# Impact of the piston velocity on the steady-state distribution of a radioactive tracer in the leaky funnel idealisation of the World Ocean

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## Introduction

According to England (1995), the "World Ocean circulation at its largest scale can be thought of as a gradual renewal or ventilation of the deep ocean by water that was once at the sea surface". Thus, estimating the age of the water as the time elapsed since leaving the ocean surface layer is likely to provide useful insight into the ventilation processes of the World Ocean. This is why the age is a popular diagnostic tool in this domain of interest.

Many methods for estimating the age of the water or understanding its distribution have ben suggested, including a number of highly idealised approaches. One of the latter is the so-called "leaky funnel metaphor" (Figure 1) (Mouchet and Deleersnijder 2008, Mouchet et al. 2012), i.e. a semi-infinite pipe with a decreasing cross-sectional area and porous walls. It rests on the following principle: "The horizontal circulation in the actual ocean may be thought to be a consequence of localized sinking and generalized upwelling" (Warren 1981).

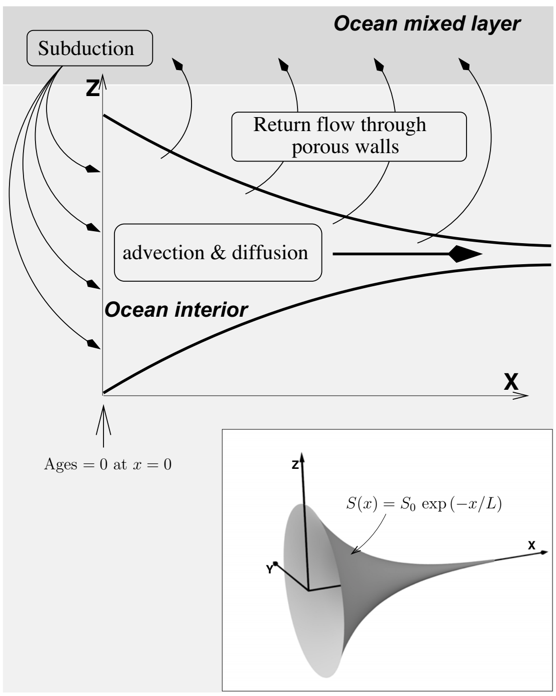


**Figure 1.** Schematic representation of the leaky funnel, which extends from $x = 0$ to $x = \infty$. The surface water enters the leaky funnel at $x = 0$, is advected at a constant speed $U$ and is subjected to diffusive processes characterised by a constant eddy coefficient $K$. The water progressively escapes the funnel through its porous walls. As is depicted in the inset, the funnel section decreases exponentially, the constant $L$ being the associated e-folding length scale. The latter may also be seen to be the mean length of the trajectory of water parcels in the funnel. This is Figure 1 of Mouchet and Deleersnijder (2008).

The leaky funnel metaphor has distinct advantages over some other highly idealised models of the World Ocean. As opposed to the very popular "great ocean conveyor", it leads to quantitative estimates of a number of variables. Semi-infinite pipes with a constant cross-sectional area, though widely used, allow for analytical solutions to be arrived at, but prevent domain-averaged values from being derived, since the volume of the domain is infinite. The volume of the so-called Munk loop (Munk 1966) is finite, but no diffuse return of the water toward the surface can be represented. The leaky-funnel model is plagued by none of the aforementioned shortcomings.

The leaky funnel model was conceived as a tool for providing a scaling of the mean age of the water in the World Ocean based on a velocity scale $U$, a typical value of the diffusivity $K$ and a relevant length scale $L$ (Mouchet and Delersnijder 2008). Then, this idealisation was seen to be able to deal successfully with the age of radioactive tracers (Mouchet and Delersnijder 2008) and represent the water age distribution with an astonishing accuracy (Mouchet et al. 2012). In all of the aforementioned studies the entrance of the leaky funnel was assumed to be the bottom of the surface mixed layer, where Dirichlet conditions were prescribed.

Hereinafter, an attempt is made to switch to a Robin boundary condition and apply it to a hypothetical radioactive tracer at a steady state. A word of caution is in order: since the entrance of the leaky funnel is located at the bottom of the ocean mixed layer rather than the ocean-atmosphere interface, the exchange coefficient appearing in the Robin boundary condition used below is not entirely equivalent to the traditional piston velocity that is meant to represent exchanges through the ocean-atmosphere interface.

## Steady-state governing equations

Let the space variable $x$ represent the distance to the entrance of the leaky funnel, with $0 \le x < \infty$. Assume that $S(x)$ and $C(x)$ denote the cross-sectional area of the pipe and the steady-state concentration of a radioactive tracer, respectively. The equation obeyed by the latter is (Mouchet and Deleersnijder 2008)

$$-\underbrace{\frac{d}{dx}\left(-SK\frac{dC}{dx}+SUC\right)}_{\substack{\text{diffusive+advective flux}\\ \text{in the}\\ \text{along-flow direction}}} + \underbrace{\frac{d(SU)}{dx}C}_{\substack{\text{escape through}\\ \text{porous walls}}} + \underbrace{(-\gamma SC)}_{\substack{\text{radioactive}\\ \text{decay term}}} = 0 \ , \qquad (1)$$

where the positive constant $\gamma$ is the rate of decay of the tracer under consideration. The tracer flux entering the domain is parameterised by means of the Robin boundary condition

$$\left[-K\frac{dC}{dx}+UC\right]_{x=0} = \left[\chi(C^a - C)+UC\right]_{x=0} \ , \qquad (2)$$

where the constant $C^a$ represents a value of the tracer concentration that is related to the atmospheric concentration, while the positive constant $\chi$ plays a role rather similar to that of the piston velocity in proper atmosphere-ocean boundary conditions. For simplicity, $\chi$ will be termed "piston velocity" in the present working note.

The boundary condition (2) simplifies to

$$\left[-K\frac{dC}{dx}\right]_{x=0} = \left[\chi(C^a - C)\right]_{x=0} \ . \qquad (3)$$

Clearly, at the entrance of the leaky funnel, the concentration is relaxed to the reference value $C^a$. The tracer flux entering the domain consists of an advective part, $[SUC]_{x=0}$, and a diffusive contribution, $\left[-SK\,dC/dx\right]_{x=0}$. The ratio of the diffusive flux to the total flux,

$$\omega \equiv \frac{\left[-SK\dfrac{dC}{dx}\right]_{x=0}}{\left[-SK\dfrac{dC}{dx}+SUC\right]_{x=0}} = \frac{\left[\chi(C^a - C)\right]_{x=0}}{\left[\chi(C^a - C)+UC\right]_{x=0}} \ , \tag{4}$$

is a variable that is worth investigating.

As in the previous studies involving the leaky funnel metaphor, the diffusivity $K$ and the water velocity $U$ are assumed to be positive constants. Then, for water to escape through the porous walls (and take tracer particles along with it), it is necessary that $S(x)$, the cross-sectional area of the pipe, decay monotonously, i.e. $dS/dx < 0$.

At the entrance of the domain, the cross-sectional area is denoted $S_0$. Then, if the following expression is adopted

$$S(x) = S_0 \, e^{-x/L} \ , \tag{5}$$

analytical solutions will be arrived rather easily. This is mainly because the governing equation (1) simplifies to an ODE with constant coefficients, i.e.

$$K\frac{d^2C}{dx^2} - V\frac{dC}{dx} - \gamma C = 0 \ , \tag{6}$$

where the equivalent velocity $V$ is

$$V = U + \frac{K}{L} \ . \tag{7}$$

Though the pipe is semi-infinite, its volume is finite and is readily seen to be

$$\Omega \equiv \int_0^\infty S(x)\,dx = \int_0^\infty S_0\, e^{-x/L}\,dx = S_0 L \ . \tag{8}$$

Therefore, the domain-averaged tracer concentration reads

$$\bar{C} \equiv \frac{1}{\Omega}\int_0^\infty S(x)C(x)\,dx = \frac{1}{L}\int_0^\infty e^{-x/L}\,C(x)\,dx \ . \tag{9}$$

It is also worth introducing the volumetric distribution of the tracer concentration, $\Phi(C)$. The latter is defined as follows: the fraction of the volume of the domain of interest in which the tracer concentration lies in the interval $[C, C+\delta C]$ tends to $\Phi(C)\delta C$ as $\delta C \to 0$. Accordingly, the distribution function must satisfy the constraint

$$\int_0^\infty \Phi(C)\,dC = 1 \ . \tag{10}$$

**Dimensionless formulation of the problem**

It is convenient to reformulate the problem under consideration by having recourse to dimensionless variables. The appropriate dimensionless space coordinate, tracer concentration and volumetric distribution function are

$$\sigma = \frac{x}{L} \ , \ \zeta = \frac{C}{C^a} \ , \ \theta = C^a \Phi \ . \tag{11}$$

Then, a number of dimensionless parameters naturally arise, which may all be viewed as Peclet numbers. Indeed, they all are of the form $\mathcal{V}\mathcal{L}/K$, where $\mathcal{V}$ and $\mathcal{L}$ denote the relevant velocity and length scales. By setting $\mathcal{V}=U$ and $\mathcal{L}=L$, the usual Peclet number is obtained, i.e.

$$Pe = \frac{UL}{K} . \tag{12}$$

The modified Peclet number is associated with the velocity scale *V*:

$$Pe' = \frac{VL}{K} = Pe + 1 . \tag{13}$$

The modified Jenkins number (Mouchet and Deleersnijder 2008) is associated with the modified velocity *V* and the length scale $V/\gamma$ (i.e. the distance travelled by a particle moving at speed *V* during the time interval that is equal to the mean life of the radioactive tracer, $1/\gamma$):

$$Je' = \frac{V^2}{K\gamma} . \tag{14}$$

Finally, the dimensionless piston velocity is taken to be

$$\lambda = \frac{\chi L}{K} . \tag{15}$$

Using the aforementioned dimensionless variables and parameters, the differential problem to be solved reads

$$\frac{1}{Pe'}\frac{d^2\zeta}{d\sigma^2} - \frac{d\zeta}{d\sigma} - \frac{Pe'}{Je'}\zeta = 0 , \tag{16}$$

$$\left[\frac{d\zeta}{d\sigma} + \lambda(1-\zeta)\right]_{\sigma=0} = 0 . \tag{17}$$

Then, the dimensionless concentration reads

$$\zeta(\sigma) = \frac{\lambda\, e^{-\mu\sigma}}{\mu+\lambda} \tag{18}$$

with

$$\mu = \frac{Pe'}{2}\left(\sqrt{1+4/Je'}-1\right) . \tag{19}$$

**Properties of the analytical solution**

Below, in the course of the analysis of the properties of the analytical solution (18), the Peclet number will remain equal to 3, i.e. a value in accordance with the findings of Mouchet and Deleersnijder (2008). Accordingly the modified Peclet number will also be fixed, and, hence, will be equal to $Pe' = Pe + 1 = 4$. Then, the sensitivity of the solution to the values of the other dimensionless parameters will be examined.

The concentration decreases exponentially as a function of the distance to the entrance of the domain. The associated e-folding length scale is independent of the piston velocity, but, as expected, depends on the decay rate of the tracer. The larger the decay rate (i.e. the smaller the modified Jenkins number $Je'$), the smaller the e-folding length scale and the tracer concentration (Figure 2):

$$\frac{\partial \zeta}{\partial Je'} > 0 \quad \Rightarrow \quad \frac{\partial C}{\partial \gamma} < 0 \ . \tag{20}$$

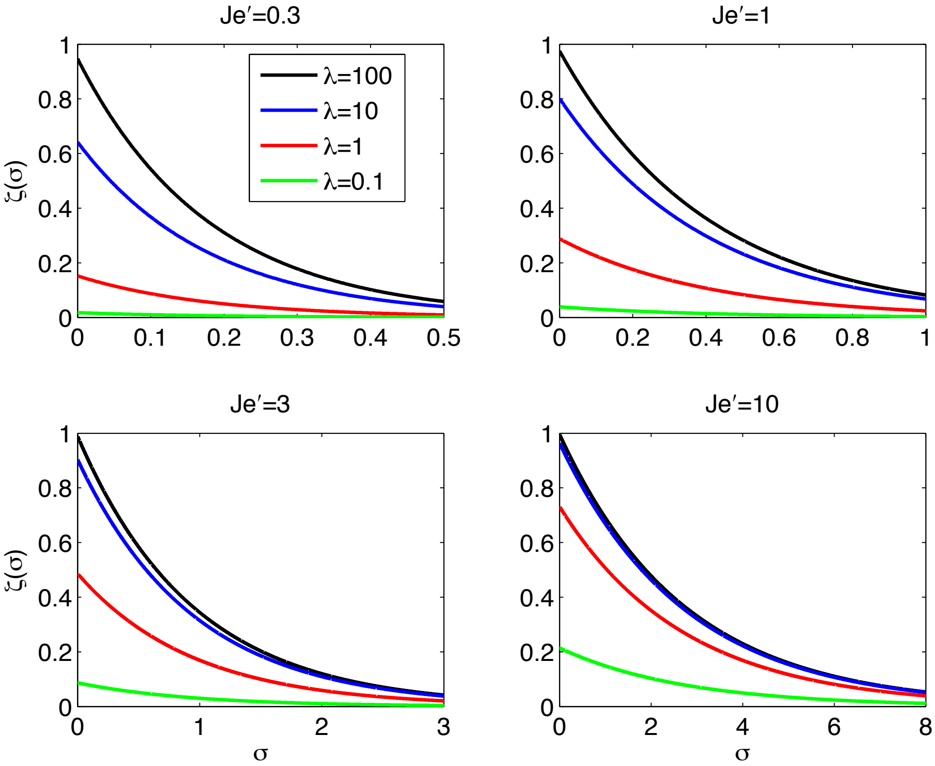


**Figure 2.** The dimensionless concentration, $\zeta$, as a function of the dimensionless distance to the entrance, $\sigma$, for various values of the modified Jenkins number, $Je'$, and the dimensionless piston velocity, $\lambda$. For all cases considered, the Peclet number, $Pe = UL/K$, is equal to 3.

The concentration increases as the piston velocity increases:

$$\frac{\partial \zeta}{\partial \lambda} > 0 \quad \Rightarrow \quad \frac{\partial C}{\partial \chi} > 0 \ . \tag{21}$$

In the limit $\lambda \to \infty$, the Robin boundary condition degenerates into a Dirichlet boundary condition with $\zeta(0) = 1$:

$$\zeta(\sigma) \sim \left(1 - \frac{\mu}{\lambda}\right) e^{-\mu\sigma} \ , \quad \lambda \to \infty \ . \tag{22}$$

The domain-averaged concentration is readily seen to be

$$\bar{\zeta} \equiv \int_0^\infty e^{-\sigma} \zeta(\sigma)\, d\sigma = \frac{\lambda}{(\mu+1)(\mu+\lambda)} \ , \tag{23}$$

and is illustrated in Figure 3.

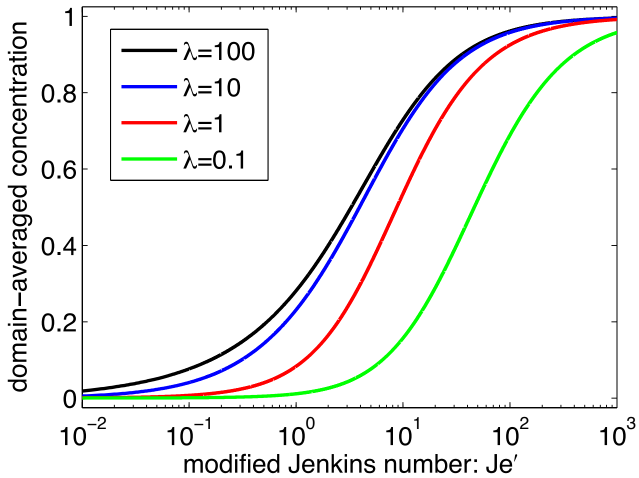


**Figure 3.** The domain-averaged dimensionless concentration, $\bar{\zeta}$ , as a function of the modified Jenkins number, $Je'$ , for various values of the modified Jenkins number, $Je'$ , and the dimensionless piston velocity, $\lambda$ . For all cases considered, the Peclet number, $Pe = UL/K$ , is equal to 3.

At the entrance of the domain, the ratio (4) of the diffusive flux to the total flux is

$$\omega = \frac{\mu}{\mu + Pe} \tag{24}$$

Surprisingly, this ratio does not depend on the piston velocity! A convincing physical explanation of this remarkable property has yet to be found.

## Volumetric distribution of the tracer concentration

Any (dimensionless) elemental volume satisfies

$$\delta\Omega = e^{-\sigma}\,\delta\sigma = e^{-\sigma}\left(-\frac{\partial\sigma}{\partial\zeta}\right)\delta\zeta \ , \tag{25}$$

Bearing in mind that the dimensionless volume of the domain is equal to unity, the (dimensionless) volumetric distribution of the tracer concentration is

$$\theta(\zeta) = e^{-\sigma}\left(-\frac{\partial\sigma}{\partial\zeta}\right) . \tag{26}$$

According to the solution (18), the space variable is related to the concentration by the expression

$$\sigma = \frac{1}{\mu}\log\frac{\lambda}{(\mu+\lambda)\zeta} \ , \tag{27}$$

which implies

$$-\frac{\partial\sigma}{\partial\zeta} = \frac{1}{\mu\zeta} \tag{28}$$

and

$$e^{-\sigma} = \left[\frac{(\mu+\lambda)\zeta}{\lambda}\right]^{\frac{1}{\mu}} . \tag{29}$$

Next, by substituting (28)-(29) into (26), the final expression of the (dimensionless) volumetric distribution of the tracer concentration is arrived at, i.e.

$$\theta(\zeta) = \begin{cases} \dfrac{1}{\mu}\left[\dfrac{(\mu+\lambda)}{\lambda}\right]^{\frac{1}{\mu}} \zeta^{-1+1/\mu} , & 0<\zeta<\lambda/(\mu+\lambda) \\ 0 , \quad \lambda/(\mu+\lambda)<\zeta \end{cases} \tag{30}$$

Finally it is readily seen that the distribution function satisfies

$$\int_0^\infty \theta(\zeta)\, d\zeta = \int_0^{\frac{\lambda}{\mu+\lambda}} \theta(\zeta)\, d\zeta = 1 , \tag{31}$$

which is the dimensionless counterpart of (10).

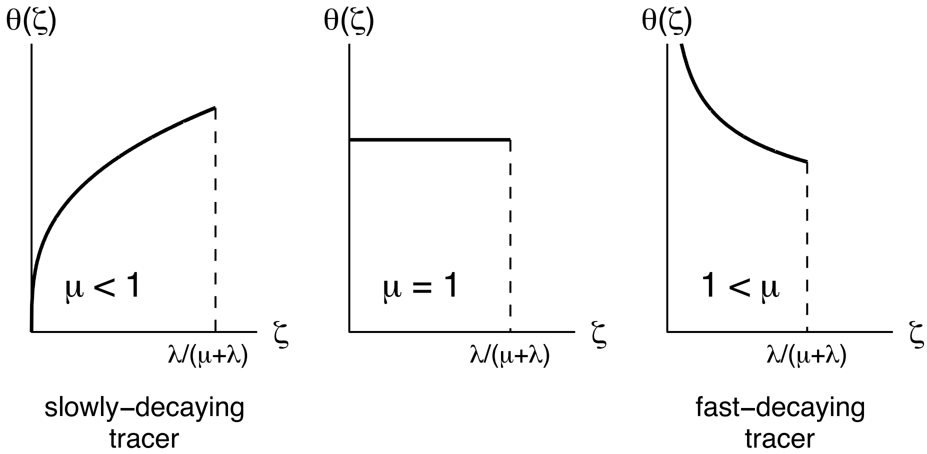


**Figure 4.** Sketch of the volumetric distribution of the tracer concentration, $\theta(\zeta)$, for various values of $\mu$.

The behaviour of the distribution function depends on the value of $\mu$ (Figure 4):

$$\frac{d\theta}{d\zeta} = \begin{cases} >0 , \quad 0<\mu<1 \quad \text{(slowly-decaying tracer)} \\ 0 , \quad \mu=1 \\ <0 , \quad 1<\mu \quad \text{(fast-decaying tracer)} \end{cases} \tag{32}$$

For a slowly-decaying tracer ($0<\mu<1$), the distribution function is such that $\theta(0)=0$, suggesting that the fraction of the volume of the domain where the tracer concentration is small is negligible. On the contrary, for a tracer exhibiting a high rate of decay ($1<\mu$), the region in which the tracer concentration is small is rather large, which is in agreement with

$$\lim_{\zeta\to 0} \theta(\zeta) = \infty . \tag{33}$$

That, in this case, the distribution function is singular at $\zeta = 0$ does not prevent the constraint (31) from being satisfied, since the distribution function remains integrable.

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