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Estimation of a bivariate conditional copula when a variable is subject to random right censoring

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Abstract

This paper is concern with studying the dependance structure of a random pair (Y_1, Y_2) conditionally upon a covariate X when the variable variable Y_1 is subject to random right censoring. The dependence structure is described by a conditional copula, defined as the function $C_x : [0, 1]^2 \rightarrow [0, 1]$ such that $\mathbb{P}(Y_1 \leq y_1, Y_2 \leq y_2 | X = x) = C_x\{\mathbb{P}(Y_1 \leq y_1 | X = x), \mathbb{P}(Y_2 \leq y_2 | X = x)\}$. In this paper, we propose a procedure to estimate the conditional copula when the variable Y_1 is censored. We establish the asymptotic properties of the proposed estimator. Its finite sample behavior is then investigate in a numerical study. The methodology is illustrated through a real data example featuring patients with malignant melanoma.

Keywords: Survival analysis, conditional copula, functional delta method, Kendall's tau, Spearman's rho, weak convergence.

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1. Introduction

Copulas have become a popular tool to model dependence. Recently, many works in this field have been concerned with capturing the influence of a covariate $X \in \mathbb{R}$ on the dependence structure of a vector of interest $(Y_1, Y_2) \in \mathbb{R}^2$. An example is given in Gijbels et al. (2011), where a copula function is used to illustrate how the relationship between the life expectancy of men (Y_1) and women (Y_2) varies with the gross domestic product (X). To describe this copula function, consider the conditional joint distribution of (Y_1, Y_2) given $X = x$, for a real number x , given by $F_x(y_1, y_2) = \mathbb{P}(Y_1 \leq y_1, Y_2 \leq y_2 | X = x)$. The conditional marginal distributions of Y_1 and Y_2 given $X = x$ are obtained from F_x via $F_{1x}(y) = \lim_{w \rightarrow \infty} F_x(y, w)$ and $F_{2x}(y) = \lim_{w \rightarrow \infty} F_x(w, y)$. If F_{1x} and F_{2x} are continuous, then Sklar's theorem ensures that there exists a unique copula $\mathbb{C}_x : [0, 1]^2 \rightarrow [0, 1]$ such that $F_x(y_1, y_2) = \mathbb{C}_x\{F_{1x}(y_1), F_{2x}(y_2)\}$. Conversely, the copula associated to the bivariate conditional distribution F_x can be extracted from the formula

$$\mathbb{C}_x(u_1, u_2) = F_x\{F_{1x}^{-1}(u_1), F_{2x}^{-1}(u_2)\}. \quad (1)$$

The bivariate function $\mathbb{C}_x$ is called the *conditional copula* and contains all the dependence feature of (Y_1, Y_2) given a fixed value taken by the covariate. For that reason, it is an important task to be able to estimate $\mathbb{C}_x$.

The topic of modeling and estimating conditional copula models have recently gained momentum, since the pioneer work of Patton (2006). For example, models specifying a functional connection between the covariate and a parametric copula were studied in Jondeau & Rockinger (2006) and Patton (2006). A nonparametric estimation procedure for this functional connection was proposed in Acar et al. (2011) while Abegaz et al. (2012) have considered an extension of this method to the case of unknown conditional marginal distributions. Assuming the availability of an i.i.d sample, a nonparametric approach has been investigated in Veraverbeke et al. (2011) and Gijbels et al. (2011), and a bootstrap method suitable for this estimation procedure was developed in Omelka et al. (2013).

However, all of the previously-mentioned estimation strategies rely on the full knowledge of the random variables $(Y_{11}, Y_{21}, X_1), \dots, (Y_{1n}, Y_{2n}, X_n)$ and therefore reveal unsatisfactory when the data are incomplete. Amongst others, the right censoring scheme is a source of incompleteness that frequently appears in medical studies and clinical trials. On this matter, this occurs in

a dataset, to be detailed later, featuring patients with malignant melanoma followed after their surgery for skin tumor.

For the unconditional case, the parametric and semiparametric estimation for the copula functions are studied by Shih & Louis (1995). A nonparametric estimation of the distribution functions for censored data are considered by Dabrowska (1988) and Akritas & Van Keilegom (2003) among others. The nonparametric copula estimation is investigated recently by Gribkova & Lopez (2015) under different scenarios of censoring. Also, they have proposed Goodnees-of-fit procedure for copula when the data are right censored data.

The purpose of this work is to propose a proper methodology designed to estimate the conditional copula when a variable is subject to random right censoring. In that case, the true event time is not recorded for one component but instead a smaller time is observed. To be more specific, assume Y_2 is completely observed, and that the observed random variables are $(T_i, Y_{2i}, X_i, \delta_i)$, where

$$T_i = \min(Y_{1i}, C_i) \quad \delta_i = \mathbb{I}\{Y_{1i} \leq C_i\}, \quad i = 1, \dots, n,$$

and $C_1, \dots, C_n$ are independent and non-negative censoring variables. Hereafter, the conditional distribution function of C_i given $X_i = x_i$ will be denoted $G_{x_i}(t) = \mathbb{P}\{C_i \leq t \mid X_i = x_i\}$. Assuming that (Y_{1i}, Y_{2i}) are independent from C_i conditional upon x_i implies that $F_{2x_i} - H_{x_i} = (F_{2x_i} - F_{x_i})(1 - G_{x_i})$, where $H_{x_i}(t, y) = \mathbb{P}\{T_i \leq t, Y_{2i} \leq y \mid X_i = x_i\}$ is the distribution function of the survival times with marginals H_{1x_i} and H_{2x_i} .

This paper is organized as follows. In Section 2, we propose an estimator for the conditional copula in presence of censoring. This estimator relies on a nonparametric estimator for the joint conditional distribution, which is another original contribution of this paper. In Section 3, we investigate the asymptotic properties of these estimators by providing an asymptotic i.i.d. representation for the conditional distribution estimator, and by identifying the weak limit of properly re-scaled version of these estimators. A simulation study showing the performance of the conditional copula estimation procedure is presented in Section 4. In Section 5, we apply this methodology to the melanoma dataset to illustrate the influence of tumour thickness on the relationship between the survival time and the age of a patient after surgery. All the required assumptions and conditions for the theoretical validity of the results presented in Section 3 are provided in Section 6. The proofs are given in the Appendix.

2. An inverse-conditional-probability-of-censoring estimator for $\mathbb{C}_x$

Let $(Y_{11}, Y_{21}, C_1, x_1), \dots, (Y_{1n}, Y_{2n}, C_n, x_n)$ be a random sample of independent and identically distributed vectors. In this work, for the theoretical developments, we consider the fixed design, which means the case where X is not random. The results can be extended from a fixed design to a random design by using quite similar arguments in the appendix and simply replacing x_i by X_i and using $O_{\mathbb{P}}$ and $o_{\mathbb{P}}$ instead of O and o in the assumptions stated in the appendix.

As previously mentioned, the estimation of $\mathbb{C}_x$ from i.i.d. observations has been considered by Veraverbeke et al. (2011) and Gijbels et al. (2011). In fact, from independent (and fully observed) random variables $(Y_{11}, Y_{21}, x_1), \dots, (Y_{1n}, Y_{2n}, x_n)$, consider the estimator of the joint conditional distribution F_x given by

$$F_{xh}(y_1, y_2) = \sum_{i=1}^n w_{ni}(x, h) \mathbb{I}(Y_{1i} \leq y_1, Y_{2i} \leq y_2),$$

where $w_{n1}(x, h), \dots, w_{nn}(x, h)$ are non-negative kernel-based weight functions that smooth the covariate and $h = h_n$ is a bandwidth parameter that typically depends on the sample size. Popular choices for these functions are the Nadaraya-Watson and the Local-Linear weights given respectively by

$$w_{ni}^{\text{NW}}(x, h) = \frac{K\left(\frac{x_i - x}{h}\right)}{S_{n,0}} \quad \text{and} \quad w_{ni}^{\text{LL}}(x, h) = \frac{K\left(\frac{x_i - x}{h}\right) \{S_{n,2} - \left(\frac{x_i - x}{h}\right) S_{n,1}\}}{S_{n,0}S_{n,2} - S_{n,1}^2},$$

where K is a symmetric and continuously differentiable kernel density function on $[-1, 1]$ and for $j \in \{0, 1, 2\}$,

$$S_{n,j} = \sum_{i=1}^n \left(\frac{x_i - x}{h}\right)^j K\left(\frac{x_i - x}{h}\right).$$

The conditional empirical marginal distributions extracted from F_{xh} are simply

$$F_{1xh}(y) = \lim_{w \rightarrow \infty} F_{xh}(y, w) \quad \text{and} \quad F_{2xh}(y) = \lim_{w \rightarrow \infty} F_{xh}(w, y).$$

From representation (1), a natural plug-in estimator of $\mathbb{C}_x$ is given by

$$\begin{aligned} \mathbb{C}_{xh}(u_1, u_2) &= F_{xh} \{F_{1xh}^{-1}(u_1), F_{2xh}^{-1}(u_2)\} \\ &= \sum_{i=1}^n w_{ni}(x, h) \mathbb{I} \{Y_{1i} \leq F_{1xh}^{-1}(u_1), Y_{2i} \leq F_{2xh}^{-1}(u_2)\}, \end{aligned} \quad (2)$$

where for $j = 1, 2$, $F_{jxh}^{-1}(u) = \inf\{y \in \mathbb{R} : F_{jxh}(y) \geq u\}$ is the left-continuous generalized inverse of F_{jxh} .

The goal of this section is to propose an estimator for the conditional copula in order to take into account the presence of censoring on the variable Y_1 . To do this, we need an estimator for the conditional distribution function F_x . In the unconditional context (i.e without a covariate), the non-parametric estimation of the bivariate distribution of (Y_1, Y_2) in presence of censoring have been studied by many authors, see for example Dabrowska (1988), Akritas (1994) and Akritas & Van Keilegom (2003). However, to the best of our knowledge, the nonparametric estimation of F_x have never been addressed and hence is an original contribution of the present paper.

To built our estimator for F_x , we use a similar idea as the one originally exposed in Robins & Rotnitzky (1992). To compensate for the presence of censoring, each uncensored observation receives an extra weight equal to its inverse probability of failiure. This idea is motivated by the fact that

$$\mathbb{E} \left\{ \mathbb{I}(T \leq t, Y_2 \leq y) \frac{\delta}{1 - G_x(T-)} | X = x \right\} = F_x(t, y).$$

Hence, provided G_x is known, one could estimate F_x with

$$\sum_{i=1}^n w_{ni}(x, h) \mathbb{I}(T_i \leq t, Y_2 \leq y) \frac{\delta_i}{1 - G_x(T_i-)} . \quad (3)$$

As G_x is unknown, we simply replace it with the conditional Kaplan-Meier estimator for the censoring variable C . This estimator is given by:

$$G_{xg}(t) = 1 - \prod_{T_{(i)} \leq t} \left\{ 1 - \frac{w_{n[i]}(x, g)}{1 - \sum_{k=1}^{i-1} w_{n[k]}(x, g)} \right\}^{1-\delta_{[i]}} ,$$

where $T_{(1)} \leq \dots \leq T_{(n)}$ are the ordered T_i , and $\delta_{[i]}$ and $w_{n[i]}(x, h)$ are respectively the corresponding δ_i and $w_{ni}(x, h)$. Here, $g = g_n$ is an auxiliary bandwidth parameter that may differ from h . The resulting estimator for the conditional distribution function F_x is then given by

$$F_{xh}^{(rc)}(t, y) = \sum_{i=1}^n \mathbb{I}\{T_i \leq t, Y_{2i} \leq y\} \frac{w_{ni}(x, h) \delta_i}{1 - G_{xg}(T_i-)} .$$

Although $F_{xh}^{(\text{rc})}$ depends on the bandwidth g , the latter is omitted for notational simplicity. This estimator can be seen as a conditional bivariate analogue to the inverse-probability-of-censoring estimator proposed in Robins & Rotnitzky (1992). Notice that when no censoring occurs, $F_{xh}^{(\text{rc})}(t, y)$ is equal to F_{xh} .

The marginal distributions of $F_{xh}^{(\text{rc})}$ are simply

$$F_{1xh}^{(\text{rc})}(t) = \lim_{y \rightarrow \infty} F_{xh}^{(\text{rc})}(t, y) \quad \text{and} \quad F_{2xh}^{(\text{rc})}(t) = \lim_{y \rightarrow \infty} F_{xh}^{(\text{rc})}(t, y).$$

When $g = h$, we can show that $F_{1xh}^{(\text{rc})}(t)$ coincides with the conditional Kaplan-Meier estimator for the survival time Y_1 introduced in Beran (1981) (see Appendix D).

Following Equation (1), a plug-in estimator for the conditional copula could be defined as

$$F_{xh}^{(\text{rc})}(u, v) = F_{xh}^{(\text{rc})} \left\{ F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{(\text{rc})-1}(v) \right\}$$

However, this estimator does not properly take advantage of the fact that Y_2 is completely observed. Instead of using $F_{2xh}^{(\text{rc})}$, consider the estimator

$$F_{2xh}(y) = \sum_{i=1}^n \mathbb{I}(Y_{2i} \leq y) w_{ni}(x, h).$$

Note that this expression does not, in general, coincide with $F_{2xh}^{(\text{rc})}$. Nevertheless, F_{2xh} uses all of the available knowledge related to F_{2x} . Then, one proceeds to the estimation of $\mathbb{C}_x$ with

$$\begin{aligned} \mathbb{C}_{xh}^{(\text{rc})}(u, v) &= F_{xh}^{(\text{rc})} \{ F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v) \} \\ &= \sum_{i=1}^n \mathbb{I}\{T_i \leq F_{1xh}^{(\text{rc})-1}(u), Y_{2i} \leq F_{2xh}^{-1}(v)\} \frac{w_{ni}(x, h) \delta_i}{1 - G_{xg}(T_i -)}. \end{aligned}$$

When all the survival times are observed, it follows that $\mathbb{C}_{xh}^{(\text{rc})}$ is equal to $\mathbb{C}_{xh}$. Moreover, upon setting all the weight functions $w_{ni}(x, \cdot)$ equal to n^{-1} , we retrieve a very similar estimator as the one proposed in Gribkova & Lopez (2015) to estimate the (unconditional) copula. The only difference is in the estimation of the second marginal distribution.

3. Main theoretical results

The aim of this section is to investigate the large sample behaviour of the processes

$$\mathbb{F}_{xh}^{(\text{rc})} = \sqrt{nh}(F_{xh}^{(\text{rc})} - F_x) \quad \text{and} \quad \mathbb{B}_{xh}^{(\text{rc})} = \sqrt{nh}(\mathbb{C}_{xh}^{(\text{rc})} - C_x).$$

To this end, we first provide in Theorem 3.1 an asymptotic i.i.d representation for $\mathbb{F}_{xh}^{(\text{rc})}$. Then, based on this representation, we obtain a weak convergence result for $\mathbb{F}_{xh}^{(\text{rc})}$ in Corollary 3.2. Finally, the weak limit of $\mathbb{F}_{xh}^{(\text{rc})}$ allows us to establish, in Proposition 3.5, the weak convergence of $\mathbb{B}_{xh}^{(\text{rc})}$.

3.1. Asymptotic i.i.d representation for $\mathbb{F}_{xh}^{(\text{rc})}$

For any distribution function L , let τ_L be the right endpoint of its support, i.e $\inf\{t : L(t) = 1\}$, and write $\bar{\tau}_x = \min\{\tau_{F_{1x}}, \tau_{G_x}\}$. It is a well known problem in life time analysis that the tail support of the distribution of a random variable may not be identifiable due to right censoring (see Stute (1994)). This occurs when the support of the censoring variable is included in the support of the variable of interest, i.e when $\tau_{C_x} < \tau_{F_x}$. As a consequence, we cannot hope to infer on the conditional distribution beyond $\bar{\tau}_x$. Nevertheless, we next establish the asymptotic behavior of $\mathbb{F}_{xh}^{(\text{rc})}$ over any closed subset included in $[0, \bar{\tau}_x] \times \mathbb{R}$.

Hereafter, the sub-distribution function of the uncensored observations will be denoted by $H_{x_i}^u(t, y) = \mathbb{P}(T_i \leq t, Y_{2i} \leq y, \delta_i = 1 \mid X_i = x_i)$. Also, we write $H_{1x_i}(t) = \lim_{y \rightarrow \infty} H_{x_i}(t, y)$, $H_{1x_i}^u(t) = \lim_{y \rightarrow \infty} H_{x_i}^u(t, y)$ and $H_{1x_i}^c(t) = \mathbb{P}(T_i \leq t, \delta_i = 0 \mid X_i = x_i)$.

To identify the asymptotic i.i.d representation for $\mathbb{F}_{xh}^{(\text{rc})}$, we introduce the following random functions

$$\mathcal{J}_{ix}^{(1)}(t, y) = \frac{\mathbb{I}(T_i \leq t, Y_{2i} \leq y, \delta_i = 1)}{1 - G_x(T_i)} - F_x(t, y)$$

and

$$\begin{aligned} \mathcal{J}_{ix}^{(2)}(t, y) = & \int_0^t \frac{\mathbb{I}(T_i \leq v) - H_{1x}(v)}{\{1 - H_{1x}(v)\}^2} v_x(t, y, v) dH_{1x}^c(v) \\ & + \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 0) - H_{1x}^c(v)}{\{1 - H_{1x}(v)\} \{1 - G_x(v)\}} h_x^u(v, y) dv \\ & - \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta = 0) - H_{1x}^c(v)}{\{1 - H_{1x}(v)\}^2} v_x(t, y, v) dH_{1x}(v), \end{aligned}$$

where $h_x^u(v, y) = \frac{\partial}{\partial v} H_x^u(v, y)$ and $v_x(t, y, v) = \int_v^t \frac{h_x^u(z, y) dz}{1 - G_x(z)} = F_x(t, y) - F_x(v, y)$. The required assumptions for the next theorem can be found in Section 6.

Theorem 3.1. *Suppose that $\frac{nh^5}{\log(n)} = O(1)$, $\max(g, h) \rightarrow 0$, $\frac{ng^5}{\log(n)} = O(1)$ and $\frac{h}{g} = O(1)$. Assume Conditions W_1 – W_5 are satisfied, and suppose that Assumptions $(\mathcal{C}_1)$ to $(\mathcal{C}_6)$ are fulfilled for $F_x, H_x, H_x^u, H_{1x}^c$ and G_x . For any $0 < \mathfrak{t} < \bar{\tau}_x$, write $\mathcal{T}_{\mathfrak{t}} = [0, \mathfrak{t}] \times \mathbb{R}$. Then, uniformly in $(t, y) \in \mathcal{T}_{\mathfrak{t}}$, we have*

$$\mathbb{F}_{xh}^{(rc)} = \sqrt{nh} \sum_{i=1}^n \left\{ w_{ni}(x, h) \mathcal{J}_{ix}^{(1)} + w_{ni}(x, g) \mathcal{J}_{ix}^{(2)} \right\} + o_{\mathbb{P}}(1).$$

In some way, Theorem 3.1 decomposes the random function $\mathbb{F}_{xh}^{(rc)}$ into two components. The first component can be associated to the estimation of a conditional distribution function provided that the conditional probability of censoring is known. In other words, it appears as a mildly modified and properly re-scaled version of the random function presented in Equation (3). From this perspective, the second component appears as a consequence of estimating the conditional probability of censoring.

We note that when no censoring occurs, then $\mathcal{J}_{ix}^{(2)}(t, y) = 0$ and $\mathcal{J}_{ix}^{(1)}(t, y) = \mathbb{I}(Y_{1i} \leq t, Y_{2i} \leq y) - F_x(t, y)$. Therefore,

$$\begin{aligned} F_{xh}^{(rc)}(t, y) - F_x(t, y) &= \sum_{i=1}^n w_{ni}(x, h) \mathcal{J}_{ix}^{(1)}(t, y) \\ &= F_{xh}(t, y) - F_x(t, y). \end{aligned}$$

Next, as y goes to infinity, one has $F_{xh}^{(rc)}(t, y) - F_x(t, y) = F_{1xh}^{(rc)}(t, y) - F_{1x}(t, y)$. Also, when $g = h$, $F_{1xh}^{(rc)}$ is equal to the conditional Kaplan-Meier estimator introduced in Beran (1981). Hence, the random function $\sqrt{nh} \{F_{1xh}^{(rc)} - F_{1x}\}$ reduces to the conditional Kaplan-Meier process studied in Van Keilegom & Veraverbeke (1997). As expected, it is shown, in Appendix D.2, that

$$\begin{aligned}
& \lim_{y \rightarrow \infty} \left\{ \sum_{i=1}^n w_{ni}(x, h) \mathcal{J}_{ix}^{(1)}(t, y) + \mathcal{J}_{ix}^{(2)}(t, y) \right\} \\
&= \sum_{i=1}^n w_{ni}(x, h) \{1 - F_{1x}(t)\} \left[\int_0^t \frac{\mathbb{I}(T_i \leq v) - H_{1x}(v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}^u(v) \right. \\
&\quad \left. + \frac{\mathbb{I}(T_i \leq t, \delta_i = 1) - H_{1x}^u(t)}{1 - H_{1x}(t)} \right. \\
&\quad \left. - \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 1) - H_{1x}^u(v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) \right],
\end{aligned}$$

which coincides with the asymptotic i.i.d representation given in Van Keilegom & Veraverbeke (1997).

3.2. Weak convergence of $\mathbb{F}_{xh}^{(rc)}$

In view of Theorem 3.1, the large sample behavior of $\mathbb{F}_{xh}^{(rc)}$ will essentially depend on the conditions imposed on the weight functions and on the bandwidth parameters h and g . To establish its weak limit, we consider the mean zero gaussian processes $\mathbb{J}_x^{(1)}$ and $\mathbb{J}_x^{(2)}$ with covariance function

$$\text{Cov}\{\mathbb{J}_x^{(1)}(t, y), \mathbb{J}_x^{(1)}(t', y')\} = K_4 \left[\int_0^{t \wedge t'} \frac{f_x(v, y \wedge y')}{1 - G_x(v)} dv - F_x(t, y) F_x(t', y') \right]$$

and

$$\text{Cov}\{\mathbb{J}_x^{(2)}(t, y), \mathbb{J}_x^{(2)}(t', y')\} = K_4 \int_0^{t \wedge t'} \frac{v_x(t \wedge t', y, v) v_x(t \wedge t', y', v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}^u(v),$$

where the constant K_4 is defined in Assumption W_4 and $f_x(t, y) = \frac{\partial}{\partial t} F_x(t, y)$. Moreover, let

$$\begin{aligned}
b_x^{(1)}(t, y) &= K_2 \dot{F}_x(t, y) + \frac{K_3}{2} \ddot{F}_x(t, y) \\
&\quad - \int_0^t \frac{K_2 \dot{G}_x(v) + \frac{K_3}{2} \ddot{G}_x(v)}{1 - G_x(v)} f_x(v, y) dv
\end{aligned}$$

and

$$b_x^{(2)}(t, y) = \int_0^t \frac{K_2 \dot{H}_{1x}(v) + \frac{K_3}{2} \ddot{H}_{1x}(v)}{1 - H_{1x}(v)} v_x(t, y, v) dH_x^c(v) \\ + \int_0^t \frac{v_x(t, y, v)}{\{1 - H_{1x}(v)\}} d \left\{ K_2 \dot{G}_x^u(v) + \frac{K_3}{2} \ddot{H}_x^c(v) \right\}.$$

In the latter, the constants K_2 – K_3 are given in Assumption W_2 – W_3 . On one hand, the deterministic function $b_x^{(1)}$ will appear as the asymptotic bias of the process $\sum_{i=1}^n w_{ni}(x, h) \mathcal{J}_{ix}^{(1)}$. Therefore, it would have been present even if G_{xg} was replaced with G_x in the definition of $F_{xh}^{(rc)}$. On the other hand, $b_x^{(2)}$ will emerge as the bias related to the estimation of the conditional probability of censoring.

Corollary 3.2. *Suppose that the assumptions of Theorem 3.1 are met. For any $0 < t < \bar{\tau}_x$, write $\mathcal{T}_t = [0, t] \times \mathbb{R}$.*

- (a) *If $hg^{-1} \rightarrow 0$, and in addition if $\sqrt{nh}g^2 \rightarrow K_5$ for some $K_5 > 0$, then $\mathbb{F}_{xh}^{(rc)}$ converges weakly in $l^\infty(\mathcal{T}_t)$ to $\mathbb{J}_x^{(1)} + K_5 b_x^{(2)}$ over $\mathcal{T}_t$.*
- (b) *If $g = h$, and in addition if $\sqrt{nh}h^5 \rightarrow K_6$ for some $K_6 > 0$, then $\mathbb{F}_{xh}^{(rc)}$ converges weakly to a gaussian process having the representation $\mathbb{J}_x := \mathbb{J}_x^{(1)} + \mathbb{J}_x^{(2)} + K_6 \{b_x^{(1)} + b_x^{(2)}\}$ over $\mathcal{T}_t$, with*

$$\text{Cov}\{\mathbb{J}_x^{(1)}(t, y), \mathbb{J}_x^{(2)}(t', y')\} \\ = \int_0^{t'} \frac{v_x(t', y', v)}{\{1 - H_{1x}(v)\}^2} \{F_x(t \wedge v, y) - H_{1x}(v)F_x(t, y)\} dH_x^c(v) \\ - F_x(t, y) \int_0^{t'} \frac{H_x^c(v)}{\{1 - H_{1x}(v)\} \{1 - G_x(v)\}} h_x^u(v, y') dv \\ + F_x(t, y) \int_0^{t'} \frac{H_x^c(v)}{\{1 - H_{1x}(v)\}^2} v_x(t', y', v) dH_{1x}(v).$$

Remark 3.3. *In view of Part (a) of Corollary 3.2, the impact of estimating the conditional probability of censoring is negligible, provided that the bandwidth g is asymptotically larger than h . However, it is still required that $ng^5 < \infty$, which means that g must not exceed $n^{-1/5}$. Also, because $h/g \rightarrow 0$, we have $h \sim o(n^{-1/5})$. Therefore, choosing g larger than h excludes the optimal bandwidth parameter order for h in terms of mean squared error.*

When the probability of censoring is 0, the term $\mathbb{J}_x^{(2)}$ is not present in the weak limit of $\mathbb{F}_{xh}^{(rc)}$. Therefore, the asymptotic covariance function of $\mathbb{F}_{xh}^{(rc)}$ becomes

$$\begin{aligned} \text{Cov}\{\mathbb{J}_x^{(1)}(t, y), \mathbb{J}_x^{(1)}(t', y')\} &= K_4 \left[\int_0^{t \wedge t'} f_x(v, y \wedge y') dv - F_x(t, y)F_x(t', y') \right] \\ &= K_4 [F_x(v, y \wedge y') - F_x(t, y)F_x(t', y')], \end{aligned}$$

which corresponds to the asymptotic variance of the process $\sqrt{nh}(F_{xh} - F_x)$ in the context of complete i.i.d data (see e.g. Veraverbeke et al. (2011)). Moreover, the bias reduces to

$$K_2 \dot{F}_x(t, y) + \frac{K_3}{2} \ddot{F}_x(t, y),$$

which matches the asymptotic bias of the process $\sqrt{nh}(F_{xh} - F_x)$.

Remark 3.4. *As mentioned in Remark 3.3, using two different bandwidth parameters in the estimation of the conditional distribution excludes the theoretical optimal order for h . Nevertheless, this implies that the bias related to the estimation of F_x provided G_x is known, namely $b_x^{(1)}$, becomes negligible. Hence, in some cases, one might obtain a bias reduction at the cost of excluding the optimal order for h . Note, however, that the same dilemma traditionally occurs in nonparametric density estimation, referring to the decision to under-smooth or not.*

3.3. Weak convergence of $\mathbb{B}_{xh}^{(rc)}$

The next result states the weak limit of the conditional copula estimator under random censoring.

Proposition 3.5. *Suppose that the assumptions in Theorem 3.1 are satisfied, and assume Condition $(\mathcal{D})$, given in the appendix, regarding the partial derivatives of the conditional copula, is satisfied. For any $0 < \mathfrak{t} < \bar{\tau}_x$ and by denoting $\tilde{\mathcal{T}}_{\mathfrak{t}} = [0, H_{1x}(\mathfrak{t})] \times [0, 1]$, we have,*

- (a) *If $hg^{-1} \rightarrow 0$, and $\sqrt{nh}g^2 \rightarrow K_5$ for some $K_5 > 0$, then $\mathbb{B}_{xh}^{(rc)}$ converges weakly in $l^\infty(\tilde{\mathcal{T}}_{\mathfrak{t}})$ to a gaussian process with the following representation:*

$$\alpha_x^{(rc)}(u, v) - C_x^{[1]}(u, v)\alpha_x^{(rc)}(u, 1) - C_x^{[2]}(u, v)\alpha_x^{(rc)}(1, v),$$

where $\alpha_x^{(rc)}(u, v) = \mathbb{J}_x^{(1)}\{F_{1x}^{-1}(u), F_{2x}^{-1}(v)\} + K_5 b_x^{(2)}\{F_{1x}^{-1}(u), F_{2x}^{-1}(v)\}$.

(b) If $h = g$ and $\sqrt{nh^5} \rightarrow K_6$ for some $K_6 > 0$, then $\mathbb{B}_{xh}^{(rc)}$ converges weakly in $l^\infty(\tilde{\mathcal{T}}_1)$ to a gaussian process with the following representation

$$\beta_x^{(rc)}(u, v) - C_x^{[1]}(u, v)\beta_x^{(rc)}(u, 1) - C_x^{[2]}(u, v)\beta_x^{(rc)}(1, v),$$

where $\beta_x^{(rc)}(u, v) = \mathbb{J}_x\{F_{1x}^{-1}(u), F_{2x}^{-1}(v)\}$.

As pointed out in Section 3.2, when all the survival times are completely observed, the term $\mathbb{J}_x^{(2)}$ reduces to 0. In this case, the covariance structure of the limit process $\alpha_x^{(rc)}$ matches the one of the conditional copula process $\sqrt{nh}(\mathbb{C}_{xh} - \mathbb{C}_x)$ found in Veraverbeke et al. (2011).

4. Simulation study

The nonparametric estimation of the conditional copula involves a choice for the weight functions $w_{n1}(x, h), \dots, w_{nn}(x, h)$ that fulfills the required assumptions listed in Section 6.2. It is shown in Omelka et al. (2013) that the requirements W_1 – W_5 are satisfied, among others, by the Nadaraya–Watson and local linear weights, given in Section 2.

The simulation results that will be reported here have been obtained using the local linear weights with the triweight function $K(y) = 35(1 - y^2)^3 \mathbb{I}(|y| \leq 1)/32$. When it happens, negative weights are taken to be zero and the remaining weights are simply re-scaled in order that they sum to one. As pointed out in Omelka et al. (2013), this modification is asymptotically negligible. Finally note that all the numerical experiments were also run using the Nadaraya–Watson kernel. As the results were very similar, they are not presented here.

The primary aim of this section is to evaluate the performance of the proposed conditional copula estimator with respect to the percentage of censoring, the influence of the covariate on the dependance and the effect of the sample size. This performance is evaluated by considering the *average squared bias* (ASB) and the *average variance* (AV). To be specific, if $\hat{\mathbb{C}}_x$ is some estimator of $\mathbb{C}_x$, then

$$\text{ASB}(\hat{\mathbb{C}}_x) = \frac{1}{K^2} \sum_{i,j=1}^K \left\{ \mathbb{E}(\hat{\mathbb{C}}_x(u_i, u_j)) - \mathbb{C}_x(u_i, u_j) \right\}^2$$

and

$$\text{AV}(\widehat{\mathbb{C}}_x) = \frac{1}{K^2} \sum_{i,j=1}^K \mathbb{E} \left\{ \widehat{\mathbb{C}}_x^2(u_i, u_j) - \left(\mathbb{E}(\widehat{\mathbb{C}}_x(u_i, u_j)) \right)^2 \right\}.$$

The latter have been estimated from 1 000 replicates under each of the scenario considered for $x = 0.5$ with $n = 250$ and $n = 1\,000$ and $K = 15$.

Also, the nonparametric estimation of $\mathbb{C}_x$ requires a choice for either one or two bandwidth parameters. Indeed, an interesting aspect of Proposition 3.5 is that the limiting distribution of the copula process $\mathbb{B}_{xh}^{(rc)}$ differs in the case where $g = h$ and $\frac{h}{g} \rightarrow 0$. The secondary aim of this section is to evaluate the impact of using a single or two bandwidth parameters in the estimation of $\mathbb{C}_x$. In the following, we denote by $\mathbb{C}_{xh}^{(rc,1)}$ and $\mathbb{C}_{xh}^{(rc,2)}$ the estimators resulting from the choices $g = h$ and $g \neq h$ respectively. Upon setting $g = h \times 0.25 \log(n)$ for $\mathbb{C}_{xh}^{(rc,2)}$, their performance are compared for different values of h .

The covariate is generated from the standard normal and the estimation of the conditional copula is evaluate at $x = 0.5$. The copula which joins the marginals is either a normal copula $\mathbb{C}_\varrho^N$ or a Clayton copula $\mathbb{C}_\gamma^{\text{CL}}$. Theses are defined for $-1 < \varrho < 1$ and $\theta > 0$ by

$$\mathbb{C}_\varrho^N(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \varphi_\varrho(y, z) \, dz \, dy,$$

and

$$\mathbb{C}_\gamma^{\text{CL}}(u, v) = (u^{-\gamma} + v^{-\gamma} - 1)^{-1/\gamma},$$

where φ_ϱ is the bivariate standard Normal density with correlation ϱ and Φ is the standard Normal distribution. Their parameters will be set to vary with X in the following way. In fact, since the copula parameters are sometimes hard to interpret, it is convenient to quantify the dependence in a bivariate random vector using its corresponding value of Kendall's tau. For any copula $\mathbb{C}$, its associate Kendall's tau can be written as

$$\mathfrak{T}(\mathbb{C}) = 4 \int_0^1 \int_0^1 \mathbb{C}(u_1, u_2) \, d\mathbb{C}(u_1, u_2) - 1. \quad (4)$$

For a given conditional copula $\mathbb{C}_x$, the conditional Kendall's tau is simply $\mathfrak{T}_x = \mathfrak{T}(\mathbb{C}_x)$. Both copulas are parameterized in such a way that $\mathfrak{T}_x = \text{const} \times \{\Phi(x) - 0.2\}^2$. From the relationships between Kendall's tau and the

parameters of the Normal and Clayton copulas, this can be done by setting

$$\varrho(x) = \sin\left(\frac{\mathfrak{T}_x \times \pi}{2}\right) \quad \text{and} \quad \gamma(x) = \frac{2\mathfrak{T}_x}{1 - \mathfrak{T}_x}.$$

This means that the triplet (Y_{1i}, Y_{2i}, x_i) is obtained by considering either $\varrho(x_i)$ or $\gamma(x_i)$ combined with the corresponding copula. The constant is chosen so that $\mathfrak{T}_x \in \{0, .1, .25, .35\}$.

As discussed at the beginning of Section 3.1, the tail support of the distribution of a random variable may not be identifiable due to right censoring when the support of the censoring variable is included in the support of the variable of interest, i.e when $\tau_{C_x} < \tau_{F_x}$. To evaluate the impact on the estimation of $\mathbb{C}_x$, the cases where $\tau_{C_x} = \tau_{F_x}$ and $\tau_{C_x} < \tau_{F_x}$ are examined separately in the following two sections.

4.1. $\tau_{C_x} = \tau_{F_x}$

Here, we have considered the case where $\tau_{C_x} = \tau_{F_x} = \infty$. To do this, the marginal distributions of Y_{1i} and Y_{2i} are generated from the exponential distribution with mean given by

$$\lambda_{x_i} = a\{1 + \Phi(x_i) + \Phi(x_i)^2\},$$

i.e

$$F_{jx_i}(y) = \mathbb{P}\{Y_{ji} \leq y \mid X = x_i\} = 1 - e^{-\frac{y}{\lambda_{x_i}}}.$$

The censoring variable C_i is also picked as an exponential but with mean $c\{1 + \Phi(x_i) + \Phi(x_i)^2\}$. Hence, the probability of censoring conditional on $X = x$, denoted θ thereafter, is simply $\frac{a}{a+c}$. The results are reported for $a = 5$ and $\theta \in \{.2, .4, .6\}$ in Table D.1 and D.2.

4.2. $\tau_{C_x} < \tau_{F_x}$

Here, we have considered $\tau_{F_x} = \infty$ and $\tau_{C_x} < \infty$. In that case, the marginal distributions of Y_{1i} and Y_{2i} are generated from the exponential distribution with mean λ_{x_i} . The censoring variable C_i is generated from the uniform distribution over $[0, \text{const} \times \lambda_{x_i}]$. We can show that in this case, the percentage of censoring is given by

$$\theta = \frac{1}{\text{const}} \times (1 - e^{-\text{const}}).$$

The constant is chosen so that $\theta \in \{0.2, 0.4\}$. We have also cover the scenario $\theta = 0$, which corresponds to the situation when all the survival times are observed. The results are reported in Table D.3 and D.4.

4.3. Comments on the simulations results

From the obtained results it can be seen that, globally, when the association between Y_1 and Y_2 increases, the bias increases whereas the variance decreases slightly. Another interesting finding is that there is no significant difference between the results obtained with a single bandwidth ($g = h$) and double bandwidth ($g \neq h$), except for a large sample size ($n = 1000$). In this case, double bandwidth reduces the bias substantially without increasing the variance of the resulting estimator. The results from the Normal and Clayton copula are quite similar. Also, and as expected, increasing the percentage of censoring decreases the performances of the copula estimator both in terms of bias and variance. The opposite is observed regarding the effect of the sample size. A larger (smaller) bandwidth is needed when censoring (sample size) increase. Notice that we obtain much more accurate information on the conditional copula when $\tau_{C_x} = \tau_{F_x}$. When $\tau_{C_x} < \tau_{F_x}$, one needs a large sample size to get accurate estimates otherwise the results should be interpreted with care especially when the percentage of censoring is high. Finally, as for any kernel based estimator, we can see that a large bandwidth, typically, leads to a larger bias and smaller variance. This becomes clear with large sample size (see the results for $n = 1000$).

5. Illustrative example

In this section we consider a dataset that was used in Andersen et al. (1993). This data contains information on 205 patients with malignant melanoma that were followed for a period up to 15 years. The main variable of interest is Y_1 : the survival times after surgery for skin tumour. Other measured quantities include Y_2 : the age of the patient and X : the tumour thickness in *mm*. 134 patients were alive by the end of the follow-up period and 14 patients died of causes unrelated to melanoma. These patients are censored (*status* = 0) at their last observed duration time or death time. All the remaining patients died from melanoma and so they are uncensored (*status* = 1). The typical objective of such studies is to assess the effect of risk factors (like age and tumour thickness) on survival time. This is done usually by constructing a regression model with Y_1 as response and Y_2 and X as covariates. Before attempting to model the relations between these variables, it may be helpful to measure the strength of the relationship between them using model-free tools.

Kendall's tau is a popular coefficient that measure the concordance-discordance between two random variables. This coefficient lies in $[-1, 1]$ and is equal to zero for independent random variables. In contrast to the well-known Pearson correlation coefficient, Kendall's tau does not require knowledge of the parametric form of the marginal distributions. For more details, see Nelsen (2006). A conditional version of this coefficient was suggested by Gijbels et al. (2011). In terms of copula, the population version of the conditional Kendall's tau of (Y_1, Y_2) given $X = x$ is

$$\mathfrak{T}(\mathbb{C}_x) = 4 \int_0^1 \int_0^1 \mathbb{C}_x(u_1, u_2) d\mathbb{C}_x(u_1, u_2) - 1. \quad (5)$$

A natural way to estimate this coefficient is to replace the unknown quantity $\mathbb{C}_x$ in the above expression by its nonparametric estimator $\hat{\mathbb{C}}_x$ given by (4). This can be expressed as

$$\mathfrak{T}(\hat{\mathbb{C}}_x) = 4 \int_0^{H_{1x}(\mathbf{t})} \int_0^1 \hat{\mathbb{C}}_x(u_1, u_2) d\hat{\mathbb{C}}_x(u_1, u_2) - 1. \quad (6)$$

Except the case when $H_{1x}(\mathbf{t}) = 1$, the truncation in the integral above is needed because $\hat{\mathbb{C}}_x$ is inconsistent outside $[0, H_{1x}(\mathbf{t})] \times [0, 1]$, see Proposition 3.5 above. Unfortunately the quantity $H_{1x}(\mathbf{t})$ is unknown and there is no obvious way to estimate it without imposing some restrictive assumptions on the data generating process. In practice one may consider (6) without the truncation, but then the results should be interpreted with care.

Figure 1 (a) shows the scatter plot of the observed survival times on the y axis and age values on the x axis using different symbols for censored/uncensored observations and different colors for tumour thickness. From this figure it can be seen that there is a relationship between time and age : the survival time has tendency to decrease with increasing age. This tendency is not very strong as the estimated unconditional (global) Kendall's tau is only of -0.13 . Figure 1 (b) shows the estimated conditional Kendall's tau between time and age given thickness. The dashed curve corresponds to the estimator, say $\mathfrak{T}(\hat{\mathbb{C}}_x^{ic})$, obtained ignoring censoring, i.e. we consider all observed times as exact, and the solid curve is the estimator $\mathfrak{T}(\hat{\mathbb{C}}_x)$ obtained using our method that takes into account censoring. For both estimators a bandwidth $h = 0.93$ was used. We can see that while the estimated conditional Kendall's tau coefficients remain negative their magnitude changes with thickness. When the latter increases, the absolute value of $\mathfrak{T}(\hat{\mathbb{C}}_x)$ slightly

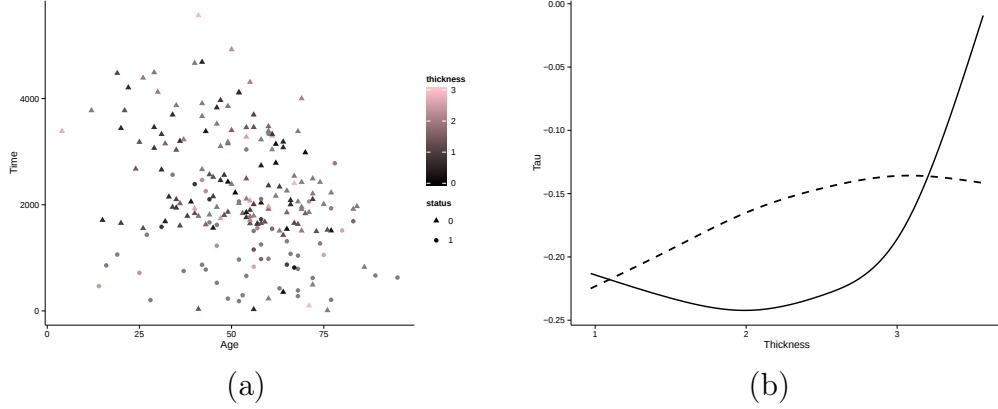


Figure 1: (a) Scatter plot of time versus age (1 $\equiv$ uncensored, 0 $\equiv$ censored); (b) Conditional Kendall's τ as function of thickness estimated (i) taking into account censoring (solid curve) and (ii) considering censored observations as exact survival times (dashed curve).

increases to reach its maximum value of 0.242 when tumor thickness is 2mm and then it starts decreasing rapidly to reach 0. So unlike the global Kendall's tau which measures only the "average" association between time and age, $\mathfrak{T}(\hat{C}_x)$ gives us a more precise picture about this association accounting for the effect of tumor thickness. From the figure it seems that, except for large tumor thickness, the "uncorrected" estimator $\mathfrak{T}(\hat{C}_x^{ic})$ underestimates the strength of association between time and age.

6. Assumptions

6.1. (sub-)distribution functions

Smoothness conditions over $F_x, H_x, H_x^u, H_{1x}^c$ and G_x are needed in the proof of Theorem 3.1. We formulate the conditions for a general (sub-)distribution function L_x .

- (C₁) $\dot{L}_x(t, y) = \frac{\partial}{\partial x} L_x(t, y)$ exists and is continuous over $V(x) \times \mathcal{T}_t$, where $V(x)$ is a neighborhood of x ;
- (C₂) $L_x^{(1)}(t, y) = \frac{\partial}{\partial t} L_x(t, y)$ and $L_x^{(2)}(t, y) = \frac{\partial}{\partial y} L_x(t, y)$ exist and are continuous over $V(x) \times \mathcal{T}_t$;
- (C₃) $\ddot{L}_x(t, y) = \frac{\partial^2}{\partial x^2} L_x(t, y)$ exist and is continuous over $V(x) \times \mathcal{T}_t$;

- (C₄) $L_x^{(1,1)}(t, y) = \frac{\partial^2}{\partial t^2} L_x(t, y)$, $L_x^{(1,2)}(t, y) = \frac{\partial^2}{\partial t \partial y} L_x(t, y)$ and $L_x^{(2,2)}(t, y) = \frac{\partial^2}{\partial y^2} L_x(t, y)$ exist and are continuous over $V(x) \times \mathcal{T}_t$;
- (C₅) $\dot{L}_x^{(1)}(t, y) = \frac{\partial^2}{\partial t \partial x} L_x(t, y)$ and $\dot{L}_x^{(2)}(t, y) = \frac{\partial^2}{\partial y \partial x} L_x(t, y)$ exist and are continuous over $V(x) \times \mathcal{T}_t$;
- (C₆) $\ddot{L}_x^{(1,2)}(t, y) = \frac{\partial^3}{\partial t \partial y \partial x} L_x(t, y)$ and $\ddot{L}_x^{(1,2)}(t, y) = \frac{\partial^4}{\partial t \partial y \partial x^2} L_x(t, y)$ exist and are continuous over $V(x) \times \mathcal{T}_t$;

The following assumption is needed to guaranties the weak convergence of $\mathbb{B}_{xh}^{(\text{rc})}$.

- (D). The partial derivatives $\mathbb{C}_x^{[1]}(u, v) = \partial \mathbb{C}_x(u, v) / \partial u$ and $\mathbb{C}_x^{[2]}(u, v) = \partial \mathbb{C}_x(u, v) / \partial v$ exist and are continuous on $(0, 1) \times [0, 1]$ and $[0, 1] \times (0, 1)$ respectively.

6.2. Weight functions

Assumptions W_1 – W_5 below are required to establish the asymptotic behavior of $\sqrt{nh}\{F_{xh}^{\text{rc}} - F_x\}$ stated in Section 3.

$$W_1. \sqrt{nh} \max_{1 \leq i \leq n} |w_{ni}(x, h)| = o(1);$$

$$W_2. \sqrt{nh} \left| \sum_{i=1}^n w_{ni}(x, h)(x_i - x) - h^2 K_2 \right| = o(1) \text{ for some } K_2 = K_2(x) \in (0, \infty);$$

$$W_3. \sqrt{nh} \left| \sum_{i=1}^n w_{ni}(x, h)(x_i - x)^2 / 2 - h^2 K_3 \right| = o(1) \text{ for some } K_3 = K_3(x) \in (0, \infty);$$

$$W_4. nh \sum_{i=1}^n \{w_{ni}(x, h)\}^2 - K_4 = O(1) \text{ for some } K_4 = K_4(x) \in (0, \infty);$$

$$W_5. \max_{i \in I_{nx}} x_i - \min_{i \in I_{nx}} x_i = o(1), \text{ where } I_{nx} = \{i : w_{ni}(x, h) > 0\}.$$

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Appendix A. Proofs of main results

In this section, all the expectations of the form $\mathbb{E}\{f(T_i, Y_{2i}, C_i)\}$ have to be understood as taken conditional upon $X = x_i$. Formally, for any $1 \leq i \leq n$,

$$\mathbb{E}\{f(T_i, Y_{2i}, C_i)\} = \int \int f(t, y, c) dF_{x_i}(t, y) dG_{x_i}(c),$$

whenever the left-hand side of the integral exists.

Appendix A.1. I.i.d representation for $\sqrt{nh}(F_{xh}^{(rc)} - F_x)$

We start by observing that $F_{xh}^{(rc)} = \sum_{i=1}^n \delta_i f(t, y, T_{1i}, Y_{2i}, G_{xg})$, where for any $(t, y) \in \mathcal{T}_t$, $(v, v') \in \mathbb{R}^2$ and for any function $G : \mathbb{R} \rightarrow [0, 1]$:

$$f(t, y, v, v', G) = \frac{\mathbb{I}(v \leq t, v' \leq y)}{1 - G(v)}.$$

To provide an i.i.d representation for $\sqrt{nh}(F_{xh}^{(rc)} - F_x)$, we apply the ideas of van der Vaart & Wellner (2007). To this end, we introduce the operator $\mathbb{E}(\cdot)$ defined over the set of random variables of the form $\delta_1 f(t, y, T_1, Y_{21}, G), \dots, \delta_n f(t, y, T_n, Y_{2n}, G)$ such that

$$\mathbb{E}\{\delta_i f(t, y, T_i, Y_{2i}, G)\} = \int f(t, y, v, v', G) dH_{x_i}^u(v, v'),$$

whenever the right-hand side of the integral exists. Observe that when the function G is fixed (non random), it follows that

$$\mathbb{E}\{\delta_i f(t, y, T_i, Y_{2i}, G)\} = \mathbb{E}\{\delta_i f(t, y, T_i, Y_{2i}, G)\}.$$

Then, we consider the following decomposition :

$$\sqrt{nh}(F_{xh}^{(rc)} - F_x) = \{A_{xh} - \mathbb{E}(A_{xh})\} + B_{xh} + \mathbb{E}(A_{xh}),$$

where

$$A_{xh}(t, y) =$$

$$\sqrt{nh} \sum_{i=1}^n \mathbb{I}(T_i \leq t, Y_{2i} \leq y) w_{ni}(x, h) \delta_i \left\{ \frac{1}{1 - G_{xg}(T_i)} - \frac{1}{1 - G_x(T_i)} \right\}$$

and

$$\begin{aligned} B_{xh}(t, y) &= \sqrt{nh} \left\{ \sum_{i=1}^n \mathbb{I}(T_i \leq t, Y_{2i} \leq y) w_{ni}(x, h) \frac{\delta_i}{1 - G_x(T_i)} - F_x(t, y) \right\} \\ &= \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \mathcal{J}_{ix}^{(1)}(t, y). \end{aligned}$$

Firstly, we remark that B_{xh} is, by construction, an i.i.d representation. Secondly, we show, in Appendix A.1.1, that $A_{xh} - \mathbb{E}(A_{xh})$ is asymptotically negligible. Finally, in Appendix A.1.2, we provide an i.i.d representation for $\mathbb{E}(A_{xh})$.

Appendix A.1.1. Asymptotic negligibility of $A_{xh} - \mathbb{E}(A_{xh})$

Recall that $\mathcal{T}_t = [0, t] \times \mathbb{R}$, where $0 < t < \tau_{H_x}$. Let $\epsilon > 0$ such that $G_x(t) + \epsilon < 1$, and set $t_\epsilon = G_x(t) + \epsilon$. Introduce the set of functions $\mathcal{G}_{t_\epsilon} := \{G : \mathbb{R} \rightarrow [0, 1] \text{ nondecreasing and } G(t) < t_\epsilon\}$. For $(t, y, G) \in \mathcal{T}_t \times \mathcal{G}_{t_\epsilon}$, define the stochastic processes

$$Z_{hi}(t, y, G) = \sqrt{nh} \mathbb{I}(T_i \leq t, Y_{2i} \leq y) w_{ni}(x, h) \frac{\delta_i}{1 - G(T_i)}$$

and $Z_{xh} = \sum_{i=1}^n Z_{hi}$. The process Z_{xh} can be viewed as a process indexed by the family of functions from $\mathbb{R}^2 \times \{0, 1\} \rightarrow \mathbb{R}$ given by

$$\mathcal{F} = \left\{ (v, v', w) \mapsto \mathbb{I}(v \leq t, v' \leq y) \frac{w}{1 - G(v)}, (t, y) \in \mathcal{T}_t, G \in \mathcal{G}_{t_\epsilon} \right\}.$$

Hence, each function $f \in \mathcal{F}$ may be formally identified by a triplet (t, y, G) . The introduction of the process Z_{xh} is motivated by the fact that $A_{xh}(t, h) = Z_{xh}(t, y, G_{xg}) - Z_{xh}(t, y, G_x)$. While the ϵ -enlargement in the definition of the class $\mathcal{G}_{t_\epsilon}$ might appear overdone, it is however required to guaranty that G_{xg} asymptotically fits into $\mathcal{G}_{t_\epsilon}$.

Finally, we equip the index set $\mathcal{F}$ with a semimetric ρ_{F_x} defined for $f = (t, y, G)$ and $f' = (t', y', G')$ as

$$\rho_{F_x}(f, f') = |F_{1x}(t) - F_{1x}(t')| + |F_{2x}(y) - F_{2x}(y')| + \sup_{z \in [0, t]} |G(z) - G'(z)|.$$

Notice that $(\mathcal{F}, \rho)$ is totally bounded for ρ_{F_x} , as $\mathbb{I}\{v \leq t, v' \leq y\} \frac{w}{1 - G(v)} \leq \frac{\mathbb{I}\{v \leq t\}}{1 - G(v)} \leq \frac{1}{1 - t_\epsilon} < \infty$. Moreover, $\mathfrak{F} := \frac{1}{1 - t_\epsilon}$ is an envelope function for $\mathcal{F}$.

In fact, as Assumptions W_1 - W_4 - W_6 and $\mathcal{C}_1$ are satisfied, we conclude from Appendix C.1 that the process $\bar{Z}_{xh} := Z_{xh} - \mathbb{E}Z_{xh}$ indexed by $(\mathcal{F}, \rho_{F_x})$ is asymptotically ρ_{F_x} -equicontinuous. This implies that for any $\eta > 0$ and $\eta' > 0$, there exists $\delta > 0$ such that

$$\limsup_{n \rightarrow \infty} \mathbb{P}^* \left\{ \sup_{\rho_{F_x}(f, f') < \delta} |\bar{Z}_n(f') - \bar{Z}_n(f)| > \eta \right\} < \eta'.$$

For a particular choice of weight system $w_{n1}, \dots, w_{nn}$, it is shown in Van Keilegom & Veraverbeke (1996) that as long as conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_3)$ are satisfied for H_x , G_x and H_x^c , then there exist constants $C_1, C_2 > 0$ that depends on the weight functions such that whenever $\epsilon > \max\{C_1 g^2, C_2 \frac{1}{\sqrt{ng}}\}$, we have

$$\mathbb{P} \left\{ \sup_{t \in [0, \mathfrak{t}]} |G_{xg}(t) - G_x(t)| > \epsilon \right\} \leq C_3 n g \epsilon e^{-C_4 n g \epsilon^2} \quad (\text{A.1})$$

for some constants C_3 and C_4 that rely on $\mathfrak{t}$ and the weights. It can be shown for instance by using Lemma 3 in Omelka et al. (2013) that this result still holds for general $w_{n1}, \dots, w_{nn}$, at the cost of perhaps enlarging the constants provided this weight system satisfy assumptions W_1 - W_5 . Hence, we obtain

$$\sup_{t \in [0, \mathfrak{t}]} |G_{xg}(t) - G_x(t)| = O((ng)^{-1/2} (\log n)^{1/2}) \quad \mathbb{P} - \text{a.s.} \quad (\text{A.2})$$

In view of Equation (A.2), we conclude that for any $\epsilon > 0$, $G_{xg} \in \mathcal{G}_\epsilon$ with probability 1. Moreover, as $\rho_{F_x}\{(t, y, G_{xg}), (t, y, G_x)\} = \sup_{t \in [0, \mathfrak{t}]} |G_{xg}(t) - G_x(t)|$, we deduce that for sufficiently large n :

$$\sup_{(t, y) \in \mathcal{T}_\mathfrak{t}} \left| \sqrt{nh} [A_{xh}(t, y) - \mathbb{E}\{A_{xh}(t, y)\}] \right| \leq \sup_{\rho_{F_x}(f, f') < \delta} |\bar{Z}_n(f') - \bar{Z}_n(f)| \xrightarrow{\mathbb{P}} 0.$$

which concludes the proof.

Appendix A.1.2. Asymptotic representation of $\mathbb{E}(A_{xh})$

Note that the random variable $D_{xh}(t, y) = \mathbb{E}\{A_{xh}(t, y)\}$ can be rewritten as

$$\begin{aligned} D_{xh}(t, y) &= \sqrt{nh} \int_0^t \int_0^y \frac{G_{xg}(v) - G_x(v)}{\{1 - G_x(v)\}\{1 - G_{xg}(v)\}} \sum_{i=1}^n w_{ni}(x, h) dH_{x_i}^u(v, v') \\ &= \sqrt{nh} \int_0^t \frac{G_{xg}(v) - G_x(v)}{\{1 - G_x(v)\}\{1 - G_{xg}(v)\}} \sum_{i=1}^n w_{ni}(x, h) \int_0^y dH_{x_i}^u(v, v') \\ &= \sqrt{hg^{-1}} \int_0^t \frac{\mathbb{G}_{xg}(v)}{\{1 - G_x(v)\}\{1 - G_{xg}(v)\}} \sum_{i=1}^n w_{ni}(x, h) h_{x_i}^u(v, y) dv, \end{aligned}$$

where $h_{x_i}^u(v, y) = \frac{\partial}{\partial v} H_{x_i}^u(v, y)$ and $\mathbb{G}_{xg} = \sqrt{ng}\{G_{xg} - G_x\}$. Next, we write

$$\tilde{D}_{xh}(t, y) = \sqrt{hg^{-1}} \int_0^t \frac{\mathbb{G}_{xg}(v)}{\{1 - G_x(v)\}^2} \sum_{i=1}^n w_{ni}(x, h) h_{x_i}^u(v, y) dv$$

and we introduce for any function $g \in l^\infty([0, \mathfrak{t}])$ the functional $\Lambda_x : l^\infty([0, \mathfrak{t}]) \rightarrow l^\infty(\mathcal{T}_t)$ defined as

$$\Lambda_x(g)(t, y) = \int_0^t \frac{g(v)}{\{1 - G_x(v)\}^2} h_x^u(v, y) dv.$$

The asymptotic representation of D_{xh} will follow from the representation of $\sqrt{hg^{-1}}\Lambda_x(\mathbb{G}_{xg})$ and the asymptotic negligibility of the two terms $\tilde{D}_{xh} - \sqrt{hg^{-1}}\Lambda_x(\mathbb{G}_{xg})$ and $D_{xh} - \tilde{D}_{xh}$.

For the asymptotic representation of $\sqrt{hg^{-1}}\Lambda_x(\mathbb{G}_{xg})$, let

$$\begin{aligned} g_{ix}(t) = \{1 - G_x(t)\} & \left[\int_0^t \frac{\mathbb{I}(T_i \leq v) - H_{1x}(v)}{\{1 - H_{1x}(v)\}^2} dH_x^c(v) \right. \\ & + \frac{\mathbb{I}(T_i \leq t, \delta_i = 0) - H_x^c(t)}{1 - H_{1x}(t)} \\ & \left. - \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 0) - H_x^c(v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) \right]. \end{aligned}$$

It can be shown for instance by using Lemma 3 of Omelka et al. (2013) that the result stated in Theorem 2.1 in Van Keilegom & Veraverbeke (1997) still holds as long as conditions $(\mathcal{C}_1)$ – $(\mathcal{C}_5)$ are satisfied for G_x and G_x^u , provided assumptions W_1 – W_5 on the weight functions are fulfilled. Hence, we conclude that, uniformly in $t \in [0, \mathfrak{t}]$, $\sqrt{hg^{-1}}\mathbb{G}_{xg} = \sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) g_{ix} + o_{a.s.}(1)$ as long as $\frac{ng^5}{\log(n)} < \infty$. As the map $\Lambda_x(\cdot)$ is linear and continuous, the continuous mapping theorem implies that

$$\sqrt{hg^{-1}}\Lambda_x(\mathbb{G}_{xg}) = \sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) \Lambda_x(g_{ix}) + o_{a.s.}(1).$$

Furthermore, from switching the order of integration and further computations, we show that $\Lambda_x(g_{ix})(t, y) = \mathcal{J}_{ix}^{(2)}(t, y)$.

To show the asymptotic negligibility of $\tilde{D}_{xh} - \sqrt{hg^{-1}}\Lambda_x(\mathbb{G}_{xg})$, we observe that for any $(t, y) \in \mathcal{T}_t$,

$$\begin{aligned}\tilde{D}_{xh}(t, y) - \sqrt{hg^{-1}}\Lambda_x(\mathbb{G}_{xg})(t, y) &= \int_0^t \frac{G_{xg}(v) - G_x(v)}{\{1 - G_x(v)\}^2} R_{xh}(v, y) dv \\ &\leq \frac{\sup_{v \in [0, t]} |G_{xg}(v) - G_x(v)|}{\{1 - G_x(t)\}^2} \int_0^t |R_{xh}(v, y)| dv,\end{aligned}$$

where

$$R_{xh}(v, y) = \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \{h_x^u(v, y) - h_{x_i}^u(v, y)\}.$$

Condition $(\mathcal{C}_6)$ together with W_5 allows the Taylor expansion

$$\begin{aligned}R_{xh}(v, y) &= \dot{h}_x^u(v, y) \sqrt{nh} \sum_{i=1}^n (x - x_i) w_{ni}(x, h) \\ &\quad - \frac{1}{2} \sqrt{nh} \sum_{i=1}^n (x - x_i)^2 w_{ni}(x, h) \ddot{h}_{z_i}^u(v, y),\end{aligned}$$

where z_i lies between x_i and x . Hence, from Assumptions W_2, W_3, W_6 and $(\mathcal{C}_6)$, we obtain that

$$\sqrt{nh} \int_0^t |R_{xh}(u, v)| dv = O(1).$$

In view of the previous discussion, we use Equation (A.2) to obtain that uniformly in $(t, y) \in \mathcal{T}_t$, $\tilde{D}_{xh} - \sqrt{hg^{-1}}\Lambda_x(\mathbb{G}_{xg}) = o_{a.s.}(1)O(1)$.

Finally, notice that

$$\begin{aligned}D_{xh} - \tilde{D}_{xh} &= \sqrt{nh} \int_0^t \frac{\{G_{xh}(v) - G_x(v)\}^2}{\{1 - G_x(v)\}^2 \{1 - G_{xh}(v)\}} \sum_{i=1}^n w_{ni}(x, h) h_{x_i}^u(v, y) dv \\ &\leq \sqrt{nh} \frac{\sup_{v \in [0, t]} \{G_{xg}(v) - G_x(v)\}^2}{\{1 - G_x(t)\}^2 \{1 - G_{xg}(t)\}} \int_0^t \left| \sum_{i=1}^n w_{ni}(x, h) h_{x_i}^u(v, y) \right| dv.\end{aligned}$$

Hence, using Equation (A.2), we deduce that the latter is $o_{a.s.}(1)$.

Appendix A.2. Weak convergence of $\sqrt{nh}\{F_{xh}^{(rc)} - F_x\}$

From Theorem 3.1, we have

$$\sqrt{nh}(F_{xh}^{(rc)} - F_x) = \sqrt{nh} \sum_{i=1}^n \left\{ w_{ni}(x, h) \mathcal{J}_{ix}^{(1)} + w_{ni}(x, g) \mathcal{J}_{ix}^{(2)} \right\} + o_{\mathbb{P}}(1).$$

We start by showing that the sequence $\sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \mathcal{J}_{ix}^{(1)}$ is asymptotically gaussian. Then, regarding the assumptions over the bandwidth h and g , we discuss the asymptotic representation of

$$\sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) \mathcal{J}_{ix}^{(2)} \quad \text{and} \quad \sqrt{nh} \sum_{i=1}^n \{w_{ni}(x, h) \mathcal{J}_{ix}^{(1)} + w_{ni}(x, g) \mathcal{J}_{ix}^{(2)}\}.$$

Appendix A.2.1. Weak convergence of $\sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \mathcal{J}_{ix}^{(1)}$

Let's prove that the process $\mathbb{J}_{xh}^{(1)} := \sqrt{nh} \sum_{i=1}^n \mathcal{J}_{ix}^{(1)}$ converges to the gaussian process $\mathbb{J}_x^{(1)} + Kb_x^{(1)}$ over $\mathcal{T}$.

First, the tightness of the sequence $\mathbb{J}_{xh}^{(1)}$ can be checked using similar arguments as in Appendix A.1.1. Second, from direct computation,

$$\begin{aligned} & \mathbb{E} \left\{ \mathbb{J}_{xh}^{(1)}(t, y) \right\} \\ &= \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \mathbb{E} \left\{ \mathbb{E}(\delta_i \mid Y_{1i}, Y_{2i}) \frac{\mathbb{I}(Y_{1i} \leq t, Y_{2i} \leq y)}{1 - G_x(Y_{1i})} - F_x(t, y) \right\} \\ &= \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \mathbb{E} \left\{ \frac{1 - G_{x_i}(Y_{1i})}{1 - G_x(Y_{1i})} \mathbb{I}(Y_{1i} \leq t, Y_{2i} \leq y) - F_x(t, y) \right\} \\ &= \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \{F_{x_i}(t, y) - F_x(t, y)\} + o(1) \\ &+ \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) (x_i - x) \mathbb{E} \left\{ \frac{-\dot{G}_x(Y_{1i})}{1 - G_x(Y_{1i})} \mathbb{I}(Y_{1i} \leq t, Y_{2i} \leq y) \right\} \\ &+ \frac{1}{2} \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) (x_i - x)^2 \mathbb{E} \left\{ \frac{-\ddot{G}_x(Y_{1i})}{1 - G_x(Y_{1i})} \mathbb{I}(Y_{1i} \leq t, Y_{2i} \leq y) \right\}, \quad (\text{A.3}) \end{aligned}$$

where last equality follows from a Taylor expansion of the function $z \mapsto 1 - G_z$ around $z = x$ together with the fact that from Assumptions $\mathcal{C}_1$ and $\mathcal{C}_3$, $G_x, \dot{G}_x$

and $\ddot{G}_x$ are uniformly continuous over $\mathcal{T}_t$. We calculate the first term. In fact,

$$\begin{aligned} \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \{F_{x_i}(t, y) - F_x(t, y)\} \\ = \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) (x_i - x) \dot{F}_x(t, y) \\ + \frac{1}{2} \ddot{F}_x(t, y) \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) (x_i - x)^2 + o(1), \quad (\text{A.4}) \end{aligned}$$

where last equality follows from a Taylor expansion of the function $z \mapsto F_z$ around $z = x$ together with the fact that $F_x, \dot{F}_x$ and $\ddot{F}_x$ are uniformly continuous over $V(x) \times \mathcal{T}_t$, where $V(x)$ is some neighborhood of x , as stated from Assumptions $\mathcal{C}_1$ - $\mathcal{C}_3$. From Assumptions W_2 and W_3 , Equation (A.4) is equal to

$$\sqrt{nh}h^2 \left\{ K_2 \dot{F}_x(t, y) + \frac{K_3}{2} \ddot{F}_x(t, y) + o(1) \right\}.$$

where the constants K_2 and K_3 are defined via Assumptions W_2 and W_3 . As $\sqrt{nh}h^2 \rightarrow K$, Equation (A.4) reduces to

$$K \left\{ K_2 \dot{F}_x(t, y) + \frac{K_3}{2} \ddot{F}_x(t, y) \right\} + o(1).$$

Next, for the second and third terms of Equation (A.3), we denote

$$B_z^{(1)}(t, y) = \int_0^t \frac{-\dot{G}_x(v)}{1 - G_x(v)} f_z(v, y) dv$$

and

$$B_z^{(2)}(t, y) = \int_0^t \frac{-\ddot{G}_x(v)}{1 - G_x(v)} f_z(v, y) dv.$$

The second and third terms of Equation (A.3) can be re-written as

$$\begin{aligned} \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) (x_i - x) B_{x_i}^{(1)}(t, y) + \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) (x_i - x)^2 B_{x_i}^{(2)}(t, y) \\ = K \left\{ K_2 B_x^{(1)}(t, y) K_3 + o(1) B_x^{(2)}(t, y) \right\} + o(1). \quad (\text{A.5}) \end{aligned}$$

Putting together Equations (A.3), (A.4) and (A.5) concludes to show that

$$\mathbb{E} \left\{ \mathbb{J}_{xh}^{(1)} \right\} = K b_x^{(1)} + o(1).$$

Next, as Assumption W_4 is satisfied, we use a similar strategy to obtain

$$\begin{aligned}
& \text{Cov}\{\mathbb{J}_{xh}^{(1)}(t, y), \mathbb{J}_{xh}^{(1)}(t', y')\} \\
&= nh \sum_{i=1}^n w_{ni}(x, h)^2 \left[\mathbb{E} \left\{ \frac{\mathbb{I}(T_i \leq t \wedge t', Y_{2i} \leq y \wedge y') \delta_i}{\{1 - G_{x_i}(T_i)\}^2} \right\} \right. \\
&\quad \left. - \mathbb{E} \left\{ \frac{\mathbb{I}(T_i \leq t, Y_{2i} \leq y) \delta_i}{1 - G_{x_i}(T_i)} \right\} \mathbb{E} \left\{ \frac{\mathbb{I}(T_i \leq t', Y_{2i} \leq y') \delta_i}{1 - G_{x_i}(T_i)} \right\} \right] \\
&= K_4 \left\{ \int_0^{t \wedge t'} \frac{f_x(v, y \wedge y')}{1 - G_x(v)} dv - F_x(t, y) F_x(t', y') \right\} + o(1).
\end{aligned}$$

In view of the tightness of $\mathbb{J}_{xh}^{(1)}$, the fact that

$$\text{Cov}\{\mathbb{J}_{xh}^{(1)}(t, y), \mathbb{J}_{xh}^{(1)}(t', y')\} \rightarrow \text{Cov}\{\mathbb{J}_x^{(1)}(t, y), \mathbb{J}_x^{(1)}(t', y')\},$$

and using Theorem 2.11.1 of van der Vaart & Wellner (1996), we conclude that $\mathbb{J}_{xh}^{(1)}$ converges weakly to a gaussian process whose representation matches the one of $\mathbb{J}_x^{(1)} + Kb_x^{(1)}$.

Appendix A.2.2. Asymptotic representation of $\sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) \mathcal{J}_{ix}^{(2)}$

From the definition of g_{ix} and of $\Lambda_x(\cdot)$ in Appendix A.1.2, we can re-write

$$\sum_{i=1}^n w_{ni}(x, g) \mathcal{J}_{ix}^{(2)} = \sum_{i=1}^n w_{ni}(x, g) \Lambda_x(g_{ix}) = \Lambda_x \left\{ \sum_{i=1}^n w_{ni}(x, g) g_{ix} \right\}.$$

Now because $\Lambda_x(\cdot)$ is linear, we deduce that

$$\mathbb{E} \left[\Lambda_x \left\{ \sum_{i=1}^n w_{ni}(x, g) g_{ix} \right\} \right] = \Lambda_x \left[\mathbb{E} \left\{ \sum_{i=1}^n w_{ni}(x, g) g_{ix} \right\} \right].$$

From Van Keilegom & Veraverbeke (1997), using Assumptions W_2 – W_3 together with $\mathcal{C}_1$ and $\mathcal{C}_3$, we obtain that

$$\begin{aligned}
\mathbb{E} \left\{ \sum_{i=1}^n w_{ni}(x, g) g_{ix}(t) \right\} &= g^2 \{1 - G_x(t)\} \int_0^t \frac{K_2 \dot{H}_{1x}(s) + \frac{K_3}{2} \ddot{H}_{1x}(s)}{\{1 - H_{1x}(s)\}^2} dH_{1x}^c(s) \\
&\quad + \frac{d\{K_2 \dot{H}_x^c(s) + \frac{K_3}{2} \ddot{H}_x^c(s)\}}{\{1 - H_{1x}(s)\}} + o(g^2).
\end{aligned}$$

Hence, integrating by parts leads to:

$$\Lambda_x \left[\mathbb{E} \left\{ \sum_{i=1}^n w_{ni}(x, g) g_{ix} \right\} \right] (t, y) = g^2 \mathbb{E} \{ \mathbb{J}_x^{(2)}(t, y) \} + o(g^2). \quad (\text{A.6})$$

Furthermore, provided Assumption W_4 is satisfied, Van Keilegom & Veraverbeke (1997) also gives us

$$\begin{aligned} \text{Cov} \{ \sum_{i=1}^n w_{ni}(x, g) g_{ix}(t), \sum_{i=1}^n w_{ni}(x, g) g_{ix}(t') \} \\ = \frac{K_4}{ng} \{1 - G_x(t)\} \{1 - G_x(t')\} \int_0^{t \wedge t'} \frac{dH_x^c(s)}{\{1 - H_{1x}(s)\}^2} + o\left(\frac{1}{ng}\right). \end{aligned}$$

Then,

$$\begin{aligned} \text{Cov} [\Lambda_x \{ \sum_{i=1}^n w_{ni}(x, g) g_{ix} \} (t, y), \Lambda_x \{ \sum_{i=1}^n w_{ni}(x, g) g_{ix} \} (t', y')] \\ = \int_0^t \int_0^{t'} \frac{\text{Cov} \{ g_{ix}(v), g_{ix}(v') \}}{\{1 - G_x(v)\}^2 \{1 - G_x(v')\}^2} h_x^u(v, y) h_x^u(v', y') dv dv' \\ = \frac{K_4}{ng} \int_0^{t \wedge t'} \int_s^t \int_s^{t'} \frac{h_x^u(v, y) h_x^u(v', y') dv dv'}{\{1 - G_x(v)\} \{1 - G_x(v')\}} \frac{dH_{1x}^c(s)}{\{1 - H_{1x}(s)\}^2} + o\left(\frac{1}{ng}\right). \end{aligned} \quad (\text{A.7})$$

From the fact that $\int_s^t \frac{h_x^u(v, y)}{\{1 - G_x(v)\}} dv = \int_s^t f_x(v, y) dv = F_x(t, y) - F_x(s, y)$, the non negligible term of Equation (A.7) is equal to $\text{Cov} \{ \mathbb{J}_x^{(2)}(t, y), \mathbb{J}_x^{(2)}(t', y') \}$. Since $\sqrt{nh}g^2 < \infty$, and $\frac{h}{g} = O(1)$, we deduce from Equations (A.6) and (A.7) that the sequence $\sum_{i=1}^n w_{ni}(x, g) \mathcal{J}_{ix}^{(2)}$ is asymptotically tight on $\mathcal{T}_t$.

Appendix A.2.3. Asymptotic normality of $\sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \{ \mathcal{J}_{ix}^{(1)} + \mathcal{J}_{ix}^{(2)} \}$

Upon setting $\mathcal{J}_{xh} := \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \{ \mathcal{J}_{ix}^{(1)} + \mathcal{J}_{ix}^{(2)} \}$, we compute

$$\begin{aligned} \text{Cov} \{ \mathcal{J}_{xh}(t, s), \mathcal{J}_{xh}(t', s') \} &= nh \sum_{i=1}^n w_{ni}(x, h)^2 \text{Cov} \{ \mathcal{J}_{ix}^{(1)}(t, s), \mathcal{J}_{ix}^{(1)}(t', s') \} \\ &+ nh \sum_{i=1}^n w_{ni}(x, h)^2 \text{Cov} \{ \mathcal{J}_{ix}^{(1)}(t, s), \mathcal{J}_{ix}^{(2)}(t', s') \} \\ &+ nh \sum_{i=1}^n w_{ni}(x, h)^2 \text{Cov} \{ \mathcal{J}_{ix}^{(1)}(t', s'), \mathcal{J}_{ix}^{(2)}(t, s) \} \\ &+ nh \sum_{i=1}^n w_{ni}(x, h)^2 \text{Cov} \{ \mathcal{J}_{ix}^{(2)}(t, s), \mathcal{J}_{ix}^{(2)}(t', s') \}. \end{aligned}$$

We obtain

$$\begin{aligned}
\sum_{i=1}^n w_{ni}(x, h)^2 \text{Cov}\{\mathcal{J}_{ix}^{(1)}(t, y), \mathcal{J}_{ix}^{(2)}(t', y')\} &= o_{\mathbb{P}}\left(\frac{1}{nh}\right) \\
&- \frac{K_4}{nh} \{F_{2x}(y') - F_x(t', y')\} \int_0^{t'} \frac{F_x(t \wedge v, y) - H_{1x}(v)F_x(t, y)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^c(v) \\
&- \frac{K_4}{nh} F_x(t, y) \int_0^{t'} \frac{1 - H_{1x}^u(v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} h_x^u(v, y') dv - F_x(t, y)F_x(t', y') \\
&+ \frac{K_4}{nh} F_x(t, y) \{F_{2x}(y') - F_x(t', y')\} \int_0^{t'} \frac{H_{1x}^u(v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}(v).
\end{aligned}$$

Hence, $\text{Cov}\{\mathcal{J}_{xh}(t, s), \mathcal{J}_{xh}(t', s')\} = \text{Cov}\{\mathcal{J}_x(t, s), \mathcal{J}_x(t', s')\} + o(1)$. The weak convergence follows from similar arguments as the ones expose in Appendix A.1.1.

Appendix B. Weak convergence of $\mathbb{B}_{xh}^{(\text{rc})}$

This section follows along the lines of Theorem 1 in Veraverbeke et al. (2011). First, one decompose

$$\sqrt{nh}(\mathbb{C}_{xh}^{(\text{rc})} - C_x) = \{\tilde{A}_{xh} - \mathbb{E}(\tilde{A}_{xh})\} + \{\tilde{A}'_{xh} - \mathbb{E}(\tilde{A}'_{xh})\} + \tilde{B}_{xh} + \mathbb{E}(\tilde{A}_{xh}) + \mathbb{E}(\tilde{A}'_{xh}),$$

where

$$\begin{aligned}
\tilde{A}_{xh}(u, v) &= A_{xh} \left\{ F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(u) \right\}, \\
\tilde{A}'_{xh}(u, v) &= \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \left[\mathcal{J}_{ix}^{(1)} \{ F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v) \} \right. \\
&\quad \left. - \mathcal{J}_{ix}^{(1)} \{ F_{1x}^{-1}(u), F_{2x}^{-1}(v) \} \right],
\end{aligned}$$

and

$$\tilde{B}_{xh}(u, v) = \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) \mathcal{J}_{ix}^{(1)} \{ F_{1x}^{-1}(u), F_{2x}^{-1}(v) \}$$

It will now be shown that both $\{\tilde{A}_{xh} - \mathbb{E}(\tilde{A}_{xh})\}$ and $\{\tilde{A}'_{xh} - \mathbb{E}(\tilde{A}'_{xh})\}$ are asymptotically negligible. Then, it will be establish that $\mathbb{E}(\tilde{A}_{xh})$ asymptotically equals

$$\sqrt{nh} \sum_{i=1}^n w_{ni}(g, h) \mathcal{J}_{ix}^{(2)}(F_{1x}^{-1}(u), F_{2x}^{-1}(v)).$$

Finally, the asymptotic representation of $\mathbb{E}(\tilde{A}'_{xh})$ will be given.

Appendix B.1. Asymptotic negligibility of $\sqrt{nh}\{\tilde{A}_{xh} - \mathbb{E}(\tilde{A}_{xh})\}$ and $\sqrt{nh}\{\tilde{A}'_{xh} - \mathbb{E}(\tilde{A}'_{xh})\}$

First, a weak consequence of Corollary 3.2 is that uniformly in $(t, y) \in \mathcal{T}_{\mathbf{t}}$:

$$F_{1xh}^{(\text{rc})}(t) - F_{1x}(t) = o_{\mathbb{P}}(1) \quad \text{and} \quad F_{2xh}(y) - F_{2x}(y) = o_{\mathbb{P}}(1).$$

Second, it follows from assumption W_1 that, for any sufficiently small $\epsilon > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P} \left\{ F_{1x}^{-1}(u - \epsilon) \leq F_{1xh}^{(\text{rc})-1}(u) \leq F_{1x}^{-1}(u + \epsilon), u \in [0, H_{1x}(\mathbf{t})] \right\} &= 1, \\ \lim_{n \rightarrow \infty} \mathbb{P} \left\{ F_{2x}^{-1}(v - \epsilon) \leq F_{2xh}^{-1}(v) \leq F_{1x}^{-1}(v + \epsilon), v \in [0, 1] \right\} &= 1. \end{aligned} \quad (\text{B.1})$$

Now we recall the definition of Z_{xh} in Lemma *Appendix C.1*, and we note that

$$\tilde{A}_{xh}(u, v) = Z_{xh}\{F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v), G_{xg}\} - Z_{xh}\{F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v), G_x\}$$

and

$$\tilde{A}'_{xh}(u, v) = Z_{xh}\{F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v), G_x\} - Z_{xh}\{F_{1xh}^{-1}(u), F_{2x}^{-1}(v), G_x\}.$$

For $f_1 = (F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v), G_{xh})$, $f_2 = (F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v), G_x)$ and $f_3 = (F_{1x}^{-1}(u), F_{2x}^{-1}(v), G_x)$, it follows from Equations (A.2) and (B.1) that

$$\rho_{F_x}(f_1, f_2) \rightarrow 0 \quad \text{and} \quad \rho_{F_x}(f_2, f_3) \rightarrow 0.$$

The negligibility of $\tilde{A}_{xh} - \mathbb{E}(\tilde{A}_{xh})$ and $\tilde{A}'_{xh} - \mathbb{E}(\tilde{A}'_{xh})$ is then ensured by Lemma *Appendix C.1* and identical arguments as the ones use in the end of *Appendix A.1.1*.

Appendix B.2. Asymptotic representation of $\mathbb{E}(\tilde{A}_{xh})$

For any small $\epsilon > 0$ such that $H_{1x}(\mathbf{t} + \epsilon) < 1$, one obtains from Equation (B.1) that uniformly in $u \in [0, H_{1x}(\mathbf{t})]$ and with probability 1, $F_{1xh}^{(\text{rc})-1}(u) \in [0, \mathbf{t} + \epsilon]$. Thus, we get from *Appendix A.1.2* that uniformly in $(u, v) \in \tilde{\mathcal{T}}_{\mathbf{t}}$,

$$\mathbb{E}(\tilde{A}_{xh}) = \sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) \mathcal{J}_{xh}^{(2)} \left\{ F_{1xh}^{(\text{rc})-1}(u), F_{2xh}^{-1}(v) \right\} + o_{\mathbb{P}}(1).$$

Now we let $t_{xh} = F_{1xh}^{(rc)-1}(u)$, $t_x = F_{1x}^{-1}(u)$, and we define y_{xh} and y_x analogously. Recall, from Appendix A.1.2, that

$$\begin{aligned}\mathcal{J}_{xh}^{(2)}(t_{xh}, y_{xh}) &= \Lambda_x(g_{ix})(t_{xh}, y_{xh}) \\ &= \int_0^{t_{xh}} \frac{g_{ix}(v)}{\{1 - G_x(v)\}^2} h_x^u(v, y_{xh}).\end{aligned}$$

Hence, from the mean value theorem, we have

$$\mathcal{J}_{xh}^{(2)}(t_{xh}, y_{xh}) - \mathcal{J}_{xh}^{(2)}(t_x, y_{xh}) = \frac{g_{ix}(t_{xh}^*)}{\{1 - G_x(t_{xh}^*)\}^2} h_x^u(t_{xh}^*, y_{xh})(t_{xh} - t_x),$$

for some t_{xh}^* between t_{xh} and t_x . Further, since the g_{ix} 's are bounded random variables, and since $\frac{h}{g} = O(1)$, we can show that

$$\sup_{t, y \in \mathcal{T}_t} \left| \sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) \frac{g_{ix}(t)}{\{1 - G_x(t)\}^2} h_x^u(t, y) \right| = O_{\mathbb{P}}(1).$$

Thus, Equation (B.1) implies that

$$\sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) \left\{ \mathcal{J}_{xh}^{(2)}(t_{xh}, y_{xh}) - \mathcal{J}_{xh}^{(2)}(t_x, y_{xh}) \right\} = o_{\mathbb{P}}(1).$$

Similarly, we have

$$\sqrt{nh} \sum_{i=1}^n w_{ni}(x, g) \left\{ \mathcal{J}_{xh}^{(2)}(t_x, y_{xh}) - \mathcal{J}_{xh}^{(2)}(t_x, y_x) \right\} = o_{\mathbb{P}}(1).$$

Appendix B.3. Asymptotic representation of $\mathbb{E}(A'_{xh})$

From the definition of $\mathbb{E}(\cdot)$, we can write

$$\mathbb{E}(A'_{xh})(u, v) = \sqrt{nh} \sum_{i=1}^n w_{ni}(x, h) F_{x_i} \left\{ F_{1xh}^{(rc)-1}(u), F_{2xh}^{-1}(v) \right\} - C_x(u, v).$$

Conditions $\mathcal{C}_1$, $\mathcal{C}_3$ and W_2 – W_3 allow to mimick the proof of Theorem 1 in Veraverbeke et al. (2011) to obtain, uniformly in $(u, v) \in \tilde{\mathcal{T}}_t$, that

$$\mathbb{E}(A'_{xh})(u, v) = \sqrt{nh} \left(C_x \left[F_{1x} \{ F_{1xh}^{(rc)-1}(u) \}, F_{2x} \{ F_{2xh}^{-1}(v) \} \right] - C_x(u, v) \right).$$

Substituting v by 1 in the previous Equation with the asymptotic negligibility of $\tilde{A}'_{xh} - \mathbb{E}A'_{xh}$ leads to

$$\sqrt{nh} \left[F_{1x} \left\{ F_{1xh}^{(rc)-1}(u) \right\} - u \right] = -\sqrt{nh} \left[F_{1xh}^{(rc)} \left\{ F_{1x}^{-1}(u) \right\} \right] - u.$$

Moreover, from Veraverbeke et al. (2011), we have

$$\sqrt{nh} \left[F_{2x} \left\{ F_{2xh}^{-1}(v) \right\} - v \right] = -\sqrt{nh} \left[F_{2xh} \left\{ F_{2x}^{-1}(v) \right\} \right] - v.$$

Then, following the end of the proof of Theorem 1 in Veraverbeke et al. (2011) yields, uniformly over $\tilde{\mathcal{T}}_t$, to

$$\begin{aligned} \mathbb{E}(A'_{xh})(u, v) &= -\sqrt{nh} \mathbb{C}_x^{[1]}(u, v) \left[F_{1xh}^{(rc)} \left\{ F_{1x}^{-1}(u) \right\} \right] - u \\ &\quad - \sqrt{nh} \mathbb{C}_x^{[2]}(u, v) \left[F_{2xh} \left\{ F_{2x}^{-1}(v) \right\} \right] - v. \end{aligned}$$

Appendix C. Auxiliary Lemma

The following lemma is required in Appendix A.1.1 to establish the i.i.d representation for $\mathbb{F}_{xh}^{(rc)}$.

Lemma Appendix C.1. *Recall the definition of Z_{xh} , $\mathcal{F}$ and ρ_{F_x} at the beginning of Appendix A.1.1. Suppose Assumptions W_1, W_4 and W_6 are satisfied, and that the maps $z \mapsto F_{1z}$ and $z \mapsto F_{2z}$ are uniformly continuous for all z in a neighborhood of x . Then, process $\bar{Z}_{xh} := Z_{xh} - \mathbb{E}Z_{xh}$ indexed by $(\mathcal{F}, \rho_{F_x})$ is asymptotically ρ_{F_x} -equicontinuous.*

Proof. From Theorem 2.11.1 of van der Vaart & Wellner (1996), we can conclude that Z_{xh} is tight if the following requirements hold:

- ($\mathcal{R}_1$) $\sum_{i=1}^n \mathbb{E} (\|Z_{hi}\|_{\mathcal{F}}^2) \mathbb{I} \{\|Z_{hi}\|_{\mathcal{F}} > \eta\} \rightarrow 0$ for any $\eta > 0$;
- ($\mathcal{R}_2$) $\sup_{\rho_{F_x}(f, f') < \delta_n} \sum_{i=1}^n \mathbb{E} \left[\{Z_{hi}(f) - Z_{hi}(f')\}^2 \right] \rightarrow 0$ for every $\delta_n \downarrow 0$.
- ($\mathcal{R}_3$) $\int_0^{\delta_n} \{\log N(\epsilon, \mathcal{F}, d_n)\}^{1/2} d\epsilon \xrightarrow{\mathbb{P}} 0$ for every $\delta_n \downarrow 0$, where $N(\epsilon, \mathcal{F}, d_n)$ is the covering number of the set $\mathcal{F}$ with respect to the random semi-metric

$$d_n^2(f, f') = \sum_{i=1}^n \{Z_{hi}(f) - Z_{hi}(f')\}^2.$$

In the latter, $\|\cdot\|_{\mathcal{F}}$ stands for the supremum norm over $\mathcal{F}$.

($\mathcal{R}_1$): Because $Z_{hi}(f) = 0$ whenever $Y_{1i} > t$, and since G is nondecreasing, we have

$$Z_{hi}(f) \leq \sqrt{nh} \frac{w_{ni}(x, h)}{1 - G(t)} \leq \sqrt{nh} \frac{w_{ni}(x, h)}{1 - t_\epsilon}.$$

From Assumption W_1 , it follows that the latter is $o(1)$. Hence, for any $\eta > 0$, one can find $N > 0$ such that for all $n \geq N$: $\max_{1 \leq i \leq n} Z_{hi}(f) < \eta$, proving that requirement $\mathcal{R}_1$ is fulfilled.

($\mathcal{R}_2$): Assume wlog that $t \leq t'$ and $y \leq y'$. It is useful to note that when $\delta_i = 1$, it follows that $\mathbb{I}(T_i \leq t)\delta_i = \mathbb{I}(Y_{1i} \leq t)$ whereas $\mathbb{I}(T_i \leq t)\delta_i = 0$ if $\delta_i = 0$. As a consequence, $\mathbb{I}(T_i \leq t)\delta_i \leq \mathbb{I}(Y_{1i} \leq t)$. Then,

$$\begin{aligned} & Z_{hi}(f') - Z_{hi}(f) \\ &= \sqrt{nh} \mathbb{I}(t < T_i \leq t', y < Y_{2i} \leq y') \frac{\delta_i w_{ni}(x, h)}{1 - G'(T_i)} \\ &\quad + \sqrt{nh} \mathbb{I}(T_i \leq t, Y_{2i} \leq y) \delta_i w_{ni}(x, h) \frac{G(T_i) - G'(T_i)}{\{1 - G(T_i)\} \{1 - G'(T_i)\}} \\ &\leq \sqrt{nh} \frac{w_{ni}(x, h)}{1 - t_\epsilon} \left\{ \mathbb{I}(t < Y_{1i} \leq t', y < Y_{2i} \leq y') \delta_i + \|G - G'\|_{[0, t]} \right\}. \end{aligned}$$

From the last equation, one directly obtains

$$\begin{aligned} & \sup_{\rho_{F_x}(f, f') < \delta_n} \sum_{i=1}^n \mathbb{E} \left\{ (Z_{hi}(f') - Z_{hi}(f))^2 \right\} \\ &\leq \sup_{\rho_{F_x}(f, f') < \delta_n} nh \sum_{i=1}^n \frac{w_{ni}(x, h)^2}{(1 - t_\epsilon)^2} \{ |F_{1x_i}(t') - F_{1x_i}(t)| + |F_{2x_i}(y') - F_{2x_i}(y)| \} \\ &\quad + nh \sum_{i=1}^n \frac{w_{ni}(x, h)^2}{(1 - t_\epsilon)^2} (\|G - G'\|_{[0, t]} + \|G - G'\|_{[0, t]}^2) \\ &\leq \sup_{\rho_{F_x}(f, f') < \delta_n} \max_{i \in I_{nx}} [|F_{1x_i}(t') - F_{1x_i}(t)| + |F_{2x_i}(y') - F_{2x_i}(y)|] \\ &\quad \times nh \sum_{i=1}^n \frac{w_{ni}(x, h)^2}{(1 - t_\epsilon)^2} + nh \sum_{i=1}^n \frac{w_{ni}(x, h)^2}{(1 - t_\epsilon)^2} (\|G - G'\|_{[0, t]} + \|G - G'\|_{[0, t]}^2). \end{aligned}$$

From Assumption W_4 , we obtain that $nh \sum_{i=1}^n w_{ni}(x, h)^2 = O(1)$. Moreover, as Assumption W_6 holds together with the uniform continuity of the maps

$z \mapsto F_{1z}$ and $z \mapsto F_{2z}$, we deduce that the latter display is bounded by $O(1)\{2\delta_n + o(1)\}$. Hence, requirement $\mathcal{R}_2$ is fulfilled as $\delta_n \rightarrow 0$.

($\mathcal{R}_3$): To show the last requirement, the goal is to apply Lemma 2.11.6 of van der Vaart & Wellner (1996). To do this, three conditions must be verified. First, we rewrite

$$\{Z_{hi}(f') - Z_{hi}(f)\}^2 = \int \left\{ \frac{\mathbb{I}(v \leq t, v' \leq y)}{1 - G(v)} - \frac{\mathbb{I}(v \leq t', v' \leq y')}{1 - G'(v)} \right\}^2 d\mu_{ni},$$

where $\mu_{ni} = nhw_{ni}(x, h)^2 \delta_i \mathbb{I}(v = Y_{1i}, v' = Y_{2i})$. Hence, the process Z_n is *measurelike* with respect to the random measure μ_{ni} (see van der Vaart & Wellner (1996), Section 2.11). Second, as Assumption W_4 holds, $\sum_{i=1}^n \int \mathfrak{F} d\mu_{ni} = nh \sum_{i=1}^n \frac{w_{ni}(x, h)^2}{1 - t_\epsilon} = O(1)$. Third, it is required to show that the class $\mathcal{F}$ satisfy the uniform entropy condition (2.11.5) of van der Vaart & Wellner (1996). To do this, let

$$\mathcal{F}_1 = \{(v, v', w) \mapsto \mathbb{I}\{v \leq t, v' \leq y\}, (t, y) \in \mathcal{T}_t, w \in \{0, 1\}\}$$

and $\mathcal{F}_2$ be the class of monotone and bounded functions over $[0, \frac{1}{1-t_\epsilon}]$. Now we observe that $\mathcal{F} \subset \mathcal{F}_1 \mathcal{F}_2 = \{f = f_1 f_2, f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2\}$. As $\mathcal{F}_1$ is a VC-class and $\mathcal{F}_2$ is a VC-hull class for sets with envelope functions respectively $\mathfrak{F}_1 = 1$ and $\mathfrak{F}_2 = \frac{1}{1-t_\epsilon}$, an application Lemma 2.6.20 of van der Vaart & Wellner (1996) allows to conclude that $\mathcal{F}_1 \mathcal{F}_2$ is VC-hull class for sets with envelope function $\mathfrak{F}_1 \times \mathfrak{F}_2 = \mathfrak{F}$. Therefore, the uniform entropy condition is fulfilled. As a result, the conclusion Lemma 2.11.6 of van der Vaart & Wellner (1996) applies which proves the requirement $\mathcal{R}_3$. $\square$

Appendix D. Auxiliary results

The following Proposition establishes that $F_{1xh}^{(rc)}$ coincide with the conditional Kaplan-Meier estimator for F_{1x} when the bandwidth parameters g and h in the definition of $F_{1xh}^{(rc)}$ are equal.

Proposition Appendix D.1. *Consider the conditional Kaplan-Meier esti-*

mator given by:

$$\begin{aligned} F_{1xh}^{(KM)}(t) &= 1 - \left\{ \prod_{T_{(i)} \leq t} \left(1 - \frac{w_{n[i]}(x, h)}{1 - F_{1xh}(T_{(i)} -)} \right)^{\delta_{[i]}} \right\} \\ &= 1 - \left\{ \prod_{T_{(i)} \leq t} \left(1 - \frac{w_{n[i]}(x, h)}{1 - \sum_{k=1}^{i-1} w_{n[k]}(x, h)} \right)^{\delta_{[i]}} \right\}. \end{aligned}$$

Then, for any $t \in \mathbb{R}$, $F_{1xh}^{(KM)}(t) = F_{1xh}^{(rc)}(t)$ when the bandwidth parameters g and h in the definition of $F_{1xh}^{(rc)}$ are equal.

Proof. First set $T_{(0)} = 0$ and $T_{n+1} = \infty$. Second, in order to prove the result, it suffices to show that for any $0 \leq k \leq n+1$, $F_{1xh}^{(rc)}(T_{(k)}) = F_{1xh}^{(KM)}(T_{(k)})$. To do this, we use the following induction argument.

Basis: Trivially, $F_{1xh}^{(rc)}(0) = F_{1xh}^{(KM)}(0) = 0$. Moreover,

$$F_{1xh}^{(rc)}(T_{(1)}) = \frac{\delta_{[1]} w_{n[1]}(x, h)}{(1 - w_{n[1]}(x, h))^{1 - \delta_{[1]}}} = \delta_{[1]} w_{n[1]}(x, h)$$

and

$$F_{1xh}^{(KM)}(T_{(1)}) = 1 - (1 - w_{n[1]}(x, h))^{\delta_{[1]}} = \delta_{[1]} w_{n[1]}(x, h).$$

Hence, the Basis step is verified.

Induction step: Assuming that the equality $F_{1xh}^{(rc)}(t) = F_{1xh}^{(KM)}(t)$ holds for $t = T_{(0)}$ up to $t = T_{(k)}$, let's show that the equality is verified for $t = T_{(k+1)}$. From direct computations,

$$\begin{aligned} F_{1xh}^{(KM)}(T_{(k+1)}) &= 1 - \prod_{i=1}^k \left(1 - \frac{w_{n[i]}(x, h)}{1 - F_{1xh}(T_{(i-1)})} \right)^{\delta_{[i]}} \left(1 - \frac{w_{n[k+1]}(x, h)}{1 - F_{1xh}(T_{(k)})} \right)^{\delta_{[k+1]}} \\ &= 1 + \{F_{1xh}^{(rc)}(T_{(k)}) - 1\} \left(1 - \frac{w_{n[k+1]}(x, h)}{1 - F_{1xh}(T_{(k)})} \right)^{\delta_{[k+1]}}, \end{aligned}$$

where the latter equation follows from the induction hypothesis. If $\delta_{[k+1]} = 0$, we use the induction hypothesis to obtain

$$F_{1xh}^{(\text{KM})}(T_{(k+1)}) = F_{1xh}^{(\text{KM})}(T_{(k)}) = F_{1xh}^{(\text{rc})}(T_{(k)}) = F_{1xh}^{(\text{rc})}(T_{(k+1)}).$$

Otherwise, if $\delta_{[k+1]} = 1$, then

$$\begin{aligned} F_{1xh}^{(\text{KM})}(T_{(k+1)}) &= F_{1xh}^{(\text{rc})}(T_{(k)}) + \{1 - F_{1xh}^{(\text{rc})}(T_{(k)})\} \left(\frac{w_{n[k+1]}(x, h)}{1 - F_{1xh}(T_{(k+1)})} \right) \\ &= F_{1xh}^{(\text{rc})}(T_{(k)}) + \{1 - F_{1xh}^{(\text{KM})}(T_{(k)})\} \left(\frac{w_{n[k+1]}(x, h)}{1 - F_{1xh}(T_{(k+1)})} \right). \end{aligned}$$

From the identity $\{1 - F_{1xh}^{(\text{KM})}(T_{(k)})\}\{1 - G_{1xh}(T_{(k)})\} = 1 - F_{1xh}(T_{(k)})$, we deduce that

$$F_{1xh}^{(\text{KM})}(T_{(k+1)}) = F_{1xh}^{(\text{rc})}(T_{(k)}) + \frac{w_{n[k+1]}(x, h)}{1 - G_{1xh}(T_{(k)})}.$$

The proof follows from the fact that $G_{1xh}(T_{(k)}) = G_{1xh}(T_{(k+1)})$ since $\delta_{[k+1]} = 1$. □

The next Lemma shows that, upon setting $y = \infty$, the i.i.d representation for $\mathbb{F}_{xh}^{(\text{rc})}$ found in Theorem 3.1 is the same as the one obtained in Van Keilegom & Veraverbeke (1997) in the case where the bandwidths g and h required for $F_{xh}^{(\text{rc})}$ are equal.

Lemma Appendix D.2. *The following identity holds :*

$$\begin{aligned} &\lim_{y \rightarrow \infty} \{\mathcal{J}_{ix}^{(1)}(t, y) + \mathcal{J}_{ix}^{(2)}(t, y)\} \\ &= \{1 - F_{1x}(t)\} \left[\int_0^t \frac{\mathbb{I}(T_i \leq v) - H_{1x}(v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}^u(v) \right. \\ &\quad \left. + \frac{\mathbb{I}\{T_i \leq t, \delta_i = 1\} - H_{1x}^u(t)}{1 - H_{1x}(t)} \right. \\ &\quad \left. - \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 1) - H_{1x}^u(v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) \right]. \end{aligned}$$

Proof. First, since $h_x^u(v, \infty) = dH_{1x}^u(v)$, we deduce that

$$v_x(t, \infty, v) = \int_v^t \frac{h_x^u(v, \infty) dv}{1 - G_x(v)} = F_{1x}(t) - F_{1x}(v). \quad (\text{D.1})$$

In order to show the result, we first deal with the terms that contains the T_i 's in the sum $\lim_{y \rightarrow \infty} \{\mathcal{J}_{ix}^{(1)}(t, y) + \mathcal{J}_{ix}^{(2)}(t, y)\}$. In view of Equation (D.1), we have

$$\frac{\mathbb{I}\{T_i \leq t, \delta_i = 1\}}{1 - G_x(T_i)} \quad (\text{D.2})$$

$$+ \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} [F_{1x}(t) - F_{1x}(v)] dH_{1x}^c(v) \quad (\text{D.3})$$

$$+ \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 0)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^u(v) \quad (\text{D.4})$$

$$- \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 0)}{\{1 - H_{1x}(v)\}^2} [F_{1x}(t) - F_{1x}(v)] dH_{1x}(v). \quad (\text{D.5})$$

Beginning with (D.3), from the relationship $1 - H_{1x}(v) = \{1 - F_{1x}(v)\}\{1 - G_x(v)\}$, we show that

$$\begin{aligned} (\text{D.3}) &= -\{1 - F_{1x}(t)\} \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}^c(v) \\ &\quad + \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} \{1 - F_{1x}(v)\} dH_{1x}^c(v) \\ &= -\{1 - F_{1x}(t)\} \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}^c(v) \\ &\quad + \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^c(v). \end{aligned}$$

Therefore,

$$\begin{aligned} (\text{D.3}) &= \\ &= -\{1 - F_{1x}(t)\} \left[\int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) - \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}^u(v) \right] \\ &\quad + \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}(v) \\ &\quad - \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^u(v). \end{aligned}$$

because

$$H_{1x}^c(v) = H_{1x}(v) - H_{1x}^u(v).$$

Now, from the fact that $\mathbb{I}(T_i \leq v, \delta_i = 0) = \mathbb{I}(T_i \leq v) - \mathbb{I}(T_i \leq v, \delta_i = 1)$, we obtain

$$(D.4) = \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^u(v) - \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 1)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^u(v).$$

Then, from the identity $H_{1x}^u(v) = \int_0^v \{1 - G_x(t-)\} dF_{1x}(t)$, integration by parts leads to

$$\begin{aligned} (D.4) &= \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^u(v) + \frac{\mathbb{I}\{T_i \leq t, \delta_i = 1\}}{1 - G_x(t)} \\ &\quad - \frac{\mathbb{I}\{T_i \leq t, \delta_i = 1\}}{1 - G_x(T_i)} - \int_0^t \frac{\mathbb{I}\{T_i \leq v, \delta_i = 1\}}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) \\ &= \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}^u(v) \\ &\quad + \{1 - F_{1x}(t)\} \frac{\mathbb{I}\{T_i \leq t, \delta_i = 1\}}{1 - H_{1x}(t)} - \frac{\mathbb{I}\{T_i \leq t, \delta_i = 1\}}{1 - G_x(T_i)} \\ &\quad - \int_0^t \frac{\mathbb{I}\{T_i \leq v, \delta_i = 1\}}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}(v). \end{aligned}$$

Proceeding similarly, we obtain that

$$\begin{aligned} (D.5) &= \{1 - F_{1x}(t)\} \left[\int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) - \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 1)}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) \right] \\ &\quad - \int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}(v) \\ &\quad + \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 1)}{\{1 - H_{1x}(v)\}\{1 - G_x(v)\}} dH_{1x}(v). \end{aligned}$$

Then, adding these terms leads to

$$\begin{aligned} (D.2) + \dots + (D.5) &= \{1 - F_{1x}(t)\} \left[\int_0^t \frac{\mathbb{I}(T_i \leq v)}{\{1 - H_{1x}(v)\}^2} dH_{1x}^u(v) \right. \\ &\quad \left. + \frac{\mathbb{I}\{T_i \leq t, \delta_i = 1\}}{1 - H_{1x}(t)} - \int_0^t \frac{\mathbb{I}(T_i \leq v, \delta_i = 1)}{\{1 - H_{1x}(v)\}^2} dH_{1x}(v) \right], \end{aligned}$$

which concludes the proof.

Table D.1: Average integrated square bias (AISB $\times 10^4$) estimated from 1 000 replicates of for $\mathbb{C}_{xh}^{(rc)}$ with $n = 250$ and $n = 1\,000$ in the case $\tau_{C_x} = \tau_{F_x} = \infty$. Upper panel: Normal Copula. Bottom panel : Clayton Copula.

θ	h	$\mathfrak{T}_x = .1$				$\mathfrak{T}_x = .25$				$\mathfrak{T}_x = .35$			
		$n = 250$		$n = 1000$		$n = 250$		$n = 1000$		$n = 250$		$n = 1000$	
		$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$
20%	0.9	1.420	0.510	0.223	0.023	1.403	0.481	0.232	0.028	1.515	0.536	0.286	0.051
	1.2	0.356	0.167	0.014	0.009	0.361	0.178	0.022	0.028	0.471	0.264	0.057	0.057
	1.5	0.138	0.116	0.021	0.030	0.138	0.118	0.057	0.073	0.237	0.212	0.102	0.115
	1.8	0.049	0.050	0.047	0.049	0.083	0.083	0.106	0.111	0.188	0.185	0.164	0.168
	2.1	0.038	0.037	0.085	0.087	0.091	0.090	0.161	0.167	0.216	0.214	0.225	0.231
	2.4	0.049	0.048	0.095	0.096	0.114	0.114	0.189	0.195	0.254	0.251	0.265	0.270
	2.7	0.063	0.064	0.120	0.121	0.159	0.163	0.234	0.240	0.296	0.299	0.319	0.324
	0.9	1.707	1.464	0.532	0.050	1.525	1.282	0.535	0.045	1.553	1.304	0.609	0.068
	1.2	0.973	0.639	0.078	0.032	0.859	0.537	0.072	0.039	0.948	0.604	0.119	0.074
	1.5	0.535	0.405	0.054	0.035	0.435	0.311	0.072	0.067	0.530	0.392	0.149	0.131
	1.8	0.390	0.374	0.060	0.056	0.318	0.302	0.114	0.111	0.430	0.409	0.214	0.204
	2.1	0.329	0.321	0.079	0.077	0.300	0.286	0.145	0.151	0.457	0.432	0.270	0.265
	2.4	0.291	0.280	0.088	0.084	0.273	0.258	0.190	0.186	0.448	0.425	0.340	0.328
	2.7	0.259	0.249	0.090	0.087	0.291	0.277	0.210	0.208	0.487	0.466	0.384	0.375
	0.9	10.841	8.653	1.551	1.059	9.378	7.289	1.271	0.788	8.699	6.667	1.186	0.683
	1.2	5.339	4.948	0.923	0.764	4.297	3.935	0.699	0.538	3.915	3.566	0.662	0.480
	1.5	4.465	4.420	0.881	0.733	3.342	3.283	0.715	0.566	2.959	2.888	0.743	0.571
	1.8	4.441	4.230	0.752	0.758	3.329	3.130	0.618	0.590	2.968	2.762	0.723	0.662
	2.1	3.699	3.680	0.687	0.691	2.764	2.713	0.627	0.601	2.602	2.530	0.798	0.739
	2.4	3.705	3.641	0.791	0.743	2.622	2.538	0.733	0.659	2.423	2.315	0.950	0.844
	2.7	3.607	3.502	0.824	0.787	2.673	2.561	0.782	0.721	2.550	2.424	1.053	0.961
60%	0.9	1.324	0.442	0.185	0.024	1.567	0.618	0.214	0.034	1.563	0.432	0.486	0.083
	1.2	0.325	0.142	0.031	0.014	0.191	0.170	0.061	0.076	0.368	0.305	0.163	0.191
	1.5	0.141	0.085	0.047	0.049	0.161	0.135	0.174	0.180	0.323	0.325	0.318	0.347
	1.8	0.061	0.055	0.104	0.106	0.178	0.179	0.283	0.292	0.417	0.419	0.488	0.529
	2.1	0.060	0.060	0.121	0.123	0.248	0.252	0.346	0.353	0.561	0.567	0.622	0.634
	2.4	0.085	0.084	0.156	0.157	0.302	0.304	0.441	0.450	0.656	0.662	0.772	0.786
	2.7	0.097	0.096	0.223	0.226	0.368	0.370	0.551	0.560	0.792	0.797	0.915	0.930
	0.9	2.746	1.376	0.608	0.158	2.282	1.086	0.527	0.087	2.250	1.266	0.431	0.118
	1.2	0.864	0.600	0.072	0.039	0.910	0.606	0.130	0.085	0.925	0.506	0.228	0.191
	1.5	0.681	0.503	0.059	0.054	0.498	0.378	0.152	0.154	0.528	0.485	0.318	0.324
	1.8	0.369	0.330	0.101	0.098	0.363	0.339	0.254	0.258	0.551	0.532	0.488	0.497
	2.1	0.247	0.226	0.118	0.116	0.338	0.331	0.297	0.303	0.616	0.611	0.587	0.598
	2.4	0.295	0.284	0.131	0.130	0.402	0.393	0.372	0.378	0.716	0.710	0.714	0.724
	2.7	0.350	0.337	0.178	0.178	0.487	0.471	0.455	0.464	0.873	0.864	0.850	0.867
	0.9	11.450	7.530	1.866	1.264	9.481	6.093	1.555	1.000	9.294	5.838	2.023	0.767
	1.2	6.922	5.600	0.976	0.885	5.650	4.892	0.871	0.709	4.649	3.984	0.785	0.662
	1.5	4.734	4.604	0.787	0.729	4.327	3.823	0.726	0.676	3.532	3.051	0.789	0.697
	1.8	3.968	3.849	0.841	0.814	2.822	2.675	0.756	0.711	2.421	2.351	0.946	0.891
	2.1	3.569	3.428	0.955	0.972	2.428	2.359	0.877	0.823	2.281	2.173	1.072	1.026
	2.4	3.431	3.385	0.957	0.920	2.712	2.603	0.997	0.956	2.446	2.321	1.265	1.212
	2.7	4.136	4.099	0.997	0.969	3.166	3.043	1.034	0.983	2.918	2.818	1.372	1.306

Table D.2: Average integrated variance ($\text{AIV} \times 10^4$) estimated from 1 000 replicates of for $\mathbb{C}_{xh}^{(\text{rc})}$ with $n = 250$ and $n = 1\,000$ in the case $\tau_{C_x} = \tau_{F_x} = \infty$. Upper panel: Normal Copula. Bottom panel : Clayton Copula.

θ	h	$\tau_x = .1$				$\tau_x = .25$				$\tau_x = .35$			
		n = 250		n = 1000		n = 250		n = 1000		n = 250		n = 1000	
		g = h	g $\neq$ h	g = h	g $\neq$ h	g = h	g $\neq$ h	g = h	g $\neq$ h	g = h	g $\neq$ h	g = h	g $\neq$ h
20%	0.9	25.033	9.210	14.408	3.398	26.750	9.235	15.656	3.515	27.752	9.117	16.448	3.559
	1.2	10.312	3.793	4.302	0.945	10.633	3.426	4.534	0.827	10.724	3.072	4.662	0.730
	1.5	5.537	4.444	1.934	0.807	5.385	4.176	1.968	0.723	5.187	3.902	1.959	0.640
	1.8	2.878	2.882	0.708	0.711	2.556	2.560	0.614	0.619	2.264	2.266	0.543	0.548
	2.1	2.691	2.693	0.645	0.648	2.378	2.382	0.569	0.571	2.069	2.075	0.497	0.499
	2.4	2.504	2.514	0.614	0.616	2.194	2.202	0.539	0.542	1.920	1.927	0.469	0.472
	2.7	2.438	2.449	0.589	0.591	2.106	2.115	0.509	0.511	1.812	1.819	0.446	0.448
	0.9	19.754	16.672	19.448	1.997	20.449	17.054	21.114	1.857	20.733	17.127	22.155	1.703
	1.2	12.547	6.187	5.979	1.536	12.789	5.728	6.301	1.392	12.750	5.243	6.474	1.264
	1.5	8.735	5.478	4.680	1.334	8.591	4.989	4.923	1.228	8.350	4.530	5.038	1.120
	1.8	4.810	4.798	1.172	1.166	4.347	4.339	1.052	1.051	3.932	3.923	0.952	0.956
	2.1	4.419	4.425	2.225	1.105	3.987	3.996	2.241	1.003	3.573	3.582	2.212	0.902
	2.4	4.256	4.257	1.012	1.012	3.831	3.835	0.916	0.918	3.448	3.457	0.815	0.821
	2.7	4.160	4.165	0.993	0.993	3.727	3.732	0.894	0.898	3.317	3.324	0.800	0.804
	0.9	39.515	30.505	15.744	6.265	40.965	30.816	16.583	6.046	41.665	30.771	17.002	5.784
	1.2	17.887	15.903	8.380	4.062	17.286	15.089	8.564	3.782	16.588	14.231	8.591	3.513
	1.5	14.052	13.855	6.879	3.610	13.212	13.017	6.934	3.323	12.337	12.156	6.879	3.052
	1.8	12.913	11.926	3.225	3.169	11.996	10.901	2.952	2.917	11.334	10.160	2.701	2.684
	2.1	11.578	11.564	3.018	2.959	10.467	10.472	2.714	2.681	9.569	9.577	2.435	2.428
	2.4	10.745	10.602	2.844	2.822	9.693	9.577	2.552	2.540	8.816	8.714	2.286	2.283
	2.7	10.262	10.160	2.665	2.641	9.278	9.225	2.373	2.360	8.480	8.454	2.124	2.124
40%	0.9	23.844	8.186	12.115	2.329	28.510	12.680	15.334	1.093	28.287	5.322	23.461	2.235
	1.2	10.258	3.809	6.478	0.967	5.766	4.588	3.260	0.844	8.048	4.297	5.865	0.754
	1.5	6.459	3.218	0.790	0.793	6.420	2.850	0.708	0.708	2.583	3.839	1.908	0.616
	1.8	4.062	2.967	0.714	0.716	2.582	2.591	0.631	0.634	2.283	2.287	1.854	0.556
	2.1	2.649	2.651	0.645	0.647	2.318	2.324	0.570	0.570	2.050	2.065	0.496	0.499
	2.4	2.499	2.500	0.617	0.618	2.138	2.146	0.527	0.530	1.873	1.872	0.459	0.459
	2.7	2.401	2.408	0.588	0.590	2.065	2.068	0.516	0.517	1.809	1.815	0.447	0.449
	0.9	29.534	14.413	19.150	6.339	26.838	11.294	20.597	4.213	26.139	13.168	17.852	2.914
	1.2	12.588	7.344	4.891	1.605	16.057	9.181	8.617	1.456	16.231	5.245	8.918	1.334
	1.5	9.609	5.360	3.532	1.328	8.377	4.862	1.205	1.207	5.740	4.483	1.097	1.093
	1.8	5.864	4.788	2.257	1.154	5.476	4.285	1.066	1.068	3.867	3.876	0.970	0.968
	2.1	4.466	4.469	1.079	1.078	3.979	3.963	0.973	0.973	3.525	3.534	0.885	0.889
	2.4	4.187	4.179	1.030	1.029	3.732	3.736	0.909	0.906	3.395	3.405	0.809	0.809
	2.7	4.029	4.033	0.981	0.983	3.583	3.580	0.874	0.878	3.202	3.206	0.782	0.786
	0.9	41.210	22.473	15.480	6.182	40.648	20.576	15.916	5.707	43.935	23.550	27.011	4.244
	1.2	23.991	15.355	6.393	4.213	20.477	15.180	8.445	3.795	19.801	14.016	7.162	3.434
	1.5	15.315	13.351	4.689	3.593	15.274	12.114	4.426	3.228	14.634	11.134	6.734	2.975
	1.8	12.811	11.739	3.189	3.106	11.655	10.466	2.904	2.884	9.899	9.875	2.664	2.652
	2.1	11.306	11.293	2.996	2.929	9.956	9.926	2.614	2.608	8.995	8.988	2.362	2.353
	2.4	10.645	10.561	2.833	2.795	9.248	9.253	2.542	2.533	8.522	8.470	2.272	2.256
	2.7	9.772	9.695	2.731	2.701	8.950	8.961	2.454	2.413	8.279	8.243	2.170	2.163
60%	0.9	23.844	8.186	12.115	2.329	28.510	12.680	15.334	1.093	28.287	5.322	23.461	2.235
	1.2	10.258	3.809	6.478	0.967	5.766	4.588	3.260	0.844	8.048	4.297	5.865	0.754
	1.5	6.459	3.218	0.790	0.793	6.420	2.850	0.708	0.708	2.583	3.839	1.908	0.616
	1.8	4.062	2.967	0.714	0.716	2.582	2.591	0.631	0.634	2.283	2.287	1.854	0.556
	2.1	2.649	2.651	0.645	0.647	2.318	2.324	0.570	0.570	2.050	2.065	0.496	0.499
	2.4	2.499	2.500	0.617	0.618	2.138	2.146	0.527	0.530	1.873	1.872	0.459	0.459
	2.7	2.401	2.408	0.588	0.590	2.065	2.068	0.516	0.517	1.809	1.815	0.447	0.449
	0.9	29.534	14.413	19.150	6.339	26.838	11.294	20.597	4.213	26.139	13.168	17.852	2.914
	1.2	12.588	7.344	4.891	1.605	16.057	9.181	8.617	1.456	16.231	5.245	8.918	1.334
	1.5	9.609	5.360	3.532	1.328	8.377	4.862	1.205	1.207	5.740	4.483	1.097	1.093
	1.8	5.864	4.788	2.257	1.154	5.476	4.285	1.066	1.068	3.867	3.876	0.970	0.968
	2.1	4.466	4.469	1.079	1.078	3.979	3.963	0.973	0.973	3.525	3.534	0.885	0.889
	2.4	4.187	4.179	1.030	1.029	3.732	3.736	0.909	0.906	3.395	3.405	0.809	0.809
	2.7	4.029	4.033	0.981	0.983	3.583	3.580	0.874	0.878	3.202	3.206	0.782	0.786
	0.9	41.210	22.473	15.480	6.182	40.648	20.576	15.916	5.707	43.935	23.550	27.011	4.244
	1.2	23.991	15.355	6.393	4.213	20.477	15.180	8.445	3.795	19.801	14.016	7.162	3.434
	1.5	15.315	13.351	4.689	3.593	15.274	12.114	4.426	3.228	14.634	11.134	6.734	2.975
	1.8	12.811	11.739	3.189	3.106	11.655	10.466	2.904	2.884	9.899	9.875	2.664	2.652
	2.1	11.306	11.293	2.996	2.929	9.956	9.926	2.614	2.608	8.995	8.988	2.362	2.353
	2.4	10.645	10.561	2.833	2.795	9.248	9.253	2.542	2.533	8.522	8.470	2.272	2.256
	2.7	9.772	9.695	2.731	2.701	8.950	8.961	2.454	2.413	8.279	8.243	2.170	2.163

Table D.3: Average integrated square bias (AISB $\times 10^4$) estimated from 1 000 replicates of for $\mathbb{C}_{xh}^{(rc)}$ with $n = 250$ and $n = 1\,000$ in the case $\tau_{C_x} < \tau_{F_x}$. Upper pannel: Normal Copula. Bottom pannel : Clayton Copula.

θ	h	$\tau_x = .1$				$\tau_x = .25$				$\tau_x = .35$			
		$n = 250$		$n = 1000$		$n = 250$		$n = 1000$		$n = 250$		$n = 1000$	
		$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$
0%	0.9	0.230	0.230	0.005	0.005	0.228	0.228	0.013	0.013	0.284	0.284	0.032	0.032
	1.2	0.089	0.089	0.013	0.013	0.116	0.116	0.035	0.035	0.198	0.198	0.059	0.059
	1.5	0.047	0.047	0.043	0.043	0.073	0.073	0.087	0.087	0.158	0.158	0.117	0.117
	1.8	0.030	0.030	0.075	0.075	0.083	0.083	0.143	0.143	0.167	0.167	0.183	0.183
	2.1	0.029	0.029	0.126	0.126	0.105	0.105	0.209	0.209	0.200	0.200	0.251	0.251
	2.4	0.046	0.046	0.148	0.148	0.130	0.130	0.254	0.254	0.235	0.235	0.300	0.300
	2.7	0.082	0.082	0.189	0.189	0.195	0.195	0.314	0.314	0.301	0.301	0.358	0.358
	0.9	1.454	0.729	0.608	0.036	1.437	0.696	0.644	0.039	1.543	0.761	0.739	0.065
	1.2	0.464	0.184	0.091	0.010	0.466	0.192	0.085	0.027	0.584	0.276	0.138	0.057
	1.5	0.150	0.125	0.022	0.030	0.147	0.126	0.058	0.072	0.245	0.221	0.103	0.114
	1.8	0.061	0.060	0.049	0.051	0.093	0.092	0.108	0.113	0.197	0.194	0.168	0.172
	2.1	0.049	0.047	0.086	0.087	0.103	0.100	0.159	0.163	0.228	0.224	0.225	0.229
	2.4	0.059	0.058	0.097	0.098	0.121	0.120	0.190	0.194	0.264	0.261	0.269	0.272
	2.7	0.072	0.073	0.121	0.122	0.167	0.169	0.233	0.238	0.305	0.306	0.320	0.324
	0.9	7.057	3.481	6.590	0.773	6.437	2.904	6.698	0.605	6.289	2.718	6.979	0.551
	1.2	4.061	2.231	1.035	0.642	3.590	1.823	0.909	0.515	3.547	1.744	0.931	0.491
	1.5	2.029	1.718	0.798	0.597	1.608	1.339	0.740	0.526	1.548	1.284	0.824	0.559
	1.8	1.829	1.701	0.603	0.615	1.461	1.357	0.596	0.574	1.445	1.343	0.722	0.669
	2.1	1.630	1.539	0.605	0.600	1.357	1.272	0.641	0.610	1.439	1.347	0.799	0.743
	2.4	1.689	1.563	0.642	0.611	1.374	1.260	0.712	0.658	1.448	1.332	0.933	0.852
	2.7	1.538	1.438	0.643	0.626	1.321	1.237	0.741	0.701	1.440	1.354	0.999	0.929
Clayton Copula	0.9	0.212	0.212	0.008	0.008	0.215	0.215	0.035	0.035	0.298	0.298	0.088	0.088
	1.2	0.065	0.065	0.022	0.022	0.115	0.115	0.096	0.096	0.263	0.263	0.210	0.210
	1.5	0.036	0.036	0.070	0.070	0.133	0.133	0.207	0.207	0.335	0.335	0.374	0.374
	1.8	0.049	0.049	0.142	0.142	0.215	0.215	0.345	0.345	0.464	0.464	0.576	0.576
	2.1	0.064	0.064	0.180	0.180	0.297	0.297	0.429	0.429	0.617	0.617	0.697	0.697
	2.4	0.114	0.114	0.226	0.226	0.392	0.392	0.551	0.551	0.744	0.744	0.867	0.867
	2.7	0.130	0.130	0.305	0.305	0.468	0.468	0.674	0.674	0.893	0.893	1.024	1.024
	0.9	1.610	0.533	0.354	0.047	1.841	0.452	0.632	0.040	2.768	0.560	0.364	0.087
	1.2	0.421	0.160	0.108	0.016	0.255	0.181	0.062	0.078	0.466	0.329	0.166	0.191
	1.5	0.129	0.090	0.049	0.050	0.162	0.144	0.169	0.173	0.343	0.329	0.301	0.346
	1.8	0.081	0.065	0.107	0.109	0.180	0.187	0.280	0.287	0.424	0.425	0.510	0.522
	2.1	0.070	0.068	0.127	0.128	0.252	0.252	0.342	0.349	0.566	0.570	0.611	0.623
	2.4	0.097	0.096	0.157	0.158	0.316	0.319	0.442	0.451	0.663	0.665	0.765	0.778
	2.7	0.107	0.107	0.222	0.224	0.380	0.382	0.546	0.556	0.798	0.802	0.903	0.916
	0.9	8.009	4.098	5.463	1.269	8.886	3.196	4.665	0.943	8.132	2.789	4.685	0.759
	1.2	3.588	2.221	1.990	0.706	3.255	1.710	1.865	0.691	2.935	1.569	1.852	0.690
	1.5	2.634	1.901	0.875	0.671	2.095	1.520	0.840	0.684	1.720	1.367	0.976	0.805
	1.8	1.888	1.535	0.750	0.723	1.517	1.384	0.869	0.824	1.541	1.406	1.038	0.989
	2.1	1.444	1.332	0.760	0.749	1.341	1.277	0.885	0.849	1.522	1.440	1.151	1.108
	2.4	1.649	1.531	0.756	0.752	1.485	1.400	0.998	0.974	1.634	1.548	1.288	1.250
	2.7	1.776	1.674	0.840	0.812	1.649	1.571	1.038	1.014	1.872	1.791	1.403	1.357

Table D.4: Average integrated variance ($\text{AIV} \times 10^4$) estimated from 1 000 replicates of for $\mathbb{C}_{xh}^{(\text{rc})}$ with $n = 250$ and $n = 1\,000$ in the case $\tau_{C_x} < \tau_{F_x}$. Upper pannel: Normal Copula. Bottom pannel : Clayton Copula.

θ	h	$\tau_x = .1$				$\tau_x = .25$				$\tau_x = .35$			
		$n = 250$		$n = 1000$		$n = 250$		$n = 1000$		$n = 250$		$n = 1000$	
		$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$	$g = h$	$g \neq h$
0%	0.9	3.581	3.581	0.849	0.849	3.164	3.164	0.750	0.750	2.771	2.771	0.649	0.649
	1.2	2.703	2.703	0.691	0.691	2.387	2.387	0.589	0.589	2.078	2.078	0.500	0.500
	1.5	2.442	2.442	0.583	0.583	2.109	2.109	0.509	0.509	1.790	1.790	0.436	0.436
	1.8	2.124	2.124	0.511	0.511	1.839	1.839	0.434	0.434	1.558	1.558	0.369	0.369
	2.1	1.933	1.933	0.467	0.467	1.659	1.659	0.398	0.398	1.394	1.394	0.334	0.334
	2.4	1.844	1.844	0.455	0.455	1.565	1.565	0.388	0.388	1.337	1.337	0.327	0.327
	2.7	1.820	1.820	0.423	0.423	1.542	1.542	0.354	0.354	1.282	1.282	0.300	0.300
	0.9	24.896	13.329	24.072	4.462	26.604	13.807	26.315	4.695	27.605	13.997	27.771	4.815
	1.2	12.322	3.652	10.927	0.918	12.878	3.294	11.853	0.800	13.125	2.950	12.432	0.700
	1.5	5.445	4.353	1.911	0.787	5.294	4.083	1.946	0.704	5.099	3.811	1.936	0.620
	1.8	2.832	2.831	0.682	0.685	2.498	2.500	0.590	0.594	2.197	2.200	0.520	0.524
	2.1	2.631	2.641	0.628	0.630	2.316	2.326	0.553	0.555	2.013	2.021	0.481	0.484
	2.4	2.438	2.436	0.599	0.601	2.125	2.126	0.526	0.528	1.855	1.858	0.457	0.460
	2.7	2.390	2.391	0.574	0.576	2.055	2.057	0.495	0.498	1.765	1.767	0.431	0.433
	0.9	44.135	19.743	63.494	4.405	47.572	20.341	69.854	4.422	49.689	20.560	74.010	4.372
	1.2	26.871	9.619	13.628	1.718	28.587	9.369	14.703	1.524	29.528	9.043	15.368	1.364
	1.5	8.899	6.801	9.151	1.501	8.618	6.285	9.786	1.333	8.284	5.806	10.156	1.188
	1.8	5.181	5.136	1.325	1.293	4.574	4.536	1.142	1.123	4.068	4.038	1.010	1.000
	2.1	4.700	4.698	1.278	1.252	4.118	4.120	1.105	1.088	3.630	3.641	0.963	0.953
	2.4	4.587	4.577	1.189	1.173	3.998	3.999	1.017	1.013	3.528	3.533	0.876	0.876
	2.7	4.440	4.411	1.178	1.156	3.860	3.842	0.998	0.986	3.378	3.366	0.856	0.851
Clayton Copula	0.9	3.605	3.605	0.884	0.884	3.197	3.197	0.769	0.769	2.773	2.773	0.658	0.658
	1.2	2.790	2.790	0.708	0.708	2.417	2.417	0.611	0.611	2.084	2.084	0.523	0.523
	1.5	2.332	2.332	0.577	0.577	1.996	1.996	0.496	0.496	1.717	1.717	0.425	0.425
	1.8	2.154	2.154	0.513	0.513	1.827	1.827	0.438	0.438	1.550	1.550	0.372	0.372
	2.1	1.926	1.926	0.467	0.467	1.626	1.626	0.398	0.398	1.382	1.382	0.336	0.336
	2.4	1.798	1.798	0.446	0.446	1.476	1.476	0.370	0.370	1.246	1.246	0.311	0.311
	2.7	1.764	1.764	0.419	0.419	1.463	1.463	0.354	0.354	1.244	1.244	0.301	0.301
	0.9	26.717	9.075	17.445	4.482	31.666	7.874	26.904	3.450	42.212	8.880	19.740	3.477
	1.2	12.285	3.736	11.903	0.937	8.020	4.481	6.845	0.823	11.665	5.456	7.110	0.731
	1.5	5.279	3.113	0.774	0.775	5.148	2.772	0.682	0.683	5.023	2.496	3.182	0.602
	1.8	5.078	2.892	0.688	0.690	3.727	2.527	0.613	0.615	2.192	2.203	0.535	0.537
	2.1	2.557	2.559	0.626	0.629	2.263	2.268	0.545	0.547	1.973	1.975	0.475	0.478
	2.4	2.430	2.432	0.598	0.598	2.073	2.072	0.516	0.516	1.805	1.807	0.445	0.447
	2.7	2.344	2.347	0.568	0.571	1.997	2.003	0.496	0.497	1.739	1.743	0.433	0.434
	0.9	47.333	22.684	52.828	10.644	56.433	19.019	53.310	8.996	55.804	15.543	56.266	5.505
	1.2	21.724	7.633	27.012	1.795	25.893	6.950	29.195	2.762	25.443	6.473	29.539	1.450
	1.5	13.852	5.634	9.070	1.513	12.944	4.986	9.597	1.323	10.548	4.486	9.952	1.181
	1.8	9.272	5.075	2.425	1.300	5.661	4.508	3.542	1.148	5.270	4.026	2.299	1.029
	2.1	4.781	4.750	2.330	1.219	4.180	4.170	1.096	1.085	3.714	3.688	0.971	0.964
	2.4	4.441	4.421	1.207	1.188	3.934	3.912	1.029	1.016	3.564	3.569	0.887	0.879
	2.7	4.291	4.294	1.154	1.136	3.681	3.683	0.978	0.971	3.246	3.235	0.854	0.850