

A subordinated CIR model for CVA with wrong-way risk

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Abstract. Finding an adequate model for Credit Valuation Adjustment (CVA) remains a challenging task; it needs to be both flexible and tractable. More explicitly, the expected value of the survival process has to be known in closed form (for calibration purposes), the model should be able to fit any valid CDS curve (to avoid arbitrage opportunities), should lead to large volatilities (in line with CDS options) and finally should be able to feature significant Wrong-Way Risk (WWR) impact. In this paper, we consider the time-changed CIR intensity model introduced by Mendoza-Arriaga & Linetsky. This model allows for two-sides intensity jumps and seems to imply more WWR compared to jump diffusion models. In order to avoid the correlation breakdown resulting from the time change, we work with a synchronized copy of the original exposure process.

1. CVA in a reduced form setup. We consider a fixed time horizon $T > 0$ and a probability space $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{Q})$ where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration generated by the 4-dimensional Brownian motion $W = (W^1, W^2, W^3, W^\perp)$. In this setup, $\mathbb{Q}$ represents the risk-neutral probability measure and W are risk drivers. In particular, W^1 governs the dynamics of the risk-free rate r , hence that of the bank account numéraire:

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

The second Brownian motion W^2 drives the dynamics of the portfolio price process

$$dV_t = b(V_t)dt + \sigma(V_t)dW_t^2, \quad V_0 > 0 \quad (1)$$

The coefficients b, σ are regular enough to guarantee that a unique strong solution to this SDE exists. Finally, we model the default time τ of our counterparty as a random time. It is defined as a first passage time of an increasing stochastic process $\Lambda_t := \int_0^t \lambda_s ds$, $(\lambda_s)_{s \geq 0} \geq 0$, above a unit-mean exponential random barrier $\mathcal{E}$: $\tau := \inf \{t \geq 0 : \Lambda_t \geq \mathcal{E}\}$

In this setup, the default intensity λ is driven by a Brownian motion correlated to W^2 , $W^3 := \rho W^2 + \sqrt{1 - \rho^2} W^\perp$, $\rho \in [-1, 1]$ but the threshold $\mathcal{E}$ is independent from $\mathbb{F}$. In such a reduced form setup, the complete filtration $\mathbb{G}$ is obtained by progressively enlarging $\mathbb{F}$ with $\mathbb{D}$, the natural filtration of the default indicator $D_t = \mathbb{1}_{\{\tau \leq t\}}$: $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ where $\mathcal{D}_t := \sigma(D_u, 0 \leq u \leq t)$. Hence, τ is a $\mathbb{D}$ - and a $\mathbb{G}$ -stopping time, but not a $\mathbb{F}$ -stopping time. Generally speaking however, τ and V are related one to another (via W^2). Assuming $\tau > 0$ and deterministic recovery rate R , the time-0 CVA expression reads

$$\text{CVA} = (1 - R)\mathbb{E} \left[\frac{V_\tau^+}{B_\tau} \mathbb{1}_{\{\tau \leq T\}} \right] = -(1 - R)\mathbb{E} \left[\int_0^T \frac{V_u^+}{B_u} dS_u \right] \approx -(1 - R) \frac{1}{m} \sum_{i=1}^m \sum_{k=1}^n \frac{V_{t_k}^{+, (i)}}{B_{t_k}^{(i)}} \Delta S_{t_k}^{(i)}, \quad n = \frac{T}{\delta} \quad (2)$$

where $S_t := \mathbb{E}[\mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t]$ is the Azéma supermartingale. The right-hand side results from Monte Carlo approximation, by taking the sample mean of m time-integrals discretized in n intervals of length δ .

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Observe that in the above setup, $S_t = e^{-\Lambda_t}$. Moreover, $G_t(T) := \mathbb{E}[S_T|\mathcal{F}_t] = \mathbb{Q}(\tau > T|\mathcal{F}_t)$ is the risk-neutral survival probability. Usually $G_0(T)$ is parametrized as $e^{-\int_0^T h(s)ds}$, where $h > 0$ is the hazard rate curve prevailing at time 0. In the specific case where $\lambda \perp (V, B)$, CVA depends *separately* on the expected discounted exposure and $G_0(\cdot)$. We refer to [4] for more details.

2. Reduced-form (intensity) default models. A convenient way to define the intensity process λ is to set $\lambda_t = k(X_t)$ where k is a given positive function continuous on $(0, \infty)$ and

$$dX_t = \kappa(\beta - X_t)dt + \eta\sqrt{X_t}dW_t^3, \quad X_0 = x > 0 \quad (3)$$

By doing so, the intensity process becomes (a function of) a mean-reverting square-root process X with speed of mean reversion κ , long-term mean β and volatility η , usually chosen to satisfy the Feller constraint $2\kappa\beta > \eta^2$. The time- t probability to survive up to time T implied by the model is given by

$$P(t, T) := \mathbb{E}[\mathbb{1}_{\{\tau > T\}}|\mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}[S_T|\mathcal{F}_t]}{S_t} = \mathbb{1}_{\{\tau > t\}} \frac{G_t(T)}{G_t(t)} \quad (4)$$

To avoid arbitrage opportunities, one need to make sure that $P(0, t) = G_0(t)$ for all $t > 0$.

2.1. The CIR and CIR++ intensity model. A common choice is to consider $k(x) = x$, in which case the intensity is driven by CIR dynamics. Another possibility would consist in adopting a jump diffusion setup. Adding non-negative jumps independent from W^3 in the SDE (3) increases the volatility of the intensity process. These two choices belong to the class of Affine models: the time- t survival probability curve takes the simple form

$$P^{\text{CIR}}(t, T, X_t) = \mathbb{1}_{\{\tau > t\}} A(t, T) e^{-B(t, T)X_t} \quad (5)$$

for some deterministic functions A, B (see [3] for more details). Shifting the process X in a time-dependent way does not affect the above relationship as long as the shift is deterministic. Therefore, one typically consider $\lambda_t = X_t + \psi(t)$ where X is a CIR or JCIR process and ψ is chosen such that the model and market survival probability curves coincide at inception: $P(0, t) = G_0(t)$. The corresponding models are known as CIR++ and JCIR++, depending on whether X features jumps or not.

The jumps in JCIR++ models contribute to partly destroy the correlation between V and (λ, S) but increase the variance of (λ, S) . As a result, WWR usually increases as a result of an increased covariance between credit and exposure processes (in spite of a decreased correlation). However, one cannot increase the activity of J without bounds. By doing so indeed, the calibration constraint $P(0, t) = G_0(t)$ drives the implied shift function ψ downwards. As ψ cannot take negative values, there is a strong limit on the jump rates and/or sizes that one can use while preserving the consistency of the model.

2.2. The time-changed CIR++ intensity model. The fact that JCIR++ can only jump upwards rapidly pushes ψ down. This would not be the case if the jumps could go in both directions. Yet, it is not enough just to use symmetric jumps in JCIR++: this would break the positiveness of X if J is independent from W^3 . And breaking independence would break the affine structure of the model, such that most likely analytical tractability would be lost.

One possibility consists in modeling λ as a time-changed version of a standard intensity process like X . If the stochastic clock features jumps, the resulting time-changed process would still be positive, and would feature jumps in both directions. This would provide a mean to increase the volatility of λ avoiding the implied shift ψ to become negative too quickly as the jumps activity increases. We focus on this route in the sequel.

We define the time-changed CIR (TC-CIR) model by subordinating the CIR process X_t in (3) with a jump-process $\theta_t = t + J_t$ where J_t is a compound Poisson process with exponential jumps (with jump arrival rate ω and mean of the exponential jump size distribution $1/\alpha$), independent from W . That is, we define a new process X^θ by $X_t^\theta := X_{\theta_t}$. As X^θ is no longer affine, we need to appeal the procedure developed by Mendoza-Arriaga and Linetsky [1] to get a closed formula for the survival probability. This approach is a time-changed CIR default intensity by mean of subordination in the sense of Bochner.

Based on a Cox model, it is analytically tractable by means of explicitly computed eigenfunction expansions of relevant semigroups, yielding closed-form pricing of credit-sensitive securities in particular a closed formula of the survival probability. Let's define the corresponding indicator process of D by $D_t^\theta := \mathbf{1}_{\{\tau \leq \theta_t\}}$, $t \geq 0$. To introduce the time-change filtration, we need first to define an inverse subordinator process ($L_t := \inf\{s \geq 0 : \theta_s > t\}$, $t \geq 0$). Let $\mathbb{L} = (\mathcal{L}_t)_{t \geq 0}$ be its completed natural filtration and $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ the enlarged filtration with $\mathcal{H}_t = \mathcal{G}_t \vee \mathcal{L}_t$. We then define our time-changed filtration $\mathbb{H}^\theta = (\mathcal{H}_t^\theta)_{t \geq 0}$ by $\mathcal{H}_t^\theta = \mathcal{H}_{\theta_t}$. Hence, the time-changed bivariate process $(X_t^\theta, D_t^\theta)_{t \geq 0}$ is $\mathbb{H}^\theta$ -adapted and càdlàg and is an $\mathbb{H}^\theta$ -semimartingale (see [1] for details).

Laplace transform of a Lévy subordinator

The Laplace transform of our time-change process takes the simple form

$$\mathbb{E}[e^{-u\theta_t}] = e^{-t\phi(u)} \quad , \quad \phi(u) = u \left(\frac{u + \alpha + \omega}{u + \alpha} \right).$$

Following the procedure devised in [1] (from the Doob-Meyer decomposition of D^θ), our time-changed intensity is exactly

$$\lambda_t^{\mathbb{H}^\theta} = (1 - D_t^\theta)k^\theta(X_t^\theta) \quad \text{with} \quad k^\theta(x) = k(x) + \int_{(0, \infty)} \left(1 - A(0, s)e^{-B(0, s)x}\right) \nu(ds)$$

where $\nu(ds) = \omega\alpha e^{-\alpha s}ds$ and the time-changed survival probability at time t takes the closed form

$$P^{\text{TC-CIR}}(t, T, X_t^\theta) = (1 - D_t^\theta) \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)(T-t)} f_n(0) \varphi_n(X_t^\theta)$$

where λ_n , f_n and φ_n are given in [1]. Setting the time-changed market filtration: $\mathbb{F}^\theta = (\mathcal{F}_t^\theta)_{t \geq 0}$ with $\mathcal{F}_t^\theta = \mathcal{F}_{\theta_t}$, the time-changed Azéma supermartingale reads $S_t^\theta = \mathbb{Q}(\tau > \theta_t | \mathcal{F}_t^\theta)$ and the expectation of S^θ is given by $G_0^\theta(t) = \mathbb{E}[S_t^\theta] = \mathbb{E}[1 - D_t^\theta] = \mathbb{Q}(\tau > \theta_t)$ with $G_t^\theta(T) = \mathbb{Q}(\tau > \theta_T | \mathcal{F}_t^\theta)$. The shifted time-changed CIR (TC-CIR++) model is obtained by defining the time-changed intensity process as $\lambda_t^\theta = k^\theta(X_t^\theta) + \psi(t)$ and finding ψ such that $P^{\text{TC-CIR}}(0, T, \lambda_0^\theta) = G_0^\theta(T) = G_0(T)$.

In the intensity setup, WWR results from the correlation ρ of the increments of the Brownian motions W^2, W^3 . In the time-changed setup however, the intensity is obtained by time-changing the process X : the increments of V are no longer synchronized with those of X^θ . The drop of the correlation will largely impact the WWR effect. In order to limit this side effect, we reproduce the dynamics of V on the discrete time grid with the help of a new discrete-time process.

We construct a discrete time process $\widetilde{W}^2$ as:

$$\widetilde{W}_{t_0}^2 := 0 \quad , \quad \widetilde{W}_{t_k}^2 := \widetilde{W}_{t_{k-1}}^2 + \frac{\Delta W_{\theta_{t_k}}^2}{\sqrt{\Delta \theta_{t_k}}} \sqrt{\Delta t_k} \quad , \quad \Delta W_{\theta_{t_k}}^2 := W_{\theta_{t_k}}^2 - W_{\theta_{t_{k-1}}}^2, \quad k = 1, 2, \dots \quad (6)$$

with $t_k = k\delta$, δ a fixed small time step, $\Delta t_k = t_k - t_{k-1} = \delta$, $\Delta \theta_{t_k} = \theta_{t_k} - \theta_{t_{k-1}}$. This process behaves exactly as W^2 sampled on the time grid: it has the same dynamics as a Brownian motion. The dynamics of V (approximated using an Euler scheme) can thus be equivalently described in terms of W^2 or $\widetilde{W}^2$:

$$\Delta \widetilde{V}_{t_k} = b(\widetilde{V}_{t_{k-1}}) \Delta t_k + \sigma(\widetilde{V}_{t_{k-1}}) \Delta \widetilde{W}_{t_k}^2, \quad \widetilde{V}_{t_0} = V_0, \quad k = 1, 2, \dots \quad (7)$$

The advantage of reconstructing the dynamics of V using $\widetilde{W}^2$ is that the conditional increments of $(\widetilde{V}, X^\theta)$ between any consecutive jumps in the stochastic clock θ are driven by a bivariate standard Gaussian random variable with correlation ρ . This would not be the case if W^2 were used instead.

Assuming zero interest rate ($B = 1$), the CVA formula in the time-changed model can be approximated using m paths of Monte Carlo simulations as

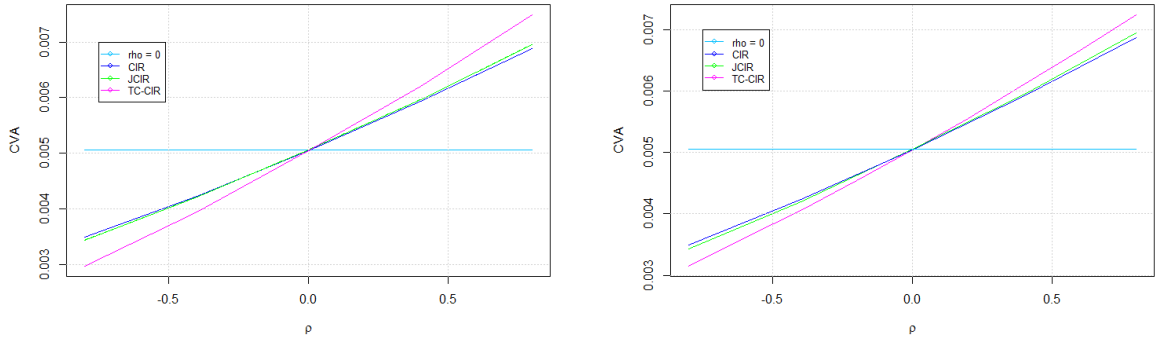
$$\text{CVA}^{\text{TC-CIR}} \approx -(1 - R) \frac{1}{m} \sum_{i=1}^m \sum_{k=1}^n \widetilde{V}_{t_k}^{+, (i)} \Delta S_{t_k}^{\theta, (i)}, \quad n = \frac{T}{\delta}. \quad (8)$$

3. Numerical experiments. In this section, we start by defining the simulation procedure of the bivariate process $(X^\theta, \tilde{V})$. The CVA is computed using standard Monte Carlo simulation and the performance of the shifted time-changed model in term of WWR is compared to the CIR++ and JCIR++ stochastic intensity models. For the sake of simplicity, the recovery rate R and the interest rate r are assumed to be constant and set to zero to put the focus and the treatment of the credit-exposure dependency.

3.1. Simulation procedure. Let's denote by $\mathcal{T} = \{0, \delta, 2\delta, \dots, T\}$ the time- t grid. Let us now sample a specific path of θ on that grid. This leads to the grid $\mathcal{T}^\theta = \{0, \theta_\delta, \theta_{2\delta}, \dots, \theta_T\}$. The simulation grid $\mathcal{T}_{fine}^\theta$ contains $\mathcal{T}^\theta$ and is completed in such a way that (after sorting), the step between two consecutive points is no greater than the chosen time step δ to keep control on the discretization error independently of the jump sizes of θ . To simulate X^θ , we simulate the CIR process X in (3) on $\mathcal{T}_{fine}^\theta$ using Euler scheme. X^θ is obtained by extracting in $\mathcal{T}_{fine}^\theta$ the corresponding values of X on the grid $\mathcal{T}^\theta$.

To obtain $\tilde{V}$, we simulate W^2 on $\mathcal{T}_{fine}^\theta$, extract its corresponding values on $\mathcal{T}^\theta$ and use (6) and (7).

3.2. Numerical results. Figure 1 compares the performances of the three models studied above in terms of WWR impact for a simple forward-type Gaussian exposure ($b(V_t) \equiv 0$, $\sigma(V_t) \equiv \sigma$ and $V_0 = 0$). We fix the CIR parameters $(\sigma, \kappa, \beta, \eta, x)$ as in [4] and search for the jump parameters (ω, α) of JCIR++ and TC-CIR++ such that $\psi \geq 0$ and the WWR impact is maximum. In this context, TC-CIR implies the largest WWR impact.



(a) CIR (0.08, 0.02, 0.161, 0.08, 0.03), JCIR (0.07, 0.08), TC-CIR (0.6, 0.512) (b) CIR (0.08, 0.35, 0.045, 0.10, 0.035), JCIR (0.07, 0.081), TC-CIR (0.6, 0.353)

Figure 1: CVA figures, 3Y Gaussian exposure. Hazard rate is $h(t) = 5\%$.

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