There are also isomorphisms

$$\mathsf{F}_i\mathsf{F}_j 1_\nu \simeq \mathsf{F}_j\mathsf{F}_i 1_\nu \qquad \text{for } |i-j| > 1, \tag{14}$$

$$\oplus_{[2]_q}\mathsf{F}_i\mathsf{F}_j\mathsf{F}_i 1_\nu \simeq \mathsf{F}_i\mathsf{F}_i\mathsf{F}_j 1_\nu \oplus \mathsf{F}_j\mathsf{F}_i\mathsf{F}_i 1_\nu \qquad \text{for } i = j \pm 1, \tag{15}$$

and the same with $\mathsf{E}_i, \mathsf{E}_j$.

Proposition 3.15. *The functors E_i and F_i are exact (and send projectives to projectives).*

3.1.6. *The categorification theorem.* Let $R_\mathfrak{b}(\nu)\text{-mod}_{\text{lf}}$ be the category of cone bounded, locally finite dimensional, graded left supermodules over $R_\mathfrak{b}(\nu)$, with degree zero morphisms. By locally finite dimensional we mean the objects, seen as graded $\Bbbk$-vector spaces, must be finite dimensional in each degree, and the cone bounded means the graded dimension is contained in some cone, so that it is an element of $\mathbb{Z}(\!(q,\Lambda)\!)$ (see [18, §5]).

In [18, §5], it is explained how the (topological) Grothendieck group of such a category can be defined as a free $\mathbb{Z}_\pi(\!(q,\Lambda)\!)$-module generated by the simple modules up to shift, where $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/\pi^2$. It is a topological module for the (q,Λ)-adic topology. By computing the projective resolutions of the simple modules, one can show that it is freely generated by the projectives as well. We denote it $\boldsymbol{G}_0(R_\mathfrak{b}(\nu))$.

We define $R_\mathfrak{b}\text{-mod}_{\text{lf}} = \bigoplus_\nu R_\mathfrak{b}(\nu)\text{-mod}_{\text{lf}}$ and we equip it with the endofunctors F_i and E_i. It is not hard to see that the functor Q descends to multiplication by $\frac{1}{q-q^{-1}} = \frac{-q}{1-q^2}$ in the Grothendieck group, after specializing $\pi = -1$ in $\mathbb{Z}_\pi$. Thus we get

$$[\oplus_{[n]_q} M] = [n]_q[M], \qquad [\oplus_{[\beta_i+k]_q} M] = \frac{\lambda_i q^k - \lambda_i^{-1}q^{-k}}{q-q^{-1}}[M].$$

Theorem 3.16. *The functors F_i and E_i induce an action of $U_q(\mathfrak{gl}_{2n})$ on the Grothendieck group of $R_\mathfrak{b}$ when specializing $\pi = -1$. With this action we have a $\mathbb{Q}(\!(q,\Lambda)\!)$-linear isomorphism*

$$\boldsymbol{G}_0(R_\mathfrak{b}) \cong M^\mathfrak{b}(\beta_{1-n},\dots,\beta_{n-1}),$$

of $U_q(\mathfrak{gl}_{2n})$ representations. Moreover, the isomorphism sends classes of projective objects to canonical basis elements and classes of simple objects to dual canonical basis elements.

Proof. By Theorem 3.14 and Proposition 3.15, the functors F_i and E_i descend to an action of $U_q(\mathfrak{gl}_{2n})$ on the Grothendieck group, giving it a structure of a weight module.

Seeing that $R_\mathfrak{b}(\nu)$ has the same decomposition into idempotents as $R(\nu)$ from [9] we can transpose the arguments in [9, § 3.2] such that $R_\mathfrak{b}(\nu)$ also categorify $U_q^-(\mathfrak{gl}_{2n})$ as a $\Bbbk$-vector space (but a priori not as a bialgebra). Since E_i act as zero on $[R_\mathfrak{p}^\beta(\varnothing)]$, it is a highest weight module with 1-dimensional highest weight space. Thus, there is a surjective morphism

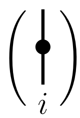
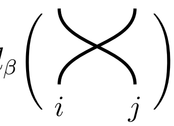
$$\gamma : M^\mathfrak{b}(\beta_{1-n},\dots,\beta_{n-1}) \to \boldsymbol{G}_0(R_\mathfrak{b}),$$

sending the highest-weight vector to $[R_\mathfrak{b}(\varnothing)]$. The module $M^\mathfrak{b}(\beta_{1-n},\dots,\beta_{n-1})$ being irreducible by the Jantzen's criterion (see [6, § 9.13] or [17] for the non-quantum case), γ must be an isomorphism. ∎

3.2. **A parabolic 2-Verma module I: the case of $\mathfrak{p} \subseteq U_q(\mathfrak{gl}_{2n})$.** We now treat the $\mathfrak{gl}_{2n}$ (the "unreduced") case. For this, we introduce a differential d_β on $R_\mathfrak{b}(\nu)$, turning it into a multigraded dg-(super)algebra, as in [20, §4.2.2] (and already in [18] and [19]). We will see how the category of dg-modules over $(R_\mathfrak{b}, d_\beta)$ yields a categorification of $M^\mathfrak{p}(\beta)$, and how we can actually interpret this more classically in terms of category of modules over $H^*(R_\mathfrak{b}, d_\beta)$. In particular, $H^*(R_\mathfrak{b}, d_\beta)$ will be presented as a cyclotomic quotient of some $\mathfrak{p}$-KLR algebra $R_\mathfrak{p}$.

3.2.1. *Dg-structure on $R_\mathfrak{b}(\nu)$.* For a given floating dot ω_i^a, the intersection with a horizontal line gives rise to an idempotent $1_{\omega_i^a}$ in $R_\mathfrak{b}$ consisting of vertical segments formed from the intersection points with the strands at the left of ω_i^a. Denote by $\mathbb{Z}[\underline{x}_{\ell_i}]$ the ring in the dots on the ℓ_i strands labeled i in $1_{\omega_i^a}$, and by $\mathbb{Z}[\underline{x}_{\ell_{i\pm1}}]$ the ring in the dots on the $\ell_{i\pm1}$ strands labeled $i \pm 1$ in $1_{\omega_i^a}$. For $r \in \mathbb{N}$ we write $h_r(\ell_i)$ for the rth complete homogeneous symmetric polynomial in the variables in $\mathbb{Z}[\underline{x}_{\ell_i}]$, and $e_r(\ell_{1\pm i})$ the rth elementary symmetric polynomial in the variables in $\mathbb{Z}[\underline{x}_{\ell_{i\pm1}}]$.

We define

d_β  $= 0,$ $\qquad$ d_β  $= 0,$

for all $i, j \in I$, and

$$d_\beta\left(\circ^{a}_{i}\right) = \begin{cases} 0, & \text{if } i = 0, \\ (-1)^{a-\ell_i}\left(\sum\limits_{r=0}^{\ell_{i\pm1}} h_{a-\ell_i+\ell_{i\pm1}+1-r}(\ell_i)e_r(\ell_{i\pm1})\right), & \text{otherwise.} \end{cases}$$

Of course by Lemma 3.6 it is in fact enough to define the differential on tight floating dots, on which d_β is given by

$$d_\beta(\omega_i) = -\delta_{i0}.$$

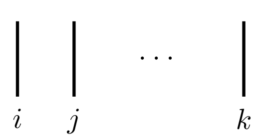
This differential has degree $q(d_\beta) = 0$, $\lambda_i(d_\beta) = -2$ for $i \neq 0$, $\lambda_0(d_\beta) = 0$, and parity 1.

Definition 3.17. Let $d_\beta : R_\mathfrak{b}(\nu) \to R_\mathfrak{b}(\nu)$ be defined as above together with the graded Leibniz rule, so that $(R_\mathfrak{b}(\nu), d_\beta)$ becomes a $\mathbb{Z} \times \mathbb{Z}$-graded dg-algebra with homological degree given by counting the number of floatings dots with subscript different from 0.

Proposition 3.18. ([20, Lemma 4.13]) *The dg-algebra $(R_\mathfrak{b}(\nu), d_\beta)$ is formal with homology concentrated in homological degree* 0.

This means there is a quasi-isomorphism $(R_\mathfrak{b}(\nu), d_\beta) \xrightarrow{\simeq} (H^*(R_\mathfrak{b}(\nu), d_\beta), 0)$. Gladly, the homology $H^*(R_\mathfrak{b}(\nu), d_\beta)$ admits a very nice description in terms of KLR-like diagrams.

Definition 3.19. Let $R_\mathfrak{p}(\nu)$ be defined as $R_\mathfrak{b}(\nu)$ but where we only admit floating dots having subscript 0. Then we define $R_\mathfrak{p}^\beta(\nu)$ to be the quotient of $R_\mathfrak{p}(\nu)$ by the two-sided ideal generated by all diagrams of the form

 $, \qquad i \neq 0.$

In other words, we kill all diagrams which at some height have a strand of label different from 0 at the left.

Proposition 3.20. *There is an isomorphism $H^*(R_{\mathfrak{b}}(\nu), d_\beta) \cong R_{\mathfrak{p}}^\beta(\nu)$.*

Proof. This is immediate by Lemma 3.6 and Proposition 3.18. □

3.2.2. *A long exact sequence.* Take $k \in I\backslash\{0\}$ and consider the short exact sequence

$$\begin{aligned}0 \to q^{-2}R_{\mathfrak{b}}(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R_{\mathfrak{b}}(\nu) \to e_{(\nu,k)}R_{\mathfrak{b}}(m+1)e_{(\nu,k)} \\ \to R_{\mathfrak{b}}(\nu)\left(\mathbb{Z}[\xi] \oplus \theta_{m+1}\mathbb{Z}[\xi]\right) \to 0,\end{aligned} \tag{16}$$

of $(R_{\mathfrak{b}}(\nu), R_{\mathfrak{b}}(\nu))$-bimodules, from Proposition 3.12. Following [20, § 4.2.3], we equip the cokernel of (16) with a differential, also denoted d_β, defined by $d_\beta(\xi^a \oplus 0) = 0$, and

$$d_\beta(0 \oplus \xi^a) = (-1)^{\nu_k} \sum_{r=0}^{\nu_{k\pm1}} \sum_{s=\nu_k}^{a+r-\nu_k} \sum_{t=0}^{s-\nu_k} \xi^t h_{s-\nu_k-t}(\nu_k) h_{a+r-s-\nu_k}(\nu_k) e_{\nu_{k\pm1}-r}(\nu_{k\pm1}).$$

This formula is given by the image of $-\tau_m \dots \tau_1 x_1^a \tau_1 \dots \tau_m$ in the projection

$$e_{(\nu,k)}R_{\mathfrak{b}}(m+1)e_{(\nu,k)} \twoheadrightarrow R_{\mathfrak{b}}(\nu)\Bbbk[\xi],$$

from the short exact sequence (16). See [19, Lemma 3.14] for explanations how to compute it.

This means that (16) can be enhanced into short exact sequence of $((R_{\mathfrak{b}}(\nu), d_\beta), (R_{\mathfrak{b}}(\nu), d_\beta))$-dg-bimodules

$$\begin{aligned}0 \to \left(q^{-2}R_{\mathfrak{b}}(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R_{\mathfrak{b}}(\nu), d_\beta\right) \to \left(e_{(\nu,k)}R_{\mathfrak{b}}(m+1)e_{(\nu,k)}, d_\beta\right) \\ \to \left(R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi] \oplus \theta_{m+1}R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi], d_\beta\right) \to 0,\end{aligned}$$

where θ_{m+1} is a shift $q^{2\nu_{k\pm1}-4\nu_k}\lambda_k^2[1]$. By the snake lemma and Proposition 3.18, it descends in homology to a long exact sequence

$$\begin{array}{c}
\cdots \longrightarrow 0 \\
\longrightarrow H^1\left(R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi] \oplus \theta_{m+1}R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi], d_\beta\right) \longrightarrow H^0\left(q^{-2}R_{\mathfrak{b}}(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R_{\mathfrak{b}}(\nu), d_\beta\right) \\
\longrightarrow H^0\left(e_{(\nu,k)}R_{\mathfrak{b}}(m+1)e_{(\nu,k)}, d_\beta\right) \longrightarrow H^0\left(R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi] \oplus \theta_{m+1}R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi], d_\beta\right) \\
\longrightarrow 0 \longrightarrow \cdots
\end{array}$$

of $\left(H^*\left(R_{\mathfrak{b}}(\nu), d_\beta\right), H^*\left(R_{\mathfrak{b}}(\nu), d_\beta\right)\right) \cong \left(R_{\mathfrak{p}}^N(\nu), R_{\mathfrak{p}}^N(\nu)\right)$-bimodules.

Proposition 3.21. *We have isomorphisms of $R_{\mathfrak{p}}^N(\nu)$-bimodules for $r_k = \nu_{k\pm1} - 2\nu_k \geqslant 0$,*

$$H^p\left(R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi] \oplus \theta_{m+1}R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi], d_\beta\right) \cong \begin{cases} \bigoplus_{t=0}^{r_k-1} q^{2t}R_{\mathfrak{p}}^\beta(\nu) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0, \end{cases}$$

and for $r_k \leqslant 0$,

$$H^p\left(R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi] \oplus \theta_{m+1}R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi], d_\beta\right) \cong \begin{cases} 0 & \text{if } p \neq 1, \\ \lambda_k^2 q^{4\nu_k - 2\nu_{k\pm1}} \bigoplus_{t=0}^{-r_k-1} q^{2t} R_{\mathfrak{p}}^\beta(\nu) & \text{if } p = 1. \end{cases}$$

Proof. Take $0 \oplus \xi^a \in R_{\mathfrak{b}}(\nu)\Bbbk[\xi] \oplus \theta_{m+1}R_{\mathfrak{b}}(\nu)\Bbbk[\xi]$. For $r_k \geqslant 0$, we notice that $d_\beta(0 \oplus \xi^a)$ yields a monic polynomial, up to sign, with leading term ξ^{r_k+a}. For $r_k \leqslant 0$, we have $d_\beta(0 \oplus \xi^a) = 0$ whenever $a < -r_k$, and $d_\beta(0 \oplus \xi^{-r_k}) = 1 \oplus 0$. □

Therefore, if $r_k = \nu_{k\pm1} - 2\nu_k \geqslant 0$, then the long exact sequence gives a short exact sequence

$$q^{-2}R_{\mathfrak{p}}^\beta(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R_{\mathfrak{p}}^\beta(\nu) \hookrightarrow e_{(\nu,k)}R_{\mathfrak{p}}^\beta(m+1)e_{(\nu,k)} \twoheadrightarrow \bigoplus_{t=0}^{r_k-1} q^{2t}R_{\mathfrak{p}}^N(\nu),$$

and if $r_k \leqslant 0$, taking the degree of the connecting homomorphism into account, it yields

$$\bigoplus_{t=0}^{-r_k-1} q^{-2t-2}R_{\mathfrak{p}}^\beta(\nu) \hookrightarrow q^{-2}R_{\mathfrak{p}}^\beta(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R_{\mathfrak{p}}^\beta(\nu) \twoheadrightarrow e_{(\nu,k)}R_{\mathfrak{p}}^\beta(m+1)e_{(\nu,k)}.$$

Clearly the left map in the first sequence splits, since the projection $e_{(\nu,k)}R_{\mathfrak{b}}(m+1)e_{(\nu,k)} \to R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi]$ was already invertible (this is obvious from the diagrammatic point of view: the splitting morphism is given by adding a strand labeled k with dots on it corresponding to the power of ξ). For the second sequence, it is not that easy to see that the injection morphism splits. One way of doing it consists in introducing A_∞ maps on $(R_{\mathfrak{b}}, \beta)$ that will do the split. However, it is probably easier to directly see it by looking at the connecting homomorphism. Indeed, for $r_k < 0$ and $a < -r_k$, the lift of the cycle $0 \oplus \xi^a \in (R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi] \oplus \theta_{m+1}R_{\mathfrak{b}}(\nu)\mathbb{Z}[\xi], d_\beta)$ is given by $\tau_m \dots \tau_1 x_1^a \omega \tau_1 \dots \tau_m \in \left(e_{(\nu,k)}R_{\mathfrak{b}}(m+1)e_{(\nu,k)}, d_\beta\right)$.

$$0 \oplus \xi^a \mapsto$$

In conclusion, both short exact sequences split into direct sums decompositions.

The short exact sequence like (16) but for the simple root $0 \in I$ can also be turned into a short exact sequence of dg-bimodules by equipping the cokernel with the trivial differential, so that it induces a similar short exact sequence on the homology.

All of this sums up to the following:

Theorem 3.22. *There is a short exact sequence of $(R_{\mathfrak{p}}^\beta(m), R_{\mathfrak{p}}^\beta(m))$-bimodules*

$$0 \to q^{-2}R_{\mathfrak{p}}^\beta(m)e_{(m-1,0)} \otimes_{m-1}^\beta e_{(m-1,0)}R_{\mathfrak{p}}^\beta(m) \to e_{(m,0)}R_{\mathfrak{p}}^\beta(m+1)e_{(m,0)} \to R_{\mathfrak{p}}^\beta(m) \otimes \Bbbk[\xi] \oplus R_{\mathfrak{p}}^\beta(m)\theta_{m+1} \otimes \Bbbk[\xi] \to 0,$$

and the following isomorphisms for $k \neq 0$:

$i)$ *if* $\nu_{k\pm1} \geqslant 2\nu_k$,

$$e_{(\nu,\pm k)}R^\beta_{\mathfrak{p}}(m+1)e_{(\nu,\pm k)} \cong \\ q^{-2}R^\beta_{\mathfrak{p}}(\nu)e_{(m-1,\pm k)} \otimes^\beta_{m-1} e_{(m-1,\pm k)}R^\beta_{\mathfrak{p}}(\nu) \oplus R^\beta_{\mathfrak{p}}(\nu) \otimes \Bbbk[\xi]/(\xi^{\nu_{k\pm1}-2\nu_k}),$$

$ii)$ *if* $\nu_{k\pm1} \leqslant 2\nu_k$,

$$q^{-2}R^\beta_{\mathfrak{p}}(\nu)e_{(m-1,\pm k)} \otimes^\beta_{m-1} e_{(m-1,\pm k)}R^\beta_{\mathfrak{p}}(\nu) \cong \\ e_{(\nu,\pm k)}R^\beta_{\mathfrak{p}}(m+1)e_{(\nu,\pm k)} \oplus q^{2(\nu_{k\pm1}-2\nu_k-1)}R^\beta_{\mathfrak{p}}(\nu) \otimes \Bbbk[\xi]/(\xi^{2\nu_k-\nu_{k\pm1}}),$$

$iii)$ *for* $i \neq j$,

$$e_{(m,i)}R^\beta_{\mathfrak{p}}(m+1)e_{(m,j)} \simeq q^{-i\cdot j}R^\beta_{\mathfrak{p}}(m)e_{(m-1,j)} \otimes^\beta_{m-1} e_{(m-1,i)}R^\beta_{\mathfrak{p}}(m).$$

The categorical Serre relations are interpreted in $R^\beta_{\mathfrak{p}}$ as the following, which is obtained by observing the direct sums of Proposition 3.23 can be interpreted as direct sums of dg-modules, so that they descend on the homology.

Proposition 3.23. *We have isomorphisms of* $(R^\beta_{\mathfrak{p}}(m+2), R^\beta_{\mathfrak{p}}(m))$*-bimodules*

$$e_{(\nu,ij)}R^\beta_{\mathfrak{p}}(m+2)e_{(\nu)} \simeq e_{(\nu,ji)}R^\beta_{\mathfrak{p}}(m+2)e_{(\nu)} \qquad \textit{for } |i-j|>1,$$

and of $(R^\beta_{\mathfrak{p}}(m+3), R^\beta_{\mathfrak{p}}(m))$*-bimodules*

$$q^{-1}e_{(\nu,iji)}R^\beta_{\mathfrak{p}}(m+2)e_{(\nu)} \oplus qe_{(\nu,iji)}R^\beta_{\mathfrak{p}}(m+2)e_{(\nu)} \\ \simeq e_{(\nu,iij)}R^\beta_{\mathfrak{p}}(m+2)e_{(\nu)} \oplus e_{(\nu,jii)}R^\beta_{\mathfrak{p}}(m+2)e_{(\nu)} \qquad \textit{for } i=j\pm1.$$

3.2.3. *Categorical action.* From now on, we will consider $R^\beta_{\mathfrak{p}}(\nu)$ as a bigraded superalgebra, with a quantum degree q and a single λ-degree (corresponding to λ_0).

Again, let $\mathsf{F}^\beta_i : R^\beta_{\mathfrak{p}}(\nu)\text{-mod} \to R^\beta_{\mathfrak{p}}(\nu+i)\text{-mod}$ be the induction functor given by adding a strand of color i to the right of a diagram and $\mathsf{E}^\beta_i : R^\beta_{\mathfrak{p}}(\nu+i)\text{-mod} \to R^\beta_{\mathfrak{p}}(\nu)\text{-mod}$ be a shift of its right adjoint:

$$\mathsf{F}^\beta_i(M) = R^\beta_{\mathfrak{p}}(\nu+i)e(\nu,i) \otimes_{R^\beta_{\mathfrak{p}}(\nu)} M,$$
$$\mathsf{E}^\beta_i(N) = q^{2\nu_i-\nu_{i\pm1}+1}\lambda^{-\delta_{i0}}e(\nu,i)N,$$

for $M \in R^\beta_{\mathfrak{p}}(\nu)\text{-mod}$ and $N \in R^\beta_{\mathfrak{p}}(\nu+i)\text{-mod}$.

A direct consequence of Theorem 3.22 and Proposition 3.23 is the following, which describes how F^β_i and E^β_i give a categorical action of $U_q(\mathfrak{gl}_{2n})$ on $\bigoplus_\nu R^\beta_{\mathfrak{p}}(\nu)\text{-mod}$.

Theorem 3.24. *There is a short exact sequence of functors*

$$0 \to \mathsf{F}^\beta_0\mathsf{E}^\beta_0 1_\nu \to \mathsf{E}^\beta_0\mathsf{F}^\beta_0 1_\nu \to \oplus_{[\beta+\nu_{\pm1}-2\nu_0]_q} 1_\nu \to 0, \tag{17}$$

and isomorphisms

$$\mathsf{E}^{\beta}_{\pm k}\mathsf{F}^{\beta}_{\pm k}1_\nu \simeq \mathsf{F}^{\beta}_{\pm k}\mathsf{E}^{\beta}_{\pm k}1_\nu \oplus_{[\nu_{k\pm 1}-2\nu_k]_q} 1_\nu \qquad \text{if } \nu_{k\pm 1} \geqslant 2\nu_k, \tag{18}$$

$$\mathsf{F}^{\beta}_{\pm k}\mathsf{E}^{\beta}_{\pm k}1_\nu \simeq \mathsf{E}^{\beta}_{\pm k}\mathsf{F}^{\beta}_{\pm k}1_\nu \oplus_{[2\nu_k-\nu_{k\pm 1}]_q} 1_\nu \qquad \text{if } \nu_{k\pm 1} \leqslant 2\nu_k, \tag{19}$$

$$\mathsf{E}^{\beta}_{i}\mathsf{F}^{\beta}_{j}1_\nu \simeq \mathsf{F}^{\beta}_{j}\mathsf{E}^{\beta}_{i}1_\nu, \tag{20}$$

for $i \neq j$ and $k \geqslant 1$. There are also isomorphisms

$$\mathsf{F}^{\beta}_{i}\mathsf{F}^{\beta}_{j}1_\nu \simeq \mathsf{F}^{\beta}_{j}\mathsf{F}^{\beta}_{i}1_\nu \qquad \text{for } |i-j|>1, \tag{21}$$

$$\oplus_{[2]_q}\mathsf{F}^{\beta}_{i}\mathsf{F}^{\beta}_{j}\mathsf{F}^{\beta}_{i}1_\nu \simeq \mathsf{F}^{\beta}_{i}\mathsf{F}^{\beta}_{i}\mathsf{F}^{\beta}_{j}1_\nu \oplus \mathsf{F}^{\beta}_{j}\mathsf{F}^{\beta}_{i}\mathsf{F}^{\beta}_{i}1_\nu \qquad \text{for } i=j\pm 1. \tag{22}$$

Proposition 3.25. *The functors E^{β}_i and F^{β}_i are exact (and send projectives to projectives).*

3.2.4. *The categorification theorem.* Let $R^{\beta}_{\mathfrak{p}}(\nu)\text{-mod}_{\mathrm{lf}}$ be the category of cone bounded, locally finite dimensional, left supermodules over $R^{\beta}_{\mathfrak{p}}(\nu)$, with degree zero morphisms. We define $R^{\beta}_{\mathfrak{p}}\text{-mod}_{\mathrm{lf}} = \bigoplus_\nu R^{\beta}_{\mathfrak{p}}(\nu)\text{-mod}_{\mathrm{lf}}$ and we equip it with the endofunctors F^{β}_i and E^{β}_i. Recall that $M^{\mathfrak{p}}(\beta) := M^{\mathfrak{p}}((\beta)^n,(0)^n)$.

Theorem 3.26. *The functors F^{β}_i and E^{β}_i induce an action of $U_q(\mathfrak{gl}_{2n})$ on the Grothendieck group of $R^{\beta}_{\mathfrak{p}}$ when specializing $\pi=-1$. With this action we have a $\mathbb{Q}((q,\lambda))$-linear isomorphism*

$$\boldsymbol{G}_0(R^{\beta}_{\mathfrak{p}}) \cong M^{\mathfrak{p}}(\beta),$$

of $U_q(\mathfrak{gl}_{2n})$ representations. Moreover, the isomorphism sends classes of projective objects to canonical basis elements and classes of simple objects to dual canonical basis elements.

Proof. By Theorem 3.24 and Proposition 3.25, the functors F^{β}_i and E^{β}_i descend to an action of $U_q(\mathfrak{gl}_{2n})$ on the Grothendieck group, giving it a structure of a weight module.

Seeing that $R_{\mathfrak{p}}(\nu)$ has the same decomposition into idempotents as $R(\nu)$ from [9] we can transpose the arguments in [9, § 3.2] such that $R_{\mathfrak{p}}(\nu)$ also categorify $U_q^-(\mathfrak{gl}_{2n})$ as $\Bbbk$-vector spaces (but not as a bialgebra). Combining this with the arguments from [7, § 6], we conclude that $\boldsymbol{G}_0(R^{\beta}_{\mathfrak{p}})$ is generated by a highest weight vector $[R^{\beta}_{\mathfrak{p}}(\varnothing)] = [\mathbb{Q}]$.

Since E_i and $\mathsf{F}_{\pm k}$ act as zero on $[R^{\beta}_{\mathfrak{p}}(\varnothing)]$, it is a highest weight module with 1-dimensional highest weight space, isomorphic to $V(0,\dots,0)$ as $U_q(\mathfrak{l})$-module. Thus, there is a surjective morphism

$$\gamma : M^{\mathfrak{p}}(\beta) \to \boldsymbol{G}_0(R^{\beta}_{\mathfrak{p}}),$$

sending the highest-weight vector to $[R^{\beta}_{\mathfrak{p}}(0)]$. The module $M^{\mathfrak{p}}(\beta)$ being irreducible by the Jantzen's criterion (see [6, § 9.13] or [17] for the non-quantum case), γ must be an isomorphism. ☐

Remark 3.27. A dg-analogue of this theorem exists and is presented in [20, §6]. However we will not need it for our purposes here.

3.2.5. *A parabolic 2-Verma module.* The parabolic 2-Verma module $\mathfrak{M}^{\mathfrak{p}}(\beta)$ is the full sub 2-category of $\mathrm{Fun}(\mathrm{R}_{\mathfrak{p}}^{\beta}\text{-mod}_{\mathrm{lf}})$ whose objects $\mathbb{1}_\mu$ are indexed by weights $\mu \in \Lambda_{n,n}^{\beta}$, 1-morphisms are compositions of locally finite, left bounded direct sums of grading shifts of the various functors F_i and E_i (in particular Q is a 1-morphism), and the 2-morphisms are grading preserving natural transformations between these functors.

Let $\dot{\mathcal{U}}(\mathfrak{sl}_n)$ denote Khovanov-Lauda and Rouquier's 2-Kac-Moody algebra from [10, 24] (which are the same by [2]). The following result is immediate.

Lemma 3.28. *There is a 2-action of $\dot{\mathcal{U}}(\mathfrak{sl}_n \times \mathfrak{sl}_n)$ on $\mathfrak{M}^{\mathfrak{p}}(\beta)$.*

The lemma implies that in particular, the categorified q-Schur algebra $\mathcal{S}(0, n)$ from [15] acts on $\mathfrak{M}^{\mathfrak{p}}(\beta)$.

3.3. **Recovering the finite dimensional irreducible $V((N)^n, (0)^n)$.** For each $N \geqslant 0$ we introduce a differential d_N on $R_{\mathfrak{p}}^{\beta}$ by declaring that d_N acts trivially on crossings and on dots, and sends a labeled floating dot ω_0^a to the following polynomial on dots

$$d_N(\omega_0^a) = (-1)^{N+a-\ell_0}\left(\sum_{r=0}^{\ell_\pm} h_{a-\ell_0+\ell_\pm+N+1-r}(\ell_0)e_r(\ell_\pm)\right), \tag{23}$$

to be placed at the same height as the floating dot. This differential has degree $\deg(d_N) = (2N, -2)$ and parity 1.

Proposition 3.29. ([20, §4.2.4]) *The differential d_N gives $R_{\mathfrak{p}}^{\beta}$ the structure of a $\mathbb{Z}$-graded dg-algebra.*

Alternatively, using the basis of $R_{\mathfrak{p}}$, one can first decompose an element of $R_{\mathfrak{p}}^{\beta}$ as a combination of diagrams with only tight floating dots and then use $d_N(\omega_1) = -x_1^N$ as definition, which is (23) specialized to a tight floating dot.

Lemma 3.30. ([20, Proposition 4.19]) *The DG-algebra $(R_{\mathfrak{p}}^{\beta}, d_N)$ is formal. Moreover, it is quasi-isomorphic to the cyclotomic KLR algebra R^{Λ}, with $\Lambda = ((0)^{n-1}, N, (0)^{n-1})$.*

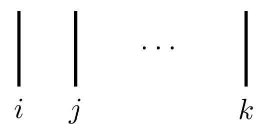
Passing to the bounded derived category $\mathcal{D}^c(R_{\mathfrak{p}}^{\beta}, d_N)$ of the category of bigraded, left, compact $(R_{\mathfrak{p}}^{\beta}, d_N)$-modules and defining functors F_i^N, E_i^N, $i = -k+1, \dots, k-1$, in the same way as F_i^{β} and E_i^{β} we have the following.

Proposition 3.31. *The functors F_i^N and E_i^N induce an action of $U_q(\mathfrak{gl}_{2n})$ on the Grothendieck group of $\mathcal{D}^c(R_{\mathfrak{p}}^{\beta}, d_N)$. With this action, $K_0(\mathcal{D}^c(R_{\mathfrak{p}}^{\beta}, d_N))$ is isomorphic to the irreducible representation of highest weight $((N)^k, (0)^k)$.*

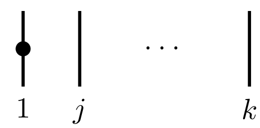
Proof. The proof is given by the Lemma 3.30 and Kang–Kashiwara's results [7, Theorem 6.2] (this is also reproved in [20] using similar arguments as in §3.2.2). ☐

3.4. **A parabolic 2-Verma module II: the case of $\bar{\mathfrak{p}} \subseteq U_q(\mathfrak{gl}_{2n-1})$.** We now treat the $\mathfrak{gl}_{2n-1}$ (the "reduced") case. Recall the $\mathfrak{gl}_{2n}$-weight $((\beta)^{n-1}, 1, (0)^{n-1})$ is translated into the $\mathfrak{sl}_{2n}$-weight $((0)^{n-2}, \beta - 1, 1, (0)^{n-2})$. By [19] and [20] we know that shifted Verma modules are given by changing the minimal integer label of the floating dots. We also know that the choice of cyclotomic quotient for the $\bar{\mathfrak{p}}$-KLR algebra determines the highest weight. Therefore, we define the following.

Definition 3.32. Let $R_{\bar{\mathfrak{p}}}$ (resp. $R_{\bar{\mathfrak{b}}}$) be defined as $R_{\mathfrak{p}}$ (resp. $R_{\mathfrak{b}}$) but where floating dots with subscript 0 are allowed to have label down to -1, and $I = \{-n+2, -n+3, \dots, n-1\}$. In this context we call tight floating dot a floating dot with label -1 at the immediate right of the left most strand. We define $R_{\bar{\mathfrak{p}}}^{\beta}$ to be the quotient of $R_{\bar{\mathfrak{p}}}$ by the two-sided ideal generated by all diagrams of the form



$i \neq 0,\ i \neq 1,$



.

As before, we can obtain $R_{\bar{\mathfrak{p}}}^{\beta}$ as the homology of $R_{\bar{\mathfrak{b}}}$ equipped with the differential sending everything to zero, except for tight floating dots

$$d_\beta(\omega_i) = \begin{cases} 0 & \text{if } i = 0, \\ -x_1 & \text{if } i = 1, \\ -1 & \text{otherwise.} \end{cases}$$

Define $\overline{\mathsf{F}}_i^{\beta} : R_{\bar{\mathfrak{p}}}^{\beta}(\nu)\text{-mod} \to R_{\bar{\mathfrak{p}}}^{\beta}(\nu + i)\text{-mod}$ and $\overline{\mathsf{E}}_i^{\beta} : R_{\bar{\mathfrak{p}}}^{\beta}(\nu + i)\text{-mod} \to R_{\bar{\mathfrak{p}}}^{\beta}(\nu)\text{-mod}$ as before

$$\overline{\mathsf{F}}_i^{\beta}(M) = R_{\bar{\mathfrak{p}}}^{\beta}(\nu + i) e(\nu, i) \otimes_{R_{\bar{\mathfrak{p}}}^{\beta}(\nu)} M,$$
$$\overline{\mathsf{E}}_i^{\beta}(N) = e(\nu, i) N \langle 2\nu_i - \nu_{i\pm1} + 1 - \delta_{i,0} + \delta_{i,1}, -\epsilon_i \rangle.$$

All results from §3.2 can be applied almost unchanged to this context. Therefore we obtain the following two theorems.

Theorem 3.33. *There is a short exact sequence of functors*

$$0 \to \overline{\mathsf{F}}_0^{\beta} \overline{\mathsf{E}}_0^{\beta} 1_\nu \to \overline{\mathsf{E}}_0^{\beta} \overline{\mathsf{F}}_0^{\beta} 1_\nu \to \oplus_{[\beta + \nu_{\pm1} - 2\nu_0 - 1]} 1_\nu \to 0, \tag{24}$$

and isomorphisms

$$\overline{\mathsf{E}}_{\pm k}^{\beta} \overline{\mathsf{F}}_{\pm k}^{\beta} 1_\nu \simeq \overline{\mathsf{F}}_{\pm k}^{\beta} \overline{\mathsf{E}}_{\pm k}^{\beta} 1_\nu \oplus_{[\nu_{k\pm1} - 2\nu_k + \delta_{k,1}]_q} 1_\nu \qquad \textit{if } \nu_{k\pm1} + \delta_{k1} \geqslant 2\nu_k, \tag{25}$$

$$\overline{\mathsf{F}}_{\pm k}^{\beta} \overline{\mathsf{E}}_{\pm k}^{\beta} 1_\nu \simeq \overline{\mathsf{E}}_{\pm k}^{\beta} \overline{\mathsf{F}}_{\pm k}^{\beta} 1_\nu \oplus_{[2\nu_k - \nu_{k\pm1} - \delta_{k1}]_q} 1_\nu \qquad \textit{if } \nu_{k\pm1} + \delta_{k1} \leqslant 2\nu_k, \tag{26}$$

$$\overline{\mathsf{E}}_i^{\beta} \overline{\mathsf{F}}_j^{\beta} 1_\nu \simeq \overline{\mathsf{F}}_j^{\beta} \overline{\mathsf{E}}_i^{\beta} 1_\nu, \tag{27}$$

for $i \neq j$ and $k \geqslant 1$. There are also isomorphisms

$$\overline{\mathsf{F}}_i^{\beta}\overline{\mathsf{F}}_j^{\beta}1_\nu \simeq \overline{\mathsf{F}}_j^{\beta}\overline{\mathsf{F}}_i^{\beta}1_\nu \qquad \textit{for } |i-j|>1, \tag{28}$$

$$\oplus_{[2]_q}\overline{\mathsf{F}}_i^{\beta}\overline{\mathsf{F}}_j^{\beta}\overline{\mathsf{F}}_i^{\beta}1_\nu \simeq \overline{\mathsf{F}}_i^{\beta}\overline{\mathsf{F}}_i^{\beta}\overline{\mathsf{F}}_j^{\beta}1_\nu \oplus \overline{\mathsf{F}}_j^{\beta}\overline{\mathsf{F}}_i^{\beta}\overline{\mathsf{F}}_i^{\beta}1_\nu \qquad \textit{for } i=j\pm 1. \tag{29}$$

Recall that $M^{\bar{\mathfrak{p}}}(\overline{\beta}) := M^{\bar{\mathfrak{p}}}((\beta)^{n-1},1,(0)^{n-1})$

Theorem 3.34. *The functors $\overline{\mathsf{F}}_i^{\beta}$ and $\overline{\mathsf{E}}_i^{\beta}$ induce an action of $U_q(\mathfrak{gl}_{2n-1})$ on the Grothendieck group of $R_{\bar{\mathfrak{p}}}^{\beta}$ when specializing $\pi=-1$. With this action we have an isomorphism*

$$\boldsymbol{G}_0(R_{\bar{\mathfrak{p}}}^{\beta}) \cong M^{\bar{\mathfrak{p}}}(\overline{\beta}),$$

of $U_q(\mathfrak{gl}_{2n-1})$ representations.

Now equip $R_{\bar{\mathfrak{p}}}^{\beta}$ with the differential $d_N(\omega_0^{-1}) = -x_1^N$. This gives a formal dg-algebra quasi-isomorphic to the cyclotomic KLR algebra R^Λ with $\Lambda = ((0)^{n-2},N,1,(0)^{n-2})$. Applying the same arguments as in §3.3 and defining $\overline{\mathsf{E}}_i^{N}$ and $\overline{\mathsf{F}}_i^{N}$ in the obvious way, we get the following proposition.

Proposition 3.35. *The functors $\overline{\mathsf{F}}_i^{N}$ and $\overline{\mathsf{E}}_i^{N}$ induce an action of $U_q(\mathfrak{gl}_{2n-1})$ on the Grothendieck group of $\mathcal{D}^c(R_{\bar{\mathfrak{p}}}^{\beta},d_N)$. With this action, $K_0(\mathcal{D}^c(R_{\bar{\mathfrak{p}}}^{\beta},d_N))$ is isomorphic to the irreducible representation of highest weight $((N+1)^{n-1},1,(0)^{n-1})$.*

4. LINK HOMOLOGY

4.1. **Braiding.** By a well-known construction due to Cautis [3], we know how to associate a chain complex in the 2-category $Kom(\dot{\mathcal{U}}(\mathfrak{sl}_n))$ of complexes of the Hom-categories of $\dot{\mathcal{U}}(\mathfrak{sl}_n)$, called a *Rickard complex*, which satisfies the braid relations up to homotopy.

We denote by $[n]$ a shift in the homological grading, which consists of shifting n units to the right the complex under consideration.

For a positive crossing $\mathcal{T}_i$ between the ith and $(i+1)$th strands the Rickard complex is

$$\mathcal{T}_i\mathbb{1}_\mu = \cdots \xrightarrow{d_{r+1}} q^r\mathsf{F}_i^{(\mu_i+r)}\mathsf{E}_i^{(r)}\mathbb{1}_\mu \xrightarrow{d_r} \cdots \xrightarrow{d_2} q\mathsf{F}_i^{(\mu_i+1)}\mathsf{E}_i\mathbb{1}_\mu \xrightarrow{d_1} \mathsf{F}_i^{(\mu_i)}\mathbb{1}_\mu, \tag{30}$$

with $\mathsf{F}_i^{(\mu_i)}\mathbb{1}_\mu$ in homological degree zero, if $\mu_i \geqslant 0$. The Rickard complex of a negative crossing is the left adjoint to $\mathcal{T}_i\mathbb{1}_\mu$ in $Kom(\dot{\mathcal{U}}(\mathfrak{sl}_n))$. There are similar formulas for $\mu_i \geqslant 0$ but we will not use them. An explicit description of the differentials can be obtained easily from Lemma 3.28 by noticing that as usual, the differential d_r in the Rickard complex (30) is given by the counit of the adjunction $F_i \dashv E_i$.

Of course, in our context the Rickard complex is always truncated at the second term, so that it looks like

$$q\mathsf{F}_i\mathsf{E}_i\mathbb{1}_\mu \xrightarrow{\varepsilon} \mathbb{1}_\mu,$$

for a positive crossing, with ε being the counit of the adjuction $\mathsf{F}_i \dashv \mathsf{E}_i$. For a negative crossing, we get

$$\mathbb{1}_\mu \xrightarrow{\eta'} q^{-1}\mathsf{F}_i\mathsf{E}_i\mathbb{1}_\mu,$$

where η' is the unit of the adjuction $\mathsf{E}_i \dashv \mathsf{F}_i$. Of course, since the $\mathfrak{gl}_{2n}$ weights around a crossing are always $(\dots, 1, 1, \dots)$, it means $\mathsf{F}_i\mathsf{E}_i\mathbb{1}_\mu \cong \mathsf{E}_i\mathsf{F}_i\mathbb{1}_\mu$ and we can thus use instead the complex

$$\mathbb{1}_\mu \xrightarrow{\eta} q^{-1}\mathsf{E}_i\mathsf{F}_i\mathbb{1}_\mu,$$

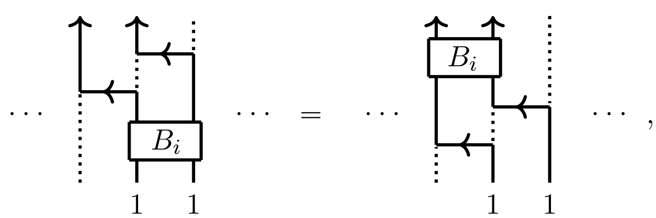
for a negative crossing, where η is now the unit of the adjunction $\mathsf{F}_i \dashv \mathsf{E}_i$, thanks to [2]. The same applies for a positive crossing. This point of view will be particularly useful in order to lift the Rickard complex in the dg-enhancement in §4.6.

Following Cautis's contruction on [3] we associate a Rickard complex $C'(b)$ in the 2-category $Kom(\mathfrak{M}^{\mathfrak{p}}(\beta))$ of complexes in the Hom-categories of $\mathfrak{M}^{\mathfrak{p}}(\beta)$ to each braid diagram b on k strands (see §3.2.5 for the definition of $\mathfrak{M}^{\mathfrak{p}}(\beta)$). This gives a braiding on the homotopy category of $Kom(\mathfrak{M}^{\mathfrak{p}}(\beta))$.

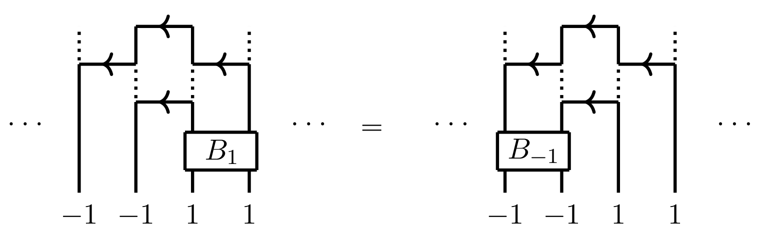
4.2. **Invariance under the Markov moves.** Closing the diagram for b consists of precomposing $C'(b)$ with the appropriate word on functors $\mathsf{F}^\beta_{-n+1}, \dots, \mathsf{F}^\beta_{n-1}$ and composing it with the apppropriate word from $\mathsf{E}^\beta_{-n+1}, \dots, \mathsf{E}^\beta_{n-1}$, following the patterns in Eq. (1). This results in a chain complex $C'(\mathrm{cl}(b))$ in $Kom(\mathfrak{M}^{\mathfrak{p}}(\beta))$, which is a complex of endofunctors of the block corresponding to the highest weight in $M^{\mathfrak{p}}((\beta)^n, (0)^n)$ that is, a complex of $\Bbbk$-spaces. For the sake of simplifying notations, we will forget the β superscript in F^β_i and E^β_i from now on.

Proposition 4.1. *The homotopy type of the chain complex $C'(\mathrm{cl}(b))$ is invariant under the Markov moves, up to parity and an overall grading shift.*

Proof. We start with the first Markov Move. Suppose we have a web in the doubled Schur algebra $\dot{S}_{q,\beta}(n-1, n)$ coming from a closure of a braid diagram. Since

(31) 

where $B_i = F_iE_i = E_iF_i$ involve the ith and the $i+1$th strands in the original braid, and we have similar relations for either downward oriented strands and for the bottom part of the braid closure, the first Markov move can be decomposed in a sequence of moves like (to avoid cluttering we have dropped the β's from the pictures, since it is clear where to place them)

(32) 

and similar for the bottom part of the closure.

Relation ([31]) requires an isomorphism of 1-morphisms in $\mathfrak{M}^{\mathfrak{p}}(\beta)$

$$\mathsf{E}_i\mathsf{E}_{i-1}\mathsf{E}_i\mathsf{F}_i\mathbb{1}_{(\dots,1,1,\dots)} \cong \mathsf{E}_{i-1}\mathsf{F}_{i-1}\mathsf{E}_i\mathsf{E}_{i-1}\mathbb{1}_{(\dots,1,1,\dots)}$$

which is proved in [21, Lemma 3.19].

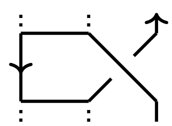
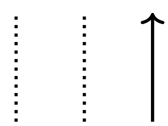
To prove relation ([32]) we write $\mathsf{L}\mathbb{1}_{(\beta-1,\beta-1,1,1)}$ and $\mathsf{R}\mathbb{1}_{(\beta-1,\beta-1,1,1)}$ for the left-hand-side and right-hand side of ([32]), respectively. We write also $\mathsf{E}_\pm$ instead of $\mathsf{E}_{\pm 1}$, and the same for $\mathsf{F}_\pm$. Using Theorem 3.24 we have the following isomorphisms

$$\begin{aligned}\mathsf{L}\mathbb{1}_{(\beta-1,\beta-1,1,1)} &= \mathsf{E}_0\mathsf{E}_+\mathsf{E}_-\mathsf{E}_0\mathsf{F}_+\mathsf{E}_+\mathbb{1}_{(\beta-1,\beta-1,1,1)}\\ &\cong \textstyle\bigoplus_{[2]_q}\mathsf{E}_0\mathsf{E}_-\mathsf{E}_0\mathsf{E}_+\mathbb{1}_{(\beta-1,\beta-1,1,1)}\\ &\cong \mathsf{E}_0\mathsf{E}_0\mathsf{E}_-\mathsf{E}_+\mathbb{1}_{(\beta-1,\beta-1,1,1)},\end{aligned}$$

and

$$\begin{aligned}\mathsf{R}\mathbb{1}_{(\beta-1,\beta-1,1,1)} &= \mathsf{E}_0\mathsf{E}_+\mathsf{E}_-\mathsf{E}_0\mathsf{F}_-\mathsf{E}_-\mathbb{1}_{(\beta-1,\beta-1,1,1)}\\ &\cong \textstyle\bigoplus_{[2]_q}\mathsf{E}_0\mathsf{E}_+\mathsf{E}_0\mathsf{E}_-\mathbb{1}_{(\beta-1,\beta-1,1,1)}\\ &\cong \mathsf{E}_0\mathsf{E}_0\mathsf{E}_+\mathsf{E}_-\mathbb{1}_{(\beta-1,\beta-1,1,1)}.\end{aligned}$$

We now prove the second Markov move. Consider diagrams D_0 and D_1^+ that differ as below.

$D_1^+ =$  , $D_0 =$  .

The complex for D_1^+ induced from the Rickard complex ([30]) is $\mathsf{E}_0\mathcal{T}_1\mathsf{F}_0\mathbb{1}_{(\beta-0,0,1)}$, which is

$$C'(D_1^+) \cong q\mathsf{E}_0\mathsf{F}_1\mathsf{E}_1\mathsf{F}_0\mathbb{1}_{(\beta-0,0,1)} \xrightarrow{d} \mathsf{E}_0\mathsf{F}_0\mathbb{1}_{(\beta-0,0,1)},$$

with the second term in homological degree zero.

Switching E_0 with F_1 and E_1 with F_0 on the term in homological degree 0 and applying the exact sequence ([17]) to $\mathsf{E}_0\mathsf{F}_0\mathbb{1}_{(\beta-0,1,0)}$ gives an isomorphism (notice that in this case $\mathsf{F}_0\mathsf{E}_0\mathbb{1}_{(\beta-0,1,0)}$ is zero)

$$\begin{aligned}q\mathsf{E}_0\mathsf{F}_1\mathsf{E}_1\mathsf{F}_0\mathbb{1}_{(\beta-0,0,1)} &\cong \lambda\mathsf{F}_1\mathsf{E}_1[\xi]\mathbb{1}_{(\beta-0,0,1)} \oplus q^2\lambda^{-1}\mathsf{F}_1\mathsf{E}_1[\xi]\mathbb{1}_{(\beta-0,0,1)}[1]\\ &\cong \lambda\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)} \oplus q^2\lambda^{-1}\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}[1].\end{aligned}$$

Applying the short exact sequence ([17]) to the term in homological degree 1 gives

$$\mathsf{E}_0\mathsf{F}_0\mathbb{1}_{(\beta-0,0,1)} \cong \lambda\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)} \oplus \lambda^{-1}\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}[1].$$

Computing with the maps used above we get that $C'(D_1^+)$ is isomorphic to the complex

$$C'(D_1^+) \cong \begin{pmatrix}\lambda\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}\\ q^2\lambda^{-1}\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}[1]\end{pmatrix} \xrightarrow{\psi} \begin{pmatrix}\lambda\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}\\ \lambda^{-1}\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}[1]\end{pmatrix},$$

with differential

$$\psi = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}$$

where $f\colon q^2\lambda^{-1}\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}[1] \cong \lambda^{-1}\xi\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}[1] \to \lambda^{-1}\Bbbk[\xi]\mathbb{1}_{(\beta-0,0,1)}[1]$ is the inclusion map. After the removal of all acyclic subcomplexes we get that $C'(D_1^+)$ is homotopy equivalent to the complex

$$C'(D_1^+) \cong 0 \to \lambda^{-1}\Bbbk\mathbb{1}_{(\beta-0,0,1)}[1].$$

We have therefore that the complexes $C'(D_1^+)$ and $\lambda^{-1}C'(D_0)[1]$ are homotopy equivalent.

An analogous computation shows that for a diagram D_1^- containing a negative crossing instead, the complexes $C'(D_1^-)$ and $q^{-2}C'(D_0)[1]$ are homotopy equivalent. Maybe explain this a bit ? ☐

Define the normalized chain complex $C(\mathrm{cl}(b)) = \Pi^{n_+}C'(\mathrm{cl}(b))[-n_-]\langle 2n_-, n_+\rangle$ Something seems weird with the homological grading/parity where as usual, $n_\pm$ is the number of positive/negative crossings in $\mathrm{cl}(b)$.

Corollary 4.2. *The homology groups $H(b)$ of $C(\mathrm{cl}(b))$ are triply-graded link invariants and its bigraded Euler characteristic is the HOMFLY-PT polynomial of the closure of b.*

A version of $\mathfrak{M}^{\mathfrak{p}}(\beta)$ for divided powers of the F_i's and E_i's could be used to construct a version of HOMFLY-PT homology for links colored by minuscule representations of $\mathfrak{gl}_N$, as the one constructed by Mackaay-Stošić-Vaz in [14] and Webster-Williamson [26]. Moreover, the differential d_N would give rise to a spectral sequence to colored $\mathfrak{gl}_N$-Khovanov-Rozansky link homology, as the one constructed by Wedrich in [27]. However, proving a version of the first exact sequence from Theorem 3.22 for divided powers might be a nontrivial problem.

4.3. $H(b)$ **is isomorphic with Khovanov-Rozansky HOMFLY-PT link homology.** We now show that our link homology is equivalent to the HOMFLY-PT link homology by Khovanov and Rozansky [12, 8] by proving that $H(L)$ is isomorphic with Rasmussen's version of HOMFLY-PT homology in [23].

Theorem 4.3. *For every braid b the homology $H(b)$ is isomorphic with Khovanov-Rozansky HOMFLY-PT link homology* $\mathrm{HKR}(b)$.

The theorem above gives us an equivalence in a weak sense. We conjecture the equivalence is in fact stronger, in the following sense. The Soergel category $\mathcal{SC}_n$ from [5] acts on $\mathfrak{M}^{\mathfrak{p}}(\beta)$, in particular, on its $(0,\dots,0,\beta,0,\dots,0)$-weight space (this action goes through the categorified q-Schur algebra $\mathcal{S}(n,n)$ (see [15, §6]), which also acts on $\mathfrak{M}^{\mathfrak{p}}(\beta)$, as explained at the end of §3.2.5). Composing with the operation of closing the braid on the top with the correct sequence of E's, following the pattern in (1), gives a functor Φ from $\mathcal{SC}_n$ to the category $Ab_{q,\lambda}$ of bigraded abelian groups (after forgeting the parity). Notice that the same pattern, but with F's, has to be used on the bottom of the diagram to create the weight on which $\mathcal{SC}_n$ acts.

Conjecture 4.4. *The functor Φ is isomorphic with the Hochschild homology functor.*

We now prove Theorem 4.3. We assume the reader is familiar with [23]. Let L be a link presented as the closure of a braid b in n strands. Recall that the process of closing b amounts to composing a word in E_i's with the Rickard complex for b (after adding n parallel strands at its right) and with a word in F_i's. Of course, the closure procedure extends canonically to webs. Let S be the (polynomial) ring in the dots on the F_0's used to form the closure of a web Γ.

Lemma 4.5. *For every web Γ, $H(\Gamma)$ is a free module over S.*

Proof. The proof follows the same reasoning as the proof of Rasmussen of an analogous result using matrix factorizations [23, Proposition 4.8] which is based on an induction scheme introduced by Wu [28, §3]. The only thing we need to check are the MOY relations 0 to III from §4.2 in [23]. MOY relations II and III are already satisfied in $\dot{\mathcal{U}}(\mathfrak{sl}_n)$, and MOY relations 0 and I are a direct consequence of the short exact sequence (17) in Theorem 3.24, when applied to the weights $(\dots,\beta-0,0,\dots)$ and $(\dots,\beta-0,1,\dots)$, since one of the terms in the exact sequence always act as the zero functor on these weights. □

Notice that in [23] the total homological grading (the powers of t in the Euler characteristic) is a linear combination of the gradings gr_h and gr_v introduced there. The minus sign occuring in the numerator of the homology of the unknot in [23] comes from a t. Since we work with bigraded superrings we can concentrate the homology of the unkot in homological degree 0. To ease the conversion between the several gradings used here and the ones used by Rasmussen in [23] and to make the terminology consistent with his, we will refer and treat the parity in our construction as a $\mathbb{Z}/2\mathbb{Z}$-grading. We will also call h-grading the homological grading in our version of $H(b)$.

Proof of Theorem 4.3. Since the braiding in both constructions is the Rickard complex, the proof follows at once from Lemma 4.5, together with the fact that both constructions satisfy the MOY relations. Our q, λ and h gradings agree with the gradings q, $2\,\mathrm{gr}_h$ and gr_v in [23]. The only difference is how Rasmussen defines the Euler characteristic, which is translated in the fact that our parity is the $\mathrm{mod}\, 2$ reduction of gr_h. □

4.4. **Khovanov-Rozansky's $\mathfrak{gl}_N$-link homologies.** Using the 2-representation of $\mathfrak{gl}_{2n}$, constructed from the cyclotomic KLR algebra $R^{((0)^{n-1},N,(0)^{n-1})}$ as input instead of $\mathfrak{M}^{\mathfrak{p}}(\beta)$, results in Khovanov and Rozansky's $\mathfrak{gl}_N$-link homology $H_N(L)$ from [11]. This follows at once from the work of Mackaay and Yonezawa [16].

4.5. **Reduced homologies.** Using the parabolic subalgebra $\bar{\mathfrak{p}} \subseteq U_q(\mathfrak{gl}_{2n-1})$ and the highest weight $((\beta)^{n-1},1,(0)^{n-1})$ in the construction above yields reduced versions $\overline{H}$ and $\overline{H}_N$ of H and H_N. All the results in the preceding subsections have analogues for the case of reduced homologies, and are proved essentially in the same way as above.

There is one subtlety to take in account when claiming the equivalence with reduced Khovanov-Rozansky homologies. Recall that in the case of the reduced versions from [11, 12, 23] the abelian groups $\overline{H}(L,i)$ and $\overline{H}_N(L,i)$ are invariants of the link L together with a marked component i. With our choice of cutting out and open a diagram of a braid closure in §2.2.4, the outermost strand in our version (the one that is cut) corresponds to the preferred component i of $\mathrm{cl}(b)$ (as described in [23]) under the isomorphism between our reduced homologies and Khovanov and Rozansky's. For the sake of completeness we state the main result for the record.

Theorem 4.6. *The homology groups $\overline{H}(b)$ of $\overline{C}(\mathrm{cl}(b))$ are triply-graded link invariants which are isomorphic to the reduced Khovanov-Rozansky homology groups when the outermost strand of $\mathrm{cl}(b)$ is taken as the preferred one. The bigraded Euler characteristic of $\overline{H}(b)$ is the normalized HOMFLY-PT polynomial of the closure of b.*

Using the cyclotomic KLR algebra $R^{((0)^{n-1},N,1,(0)^{n-1})}$ for $\mathfrak{gl}_{2n-1}$ results in a reduced version of Khovanov and Rozansky's $\mathfrak{sl}_n$-link homology.

4.6. **HOMFLY-PT to $\mathfrak{gl}_N$ spectral sequences converge at the second page.** Throughout this section we assume the reader is familiar with [23].

The differential d_N introduced in §3.2.4 can be made to anticommute with the differentials in the Rickard complex (this consists of introducing a factor of -1 to the power of the homological degree in the definition of d_N) and therefore to induce a differential on $H(b)$, also denoted d_N. Recall that the differentials d_N and d are homogeneous with respect to the q, λ, h and p gradings and

$$\deg(d_N) = (2N, -2, 0, 1), \qquad \deg(d) = (0, 0, 1, 0), \tag{33}$$

where (i, j, k, ℓ) are the various gradings in the order as above.

To any double complex (M, d', d'') we can associate two spectral sequences $\{E_r^I, d_r^I\}$ and $\{E_r^{II}, d_r^{II}\}$, which are induced by the two canonical filtrations. Moreover, $E_2^I = H(H(M, d''), d')$ and $E_2^{II} = H(H(M, d'), d'')$ and if the double complex is bounded, then both spectral sequences converge to the total homology $H(M, d' + d'')$.

For a link L presented in the form of a closure of a braid b, we form the double complex $(C(L), d, d_N)$, with total grading $\deg_h - \deg_p$. Let $\{E_r^I, d_r^I\}$ and $\{E_r^{II}, d_r^{II}\}$ be respectively the spectral sequences induced by the p and h-filtration. In other words, $\{E_r^I, d_r^I\}$ is a spectral sequence whose E_1-page is $H(L) \cong \mathrm{HKR}(L)$, which converges to $H(C(L), d+d_N) \cong \mathrm{HKR}_N(L)$. As in [23, Lemma 5.4] the differential d_k^I is homogeneous of degree $(2Nk, -2k, 1-k, k)$ and respects the grading $\deg_N = \deg_q + N \deg_\lambda$. This follows immediately from Eq. (33) (recall our λ-grading is twice the grading gr_h in [23]). The differential d_k^{II} is homogeneous of degree $(2N(1-k), 2(1-k), k, 1+k)$ and respects the grading $\deg_N$. I did not check the gradings, but is it still relevant to describe them ? We will now focus on showing the following theorem:

Theorem 4.7. *For each link L and for each $N > 0$,*

$$\mathrm{HKR}_N(L) \cong H(\mathrm{HKR}(L), d_N).$$

For this, we observe we can already define d_N from §3.3 on $R_{\mathfrak{b}}$ and it yields:

Proposition 4.8. ([20, Proposition 4.13]) *The homology of the dg-algebras $(R_{\mathfrak{b}}, d_N)$ and $(R_{\mathfrak{b}}, d_N + d_\beta)$ are concentrated in homogical degree 0.*

Moreover, d_N commutes with d_β (this is obvious from the diagrammatic description of the algebras) and we get:

Proposition 4.9. ([20, Proposition 4.19]) *The spectral sequences induced by d_N and d_β on $R_{\mathfrak{b}}$ both converge at the second page, and*

$$H(H(R_{\mathfrak{b}}, d_N), d_\beta) \cong H(H(R_{\mathfrak{b}}, d_\beta), d_N) \cong H(R_{\mathfrak{b}}, d_\beta + d_N) \cong R^{\Lambda}_{\mathfrak{gl}_{2n}}.$$

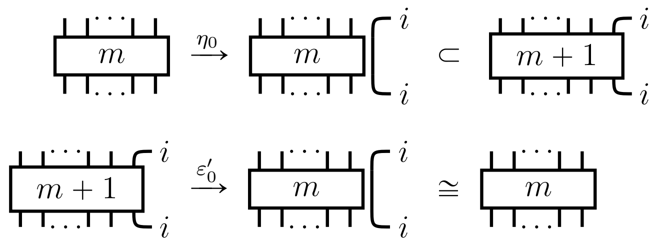
Now the main trick is to remark that we can lift the maps $\eta : \mathbb{1}_\mu \to q^{-1}\mathsf{E}_i^\beta \mathsf{F}_i^\beta \mathbb{1}_\mu$ and $\varepsilon' : q\mathsf{E}_i^\beta \mathsf{F}_i^\beta \mathbb{1}_\mu \to \mathbb{1}_\mu$ that can be used to construct the Rickard complex to maps of $R_\mathfrak{b}$-bimodules. In particular their lifts are given by (a shift) of the maps

$$\eta_0 : R_\mathfrak{b}(m) \to e_{(m,i)} R_\mathfrak{b}(m+1) e_{(m,i)}, x \mapsto x \otimes e_i$$

which is the map that add a vertical strand with label i at the right (recall F_i is given by induction that add a vertical strand with label i), and

$$\varepsilon_0' : e_{(m,i)} R_\mathfrak{b}(m+1) e_{(m,i)} \to R_\mathfrak{b}(m),$$

is given by the projection on $R_\mathfrak{b}(m)\xi^0 \subset R_\mathfrak{b}(m)\Bbbk[\xi]$ in Proposition 3.12 (by §3.2.2 this map descends in the homology w.r.t. d_β to the one that realize the isomorphism $\mathsf{F}_i^\beta \mathsf{E}_i^\beta \cong \mathsf{E}_i^\beta \mathsf{F}_i^\beta$, which we know that is the counit thanks to the 2-action of $\dot{\mathcal{U}}(\mathfrak{sl}_n \times \mathfrak{sl}_n)$). Diagrammatically, we view these maps as



All of this allows us to lift the Rickard complex as a complex of $(R_\mathfrak{b}, R_\mathfrak{b})$-bimodules (i.e. a complex in $\mathfrak{M}^\mathfrak{b}(\beta)$), which we write $(C_0(L), d)$ for a link L. Moreover, it is clear that d commutes with both d_β and d_N, since both η_0 and ϵ_0' are map of dg-bimodules.

Remark 4.10. The homology of the complex $(C_0(L), d)$ should not be an invariant of links.

Moreover, by Proposition 3.12, ε_0' is an epimorphism, and by Proposition 3.10, η_0 is a monomorphism. For the same reasons, we also get the following:

Lemma 4.11. *The map ε_0' is the left inverse of η_0'. Hence there is a decomposition*

$$e_{(m,i)} R_\mathfrak{b}(m+1) e_{(m,i)} \cong R_\mathfrak{b}(m) \oplus \ker(\varepsilon_0')$$

as $(R_\mathfrak{b}(m), R_\mathfrak{b}(m))$-bimodules, and thus a decomposition of functors

$$\mathsf{E}_i \mathsf{F}_i \mathbb{1}_\mu \cong \mathbb{1}_\mu \oplus \ker(\varepsilon_0').$$

Proposition 4.12. *The complex $(C_0(L), d)$ is homotopy equivalent to a complex concentrated in a single homological degree.*

Proof. The complex $(C_0(L), d)$ is obtained as a composition (think tensor product over $R_\mathfrak{b}$) of smaller complexes of the form $\eta_0 : \mathbb{1}_\mu \hookrightarrow q\mathsf{E}_i \mathsf{F}_i \mathbb{1}_\mu$ and $\varepsilon_0' : q^{-1}\mathsf{E}_i \mathsf{F}_i \mathbb{1}_\mu \twoheadrightarrow \mathbb{1}_\mu$ (and with the cup and cap that close the braid, but those are complexes concentrated in a single homological degree). By Lemma 4.11, they are both homotopy equivalent to the complexes $0 \to \ker(\varepsilon_0')$ and $\ker(\varepsilon_0') \to 0$, respectively. Hence, $(C_0(L), d)$ is homotopy equivalent to a composition of those complexes, which is concentrated in a single homological degree. $\square$

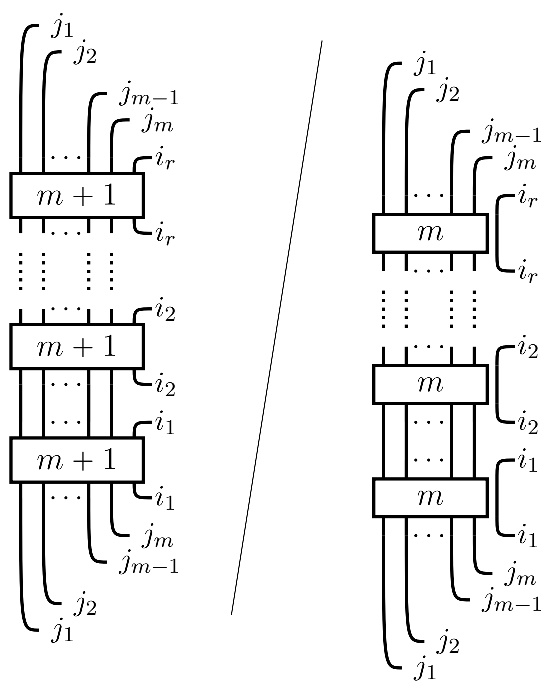
Corollary 4.13. *The homology of the complex $(C_0(L), d)$ is concentrated in a single homological degree.*

Lemma 4.14. *We have isomorphisms*

$$H(H(C_0(L), d), d_\beta) \cong H(H(C_0(L), d_\beta), d) \cong \mathrm{HKR}(L),$$
$$H(H(C_0(L), d), d_\beta + d_N) \cong H(H(C_0(L), d_\beta + d_N), d) \cong \mathrm{HKR}_N(L).$$

Proof. For both case, we have two spectral sequences. By Proposition 3.18, Proposition 4.9 and Corollary 4.13, all homology groups appearing in the first page are concentrated in single homological degree, i.e. on a single line. Therefore, we conclude the four spectral sequences converge all at the second page. □

Remark 4.15. **CHECK LABELS GREG** Note also that Lemma 4.14 gives new tools to compute the HOMFLYPY homology since $H(C_0(L), d)$ can be easily described explicitly, and thus one can construct the complex $H(H(C_0(L), d), d_\beta)$. Diagrammatically, for a braid $\sigma_{i_r} \dots \sigma_{i_1} \in B_\ell$, it is given by a shift of the dg-bimodules (over $\Bbbk$) induced by d_β on the quotient



where $m = \ell^2$ and $\boldsymbol{j} = j_1 \dots j_m$ is given by the pattern

$$01(-1)2(-2)\dots(\ell-1)(1-\ell)01(-1)\dots(\ell-2)(2-\ell)\dots01(-1)0.$$

Then, we can decompose each box as direct sum using Proposition 3.10.

Proof. of Theorem 4.7 We have

$$\begin{aligned} H(\mathrm{HKR}(L), d_N) &\cong H(H(H(C_0(L), d), d_\beta), d_N) \\ &\cong H(H(C_0(L), d), d_\beta + d_N) \\ &\cong \mathrm{HKR}_N(L), \end{aligned}$$

by the first isomorphism of Lemma 4.14, followed by Proposition 4.9, and the second isomorphism of Lemma 4.14. □

We have proved the $\mathfrak{gl}_N$-part of Conjecture 3.1 from [4], for $N > 0$. As a direct consequence, we also get

Corollary 4.16. *Both spectral sequences $\{E_k^I, d_k^I\}$ and $\{E_k^{II}, d_k^{II}\}$ converge at the second page.*

The same arguments hold for reduced homologies. We state the result for the record.

Theorem 4.17. *For a link L presented as the closure of a braid, $\overline{H}_N(L)$ is the homology of the differential $d_N\colon \overline{H}(L) \to \overline{H}(L)$.*

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Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

E-mail address: `gregoire.naisse@uclouvain.be`

Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

E-mail address: `pedro.vaz@uclouvain.be`