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Analytic aspects of locally compact groups acting on Euclidean buildings

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by

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To my family

Introduction

The research topic of my doctoral thesis explores the world of totally disconnected locally compact groups through the interaction with several geometric and harmonic analytic concepts.

The starting point of this project is the **Howe–Moore property** of a locally compact group G, namely, the vanishing at infinity of all matrix coefficients of the group's unitary representations that are without non-zero G-invariant vectors. This property is well-known to hold for connected, non-compact, simple real Lie groups, with finite center, and for their totally disconnected analogs, namely, isotropic simple algebraic groups over non Archimedean local fields and closed, topologically simple subgroups of $\mathrm{Aut}(T)$ that act 2-transitively on the boundary ∂T, where T is a bi-regular tree with valence ≥ 3 at every vertex.

The first step of my PhD project was to provide a unified proof of all the above mentioned well-known examples. This task is accomplished in Chapter 2, Section 2.1.

Theorem A. *(See Theorem 2.1.33 and Section 2.1.4) Let G be a connected, non-compact, simple real Lie group, with finite center, or an isotropic simple algebraic group over a non Archimedean local field, or a closed, topologically simple subgroup of* $\mathrm{Aut}(T)$ *that acts 2-transitively on the boundary ∂T, where T is a bi-regular tree with valence ≥ 3 at every vertex. Then G admits the Howe–Moore property.*

To prove Theorem A, we apply a general criterion related to the Howe–Moore property, which makes use of the following:

Definition B. Let G be a locally compact group and $\alpha = \{g_n\}_{n>0}$ be a sequence in G. We define

$$U_\alpha^+ := \{g \in G \mid \lim_{n\to\infty} g_n^{-1} g g_n = e\}.$$

Notice that U_α^+ is a subgroup of G. It is called **the contraction group** corresponding to α. In the same way, but using $g_n g g_n^{-1}$, we define U_α^-.

Theorem C. *(See Theorem 2.1.18) Let G be a locally compact, second countable group having the following properties:*

i) G admits a decomposition $K_1A^+K_2$, where $K_1, K_2 \subset G$ are compact subsets and A^+ is an abelian sub semi-group of G;

ii) for all sequences $\alpha = \{g_n\}_{n>0} \subset A^+$, with $g_n \to \infty$, we have $G = \overline{\langle U_\alpha^+, U_\alpha^- \rangle}$.

Then G has the Howe–Moore property.

The second step of the project aims to completely characterize locally compact groups acting on a d-regular tree and enjoying the Howe–Moore property. More precisely, we want to prove the following conjecture related to the Howe–Moore property:

Conjecture D. *Let G be a locally compact, topologically simple group acting continuously and properly by type-preserving automorphisms on a d-regular tree. If G admits the Howe–Moore property then G acts 2-transitively on the boundary of the tree.*

A good test case for this conjecture is the Burger–Mozes universal group $U(F)$, in fact the subgroup $U(F)^+ < U(F)$, with F being primitive but not 2-transitive. For the definitions of $U(F)$ and $U(F)^+$, please see Definition 2.2.5. This group $U(F)$ has the following 'universal' property, recorded by [BM00a, Prop. 3.2.2]: Let $d \geq 3$, $F < \mathrm{Sym}(\{1, \cdots, d\})$ be a transitive permutation subgroup and denote by T_d the d-regular tree. Consider a vertex transitive subgroup $H < \mathrm{Aut}(T_d)$ such that, for every vertex $x \in T_d$, the restriction of H_x to the star of x in T_d is permutation isomorphic to F, where H_x is the stabilizer of x in H. Then $H < U(F)$.

For the moment, the Conjecture D remains open and only partial results were established. Those are presented in Section 2.2 of Chapter 2. More precisely, we give the following characterization for the Burger–Mozes universal group $U(F)^+$ to admit the Howe–Moore property:

Theorem E. *(See Theorem 2.2.21) Let F be primitive, but not cyclic of prime order. Then $U(F)^+$ has the Howe–Moore property if and only if for every unitary representation $(\pi, \mathcal{H})$ of $U(F)^+$, without non-zero $U(F)^+$-invariant vectors, and for every $v \neq 0 \in \mathcal{H}$ the closed isotropy subgroup $U(F)^+_v$ is compact.*

While performing the research for Conjecture D and studying a second question proposed in my PhD project regarding **Gelfand**

pairs (see Def. 3.1.1), we introduced the notion of a **strongly regular hyperbolic automorphism** of a locally finite thick Euclidean building. This allowed us to fully characterize Gelfand pairs among locally compact groups that act continuously and properly, by type-preserving automorphisms (see Def. 1.1.7) on a locally finite thick Euclidean building. This solved, in full generality, the second question of the PhD project. We prove that admitting a Gelfand pair (G, K), where K is a compact subgroup of G, is equivalent to a strongly transitive group action (see Def. 1.1.7) on the spherical building at infinity which is also equivalent to a strongly transitive group action on the Euclidean building itself. More precisely, Chapter 3 provides the proof of the following:

Theorem F. *(See Theorems 3.1.2 and 3.3.11) Let G be a locally compact group acting continuously and properly by type-preserving automorphisms on a locally finite thick Euclidean building Δ. Then the following are equivalent:*

(i) G is strongly transitive on Δ;

(ii) G is strongly transitive on $\partial\Delta$, the spherical building at infinity of Δ;

(iii) G has no fixed point in $\partial\Delta$ and for some chamber $c \in \mathrm{Ch}(\partial\Delta)$, the stabilizer G_c acts cocompactly on Δ;

(iv) G acts cocompactly on Δ, without a fixed point in $\partial\Delta$, and there exists a compact open subgroup $K \leq G$ having finitely many chamber-orbits in $\partial\Delta$.

(v) G acts cocompactly on Δ, and there exists a compact subgroup K of G such that (G, K) is a Gelfand pair.

Combining the results of [Léc10] and [APVM13] with Theorem 3.1.2, we conclude the following important fact.

Corollary G. *(See Corollary 3.1.3) A locally compact group G acting continuously, properly and cocompactly by type-preserving automorphisms on a locally finite thick Euclidean building Δ, admits a Gelfand pair only when G acts strongly transitively on Δ. In this case, its only Gelfand pairs are of the form (G, G_x), where x is a special vertex of Δ and G_x is the stabilizer of x in G.*

Recall that, when the Euclidean building is a thick tree, the strongly transitive boundary group action is equivalent to the 2-transitive group action on the tree's boundary. We mention that for the case of thick trees the implication (v) $\Rightarrow$ (ii) of Theorem F was reported without proof in the book [FTN91, page 48].

Without giving the definitions, we recall that Gelfand pairs are useful tools for studying the spherical unitary dual of a locally compact group. Furthermore, among topologically simple locally compact groups, the only known examples admitting a Gelfand pair are again the families of 'Howe–Moore' groups studied in Chapter 2, Section 2.1; however, we do not know to what extent the Howe–Moore property relates or is equivalent to admitting a Gelfand pair. For the case of groups acting on d-regular trees a possible equivalence would imply Conjecture D, as follows from Theorem F.

In addition to the purposes mentioned above, strongly regular hyperbolic automorphisms of a locally finite Euclidean building have several other applications. By implementing their peculiar dynamical properties in the context of CAT(0) geometry, in Chapter 4 we construct compact-by-abelian subgroups inside (non necessarily discrete) closed, co-compact subgroups of type-preserving automorphisms of a locally finite general building. This answers Gromov's flat closing problem in the case of general buildings.

Theorem H (See Theorem 4.3.4)**.** *Let Δ be a locally finite building of type (W, S), with S being finite. Let G be a not necessarily discrete, closed, type-preserving subgroup of* $\mathrm{Aut}(\Delta)$*, with co-compact action on Δ. Then G contains a compact-by-$\mathbf{Z}^d$ subgroup, where d is the dimension of a maximal flat of Δ.*

Contents

Chapter 1

Preliminaries

1.1 Euclidean buildings: a brief introduction

Buildings were introduced by Jacques Tits in order to study semi-simple algebraic groups over an arbitrary field. The analogy with the real field case is the following. Recall that for a connected semi-simple real Lie group G, with finite center, we associate the Riemannian manifold G/K, which is called the symmetric space of G and where $K < G$ is the maximal compact subgroup. Many properties of G are studied through the interaction with the geometry of its corresponding symmetric space G/K. When working, for example, with a semi-simple algebraic group G over the p-adic field $\mathbf{Q}_p$, the good 'symmetric space' of G is given by the theory of Bruhat and Tits, that space being the Euclidean building associated to G. For this theory, the reader can consult [AB08, Gar97, Ron89]. In this introduction we only resume briefly the theory of Bruhat and Tits by giving the main ideas of the theory and by recalling the main definitions and concepts that are used further in this thesis.

We start with an explicit example (see [Ser80, Chap. II]). Consider the algebraic group $\mathrm{SL}(2, \mathbf{Q}_p)$, which acts by linear transformations on the 2-dimensional $\mathbf{Q}_p$-vector space denoted by V. A lattice in V is any finitely generated $\mathbf{Z}_p$-submodule of V which spans the $\mathbf{Q}_p$-vector space V. More precisely, a lattice L of V is of the form $\mathbf{Z}_p e_1 \bigoplus \mathbf{Z}_p e_2$, where $\{e_1, e_2\}$ is a basis of V. By definition, two lattices L, L' are equivalent if there exists $k \in \mathbf{Q}_p^*$ such that $L = kL'$. The equivalent class of L is denoted by $[L]$ and this is in fact the

$\mathbf{Q}_p^*$-orbit of L. Let us compare now two such equivalent classes. Let $\pi \in \mathbf{Q}_p$ be a uniformizer; for example one can take $\pi = p$. Let $[L] \neq [L']$ and we apply the invariant factor theorem for L and L'. This theorem ensures the existence of a $\mathbf{Z}_p$-basis $\{e_1, e_2\}$ of L and two integers n and m such that $\{\pi^n e_1, \pi^m e_2\}$ is a $\mathbf{Z}_p$-basis of L'. One can prove that the set $\{n, m\}$ does not depend on the chosen basis $\{e_1, e_2\}$ and that $L' \subset L$ if and only if $n, m \geq 0$. Moreover, if we change the representatives L and L', we obtain that the number $|n - m|$ is an invariant of the two classes $[L]$ and $[L']$.

To construct the 'symmetric space=Euclidean building' of $\mathrm{SL}(2, \mathbf{Q}_p)$, we consider the set X of all classes $[L]$, where L is a lattice of V. The elements of X are considered to be the vertices of the 'symmetric space'. Two vertices $[L], [L']$ are linked by an edge if and only if their corresponding invariant $|n-m|$ equals 1; this means that there exists a $\mathbf{Z}_p$-basis $\{e_1, e_2\}$ of L such that $\{e_1, \pi e_2\}$ is a $\mathbf{Z}_p$-basis of L'. The remarkable property of the graph constructed above is that X forms in fact an infinite tree on which the group $\mathrm{SL}(2, \mathbf{Q}_p)$ acts by automorphisms.

Let now consider the more general case of $\mathrm{SL}(n, \mathbf{Q}_p)$ and let V be the n-dimensional vector space over $\mathbf{Q}_p$. As in dimension 2, a lattice in V is any finitely generated $\mathbf{Z}_p$-submodule of V which spans the $\mathbf{Q}_p$-vector space V. Let X be the set of all equivalent classes $[L] := \mathbf{Q}_p^* \cdot L$ and let $[L] \neq [L']$ be two elements of X. Then, by applying the invariant factor theorem, there exists a $\mathbf{Z}_p$-basis $\{e_1, e_2, \cdots, e_n\}$ of L and n integers $\{i_1, \cdots, i_n\}$ such that $\{\pi^{i_1} e_1, \pi^{i_2} e_2, \cdots, \pi^{i_n} e_n\}$ is a $\mathbf{Z}_p$-basis of L'. Moreover, we have that the set $\{i_1, \cdots, i_n\}$ does not depend on chosen basis $\{e_1, e_2, \cdots, e_n\}$ and that $L' \subset L$ if and only if $\{i_1, \cdots, i_n\}$ are all ≥ 0. In this general case we 'link' the vertices of X in the following way. By definition, m classes $[L_1], \cdots, [L_m]$ form an m-simplex if they admit representatives $L_1, \cdots, L_m$ such that $\pi L_1 \subset L_m \subset \cdots \subset L_1$. Notice that a maximal simplex is of dimension $n-1$. Remark also the following. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of V and consider the subset $\mathcal{A}$ of X given by all classes $[L']$ such that $\{\pi^{i_1} e_1, \pi^{i_2} e_2, \cdots, \pi^{i_n} e_n\}$ is a $\mathbf{Z}_p$-basis of L', where $\{i_1, \cdots i_n\} \subset \mathbf{Z}$. The subset $\mathcal{A}$ has a special name, it is called an 'apartment' of the 'symmetric space' X of $\mathrm{SL}(n, \mathbf{Q}_p)$ and a maximal simplex is a 'chamber' of the apartment. Moreover, $\mathcal{A}$ is isometric with the Euclidean space of dimension $n-1$, the latter one being paved with the maximal simplexes, the chambers. In addition, $\mathrm{SL}(n, \mathbf{Q}_p)$ acts on this space X in a 'canonical' way, which will be explained later. Nevertheless, the importance of the above construction is coming from the fact that, in some sense, the group $\mathrm{SL}(n, \mathbf{Q}_p)$ can be 'reconstructed' by studying the 'geometry' of the 'Euclidean building' X, the main actors being the stabilizers in $\mathrm{SL}(n, \mathbf{Q}_p)$ of the apartment $\mathcal{A}$ of X and of a chamber in $\mathcal{A}$. More precisely, let $\mathrm{proj} : \mathrm{SL}(n, \mathbf{Z}_p) \to \mathrm{SL}(n, \mathbb{F}_p)$ be the natural projection. The inverse image, under the map proj, of the upper triangular subgroup of $\mathrm{SL}(n, \mathbb{F}_p)$ is, up to conjugation, the stabilizer in $\mathrm{SL}(n, \mathbf{Q}_p)$ of a chamber in X. This is denoted by B and is called the Iwahori or parahoric subgroup of $\mathrm{SL}(n, \mathbf{Q}_p)$. The stabilizer in $\mathrm{SL}(n, \mathbf{Q}_p)$ of the apartment $\mathcal{A}$ is the monomial subgroup N of $\mathrm{SL}(n, \mathbf{Q}_p)$, i.e., the subgroup consisting of all matrices with exactly one non-zero element in every row and every column. The pair (B, N) (which is called the BN-pair of $\mathrm{SL}(n, \mathbf{Q}_p)$) 'determines' (in a sense which is not aimed to be explained here) the group $\mathrm{SL}(n, \mathbf{Q}_p)$; for example, B and N generate $\mathrm{SL}(n, \mathbf{Q}_p)$, the intersection $H := B \cap N$ is a normal subgroup of N and the quotient group $W := N/H$ is called the **Weyl group** corresponding to $\mathrm{SL}(n, \mathbf{Q}_p)$.

The ideas presented above are studied and generalized (e.g., for semi-simple algebraic groups over locally compact non Archimedean fields) by the theory of buildings. For more information, the reader can consult the books [AB08, Gar97, Ron89, BT72]. We recall here only the very basic definitions and concepts of this theory.

Definition 1.1.1. A **Coxeter system** (W, S), with S finite, is a group W defined by the presentation $\langle S \mid (st)^{m(s,t)} \text{ with } s, t \in S \rangle$, where $m(s, s) = 1$ and $2 \leq m(s, t) = m(t, s) \leq \infty$ for every $s \neq t \in S$.

To a Coxeter system (W, S) one associates a vector space V, with base $\{e_s\}_{s \in S}$ and a specific symmetric bilinear form B_W defined by $B_W(e_s, e_t) := -\cos \frac{\pi}{m(s,t)}$, for every $s, t \in S$. Using the bilinear form B_W, we visualize an element $s \in S$ as acting on V as a reflection through the hyperplane $e_s^{\perp}$ that is orthogonal, with respect to B_W, to the vector e_s. We say that W acts as a reflection group on V, where the elements of S are distinct and of order 2 in W.

Definition 1.1.2. A Coxeter system (W, S) is **reducible** if S admits a partition $S = S' \sqcup S''$ with all elements of S' commuting with all elements of S'' and where S', S'' are non-empty subsets.

Therefore, W splits as a direct product $W' \times W''$ of two Coxeter groups.

In addition, any irreducible Coxeter system (W, S) is one of the following three types:

(1) (W, S) is **finite (or spherical)** if and only if B_W is positive definite; moreover, finite Coxeter system are completely classified;

(2) (W, S) is **Euclidean (or affine)** if and only if B_W is positive semidefinite but degenerate;

(3) (W, S) is **non-spherical and non-affine** (e.g., if B_W is non-degenerate, of signature $(n-1, 1)$ and $B_W(x, x) < 0$ for all x in the 'fundamental chamber C' then (W, S) is hyperbolic);

Definition 1.1.3. Let (W, S) be a Coxeter system with V being the vector space where W acts as a reflection group. Every reflection w in W determines its unique hyperplane H_w in V, which is called the **wall** in V with respect to w. Moreover, every wall H_w bounds two half-spaces V_w^-, V_w^+ of V. We make a choice of the positive half-spaces so that the intersection $\bigcap\limits_{s \in S} V_s^+$ is a simplicial cone C, called the **fundamental chamber**. Notice that its faces are the hyperplanes H_s, with $s \in S$.

Consider now the set $X := \bigcup\limits_{w \in W} wC \subset V$, called the **Tits cone**. To X we associate the 'simplicial complex' $\Sigma(W, S) := (X \backslash \{o\})/\mathbb{R}_{>0}$, which is called the **Coxeter complex** of (W, S). By the above procedure, the 'projection' of the fundamental chamber $C \subset V$ on $\Sigma(W, S)$ is also called the fundamental chamber of the Coxeter complex $\Sigma(W, S)$ and by abuse of notation is denoted with the same letter C. The fundamental chamber of $\Sigma(W, S)$ is viewed as a simplicial complex, which is of dimension $|S| - 1$, and $\Sigma(W, S)$ as **the apartment of type** (W, S). Notice that the group W still acts by reflections on $\Sigma(W, S)$, where the wall of a reflection $w \in W$ is the projection on $\Sigma(W, S)$ of the corresponding wall $H_w \subset V$.

For example, the Coxeter complex of the infinite dihedral group $D_\infty = \langle s, t | s^2 = t^2 = 1 \rangle$ is the bi-infinite line $\mathbf{R}$, where the elements of $\mathbf{Z}$ are the corresponding vertices of the complex $\Sigma(D_\infty, \{s, t\})$.

Informally, when (W, S) is irreducible, the corresponding Coxeter complex $\Sigma(W, S)$ can be imagined as one of the following three types:

(i) if (W, S) is spherical (finite) then $\Sigma(W, S)$ has a finite number of chambers which tessellate the Euclidean sphere of dimension $|S| - 1$;

(ii) if (W, S) is Euclidean (affine) then $\Sigma(W, S)$ has an infinite number of chambers which tessellate the Euclidean space of dimension $|S| - 1$;

(iii) if (W, S) is non-finite and non-Euclidean, by the Davis geometric realization, $\Sigma(W, S)$ can be viewed as a complete CAT(0) geodesic space.

We are now ready to introduce the definition of a building:

Definition 1.1.4. A **building** is a simplicial complex Δ that can be expressed as the union of sub-complexes Σ (called apartments) satisfying the following axioms:

(B1) Each apartment Σ is a Coxeter complex;

(B2) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them;

(B3) If Σ, Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

A building is called **thick** if every codimension 1 simplex is a face of at least three chambers. It is called **thin** if the building is only a Coxeter complex (i.e., one apartment).

We emphasize that throughout this thesis all buildings that are considered are endowed with the maximal system of apartments.

Notice that we can take A and B in (B3) above to be the empty simplex; hence any two apartments are isomorphic. In consequence, all apartments Σ of Δ are of the same type (W, S). If the Coxeter system (W, S) is moreover irreducible, the building Δ is called either spherical, or Euclidean, or non-spherical and non-Euclidean, accordingly to the type of (W, S).

Example 1.1.5. The most simple example of a building is the infinite tree (i.e., every vertex having valence ≥ 2). This is in fact a Euclidean building of dimension 1. The apartments are the bi-infinite lines of the tree and the chambers are represented by the edges.

Let us introduce now several other definitions and results which are specific to Euclidean Coxeter complexes $\Sigma(W, S)$ and respectively, Euclidean buildings. These are stated without proof, the intention being to give only the geometric intuition behind them.

Denote by $\mathcal{H}$ the collection of all hyperplanes in $\Sigma(W, S)$ corresponding to the reflections of W; they are also called the walls of $\Sigma(W, S)$. Two walls in $\mathcal{H}$ are in the same equivalence class if they are parallel. It can be shown that $\mathcal{H}$ contains only a finite number of equivalence classes of parallel walls (see [Gar97, Sec. 16.1]). Moreover, if we fix a chamber C in $\Sigma(W, S)$, it can be shown that there exists at least one vertex x_0 of C such that every hyperplane in $\mathcal{H}$ is parallel to a wall of $\Sigma(W, S)$ that passes through the vertex x_0. Such a vertex x_0 is called a **special vertex** of C. Let $\overline{\mathcal{H}}$ be the collection of all hyperplanes in $\mathcal{H}$ passing through x_0. The set $\overline{\mathcal{H}}$ is finite and its elements are in bijection with the equivalence classes of parallel walls in $\mathcal{H}$. Further, it can be shown that the hyperplanes in $\overline{\mathcal{H}}$ cut the Coxeter system $\Sigma(W, S)$ (in fact the corresponding Euclidean space of dimension $|S| - 1$) into a finite number of simplicial cones Q which are based in x_0; those are called the **sectors of $\mathbf{\Sigma(W, S)}$ based at $\mathbf{x_0}$**. Two sectors based at x_0 are **opposite** if they have only the point x_0 in common and they are formed by the same set of hyperplanes of $\overline{\mathcal{H}}$. Notice that the set of sectors based at x_0 tessellate the boundary at infinity of $\Sigma(W, S)$, this boundary being isometric with the Euclidean sphere of dimension $|S| - 2$.

Denote the boundary at infinity of $\Sigma(W, S)$ by $\partial\Sigma(W, S)$. In addition, it can be proved that $\partial\Sigma(W, S)$, with the tessellation given by the sectors of $\Sigma(W, S)$ based at x_0, is a spherical Coxeter complex, which moreover, does not depend on the chosen special vertex x_0. Indeed, if we consider two special vertices $x_0, y_0 \in \Sigma(W, S)$ and two sectors Q_1, Q_2 based respectively, at x_0 and y_0, with the property that every wall that forms Q_1 is parallel to a wall of Q_2, then $Q_1 \cap Q_2 \subset \Sigma(W, S)$ is either a convex (maybe empty) subset of finite Euclidean volume or a convex subset of infinite Euclidean volume. In the former case, Q_1 and Q_2 are called **opposite**, this being the definition of two opposite sectors that are based at different special vertices. In the latter case, $Q_1 \cap Q_2$ is in fact a simplicial cone of $\Sigma(W, S)$ which is 'parallel' to Q_1 and Q_2; therefore we call Q_1 and Q_2 **parallel**. Notice that parallel sectors (which are not necessarily based in the same special vertex) form an equivalence class and $\Sigma(W, S)$ contains only a finite number of these equivalence classes of parallel sectors. Moreover, for two special vertices x_0 and y_0 in $\Sigma(W, S)$, the two corresponding tessellations of $\partial\Sigma(W, S)$ with respect to the set of sectors based at x_0 and respectively, at y_0, are isometric; given thus the same spherical Coxeter complex at infinity of $\Sigma(W, S)$. Our claim follows.

Therefore, by definition we call $\partial\Sigma(W, S)$, with the tessellation described above, the **spherical apartment at infinity** of the apartment $\Sigma(W, S)$; the chambers of $\partial\Sigma(W, S)$ correspond to the equivalence classes of the parallel sectors of $\Sigma(W, S)$.

More generally, if we consider Δ to be a Euclidean building of type (W, S), by the procedure explained above applied to each apartment of Δ, it can be proved that one can associate to Δ the so-called **spherical building at infinity**; this is denoted by $\partial\Delta$ and its apartments, and respectively, its chambers gain the preposition 'at infinity of Δ'. Moreover, $\partial\Delta$ is well defined and it is in fact a genuine spherical building of dimension $|S| - 2$.

Let us now turn our discussion towards groups that act on a building Δ. We denote by $\mathrm{Aut}(\Delta)$ the group of all isometries (automorphisms) of Δ. If moreover, Δ is a locally finite building, $\mathrm{Aut}(\Delta)$ is endowed with the **compact-open topology**; this topology turns $\mathrm{Aut}(\Delta)$ into a totally disconnected locally compact group (e.g., the pointwise stabilizer in $\mathrm{Aut}(\Delta)$ of a finite number of vertices of Δ is compact and open). We mention here that $\mathrm{Aut}(\Delta)$, where Δ is a building, contains only two types of elements. More precisely, an element $g \in \mathrm{Aut}(\Delta)$ is either elliptic, i.e., fixes a least one point of Δ, or it is hyperbolic, i.e., g admits no fixed point in Δ and there exists a subset of Δ where g acts as a translation.

Example 1.1.6. The groups $\mathrm{SL}(2, \mathbf{Q}_p)$ and respectively, $\mathrm{SL}(n, \mathbf{Q}_p)$ are closed subgroup of $\mathrm{Aut}(T)$ and respectively, of $\mathrm{Aut}(\Delta)$, where T and Δ are the spaces of lattices constructed in the beginning of this section. Recall that T is in fact a (regular) tree. Both T and Δ are examples of locally finite Euclidean buildings.

Most of the groups considered in this thesis and acting on buildings enjoy the following property:

Definition 1.1.7. Let Δ be a building (e.g., a spherical building or a Euclidean building) and $G \leq \mathrm{Aut}(\Delta)$. We say that the G-action is **strongly transitive** if for any two pairs (A_1, c_1) and (A_2, c_2) consisting of an apartment A_i and a chamber $c_i \in \mathrm{Ch}(A_i)$, there exists $g \in G$ such that $g(A_1) = A_2$ and $g(c_1) = c_2$. Moreover, since buildings are colorable (i.e., the vertices of any chamber are colored differently and any chamber uses the same set of colors), we say that $g \in \mathrm{Aut}(\Delta)$ is **type-preserving** if g preserves the coloration of the building. We say that $G \leq \mathrm{Aut}(\Delta)$ is **type-preserving** if all elements of G are type-preserving.

For example, $\mathrm{SL}(n, \mathbf{Q}_p)$ acts strongly transitively on its corresponding locally compact Euclidean building.

Remark 1.1.8. Let Δ be a building of type (W, S) and G be a subgroup of $\mathrm{Aut}(\Delta)$ that acts strongly transitively. Let $\mathcal{A}$ be an apartment of Δ and consider its stabilizer subgroup $\mathrm{Stab}_G(\mathcal{A}) \leq G$ (e.g., an element $g \in \mathrm{Stab}_G(\mathcal{A})$ stabilizes the apartment $\mathcal{A}$ but not necessarily g fixes $\mathcal{A}$ pointwise). Denote by $\mathrm{Fix}_G(\mathcal{A}) \leq \mathrm{Stab}_G(\mathcal{A})$ the pointwise stabilizer of $\mathcal{A}$; its restriction to $\mathcal{A}$ is a normal subgroup of $\mathrm{Stab}_G(\mathcal{A})|_{\mathcal{A}}$. By the strong transitivity of G, we have that $(\mathrm{Stab}_G(\mathcal{A})/\mathrm{Fix}_G(\mathcal{A}))|_{\mathcal{A}} = W$. Moreover, W is the so-called **Weyl group** of G.

We end this introduction by recalling the Classification Theorem of Tits for the locally finite thick Euclidean buildings.

Theorem 1.1.9. *(See* [Ron89, Corollary 10.25]*) Every locally finite thick Euclidean building Δ that is of dimension $n \geq 3$ arises from simple algebraic groups of rank n over locally compact non Archimedean fields. Any isomorphism of two such buildings is induced by an isomorphism between the associated algebraic groups.*

For a detailed presentation of the complete proof of the classification of the Bruhat–Tits buildings (i.e., Euclidean buildings of dimension ≥ 2 whose corresponding buildings at infinity satisfy the Moufang property) we recommend the book of R. Weiss [Wei08]. We mention that every Euclidean building of dimension ≥ 3 is a Bruhat–Tits building.

1.2 The boundary at infinity and the cone topology of a $\mathrm{CAT}(0)$ space

This section is meant to recall the basic definitions and to give suggestive examples of the so-called **cone topology** to which a Euclidean building (or more generally, a CAT(0) space) is endowed; this topology is intrinsically used in most of the results presented in this PhD thesis.

Instead of giving first the definitions, we start with the most simple example: the particular case of a locally finite tree T (i.e., the degree of every vertex is finite and is ≥ 2).

Recall that T is a geodesic metric space, where every edge is considered to have length 1 and any two vertices of the tree are connected by a unique geodesic, namely, the path formed by a finite number of edges of the tree and which links the two chosen vertices. In addition, one notices that the open balls $B(x, n)$, where $x \in T$ and $n \in \mathbf{N}$, form a base of open sets for the underlying topology of the geodesic metric space T. Moreover, by the definition, T is an infinite space. Is there thus a method to 'transform' it into a compact space? For example, if one takes the degree of every vertex to be 2, then our tree T is the real line $\mathbf{R}$, where the vertices are considered to be the elements of $\mathbf{Z}$. In this case we know that $\mathbf{R} \cup \{-\infty, \infty\}$ is a compactification of $\mathbf{R}$, the set $\{-\infty, \infty\}$ represents its boundary at infinity and the corresponding open neighborhoods of this boundary are given by $\mathbf{R} \setminus \overline{B(x, r)}$, with $x \in \mathbf{R}$ and $r > 0$.

For a general locally finite tree T we can do the same. First, one needs to define the boundary at infinity. Indeed, let us fix a base vertex $x_0 \in T$ and starting from there we walk on the edges of the tree T in such a way that any edge we choose to step on is different from the previous one on which we have just finished walking. Because we are living in a tree, which is a graph without circuits, by walking in the way presented above, we cannot come back to x_0. Furthermore, if we decide not to stop, we will reach the 'infinity'. Notice that this 'infinity' is not unique and really depends on our choice made for each edge that we walked on. To conclude, **a point at infinity** of T is defined to be the 'end' of an infinite geodesic path started at the base point x_0; this path uniquely determines the end. The set of all boundary points of T is denoted by ∂T and it is called **the boundary at infinity** of T. If we look carefully, we notice that, in fact, an endpoint ξ in ∂T does not depend on the base point x_0 chosen, as any other vertex of the tree T is connected by a unique geodesic segment to the geodesic line with base point x_0 and which defines the corresponding endpoint ξ in ∂T. Therefore, the definition of ∂T is independent of the base point $x_0 \in T$ chosen.

As in the case of $\mathbf{R}$, we also want to define a topology on $T \cup \partial T$ such that the union $T \cup \partial T$ becomes a compactification of T and such that this new topology preserves the canonical metric topology on T. Therefore, consider a ball $B(x, n) \subset T$, where x is a vertex in T and $n \in \mathbf{N}$. Then take $T \setminus B(x, n)$; the remaining space is now disconnected and each of its connected components is in fact a locally finite subtree of T. Intuitively, these connected components represent 'open neighborhoods' of the points in ∂T. The definition is formalized as follows.

Definition 1.2.1. Let T be a locally finite tree and let x_0 be a vertex in T. Consider $\xi \in \partial T$ and $n \in \mathbf{N}$. Define the following subset in $T \cup \partial T$:

$$\begin{aligned} U_{x_0,n,\xi} :=\{ & y \in T \cup \partial T \mid [x_0, y) \cap [x_0, \xi) \text{ is a geodesic segment} \\ & \text{of length at least } n\}, \end{aligned}$$

where we denote by $[x_0, y)$ the geodesic segment (or the infinite geodesic line if $y \in \partial T$) joining the points x_0 and y. The set $U_{x_0,n,\xi}$ is called a **standard (open) neighborhood** of the point ξ with base point x_0, where $n > 0$ is given.

Remark 1.2.2. By the Definition 1.2.1, $U_{x_0,n,\xi} = U_{x_0,n,\eta}$, for every $\eta \in U_{x_0,n,\xi}$. Moreover, for any $m \geq n \in \mathbf{N}$ and $\xi_1, \xi_2 \in \partial T$, we have either $U_{x_0,m,\xi_1} \subset U_{x_0,n,\xi_2}$, if $[x_0, \xi_1) \cap [x_0, \xi_2)$ is a geodesic segment of length at least n, or $U_{x_0,m,\xi_1} \cap U_{x_0,n,\xi_2} = \emptyset$, in any other case.

Definition 1.2.3. Let T be a locally finite tree and let x_0 be a vertex of T. The **cone topology** $\mathcal{T}(x_0)$ on $T \cup \partial T$, with base point x_0, is defined to be the topology generated by the open balls $B(x, n)$ together with the standard open neighborhoods $U_{x_0,n,\xi}$, where $x \in T$, $\xi \in \partial T$ and $n \in \mathbf{N}$.

Regarding the base point x_0, one can prove that if we consider $y \neq x_0 \in T$, the cone topologies $\mathcal{T}(x_0)$ and $\mathcal{T}(y)$ defined on $T \cup \partial T$ are the same. This is even true in the more general setting of CAT(0) spaces (see [BH99, Chap.II.8]). For our case of locally finite trees, this can be easily checked and therefore it is left as an exercise:

Proposition 1.2.4. *Let T be a (not necessarily a locally finite) tree and let $x, y \in T$. Then the cone topologies $\mathcal{T}(x)$ and $\mathcal{T}(y)$ are the same.*

It remains to prove that the cone topology defined above turns the space $T \cup \partial T$ into a compact one:

Proposition 1.2.5. *Let T be a locally finite tree and let $x_0 \in T$. Then $T \cup \partial T$ is a compact space with respect to the cone topology $\mathcal{T}(x_0)$.*

Proof. To show this, it is enough to extract a finite cover from any collection of open set $\{U_i\}_{i \in I}$, such that I is infinite, $\bigcup\limits_{i \in I} U_i = T \cup \partial T$ and where U_i is either a ball $B(x, n)$ or a standard open neighborhood $U_{x_0,n,\xi}$, where $x \in T$, $\xi \in \partial T$, $n \in \mathbf{N}$ and $i \in I$. First of all, suppose that there exist infinite sequences $\{\xi_j\}_{j \in J \subset I}$ and $\{n_j\}_{j \in J}$ with the property that $U_j = U_{x_0,n_j,\xi_j}$, $\{U_j\}_{j \in J}$ covers ∂T and $U_{j_1} \cap U_{j_2} = \emptyset$, for every $j_1 \neq j_2 \in J$. Notice that, by Remark 1.2.2, either U_{j_1} and U_{j_2} are disjoint or, if they intersect, one is included into the second one. We claim that our supposition made above cannot be possible. Indeed, recall that our tree T is locally finite; therefore every ball $B(x_0, n)$ contains a finite set of vertices of T and, in particular, the set of all geodesic segments of length n starting with the vertex x_0 is finite. Moreover, the sequence $\{n_j\}_{j \in J}$ cannot be bounded above by a constant, as otherwise the sets $\{U_{x_0,n_j,\xi_j}\}_{j \in J}$ would not be pairwise disjoint anymore. Therefore, for every $n > 0$ there exists a geodesic segment $[x_0, x_n] \subset T$ of length n such the end vertex x_n is not contained in any of the sets from $\{U_{x_0,n_j,\xi_j}\}_{j \in J}$; this means that the same is true for any vertex of the segment (x_0, x_n). Consequently, as $n \to \infty$, we obtain a infinite geodesic line $[x_0, \xi)$ whose end ξ is not cover by any of the sets from $\{U_{x_0,n_j,\xi_j}\}_{j \in J}$. This is a contradiction with our assumption and the claim stands proven.

As a consequence, from the sequence $\{U_i\}_{i \in I}$ we can extract a finite one, say $\{U_1, ..., U_m\}$, that covers the boundary ∂T. Then, by covering the remaining finite set $T \setminus (U_1 \cup ... \cup U_m)$, we obtain our conclusion. Therefore, $T \cup \partial T$ is indeed compact with respect to the cone topology $\mathcal{T}(x_0)$. Notice that, if the tree T is not locally finite then $T \cup \partial T$ is not a compact space with respect to the cone topology. $\square$

Let us now recall the definition of a CAT(0) space and give the basic results regarding its boundary at infinity and its cone topology. All these definitions and basic facts are taken from the book [BH99]. As the aim of this section is just to give the intuition behind this theory, we do not intend to include the proofs in the general case of a CAT(0) space; the interested reader can consult thus [BH99, Chap.II.8].

Definition 1.2.6. (See [BH99, Chap.II.1, Def. 1.1]) Let X be a geodesic metric space. By definition, a **geodesic triangle** Δ consists of three points $p, q, r \in X$, which are its vertices, and a choice of three geodesic segments $[p, q], [q, r], [r, p]$ in X joining them, which are the sides of the triangle Δ. In general the geodesic segment joining two points in X is not unique.

Let Δ be a geodesic triangle and consider its comparison triangle $\overline{\Delta}$ in the 2-dimensional Euclidean space E^2. We say that Δ satisfies the CAT(0) **inequality** if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$ we have that $\operatorname{dist}_X(x, y) \leq \operatorname{dist}_{E^2}(\overline{x}, \overline{y})$. We say that X is a CAT(0) space if the CAT(0) inequality holds for any geodesic triangle.

Example 1.2.7. Examples of CAT(0) spaces, which are moreover complete metric spaces, are the n-dimensional Euclidean and Hyperbolic spaces, the Euclidean buildings as well as the general buildings.

Notice that, by the CAT(0) inequality, a CAT(0) space is uniquely geodesic.

Definition 1.2.8. (See [BH99, Chap.II.8, Def. 8.1]) Let X be a metric space. Two geodesic rays $c, c' : [0, \infty) \to X$ are said to be **asymptotic** if there exists a constant K such that $d(c(t), c'(t)) \leq K$ for all $t \geq 0$. The set ∂X of **boundary points** of X (which we call the **points at infinity**) is the set of equivalence classes of geodesic rays: two geodesic rays are equivalent if and only if they are asymptotic. A typical point of ∂X is often denoted by ξ.

Example 1.2.9. For the examples of CAT(0) spaces presented above, the boundary at infinity of the n-dimensional Euclidean space or of the n-dimensional Hyperbolic space is the usual sphere S^{n-1}. For a Euclidean building the boundary at infinity is the corresponding spherical building at infinity.

Proposition 1.2.10. *(See* [BH99, Chap.II.8, Def. 8.2]*) Let X be a complete* CAT(0) *space and let $c : [0, \infty) \to X$ be a geodesic ray issuing from the point $x \in X$. Then for every geodesic ray $x' \in X$, there is a unique geodesic ray c' which issues from x' and is asymptotic to c.*

Let us now define a standard neighborhood of a point $\xi \in \partial X$. Notice that the idea is the same as in the case of trees. However, we have to use a third parameter in order to define a standard neighborhood of a point $\xi \in X$; this is because CAT(0) spaces can contain Euclidean subspaces.

Definition 1.2.11. (See [BH99, Chap.II.8]) Let X be a complete CAT(0) space and let x_0 be a point in X. Consider $\xi \in \partial X$ and let c be the geodesic ray issuing from x_0 and corresponding to ξ. Let $r, \epsilon > 0$. Define the following subset of $X \cup \partial X$:

$$U_{x_0,r,\epsilon,\xi} := \{y \in X \cup \partial X \mid \operatorname{dist}_X(x_0, y) > r, \operatorname{dist}_X(p_r(y), c(r)) < \epsilon\},$$

where $p_r(y)$ denotes the unique point on the geodesic from x_0 to y at distance r from x_0. The set $U_{x_0,r,\epsilon,\xi}$ is called a **standard (open) neighborhood** of the point ξ with base point x_0, where $r, \epsilon > 0$ are given.

Definition 1.2.12. (See [BH99, Chap.II.8, Def. 8.6 and Ex. 8.7]) Let X be a complete CAT(0) space and let x_0 be a point of X. The **cone topology** $\mathcal{T}(x_0)$ on $X \cup \partial X$, with base point x_0, is defined to be the topology generated by the open balls $B(x, r) \subset X$ together with the standard open neighborhoods $U_{x_0,r,\epsilon,\xi}$, where $x \in X$, $\xi \in \partial X$ and $r, \epsilon > 0$.

As in the particular case of trees, for a complete CAT(0) space we have the same basic properties of the corresponding cone topology.

Proposition 1.2.13. *(See* [BH99, Chap.II.8, Prop. 8.8]*) Let X be a complete* CAT(0) *space and let $x_0, y_0 \in X$. Then the cone topologies $\mathcal{T}(x_0)$ and $\mathcal{T}(y_0)$ are the same. Moreover, if X is proper then $X \cup \partial X$ is compact with respect to the cone topology.*

Let us recall a final basic definition, that of an Alexandrov angle and which is used further in this thesis.

Definition 1.2.14. (See [BH99, Def. 1.12]) Let X be a metric space and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, we consider the comparison triangle $\overline{\Delta}(c(0), c(t), c'(t'))$ and the comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$. **The Alexandrov angle** between the geodesic paths c and c' is the number $\angle_{c,c'} \in [0, \pi]$ defined by:

$$\angle_{c,c'} := \limsup_{t,t' \to 0} \overline{\angle}_{c(0)}(c(t), c'(t')).$$

If the limit $\lim_{t,t' \to 0} \overline{\angle}_{c(0)}(c(t), c'(t'))$ exists, then we say that the angle exists in the strict sense. When X is a CAT(0) space, one can prove that the Alexandrov angle always exists.

For example, in the Euclidean space, the Alexandrov angle is equal to the usual Euclidean angle. Moreover, the angle between the incoming and outgoing germs of a geodesic at any interior point is π, or in the particular case of a tree, the angle between two geodesic segments which have a common endpoint is either 0 or π.

Chapter 2

The Howe–Moore property

2.1 A unified proof of the Howe–Moore property

2.1.1 Introduction

For a locally compact group G, the Howe–Moore property asserts that all matrix coefficients of its unitary representations, that are without non-zero G-invariant vectors, vanish at infinity. This property was first established by Howe and Moore (see [HM79]) and Zimmer (see [Zim84]), around 1977, for connected, non-compact, simple real Lie groups that are with finite center. It plays an important role in the Mostow rigidity theorem, as well as in various other rigidity results. The original proof of the Howe–Moore property uses the geometry of semi-simple real Lie groups. Nowadays the proof is more algebraic (see for example [BM00] or [WM, Theorem 10.14] for the case of $\mathrm{SL}(2, \mathbb{R})$).

In the same article, Howe and Moore also treat the case of isotropic simple algebraic groups over non Archimedean local fields (see [HM79, Theorem 5.1]). By [BT72], to such a group one associates a locally finite thick Euclidean building Δ where the group acts continuously and properly, by type-preserving automorphisms and strongly transitively (see Def. 1.1.7). For other totally disconnected analogs of semi-simple real Lie groups, namely, closed, strongly transitive subgroups of $\mathrm{Aut}(\Delta)$, where Δ is a locally finite thick Euclidean building, the study of the Howe–Moore property was initiated by Lubotzky and Mozes (see [LM92]) in the particular case of Δ being a d-regular tree T_d, with $d \geq 3$. Using the geometry of horocycles and horoballs, they prove that the index-two subgroup of $\mathrm{Aut}(T_d)$, which preserves the 2-coloring of the tree T_d, enjoys this property. A more general result, in the context of d-regular trees, was obtained by Burger and Mozes in [BM00a]. There they show that every closed, topologically simple subgroup of $\mathrm{Aut}(T_d)$ which is 2-transitive on the boundary has the Howe–Moore property. We mention that the latter class of subgroups of $\mathrm{Aut}(T_d)$ contains also non linear examples.

Regarding all the examples presented above, this section proposes to give a unified proof of the Howe–Moore property:

Theorem 2.1.1. *(See Theorem 2.1.33 and Section 2.1.4) Let G be a connected, non-compact, simple real Lie group, with finite center, or an isotropic simple algebraic group over a non Archimedean local field, or a closed, topologically simple subgroup of* $\mathrm{Aut}(T)$ *that acts* 2*-transitively on the boundary ∂T, where T is a bi-regular tree with valence ≥ 3 at every vertex. Then G admits the Howe–Moore property.*

The main ingredients of the unified proof, are:

i) the $K_1A^+K_2$ **decomposition** of these groups, where K_1, K_2 are compact subsets and A^+ is the 'maximal abelian' sub semi-group part (see Section 2.1.4);

ii) the **'contraction'** subgroups $U_\alpha^{\pm}$ (see Definition 2.1.13) with respect to hyperbolic elements, combined with basic facts regarding normal operators (see Sections 2.1.2, 2.1.2).

More precisely, the above mentioned ingredients are assembled to obtain the following criterion, used to verify the Howe–Moore property:

Theorem 2.1.2. *Let G be a second countable, locally compact group having the following properties:*

i) G admits a decomposition $K_1A^+K_2$, where $K_1, K_2 \subset G$ are compact subsets and A^+ is an abelian sub semi-group of G;

ii) for all sequences $\alpha = \{g_n\}_{n>0} \subset A^+$, with $g_n \to \infty$, we have $G = \overline{\langle U_\alpha^+, U_\alpha^- \rangle}$.

Then G has the Howe–Moore property.

The idea behind this criterion is inspired by a proof given for $\mathrm{SL}(2,\mathbb{R})$ by Dave Witte Morris (see [WM, Theorem 10.14]).

To the best of our knowledge, there are no other known examples of locally compact or even discrete groups enjoying the Howe–Moore property, beside those covered by the unified proof provided by this section. Some necessary algebraic conditions for a locally compact group admitting the Howe–Moore property are described in [CdCL+11]. Those conditions are quite elementary, but probably far from being sufficient. Even for the two classes of totally disconnected locally compact groups of Theorem 2.1.1, we still do not know to what extent the Howe–Moore property relates with the KA^+K decomposition and with the strong transitivity on the corresponding Euclidean building; in this direction, only partial results have been obtained so far.

2.1.2 Preliminaries

Basic definitions

In what follows all Hilbert spaces are assumed to be complex, not necessarily separable, and their inner product is denoted by $\langle \cdot, \cdot \rangle$. Let $B(\mathcal{H})$ be the space of all bounded linear operators $F : \mathcal{H} \to \mathcal{H}$.

Recall that the closed unit ball of a Hilbert space $\mathcal{H}$ is compact with respect to the weak topology and moreover, by the Theorem of Eberlein–Šmulian, compactness is equivalent to sequential compactness. Hence, any sequence of vectors $\{v_n\}_{n\geq 0} \subset \mathcal{H}$, with their norms being bounded above, contains a **weakly convergent** subsequence $\{v_{n_k}\}_{k>0}$ in $\mathcal{H}$. Namely, there exists a vector $v \in \mathcal{H}$ such that for any $w \in \mathcal{H}$ one has that $\lim\limits_{n_k \to \infty} \langle v_{n_k}, w \rangle = \langle v, w \rangle$.

In the same spirit, if $\mathcal{H}$ is a separable Hilbert space, it is well-known that the closed unit ball in $B(\mathcal{H})$ is compact and metrizable with respect to the weak operator topology. Thus, any sequence of operators $\{F_n\}_{n\geq 0} \subset B(\mathcal{H})$, with their operator norms being bounded above, contains a **weakly convergent** subsequence $\{F_{n_k}\}_{k>0}$ in $B(\mathcal{H})$. Namely, there exists F in $B(\mathcal{H})$ such that for all $v, w \in \mathcal{H}$ one has that $\lim\limits_{n_k \to \infty} \langle F_{n_k} v, w \rangle = \langle F v, w \rangle$

Definition 2.1.3. Let $\mathcal{H}$ be a Hilbert space. We say that $E \in B(\mathcal{H})$

is **normal** if $E^*E = EE^*$. It is well-known that this is equivalent to $\langle EE^*v, v\rangle = \langle E^*Ev, v\rangle$, for every $v \in \mathcal{H}$. Moreover, $U \in B(\mathcal{H})$ is called **unitary** if $UU^* = U^*U = I$ or, equivalently, if $\langle U\xi, U\eta\rangle = \langle \xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$ and U is onto. We denote by $\mathcal{U}(\mathcal{H})$ the group of all unitary operators of $\mathcal{H}$.

Definition 2.1.4. A **unitary representation** of a topological group G into a Hilbert space $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$, which is moreover strongly continuous: the map $g \in G \mapsto \pi(g)v \in \mathcal{H}$ is continuous for every $v \in \mathcal{H}$. Instead of $\pi : G \to \mathcal{U}(\mathcal{H})$ we often write $(\pi, \mathcal{H})$ for a unitary representation of G.

Furthermore, any two vectors $v, w \in \mathcal{H}$ define a continuous function $c_{v,w} : G \to \mathbb{C}$, given by $c_{v,w}(g) := \langle \pi(g)v, w\rangle$ and we call it the associated (v, w)**-matrix coefficient**. Moreover, we say that $c_{v,w}$ **is** $\mathrm{C}_{\mathbf{0}}$ if for every $\epsilon > 0$, the subset $\{g \in G \ : \ |c_{v,w}(g)| \geq \epsilon\}$ is compact in G.

Remark 2.1.5. (See [BdlHV08, Proposition C.4.9]) We mention that for a unitary representation $(\pi, \mathcal{H})$ of a topological group G, the Hilbert space $\mathcal{H}$ can be decomposed as a direct sum $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ of mutually orthogonal, closed and G-invariant subspaces $\mathcal{H}_i$, such that the restriction of π to $\mathcal{H}_i$ is cyclic for every i (i.e., for every i there exists a non-zero vector $v \in \mathcal{H}_i$ such that $\pi(G)v$ is dense in $\mathcal{H}_i$).

Moreover, the following lemma asserts that for 'nice' topological groups, only separable Hilbert spaces can be considered when working with unitary representations.

Lemma 2.1.6. *Let G be a separable locally compact group. Then for every cyclic unitary representation $(\pi, \mathcal{H})$ of G, the corresponding Hilbert space $\mathcal{H}$ is separable. In particular, all irreducible unitary representations of G are over separable Hilbert spaces.*

Proof. Let $(\pi, \mathcal{H})$ be a cyclic unitary representation of G and denote by v a cyclic vector of it. Then, we have that $f : G \to \mathcal{H}$, defined by $f(g) := \pi(g)v$, is a continuous function. Moreover, by the definition of a cyclic vector, we know that the linear span of $\pi(G)v$ is dense in $\mathcal{H}$.

As G is separable, let Q be a countable dense subset of it. Thus $f(Q)$ is dense in $\pi(G)v$ and therefore the linear span of $\pi(Q)v$ is also dense in $\mathcal{H}$. The conclusion thus follows. $\square$

Remark 2.1.7. The class of separable locally compact groups contains non-compact, simple real Lie groups with finite center and also closed, non-compact subgroups of $\mathrm{Aut}(\Delta)$, where Δ is a locally finite thick Euclidean building. In particular, all non-compact, simple p-adic Lie groups are separable.

The Howe–Moore property

Definition 2.1.8. Let G be a locally compact group and let $G \bigcup \{\infty\}$ be the one point compactification of G. We say that G has the **Howe–Moore property** if the set of all unitary representations of G is the union of the ones having non-zero G-invariant vectors and the ones for which all matrix coefficients are C_0. When a matrix coefficient is C_0, we also say that it **vanishes at** ∞.

Throughout this section we assume that all locally compact groups are second countable; this is to simplify our notations, by using sequences instead of nets. The second reason is given by the next remark.

Remark 2.1.9. By [CdCL$^+$11, Prop. 2.3], to prove the Howe–Moore property for a second countable, locally compact group G it is enough to verify the C_0 condition only for all irreducible, non-trivial, unitary representations of G, which, by Lemma 2.1.6, are over separable Hilbert spaces.

Moreover, the following well-known lemma shows that matrix coefficients not vanishing at ∞ give rise to particular matrix coefficients with the same property.

Lemma 2.1.10. *Let G be a second countable, locally compact group and $(\pi, \mathcal{H})$ be a unitary representation of G. Suppose there exist two non-zero vectors $v, w \in \mathcal{H}$ such that the (v, w)-matrix coefficient does not vanish at ∞. Then the (v, v)-matrix coefficient does not vanish at ∞.*

Proof. By hypothesis, there exists a sequence $\{g_n\}_{n\geq 0} \subset G$ and $\epsilon > 0$ such that $g_n \to \infty$ and $|\langle \pi(g_n)v, w\rangle| \geq \epsilon$. Thus w is not orthogonal to the Hilbert subspace generated by $\pi(G)v$. Denote this Hilbert subspace by $\overline{\langle \pi(G)v\rangle}$ and remark this is G-invariant.

Moreover, $\{\pi(g_n)v\}_{n\geq 0}$ being bounded in the norm of $\mathcal{H}$, there exist $v_0 \in \mathcal{H}$ and a subsequence $\{n_k\}_{k\geq 0}$ such that $\{\pi(g_{n_k})v\}_{k\geq 0}$

weakly converges to v_0. Thus $\lim\limits_{n_k\to\infty}\langle\pi(g_{n_k})v, w'\rangle = \langle v_0, w'\rangle$, for every $w' \in \mathcal{H}$. From here we conclude that v_0 is a non-zero vector.

Furthermore, we claim that v_0 is a vector in $\overline{\langle\pi(G)v\rangle}$. Indeed, suppose the converse that $v_0 = w_1 + w_2$, with $w_1 \in \overline{\langle\pi(G)v\rangle}$ and w_2 a non-zero vector orthogonal to $\overline{\langle\pi(G)v\rangle}$. Taking $w' = w_2$, we obtain that $\lim\limits_{n_k\to\infty}\langle\pi(g_{n_k})v, w_2\rangle = 0 = \langle v_0, w_2\rangle$, which is impossible. The claim follows and we conclude that there exists $g \in G$ such that $\langle v_0, \pi(g)v\rangle \neq 0$. For this g we obtain that $\lim\limits_{n_k\to\infty}\langle\pi(g_{n_k})v, \pi(g)v\rangle = \langle v_0, \pi(g)v\rangle$. The conclusion follows by only remarking that $|\langle\pi(g^{-1}g_{n_k})v, v\rangle| \nrightarrow 0$.

$\square$

In addition, the following easy lemma asserts that compact subsets do not count for the Howe–Moore property, given a 'polar decomposition' of the locally compact group.

Lemma 2.1.11. *Let G be a second countable, locally compact group admitting a decomposition $G = K_1AK_2$, where K_1, K_2 are compact subset and A is any subset of G. Let $(\pi, \mathcal{H})$ be a unitary representation of G. If for every sequence $\{a_n\}_{n>0} \subset A$, with $\{a_n\}_{n>0} \to \infty$, the corresponding matrix coefficients are C_0, then all matrix coefficients of G, with respect to $(\pi, \mathcal{H})$, vanish at ∞.*

Proof. Suppose there exist a sequence $\{g_n\}_{n>0} \subset G$, with $g_n \to \infty$, and two non-zero vectors $v, w \in \mathcal{H}$ such that the corresponding matrix coefficient $c_{v,w}(g_n)$ is not C_0. As $G = K_1AK_2$ one has $g_n = k_na_nh_n$, where $k_n \in K_1, h_n \in K_2$ and $a_n \in A$, for every $n > 0$. Thus, we have $a_n \to \infty$. As K_1, K_2 are compact sets and by passing to a subsequence, we can assume that $\{\pi(k_n)^{-1}w\}_{n>0}$ and $\{\pi(h_n)v\}_{n>0}$ converge, in the norm of $\mathcal{H}$, to the vectors w_0, respectively v_0. One has:

$$\begin{aligned}
|\langle\pi(k_na_nh_n)v, w\rangle| &= \left|\left\langle\pi(a_nh_n)v, \pi(k_n)^{-1}w\right\rangle - \langle\pi(a_nh_n)v, w_0\rangle\right. \\
&\quad + \left\langle\pi(h_n)v, \pi(a_n)^{-1}w_0\right\rangle \\
&\quad \left. - \left\langle v_0, \pi(a_n)^{-1}w_0\right\rangle + \langle\pi(a_n)v_0, w_0\rangle\right| \\
&\leq \|\pi(k_n)^{-1}w - w_0\| \cdot \|v\| + \|(h_n)v - v_0\| \cdot \|w_0\| \\
&\quad + |\langle\pi(a_n)v_0, w_0\rangle|.
\end{aligned}$$

As the first two terms tend to zero as $n \to \infty$, we have that the matrix coefficient $c_{v_0,w_0}(a_n)$ is not C_0. We obtain a contradiction and the conclusion follows.

$\square$

Let us add here some necessary algebraic conditions for a non-compact locally compact group to admit the Howe–Moore property; however, those conditions are far from being sufficient.

Proposition 2.1.12. *(See Prop. 3.2 in* [CdCL+11]*) Let G be a non-compact locally compact group with the Howe–Moore property. Then any proper open subgroup of G is compact and thus of infinite index in G.*

Mautner's phenomenon

One of the main ideas to verify the Howe–Moore property is to search for 'big' subgroups $H \leq G$ admitting at least one non-zero H-invariant vector in $\mathcal{H}$. The Mautner's phenomenon gives us a method to construct such subgroups. First, let us introduce the following:

Definition 2.1.13. Let G be a locally compact group. Let $\alpha = \{g_n\}_{n>0}$ be a sequence in G. This sequence α defines the set:

$$U_\alpha^+ := \{g \in G \mid \lim_{n\to\infty} g_n^{-1} g g_n = e\}.$$

Notice that U_α^+ is a subgroup of G. It is called **the contraction group** corresponding to α. Because it doesn't need to be closed in general, we denote by N_α^+ the closed subgroup $\overline{U_\alpha^+}$. In the same way, but using $g_n g g_n^{-1}$ we define U_α^- and N_α^-. When $\alpha = \{a^n\}_{n>0}$, for some $a \in G$, we simply use the notations $U_a^\pm := U_\alpha^\pm$ and $N_a^\pm := N_\alpha^\pm$.

We record the following lemma, known as Mautner's phenomenon (see [BM00, Chp. III, Thm. 1.4]):

Lemma 2.1.14. *Let G be a second countable, locally compact group and $(\pi, \mathcal{H})$ be a unitary representation. Let $\{g_n\}_{n>0}$ be a sequence in G and take $v \in \mathcal{H}$ a non-zero vector. Suppose moreover that $\{\pi(g_n)v\}_{n>0}$ weakly converges to v_0.*

Then v_0 is N_α^+-invariant.

Proof. Let $w \in \mathcal{H}$ be an arbitrary vector and take $g \in U_\alpha^+$. We have:

$$\begin{aligned} |\langle \pi(g)v_0 - v_0, w\rangle| &= \lim_{n\to\infty} |\langle \pi(gg_n)v - \pi(g_n)v, w\rangle| \\ &= \lim_{n\to\infty} |\langle \pi(g_n^{-1}gg_n)v - v, \pi(g_n^{-1})w\rangle| \\ &\le \lim_{n\to\infty} \|(\pi(g_n^{-1}gg_n) - Id_{\mathcal{H}})v\| \cdot \|w\| = 0. \end{aligned}$$

Since U_α^+ is dense in N_α^+ and π is continuous, it follows that N_α^+ fixes v_0. □

Normal operators as weak limits

This elementary lemma is a key step used in Section 2.1.3.

Lemma 2.1.15. *Let $\mathcal{H}$ be a separable Hilbert space and consider a sequence $\{U_n\}_{n>0} \subset \mathcal{U}(\mathcal{H})$ of pairwise commuting unitary operators.*

Then there exists a subsequence $\{U_{n_k}\}_{n_k}$ which weakly converges to an operator $E \in B(\mathcal{H})$. Moreover, any such weak limit is normal.

First proof. As $\{U_n\}_{n>0}$ are unitary operators, they have norm one in $B(\mathcal{H})$. Hence, there is a subsequence $\{U_{n_k}\}_{n_k}$ which weakly converges to an operator $E \in B(\mathcal{H})$. It remains to prove that E is normal. In what follows we replace the subsequence $\{U_{n_k}\}_{n_k}$ by the sequence $\{U_n\}_{n>0}$. Notice that, as the sequence $\{U_n\}_{n>0}$ converges weakly to E, the sequence $\{U_n^*\}_{n>0}$ converges weakly to E^*.

By hypothesis, the unitary operators $\{U_n\}_{n>0}$ commute pairwise, thus U_n and U_k^* commute for all $n, k > 0$. Therefore for all $v, w \in \mathcal{H}$ one has:

$$\begin{aligned} \langle EE^*v, w\rangle &= \lim_{n\to\infty} \langle U_n(E^*v), w\rangle = \lim_{n\to\infty} \langle E^*v, U_n^*w\rangle \\ &= \lim_{n\to\infty} \big(\lim_{k\to\infty} \langle U_k^*v, U_n^*w\rangle\big) = \lim_{n\to\infty} \big(\lim_{k\to\infty} \langle U_nU_k^*v, w\rangle\big) \\ &= \lim_{n\to\infty} \big(\lim_{k\to\infty} \langle U_k^*U_nv, w\rangle\big) = \lim_{n\to\infty} \langle U_nv, Ew\rangle = \langle Ev, Ew\rangle \\ &= \langle E^*Ev, w\rangle. \end{aligned}$$

□

Second proof. Another way to see this is using von Neumann algebras. More precisely, as the unitary operators $\{U_n\}_{n>0}$ commute pairwise, they generate a commutative von Neumann algebra which is closed, by definition, in the weak topology. Thus the weak limit E is also contained in this von Neumann algebra. We conclude that E is normal as operator. $\square$

The pairwise commutativity condition in Lemma 2.1.15 is essential to obtain a normal weak limit. The following is an example in that sense.

Example 2.1.16. Consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{Z}) \bigoplus \ell^2(\mathbb{Z})$. For every $n > 0$, let $U_n := \begin{pmatrix} 0 & \lambda_n \\ 1 & 0 \end{pmatrix}$, where the sequence $\{\lambda_n\}_{n>0} \subset \mathbb{R}$ is such that $\lambda_n \to 0$. It is easy to see that $\{U_n\}_{n>0}$ is a sequence of unitary operators of $\mathcal{H}$ which do not commute pairwise and whose weak limit is the operator $E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Also the weak limit of $\{U_n^*\}_{n>0}$ is the operator $E^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. One can notice that $EE^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $E^*E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The conclusion follows.

2.1.3 An algebraic criterion

This section proposes a criterion, given by Theorem 2.1.18, to verify the Howe–Moore property, over separable Hilbert spaces, for locally compact groups admitting a special decomposition. Its proof uses the idea of Dave Witte Morris (see [WM, Theorem 10.14]), which relies on a more elaborate version of the Mautner's phenomenon; for further references this idea is recorded in a separate lemma.

Lemma 2.1.17. *Let G be a second countable, locally compact group and $(\pi, \mathcal{H})$ be a unitary representation of G, without G-invariant vectors. Assume there exists a sequence $\alpha = \{g_n\}_{n>0}$ of pairwise commuting elements of G such that $g_n \to \infty$ and $G = \overline{\langle U_\alpha^+, U_\alpha^- \rangle}$.*

Then all matrix coefficients of $(\pi, \mathcal{H})$ corresponding to $\{g_n\}_{n>0}$ are C_0.

Proof. As G is second countable and using Lemma 2.1.6, we can assume, without loss of generality, that the Hilbert space $\mathcal{H}$ is separable. In particular, by restricting to a subsequence, we can consider that the sequence $\{\pi(g_n)\}_{n>0}$ weakly converges to an operator E.

By contraposition, we assume there exist two non-zero vectors $v, w \in \mathcal{H}$ (considered fixed for what follows) such that $|\langle \pi(g_n)v, w\rangle| \nrightarrow 0$. We want to prove that $(\pi, \mathcal{H})$ has a non-zero G-invariant vector. Because $\{\pi(g_n)\}_{n>0}$ weakly converges to an operator E, by Lemma 2.1.15 we have moreover that E is normal and $E \neq 0$.

We now claim that the following important property of E holds: $\pi(u_+)E\pi(u_-) = E$, for every $u_+ \in U_\alpha^+$ and every $u_- \in U_\alpha^-$.

Proof of the claim.

Let $u_- \in U_\alpha^-$. For every $\phi, \psi \in \mathcal{H}$ one has:

$$\begin{aligned}
|\langle E\pi(u_-)\phi, \psi\rangle - \langle E\phi, \psi\rangle| &= |\lim_{n\to\infty} \langle \pi(g_n u_-)\phi - \pi(g_n)\phi, \psi\rangle| \\
&= |\lim_{n\to\infty} \left\langle \pi(g_n u_- g_n^{-1})\pi(g_n)\phi - \pi(g_n)\phi, \psi\right\rangle| \\
&= |\lim_{n\to\infty} \left\langle \pi(g_n)\phi, \pi(g_n u_-^{-1} g_n^{-1})\psi - \psi\right\rangle| \\
&\leq \lim_{n\to\infty} \|\pi(g_n)\phi\| \cdot \|(\pi(g_n u_-^{-1} g_n^{-1} - e)\psi\| = 0,
\end{aligned}$$

as $\lim\limits_{n\to\infty} g_n u_- g_n^{-1} = e$. We conclude that $E\pi(u_-) = E$, for every $u_- \in U_\alpha^-$. In the same way one proves that $\pi(u_+)E = E$, for every $u_+ \in U_\alpha^+$. This proves the claim.

By the claim we obtain that, for every $\phi \in \mathcal{H}$, the vector $E(\phi)$ (respectively, $E^*(\phi)$) is U_α^+ (respectively, U_α^-) invariant. In particular, this is the case for the vectors $E(v)$ and respectively, $E^*(w)$.

Recall that E is normal and different from zero. Thus, we have that $E^*E = EE^* \neq 0$ (otherwise $0 = \langle E^*E(\phi), \phi\rangle = \|E(\phi)\|^2$ for every $\phi \in \mathcal{H}$ contradicting $E \neq 0$). In this way we are able to find a non-zero vector $\psi \in E(\mathcal{H}) \bigcap E^*(\mathcal{H}) \neq \{0\} \subset \mathcal{H}$ which is $\langle N_\alpha^+, N_\alpha^-\rangle$-invariant, so also G-invariant. The contradiction follows and the lemma stands proven. $\square$

Theorem 2.1.18. *Let G be a second countable, locally compact group having the following properties:*

i) G admits a decomposition $K_1 A^+ K_2$, where $K_1, K_2 \subset G$ are compact subsets and A^+ is an abelian sub semi-group of G;

ii) for all sequences $\alpha = \{g_n\}_{n>0} \subset A^+$, with $g_n \to \infty$, we have $G = \overline{\langle U_\alpha^+, U_\alpha^-\rangle}$.

Then G has the Howe–Moore property.

Proof. Notice that by Lemma 2.1.6 and the fact that G is also separable, it is enough to prove the Howe–Moore property only for unitary representations that are over separable Hilbert spaces.

Therefore, let $(\pi, \mathcal{H})$ be a unitary representation of G, without non-zero G-invariant vectors and where $\mathcal{H}$ a separable Hilbert space. Suppose that not all matrix coefficients of $(\pi, \mathcal{H})$ vanish at ∞.

Using Lemma 2.1.11 there exists a sequence $\{g_n\}_{n>0} \subset A^+$ and two non-zero vectors $v, w \in \mathcal{H}$ such that $g_n \to \infty$ and $| \langle \pi(g_n)v, w\rangle |$ is not tending to 0. By passing to a subsequence $\{g_{n_k}\}_{k>0}$ and using the separability of $\mathcal{H}$ (see Lemma 2.1.15), assume that $\{\pi(g_{n_k})\}_{n_k}$ weakly converges to an operator E and that $| \langle \pi(g_{n_k})v, w\rangle | > C > 0$, for every n_k and where C a constant. Applying Lemma 2.1.17 to the sequence $\alpha = \{\pi(g_{n_k})\}_{n_k}$ we obtain a contradiction. Our criterion stands proven.

□

2.1.4 Unified proof of known examples

As announced, in this section we apply the criterion provided by Theorem 2.1.18 to the classes of groups mentioned in the introduction. Thereby we deduce, in a unified fashion, that the Howe–Moore property holds for all those classes.

Real Lie groups

For the class of connected simple real Lie groups with finite centre the proof is the same as the one given in [BM00]; still we outline briefly the main ideas.

It is a well-known fact that a semi-simple real Lie group G, with finite center, has a KA^+K decomposition (called the polar decomposition) where K is a maximal compact subgroup, $A < G$ is the closed subgroup corresponding to the maximal abelian subalgebra of the Lie algebra of G and $A^+ \subset A$ the corresponding positive sub semi-group part (see [BM00, Chapter III] for more details).

Therefore, to use our criterion Theorem 2.1.18, it is enough to prove that for every sequence $\alpha = (g_n)_{n\geq 1} \subset A^+$ with $g_n \to \infty$ one has that $\overline{\langle U_\alpha^+, U_\alpha^- \rangle} = G$. More exactly, we have the following lemma (see [BM00, Chap. III, Lemma 1.8]):

Lemma 2.1.19. *i) For every $a \in A^+$, the subgroup $\overline{\langle U_a^+, U_a^- \rangle}$ is normal and non-discrete in G.*

ii) For every $\alpha = (g_n)_{n \geq 1} \subset A^+$, with $g_n \to \infty$, there exists $b \in A^+$ such that $\overline{\langle U_\alpha^+, U_\alpha^- \rangle} = \overline{\langle U_b^+, U_b^- \rangle}$; moreover $N_b^\pm = N_\alpha^\pm$.

The proof of this lemma uses the Lie algebra of G and the root system.

Euclidean building groups

Let us now turn our attention to the totally disconnected analogs of semi-simple real Lie groups; namely locally compact groups acting continuously and properly by automorphisms and strongly transitively on locally finite thick Euclidean buildings.

Notation 2.1.20. To fix the notations, let us denote a locally finite thick Euclidean building by Δ (for the definition of a building and the related concepts the reader can consult [AB08]). Fix for what follows an apartment $\mathcal{A}$ in Δ, a chamber C in $\mathcal{A}$ and a special vertex x_0 of C. By $\mathrm{Ch}(\mathcal{B})$ we denote the set of all chambers of an apartment $\mathcal{B}$ in Δ and by $\partial\mathcal{B}$ its corresponding apartment in the spherical building at infinity $\partial\Delta$.

Remark 2.1.21. Recall that the subgroup of all type-preserving automorphisms of Δ is of finite index in $\mathrm{Aut}(\Delta)$. Therefore, if a subgroup of $\mathrm{Aut}(\Delta)$ enjoys the Howe–Moore property then, by Proposition 2.1.12, that group must be type-preserving.

For the rest of this section, by Remark 2.1.21, we design G to be a closed, strongly transitive (see Definition 1.1.7) and type-preserving subgroup of $\mathrm{Aut}(\Delta)$. For most of such groups G we want to verify the two conditions given in Theorem 2.1.18. The first one is the well-known polar decomposition whose proof is recalled in the next subsection.

Polar decomposition

Let $\mathrm{Stab}_G(\mathcal{A})$, respectively $\mathrm{Fix}_G(\mathcal{A})$, be the stabilizer subgroup, respectively the pointwise stabilizer subgroup, in G of an apartment $\mathcal{A}$ of Δ.

Recall that $\mathrm{Stab}_G(\mathcal{A})/\mathrm{Fix}_G(\mathcal{A})$ is the Weyl group W, corresponding to the Euclidean building Δ and that W is generated by the reflexions through the walls of a chamber $C \in \mathrm{Ch}(\mathcal{A})$. Another well-known fact is that W contains a maximal abelian normal subgroup isomorphic to $\mathbb{Z}^m$, whose elements are Euclidean translation automorphisms of the apartment $\mathcal{A}$ and m is the Euclidean dimension of $\mathcal{A}$. Denote this maximal abelian normal subgroup by A and remark that its elements are induced by hyperbolic automorphisms of Δ. In particular, every element of A can be lifted (not in a unique way) to a hyperbolic element of G. Notice that by lifting A to G, in a consistent way, the resulting subgroup does not remain abelian in general. To construct a basis of A, let $x_0 \in \mathcal{A}$ be a special vertex and let Q be a sector in $\mathcal{A}$ based at x_0. Recall that W_{x_0} is transitive on the set of sectors of $\mathcal{A}$ that are based at x_0. This implies that for every $a \in A$, there exists $k \in W_{x_0}$ such that kak^{-1} maps x_0 in Q. Moreover, as the Euclidean dimension of $\mathcal{A}$ is m, there exists a basis $\{\gamma_1, \cdots, \gamma_m\}$ of A such that γ_i maps x_0 in Q, for every $i \in \{1, \cdots m\}$. Define now $A^+ \subset A$ to be the sub semi-group consisting of translations mapping x_0 in Q. Remark that every element of A^+ is a product of positive powers of elements from the basis $\{\gamma_1, \cdots, \gamma_m\}$. The subset A^+ is the desired 'maximal abelian' and 'positive' sub semi-group part of A.

Take now $g \in G$ and let $\mathcal{B}$ be an apartment in Δ such that $C, g(C) \in \mathrm{Ch}(\mathcal{B})$. As G is strongly transitive, there exists $k \in G$ such that $k(\mathcal{B}) = \mathcal{A}$ and $k(C) = C$ pointwise. Therefore $k \in \mathrm{Stab}_G(x_0)$. As G is type-preserving, $kg(x_0)$ is a special vertex of the same type as x_0. Therefore, using the fact that W is simply transitive on special vertices of the same type, there exists a translation automorphism a of the apartment $\mathcal{A}$ sending $kg(x_0)$ into x_0. We conclude that $akg(x_0) = x_0$ and thus $akg \in \mathrm{Stab}_G(x_0)$. By writing the element a using elements from A^+ and $\mathrm{Stab}_G(x_0)$, we have that $g \in \mathrm{Stab}_G(x_0)A^+\mathrm{Stab}_G(x_0)$. Therefore the **polar decomposition** of the group G is proven, as $\mathrm{Stab}_G(x_0)$, being the stabilizer of the vertex x_0, is compact and open in G. To conclude, we have proved the following:

Lemma 2.1.22. *Let G be a closed, strongly transitive and type-preserving subgroup of* $\mathrm{Aut}(\Delta)$. *Take $\mathcal{A}$ an apartment in Δ and x_0 be a special vertex in $\mathcal{A}$. Then* $G = \mathrm{Stab}_G(x_0)A^+\mathrm{Stab}_G(x_0)$.

Remark 2.1.23. As we already mentioned, for a general closed, strongly transitive and type-preserving subgroup G of $\mathrm{Aut}(\Delta)$, its corresponding abelian group $A < W$ does not necessarily lift to an abelian subgroup of G. Still, this is the case if we consider that G is a semi-simple algebraic group over a non Archimedean local field. By the theory of Bruhat–Tits, such a group gives rise to a Euclidean building where the group G acts by automorphisms and strongly transitively. The abelian group $A < W$ is in fact the restriction to $\mathcal{A}$ of a maximal split torus of G, which is abelian.

Related to the polar decomposition from Lemma 2.1.22, we recall another well-known decomposition of the group G.

Let c be an ideal chamber in $\partial\mathcal{A}$. Denote by

$$G_c^0 := \{g \in G \mid g(c) = c \text{ and } g \text{ fixes at least one vertex of } \Delta\} \tag{2.1}$$

the 'pointwise' stabilizer subgroup in G corresponding to $c \in \mathrm{Ch}(\partial\mathcal{A})$. Remark that G_c^0 is a closed subgroup of $\mathrm{Stab}_G(c)$ and does not contain any hyperbolic automorphisms of Δ.

Take $g \in G$ and let $\mathcal{B}$ be an apartment in Δ such that $g^{-1}(C) \in \mathrm{Ch}(\mathcal{B})$ and $c \in \mathrm{Ch}(\partial\mathcal{B})$. Remark that $\mathcal{A}$ and $\mathcal{B}$ have a sector in common, corresponding to the chamber at infinity c. Thus, by the strongly transitive action, there exists $n \in G_c^0$ such that $n(\mathcal{B}) = \mathcal{A}$. We obtain that $ng^{-1}(C) \in \mathrm{Ch}(\mathcal{A})$. As G is type-preserving, there exists a translation automorphism $a \in A$ of the apartment $\mathcal{A}$ such that $ang^{-1}(x_0) = x_0$. From here we conclude that $ang^{-1} \in \mathrm{Stab}_G(x_0)$ and thus $g \in \mathrm{Stab}_G(x_0)AG_c^0$. Therefore

$$G = \mathrm{Stab}_G(x_0)AG_c^0. \tag{2.2}$$

Levi conditions

Let us now start verifying the second condition given by Theorem 2.1.18, for a closed, strongly transitive and type-preserving subgroup G of $\mathrm{Aut}(\Delta)$. Its first step is given by the following well-known proposition; for the convenience of the reader we include its proof.

Regarding the closed subgroup G_c^0 defined in (2.1), by the strongly transitive action of G on Δ, we have that G_c^0 acts transitively on the set $\mathrm{Opp}(c)$ of all chambers opposite c of $\mathrm{Ch}(\partial\Delta)$. Indeed, let $c_1 \neq c_2 \in \mathrm{Opp}(c)$. Denote by A_1 and respectively, by A_2, the unique apartments of Δ such that $c_1, c \in \mathrm{Ch}(\partial A_1)$ and respectively, $c_2, c \in \mathrm{Ch}(\partial A_2)$. Let $Q \subset A_1 \cap A_2$ be a sector pointing to the chamber at infinity c and choose a chamber C of $\mathrm{Ch}(\Delta)$ contained in the interior of Q. Then, by the strong transitive action of G, there exists $g \in G$ such that $g(C) = C$ and $g(A_1) = A_2$. In addition, we have that $g(Q) = Q$ pointwise; therefore $g \in G_c^0$ and $g(c_1) = c_2$.

Proposition 2.1.24. *Let G be a closed, strongly transitive and type-preserving subgroup of* $\mathrm{Aut}(\Delta)$. *Let $\mathcal{A}$ be an apartment of Δ and take $c_+ \in \mathrm{Ch}(\partial\mathcal{A})$. Denote by c_- the chamber opposite c_+ in $\partial\mathcal{A}$.*

Then $G = \langle G^0_{c_+}, G^0_{c_-} \rangle$.

Before starting the proof, we state the next useful lemma.

Lemma 2.1.25. *Let G be a closed, strongly transitive and type-preserving subgroup of* $\mathrm{Aut}(\Delta)$. *Let $\mathcal{A}$ be an apartment of Δ and take $c_+ \in \mathrm{Ch}(\partial\mathcal{A})$. Denote by c_- the chamber opposite c_+ in $\partial\mathcal{A}$.*

Then $\mathrm{Stab}_G(\mathcal{A}) = \mathrm{Stab}_G(\partial\mathcal{A}) \leq \langle G^0_{c_+}, G^0_{c_-} \rangle$.

Proof. The equality $\mathrm{Stab}_G(\mathcal{A}) = \mathrm{Stab}_G(\partial\mathcal{A})$ is straightforward.

Following the proof of Lemma 3.3.5 we start with a basic observation. Let $\mathcal{A}'$ be an apartment of Δ such that the intersection $\mathcal{A} \cap \mathcal{A}'$ is a half-apartment. We claim that there is some $u \in \langle G^0_{c_+}, G^0_{c_-} \rangle$ mapping $\mathcal{A}'$ to $\mathcal{A}$ and fixing $\mathcal{A} \cap \mathcal{A}'$ pointwise. Indeed, because c_+, c_- are opposite in $\partial\mathcal{A}$, they are separated by any wall of $\mathcal{A}$. Thus, $\mathcal{A} \cap \mathcal{A}'$, being a half-apartment, contains a sector Q corresponding either to c_+, or to c_-. We apply the strong transitivity of G; there exists an element $g \in G$ fixing pointwise a chamber in the interior of the sector Q and sending $\mathcal{A}'$ to $\mathcal{A}$. Then $g \in \langle G^0_{c_+}, G^0_{c_-} \rangle$.

Let now M be a wall of $\mathcal{A}$. Let H and H' be the two half-apartments of $\mathcal{A}$ determined by M. Since Δ is thick, there exists a half-apartment H'' such that $H \cup H''$ and $H' \cup H''$ are both apartments of Δ. By the above claim, we can find an element $u \in \langle G^0_{c_+}, G^0_{c_-} \rangle$ fixing H pointwise and mapping H' to H''. Similarly, there are elements $v, w \in \langle G^0_{c_+}, G^0_{c_-} \rangle$ fixing H' pointwise and such that $v(H'') = H$ and $w(H) = u^{-1}(H')$. Now we set $r := vuw$. By construction r fixes pointwise the wall M and $r \in \langle G^0_{c_+}, G^0_{c_-} \rangle$. Moreover we have

$$r(H) = vuu^{-1}(H') = v(H') = H'$$

and

$$r(H') = vu(H') = v(H'') = H,$$

so that r swaps H and H'. It follows that r stabilizes the apartment $\mathcal{A}$ and acts on $\mathcal{A}$ as the reflection through the wall M. Since this holds for an arbitrary wall of $\mathcal{A}$ and $\mathrm{Stab}_G(\mathcal{A})/\mathrm{Fix}_G(\mathcal{A})$, being the Weyl group W, is generated by reflexions through the walls of a chamber $C \in \mathrm{Ch}(\mathcal{A})$, our conclusion follows. $\square$

Proof of Proposition 2.1.24. Take $g \in G$ and let $\mathcal{B}$ an apartment of Δ such that $g(c_+), c_+ \in \mathrm{Ch}(\partial\mathcal{B})$. Then $\mathcal{A}$ and $\mathcal{B}$ have a sector in common, corresponding to the chamber at infinity c_+. By the strong transitivity of G, take $h_1 \in G^0_{c_+}$ such that $h_1(\mathcal{B}) = \mathcal{A}$. Therefore $h_1 g(c_+) \in \mathrm{Ch}(\partial\mathcal{A})$. Recall that the strong transitivity of G on Δ implies that G acts strongly transitively on $\partial\Delta$ (see [Gar97, Section 17.1]). Applying Lemma 2.1.25, there exists $h_2 \in \mathrm{Stab}_G(\partial\mathcal{A}) \leq \langle G^0_{c_+}, G^0_{c_-}\rangle$ such that $h_2 h_1 g(c_+) = c_+$. Thus $h_2 h_1 g(\mathcal{A})$ and $\mathcal{A}$ have a sector in common corresponding to the chamber at infinity c_+. Take now $h_3 \in G^0_{c_+}$ with $h_3 h_2 h_1 g(\mathcal{A}) = \mathcal{A}$. Therefore $h_3 h_2 h_1 g \in \mathrm{Stab}_G(\mathcal{A})$ and by Lemma 2.1.25 the conclusion follows. $\square$

'Algebraic' Levi decomposition

Recall from [BW04] the following general theory. Let G be a totally disconnected locally compact group and take $a \in G$. Let

$$P_a^+ := \{g \in G \mid \{a^{-n} g a^n\}_{n\in\mathbb{N}} \text{ is bounded}\}. \tag{2.3}$$

In the same way, but using $a^n g a^{-n}$, we define P_a^-. Following [BW04, Section 3], P_a^+ is a closed subgroup of G and denote

$$U_a^+ := \{g \in G \mid \lim_{n\to\infty} a^{-n} g a^n = e\}. \tag{2.4}$$

By [BW04, Section 3], P_a^+ and U_a^+ are called the **parabolic**, respectively the **contraction**, subgroups associated to a, where in general U_a^+ is not closed. Moreover, from the algebraic point of view, we have the following **Levi decomposition**:

Theorem 2.1.26. *(See* [BW04, Proposition 3.4, Corollary 3.17]*) Let G be a totally disconnected locally compact group which is also metrizable. Let $a \in G$. Then $P_a^+ = U_a^+ M_a$, where $M_a := P_a^+ \cap P_a^-$. Moreover, U_a^+ is normal in P_a^+.*

Specialized it to our context we obtain:

Corollary 2.1.27. *Let G be a closed, strongly transitive and type-preserving subgroup of* $\mathrm{Aut}(\Delta)$*. Let $a \in G$ be a hyperbolic element. Then $P_a^+ = U_a^+ M_a$, where $M_a := P_a^+ \cap P_a^-$. Moreover, U_a^+ is normal in P_a^+.*

Remark 2.1.28. In the setting of a locally finite Euclidean building Δ, a subset $F \subset \mathrm{Aut}(\Delta)$ is called **bounded** if there exists a point $y \in \Delta$ such that the set $F \cdot y$ is a bounded subset of Δ. For equivalent definitions see [AB08, Lemma 11.32]. Those characterizations are useful and used in what follows when working with elements of P_a^+.

'Geometric' Levi decomposition

This subsection proposes to link the 'algebraic' Levi decomposition from Corollary 2.1.27 with the subgroup G_c^0 defined in (2.1), a key ingredient being given by the following:

Proposition 2.1.29. *Let G be a closed, non-compact and type-preserving subgroup of* $\mathrm{Aut}(\Delta)$*. Let $\mathcal{A}$ be an apartment in Δ and assume there exists an hyperbolic automorphism $a \in \mathrm{Stab}_G(\mathcal{A})$. Denote its attracting endpoint by $\xi_+ \in \partial\mathcal{A}$ and let $c_+ \in \mathrm{Ch}(\partial\mathcal{A})$, with $\xi_+ \in c_+$. Let $G_{c_+} := \{g \in G \mid g(c_+) = c_+\}$ and $G_{\xi_+} := \{g \in G \mid g(\xi_+) = \xi_+\}$.*

Then $G_{c_+} \leq P_a^+ = G_{\xi_+} = G_\sigma$, where σ is the unique simplex in $\partial\mathcal{A}$ such that ξ_+ is contained in the interior of σ ($\sigma = \xi_+$ if and only if ξ_+ is a vertex). In particular, we obtain that $P_a^+ \cap P_a^- = G_{(\xi_-,\xi_+)}$.

Proof. The equality $G_{\xi_+} = G_\sigma$ is immediate as G is type-preserving.

Let us first prove that $P_a^+ \subset G_{\xi_+}$.

Take $g \in P_a^+$ and let x_0 be a special vertex of $\mathcal{A}$. By the definition of P_a^+, the set $\{a^{-n}ga^n(x_0) \mid n \in \mathbb{N}\}$ is a bounded subset of Δ and thus is finite. Extract a subsequence $n_k \to \infty$ such that $a^{-n_k}ga^{n_k}(x_0) = y$, for every n_k. As G is type-preserving, the point y is also a special vertex of Δ.

Now, because $a^{n_k}(x_0) \to \xi_+$ and $a^{n_k}(y) \to \xi_+$ (in the cone topology of $\Delta \cup \partial\Delta$), one can conclude that $g(\xi_+) = \xi_+$. Therefore $g \in G_{\xi_+}$.

Let us show that $G_{\xi_+} \subset P_a^+$.

Let $g \in G_{\xi_+}$. As $g(\xi_+) = \xi_+$ there exists a geodesic ray in $\mathcal{A}$, say $[y, \xi_+)$, such that $g([y, \xi_+)) \subset \mathcal{A}$ is parallel to $[y, \xi_+)$. Moreover, we have that $\mathrm{dist}_{\mathcal{A}}(g(t), t) = \mathrm{dist}_{\mathcal{A}}(g(y), y)$, for every $t \in [y, \xi_+)$. To prove that $g \in P_a^+$ it is enough to show that the set $\{a^{-n} g a^n(z) \mid n \in \mathbb{N}\}$ is a bounded subset of Δ, for some point $z \in \mathcal{A}$. Take $z = y$. Because a is a hyperbolic element in $\mathrm{Stab}_G(\mathcal{A})$, $a^n(y) \in [y, \xi_+)$. From here, we immediately conclude that $\{a^{-n} g a^n(z) \mid n \in \mathbb{N}\}$ is indeed a bounded set.

The fact that $G_{c_+} \subset P_a^+$ is an immediate consequence of the equality $P_a^+ = G_{\xi_+}$, as G is type-preserving and $G_{c_+} \leq G_{\xi_+}$.

□

Remark 2.1.30. For a general hyperbolic element $a \in \mathrm{Stab}_G(\mathcal{A})$ we do not necessarily have the equality $P_a^+ = G_{c_+}$ in Proposition 2.1.29. However, if a is a strongly regular hyperbolic element of $\mathrm{Stab}_G(\mathcal{A})$ (see Definition 3.2.2 and Lemma 3.2.3), the equality $P_a^+ = G_{c_+}$ holds, as in this case $G_{c_+} = G_{\xi_+}$. In addition, if we take Δ to be a locally finite tree, the equality $P_a^+ = G_{c_+}$ is always verified as the chambers at infinity are just the ends of the tree, which are points.

By combining Proposition 2.1.29 and Corollary 2.1.27 we obtain the **geometric Levi decomposition**:

Corollary 2.1.31. *Let G be a closed and type-preserving subgroup of* $\mathrm{Aut}(\Delta)$. *Let $\mathcal{A}$ be an apartment in Δ and assume there exists a hyperbolic automorphism $a \in \mathrm{Stab}_G(\mathcal{A})$. Denote its attracting endpoint by $\xi_+ \in \partial\mathcal{A}$ and let σ be a simplex in $\partial\mathcal{A}$ such that $\xi_+ \in \sigma$. Let also $c_+ \in \mathrm{Ch}(\partial\mathcal{A})$ with $\xi_+ \in c_+$.*

Then $G_\sigma = U_a^+(M_a \cap G_\sigma)$, where $M_a := P_a^+ \cap P_a^-$. In particular, $G^0_{c_+} = U_a^+(M_a \cap G^0_{c_+})$. In addition U_a^+ is normal in $G^0_{c_+}$, respectivly, G_σ.

Proof. By the definition of U_a^+ one can easily verify that $U_a^+ \leq G^0_{c_+} \leq G_\sigma \leq P_a^+$. The conclusion follows by applying Corollary 2.1.27. □

Furthermore, by Proposition 2.1.24 and Corollary 2.1.31 we obtain:

Corollary 2.1.32. *Let G be a closed, strongly transitive and type-preserving subgroup of* $\mathrm{Aut}(\Delta)$. *Let $\mathcal{A}$ be an apartment of Δ and let $a \in \mathrm{Stab}_G(\mathcal{A})$ be a hyperbolic automorphism.*

Then $\overline{\langle U_a^+, U_a^- \rangle}$ is normal in G.

Proof. Let $c_-, c_+ \in \mathrm{Ch}(\partial\mathcal{A})$ be two opposite ideal chambers containing the repelling point and respectively the attracting point of the hyperbolic element a. By Proposition 2.1.24 we have that $G = \langle G^0_{c_-}, G^0_{c_+}\rangle$. Therefore, to prove that $\overline{\langle U^+_a, U^-_a\rangle}$ is normal in G, it is enough to verify that, for every $g \in G^0_{c_\pm}$, we have that $gU^\pm_a g^{-1} \in \langle U^+_a, U^-_a\rangle$. Indeed, using the decomposition of $G^0_{c_\pm}$ given by Corollary 2.1.31, and the fact that $U^\pm_a$ is normal in $G^0_{c_\pm}$, the conclusion follows. $\square$

The proof of the Howe–Moore property

Using all the ingredients presented in the above subsections, the goal here is to provide the proof of the following theorem:

Theorem 2.1.33. *Let G be an isotropic simple algebraic group over a non Archimedean local field or a closed, topologically simple subgroup of* $\mathrm{Aut}(T)$ *that acts 2-transitively on the boundary ∂T, where T is a bi-regular tree with valence ≥ 3 at every vertex. Then G admits the Howe–Moore property.*

To prove Theorem 2.1.33, in the case of algebraic groups, it remains to verify the second condition given by Theorem 2.1.18, namely, the analogue of Lemma 2.1.19. As in the case of connected simple real Lie groups with finite center, the analogue lemma uses the theory of root group datum related to Euclidean buildings; therefore, we briefly recall some definitions and state some known properties of this theory from [Wei08, AB08, Ron89]. We mention that the proof of the analogue of Lemma 2.1.19, stated in Proposition 2.1.42 for the case of Euclidean buildings, works in the same spirit as for real Lie groups.

Definition 2.1.34. Let X be a Euclidean or a spherical building. A half-apartment h of X is called a **root of** X. By abuse of notation, we use ∂h to denote the boundary wall in X which is determined by h.

Moreover, if X is a spherical building, the **root group** U_h corresponding to a root h of X is defined to be the set of all $g \in \mathrm{Aut}(X)$ such that g fixes pointwise h and also the star of every panel contained in $h \setminus \partial h$.

Definition 2.1.35. Let X be a thick spherical building. For a root h of X, we denote by $\mathcal{A}(h)$ the set of all apartments in X that contain h. We say that X **is Moufang** if, for every root h of X, the root group $U_h < \mathrm{Aut}(X)$ is transitive on $\mathcal{A}(h)$.

Recall that for a semi-simple algebraic group $\mathbb{G}$ over a non Archimedean local field, by [BT72] one associates a locally finite thick Euclidean building Δ where $\mathbb{G}$ acts by automorphisms and strongly transitively. More precisely:

Definition 2.1.36. Let $\mathbb{G}$ be a semi-simple algebraic group over a non Archimedean local field k, of rank ≥ 1. Let $G = \mathbb{G}(k)$ be the group of all k-rational points of $\mathbb{G}$. By [BT72], to the group G one associates a locally finite thick Euclidean building Δ where G acts by type-preserving automorphisms and strongly transitively. Moreover, by [BT72], the spherical building $\partial\Delta$ at infinity of Δ is Moufang. Let $G^+ := \langle U_h \leq G \mid h \text{ is a root of } \partial\Delta \rangle$. It is known that G^+ is a closed normal subgroup of G and that the factor group G/G^+ is compact (see [Mar91, Thm. 2.3.1]). We call G^+ **an isotropic simple algebraic group over the non Archimedean local field** k. From its definition, notice that G^+ acts by type-preserving automorphisms and also strongly transitively on Δ.

Proposition 2.1.37. *(See* [Wei08, Prop. 13.2 and 13.5]*) Let Δ be a Euclidean building such that its corresponding building at infinity $\partial\Delta$ is Moufang. Let $\mathcal{A}$ be an apartment of Δ and let h be a root of $\partial\mathcal{A}$. For $u \in U_h^*$ let $h_u := \mathcal{A} \cap u(\mathcal{A})$.*

Then h_u is a root of $\mathcal{A}$ such that its boundary at infinity $h_u(\infty) = h$. Moreover, u acts trivially on h_u. Conversely, let $\mathcal{H}$ be a root of $\mathcal{A}$ such that its boundary at infinity $\mathcal{H}(\infty) = h$. Then there exists $u \in U_h^*$ such that $\mathcal{H} = h_u$.

Definition 2.1.38. Let Δ be a Euclidean building such that its corresponding building at infinity $\partial\Delta$ is Moufang. Let $\mathcal{H}$ be a root of an apartment $\mathcal{A}$ of Δ and denote $h := \mathcal{H}(\infty)$ the corresponding root in $\partial\mathcal{A}$. With respect to U_h, the **affine root group corresponding to** $\mathcal{H}$ is the set $U_{\mathcal{H}} := \{u \in U_h^* \mid h_u = \mathcal{H}\} \cup \mathrm{id}$, where $h_u := \mathcal{A} \cap u(\mathcal{A})$, for $u \in U_h^*$.

Proposition 2.1.39. *(See* [BT72, Prop. 7.4.33]*) Let $\mathbb{G}$ be a semi-simple algebraic group over a non Archimedean local field k and let $G = \mathbb{G}(k)$ be the group of all k-rational points of $\mathbb{G}$. Denote by Δ the corresponding locally finite thick Euclidean building on which*

G acts by type-preserving automorphisms and strongly transitively. Let $\mathcal{H}$ be a root of Δ and let $x \in \mathcal{H}$. Then the radius of the ball of Δ around x which is fixed pointwise by $U_{\mathcal{H}} < G$ goes to infinity as the distance of x to $\partial\mathcal{H}$ goes to infinity.

From Propositions 2.1.39 and 2.1.37 we immediately obtain:

Corollary 2.1.40. *Let $\mathbb{G}$ be a semi-simple algebraic group over a non Archimedean local field k and let $G = \mathbb{G}(k)$ be the group of all k-rational points of $\mathbb{G}$. Denote by Δ the corresponding locally finite thick Euclidean building on which G acts by type-preserving automorphisms and strongly transitively. Let $\mathcal{A}$ be an apartment of Δ and let h be a root of $\partial\mathcal{A}$. Let $\alpha = \{a_n\}_{n>0} \subset \mathrm{Stab}_G(\mathcal{A})$ with $a_n \to \infty$. Assume that $a_n(x_0) \to \xi \in h \setminus \partial h$, in the cone topology, where $x_0 \in \mathcal{A}$ is some special vertex.*

Then $U_h \leq U_\alpha^+$. In particular, we obtain that $U_h \leq U_a^+$, for every hyperbolic element $a \in \mathrm{Stab}_G(\mathcal{A})$ whose attracting endpoint $\xi_+ \in h \setminus \partial h$.

Proof. As U_h is the union of its corresponding affine root groups $U_{\mathcal{H}}$, it is enough to prove that $U_{\mathcal{H}} \leq U_\alpha^+$. Indeed, let $u \in U_{\mathcal{H}}$. As $a_n(x_0) \to \xi \in h \setminus \partial h$, in the cone topology, where x_0 is a special vertex of $\mathcal{A}$, we have that the intersection of the geodesic ray $[x_0, \xi)$ with the root $\mathcal{H}$ is a geodesic ray with endpoint ξ. Therefore, for every standard open neighborhood $V \subset \mathcal{H}$ of ξ with respect to the cone topology of $\mathcal{A}$, there exists N such that $a_n(x_0) \in V \subset \mathcal{H}$, for every $n \geq N$. By Proposition 2.1.39, we immediately obtain that $u \in U_\alpha^+$. $\square$

Another important property is recorded by the following corollary.

Corollary 2.1.41. *Let $\mathbb{G}$ be a semi-simple algebraic group over a non Archimedean local field k and let $G = \mathbb{G}(k)$ be the group of all k-rational points of $\mathbb{G}$. Denote by Δ the corresponding locally finite thick Euclidean building on which G acts by type-preserving automorphisms and strongly transitively. Let $\mathcal{A}$ be an apartment of Δ and let $a \in \mathrm{Stab}_G(\mathcal{A})$ be a hyperbolic element whose attracting endpoint in $\partial\mathcal{A}$ is denoted by ξ_+. Let σ be the unique simplex in $\partial\mathcal{A}$ such that ξ_+ is contained the interior of σ and denote by $\mathrm{St}(\sigma)$ the star of σ in $\partial\mathcal{A}$.*

Then $U_a^+ = \langle U_h \mid h \text{ is a root of } \partial\mathcal{A} \text{ with } \mathrm{St}(\sigma) \subset h \rangle$.

Proof. The conclusion follows by combining Corollary 2.1.31 with the theory of root group datum [Ron89, Thm. 6.18]. $\square$

The analogue of Lemma 2.1.19 is given by the following proposition.

Proposition 2.1.42. *Let $\mathbb{G}$ be a semi-simple algebraic group over a non Archimedean local field k and let $G = \mathbb{G}(k)$ be the group of all k-rational points of $\mathbb{G}$. Denote by Δ the corresponding locally finite thick Euclidean building on which G acts by type-preserving automorphisms and strongly transitively. Let $\mathcal{A}$ be the apartment that corresponds to the abelian sub semi-group A^+ of the Weyl group W of G.*

Let $\alpha = \{a_n\}_{n>0} \subset A^+$ with $a_n \to \infty$ and such that $a_n(x_0) \to \xi \in \partial\mathcal{A}$, for some special vertex $x_0 \in \mathcal{A}$. Then there exists a hyperbolic element $b \in \mathrm{Stab}_G(\mathcal{A})$ such that $U_b^{\pm} \subset U_\alpha^{\pm}$.

Proof. Let x_0 be a fixed special vertex in the apartment $\mathcal{A}$ and let $\{\gamma_1, \cdots, \gamma_m\}$ be the basis of A^+ described in the beginning of Section 2.1.4, where m is the dimension of the Euclidean building Δ. Moreover, we can choose the basis $\{\gamma_1, \cdots, \gamma_m\}$ such that the attracting endpoint of γ_j, for every $j \in \{1, \cdots, m\}$, is a vertex in $\partial\mathcal{A}$ (i.e., it is a vertex of the chamber at infinity determined by the sector that defines A^+). Let σ be the unique simplex of $\partial\mathcal{A}$ that contains ξ in its interior ($\xi = \sigma$ if and only if ξ is a vertex in $\partial\mathcal{A}$). Let $\{\gamma_{i_1}, \cdots, \gamma_{i_l}\}$ be the set of the elements of $\{\gamma_1, \cdots, \gamma_m\}$ whose attracting endpoints determine the simplex σ.

Define $b := \gamma_{i_1} \cdot \ldots \cdot \gamma_{i_l}$ and notice that b is a hyperbolic element in $\mathrm{Stab}_G(\mathcal{A})$. Moreover, the attracting endpoint ξ_+ of b and the point ξ are contained in the interior of the simplex σ of $\partial\mathcal{A}$.

Apply Corollary 2.1.41 to b and σ. Then, by Corollary 2.1.40 we obtain that $U_b^{\pm} \subset U_\alpha^{\pm}$. $\square$

Proof of Theorem 2.1.33. By Lemma 2.1.6 and Remark 2.1.7 it is enough to verify the Howe–Moore property only for separable Hilbert spaces.

Let G be an isotropic simple algebraic group over a non Archimedean local field or a closed, topologically simple subgroup of $\mathrm{Aut}(T)$ that acts 2-transitively on the boundary ∂T, where T is a bi-regular tree with valence ≥ 3 at every vertex. For such G, we verify the two conditions of the Theorem 2.1.18. In both cases, the first condition of Theorem 2.1.18 is verified by applying Lemma 2.1.22 to G.

Let us verify the second condition of Theorem 2.1.18. Notice that, in the case of a bi-regular tree, the corresponding group A^+ of G, given by the polar decomposition of G, is generated by a single hyperbolic element $a \in G$. Therefore, the second condition follows immediately from Corollary 2.1.32 applied to G.

When G is an isotropic simple algebraic group over a non Archimedean local field, to obtain the Howe–Moore property, it is enough to verify the second condition of Theorem 2.1.18 only for a sequence $\alpha = \{a_n\}_{n>0} \subset A^+$ with $a_n \to \infty$ and such that $a_n(x_0) \to \xi \in \partial\mathcal{A}$, for some special vertex $x_0 \in \mathcal{A}$. This follows by applying to $\alpha = \{a_n\}_{n>0}$ Proposition 2.1.42 and Corollary 2.1.32. Notice that, for every hyperbolic element $b \in \mathrm{Stab}_G(\mathcal{A})$, the group $\overline{\langle U_b^+, U_b^- \rangle}$, which is normal in G, acts non-trivially on Δ. By [Tit64, Main Theorem], we obtain that for every hyperbolic element $b \in \mathrm{Stab}_G(\mathcal{A})$, the group $\overline{\langle U_b^+, U_b^- \rangle} = G$. The theorem stands proven. □

2.2 On the matrix coefficients of the universal group $U(F)^+$

2.2.1 Introduction

Known examples of locally compact groups that enjoy the Howe–Moore property, namely, the vanishing at infinity of all matrix coefficients of the group's unitary representations that are without non-zero invariant vectors, are: connected, non-compact, simple real Lie groups and their totally disconnected analogs, isotropic simple algebraic groups over non Archimedean local fields and closed, topologically simple subgroups of $\mathrm{Aut}(T)$ that act 2-transitively on the boundary ∂T, where T is a bi-regular tree with valence ≥ 3 at every vertex. We emphasize that both isotropic simple algebraic groups and 2-transitive subgroups of $\mathrm{Aut}(T)$ admit a strongly transitive action on a locally finite thick Euclidean building. The strong transitivity of this action is essentially used to show the Howe–Moore property. Whether this condition is also necessary, is an open problem, even in the particular case when the Euclidean building is a d-regular tree T_d. For T_d, the above mentioned problem is translated as:

Problem 2.2.1. *Is a closed, non-compact, type-preserving subgroup of* $\mathrm{Aut}(T_d)$*, acting co-compactly and enjoying the Howe–Moore property,* 2*-transitive on the tree's boundary?*

To approach this question a test case is proposed to be studied: the group $U(F)$, introduced by Burger and Mozes in [BM00a, Section 3] (see Definition 2.2.5 below for the groups $U(F)$ and $U(F)^+$). This group has the following 'universal' property recorded by [BM00a, Prop. 3.2.2]: Let $d \geq 3$, $F < \mathrm{Sym}(\{1, \cdots, d\})$ be a transitive permutation subgroup and consider a vertex transitive subgroup $H < \mathrm{Aut}(T_d)$ such that, for every vertex $x \in T_d$, the restriction of H_x to the star of x in T_d is permutation isomorphic to F, where H_x is the stabilizer of x in H. Then $H < U(F)$.

Apart from being universal, the group $U(F)$ enjoys a bunch of other properties (see [BM00a, Section 3] or [Ama03]). For example, it acts 2-transitively on the boundary ∂T_d if and only if F is 2-transitive on $\{1, \cdots, d\}$. Moreover, it satisfies Tits' independence property (see Definition 2.2.6), which guarantees the existence of 'enough' rotations in the group. In Section 2.2.2 below we record and explain briefly these two properties as well as many other well-known characteristics of the universal group $U(F)$. Among them, one finds the reason why the universal group $U(F)$ (or more precisely its subgroup $U(F)^+$) is suggested as the first example that should be studied towards answering Problem 2.2.1 proposed above: By [CDM11, Prop. 4.1], we know that $F < \mathrm{Sym}(\{1, \cdots, d\})$ is primitive if and only if every proper open subgroup of $U(F)^+$ is compact. The latter condition that every proper open subgroup is compact is well-known to be satisfied by any group that enjoys the Howe–Moore property; however, it is not sufficient, in general, to imply the Howe–Moore property. To conclude, we emphasize a second problem that stems from Problem 2.2.1:

Problem 2.2.2. *Is it true that the group $U(F)^+$ does not have the Howe–Moore property if F is primitive but not 2-transitive and not cyclic of prime order?*

In the case when F is cyclic of prime order, the group $U(F)^+$ is trivial; therefore it has the Howe–Moore property. To find an answer to Problem 2.2.2, we give two main different directions that could be studied.

A first direction is given by the theory of Gelfand pairs and the characterizations of the 2-transitive action on the boundary ∂T_d, recorded in Proposition 2.2.9. More precisely, when F is primitive but not 2-transitive (and not cyclic of prime order), we apply Theorem 3.1.2 and known properties of Gelfand pairs [vD09, Prop. 6.3.1] to obtain the existence of at least one non-trivial irreducible unitary representation of $U(F)^+$ whose space of K-invariant vectors is at least of dimension 2. Here we denote by K the stabilizer in $U(F)^+$ of a vertex of the d-regular tree. One direction is then to investigate further those irreducible unitary representations. This approach is intended to be studied in a future project.

The second direction is to study further algebraic properties of $U(F)^+$, when F is primitive but not 2-transitive (and not cyclic of prime order).

The structure of this section is the following. In the first place, Section 2.2.3 adds to the long list of Section 2.2.2 new equivalent conditions of the universal group $U(F)^+$ to act 2-transitively on the boundary ∂T_d. Section 2.2.4 gathers new algebraic and harmonic analytic properties of the group $U(F)^+$, when F is primitive, but not cyclic of prime order. The main goal of those properties is to prove the main result of this section. This provides a characterization of the group $U(F)^+$, with F being primitive, but not cyclic of prime order, to admit the Howe–Moore property.

Theorem 2.2.3. *(See Theorem 2.2.21) Let F be primitive, but not cyclic of prime order. Then $U(F)^+$ has the Howe–Moore property if and only if for every unitary representation $(\pi, \mathcal{H})$ of $U(F)^+$, without non-zero $U(F)^+$-invariant vectors, and for every $v \neq 0 \in \mathcal{H}$ the closed isotropy subgroup $U(F)^+_v$ is compact.*

2.2.2 Definitions and some known properties

As it was mentioned in the Introduction, the notion of the universal group $U(F)$ was initiated by Burger–Mozes in [BM00a, Section 3]. In his PhD thesis [Ama03], Amann studies this group from the point of view of its unitary representations.

This section is meant to introduce the definition and to gather some well-known related properties, from various sources, of the Burger–Mozes universal group $U(F)$.

First, let us fix the notations. Denote by $\mathcal{T}$ **a d-regular tree, with $d \geq 3$,** and by $\mathrm{Aut}(\mathcal{T})$ its full group of automorphisms.

Definition 2.2.4. Let $i : E(\mathcal{T}) \to \{1, ..., d\}$ be a function from the set $E(\mathcal{T})$ of unoriented edges of the tree $\mathcal{T}$ such that its restriction to the star $E(x)$ of every vertex $x \in \mathcal{T}$ is in bijection with $\{1, ..., d\}$. A function i with those properties is called a **legal coloring** of the tree $\mathcal{T}$.

Definition 2.2.5. Let F be a subgroup of permutations of the set $\{1, ..., d\}$ and let i be a legal coloring of $\mathcal{T}$. We define the **universal group**, with respect to F and i, to be

$$U(F) := \{g \in \mathrm{Aut}(\mathcal{T}) \mid i \circ g \circ (i|_{E(x)})^{-1} \in F, \text{ for every x} \in \mathcal{T}\}.$$

By $U(F)^+$ we denote the subgroup generated by the edge-stabilizing elements of $U(F)$. Moreover, Proposition 52 of [Ama03] tells us that the group $U(F)$ is independent of the legal coloring i of $\mathcal{T}$.

Immediately from the definition we deduce that $U(F)$ and $U(F)^+$ are closed subgroups of $\mathrm{Aut}(\mathcal{T})$. Notice that, when F is the full permutation group $\mathrm{Sym}(\{1,...,d\})$, $U(F) = \mathrm{Aut}(\mathcal{T})$ and $U(F)^+$ is simply denoted by $\mathrm{Aut}(\mathcal{T})^+$, which is an index 2, simple subgroup of $\mathrm{Aut}(\mathcal{T})$ (for the latter assertion see [Tit70]).

Before stating some main properties of the universal group, we record another definition which is given in general, for a locally finite tree, and which is used in the sequel as a key property.

Definition 2.2.6 (See [Tit70])**.** Let T be a locally finite tree and let $G \leq \mathrm{Aut}(T)$ be a closed subgroup. We say that G has **Tits' independence property** if for every edge e of T we have the equality $G_e = G_{T_1}G_{T_2}$, where T_i are the two infinite half sub-trees of T emanating from the edge e and G_{T_i} is the pointwise stabilizer of the half-tree T_i.

We mention that Tits' independence property guarantees the existence of 'enough' rotations in the group G. It is used in the work of Tits as a sufficient condition to prove simplicity of 'large' subgroups of $\mathrm{Aut}(\mathcal{T})$ (see [Tit70]). In his thesis, [Ama03, Theorem 2], Amann employes it to give a complete classification of all super-cuspidal representations of a closed subgroup in $\mathrm{Aut}(\mathcal{T})$ acting transitively on the vertices and on the boundary of $\mathcal{T}$ and having Tits' independence property. For a closed subgroup G of $\mathrm{Aut}(\mathcal{T})$ that acts transitively on the vertices and on the boundary of $\mathcal{T}$ but which does not enjoy Tits' independence property less is known about the complete classification of all its super-cuspidal representations. In contrast, for the above mentioned groups, with or without Tits' independence property, the remaining two classes of irreducible unitary representations, namely the special and respectively, the spherical ones, are completely classified in [FTN91, Chapter III, resp. Chapter II].

Following [BM00a, Ama03], we enumerate the following properties of the groups $U(F)$ and $U(F)^+$:

1) $U(F)$ and $U(F)^+$ have Tits' independence property;
2) $U(F)^+$ is trivial or simple;
3) if F acts transitively on the set $\{1,...,d\}$ then $U(F)$ acts transitively on the vertices of $\mathcal{T}$ and it is a unimodular group;

4) $U(F)$ and $U(F)^+$ act transitively on the boundary $\partial\mathcal{T}$ if and only if F is 2-transitive;

5) When F is transitive and generated by its point stabilizers, by [Ama03, Prop. 57], we have that $U(F)^+$ is edge-transitive, $U(F)^+ = U(F) \cap \mathrm{Aut}(\mathcal{T})^+$ and $U(F)^+$ is of index 2 in $U(F)$. For example, this is the case when F is primitive but not cyclic of prime order. When F is primitive and not generated by its point stabilizers, all point stabilizers of F are equal. This implies that all point stabilizers of F are just the identity, that F is cyclic of prime order and that the group $U(F)^+$ is trivial. Under these hypotheses, we obtain that $U(F)$ is a nontrivial, discrete subgroup of $\mathrm{Aut}(\mathcal{T})$.

Regarding the 2-transitive action on the boundary $\partial\mathcal{T}$ of a group $G \leq \mathrm{Aut}(\mathcal{T})$, we have the following general well-known equivalences.

Proposition 2.2.7. *(See* [FTN91], [BM00a, Lemma 3.1.1]*) For a closed subgroup $G \leq \mathrm{Aut}(\mathcal{T})$ the following are equivalent:*

i) G_x is transitive on $\partial\mathcal{T}$, for every $x \in \mathcal{T}$;

ii) G is non-compact and there exists $x \in \mathcal{T}$ such that G_x acts transitively on $\partial\mathcal{T}$;

iii) G is non-compact and transitive on $\partial\mathcal{T}$;

iv) G is 2-transitive on $\partial\mathcal{T}$.

Remark 2.2.8. (See [FTN91, Prop. 10.2]) One can prove that if a closed, non-compact subgroup $G \leq \mathrm{Aut}(\mathcal{T})$ is transitive on the boundary $\partial\mathcal{T}$, then either G is transitive on the vertices of $\mathcal{T}$ or it has exactly two G-vertex-orbits in $\mathcal{T}$.

The following proposition gives us a slightly different, but equivalent, description of a 2-transitive action on $\partial\mathcal{T}$.

Proposition 2.2.9. *(See Theorem 3.1.2, Theorem. 3.3.11) Let G be a closed, non-compact subgroup of* $\mathrm{Aut}(\mathcal{T})$ *of type-preserving automorphisms. The following are equivalent:*

i) G is strongly transitive on $\mathcal{T}$, where $\mathcal{T}$ is viewed as a 1-*dimensional Euclidean building;*

ii) G is 2-transitive on $\partial\mathcal{T}$;

iii) G has no fixed point in $\partial\mathcal{T}$ and for some point $\xi \in \partial\mathcal{T}$, the stabilizer G_ξ acts co-compactly on $\mathcal{T}$;

iv) G acts co-compactly on $\mathcal{T}$, without a fixed point in $\partial\mathcal{T}$, and there exists a compact open subgroup in G having finitely many orbits on ∂T;

v) G acts co-compactly on $\mathcal{T}$, without a fixed point in $\partial\mathcal{T}$, and every compact open subgroup has finitely many orbits on ∂T;

vi) G acts co-compactly on $\mathcal{T}$ and (G, G_x) is a Gelfand pair, where x is a vertex of $\mathcal{T}$.

As explained in the Introduction, the following proposition states an important property regarding the proper open subgroups of $U(F)^+$. This property is a useful tool employed also in the proofs of the results obtained in Section 2.2.4.

Proposition 2.2.10. *(See* [CDM11, Prop. 4.1]*) The group F is primitive if and only if every proper open subgroup of $U(F)^+$ is compact.*

To simplify the notations, for the rest of this chapter we refer to the group $U(F)^+$ simply by $\mathbb{G}$, where F is a group of permutations of $\{1, ..., d\}$. From now on, consider also fixed a legal coloring $i : \mathcal{T} \to \{1, ..., d\}$ of the tree $\mathcal{T}$.

2.2.3 New characterizations of 2-transitivity

In addition to property 4) from Section 2.2.2, the next propositions provide new characterizations of the 2-transitivity of the $\mathbb{G}$-action on the boundary $\partial\mathcal{T}$. These characterizations are expressed in terms of the subgroups $\mathbb{G}_\xi$ and the existence of hyperbolic elements in $\mathbb{G}_\xi$, where $\xi \in \partial\mathcal{T}$.

Proposition 2.2.11. *$\mathbb{G}$ is transitive on the boundary $\partial\mathcal{T}$ if and only if for every $\xi \in \partial\mathcal{T}$, $\mathbb{G}_\xi$ contains a hyperbolic element.*

Proof. '$\Rightarrow$' This implication is easy and follows from Propositions 2.2.7 and 2.2.9.

'$\Leftarrow$' Suppose that for every $\xi \in \partial\mathcal{T}$ the group $\mathbb{G}_\xi$ contains a hyperbolic element. We want to show that F is 2-transitive, which by property 4) from Section 2.2.2 is equivalent to the transitivity of $\mathbb{G}$ on the boundary $\partial\mathcal{T}$.

Suppose that F is not 2-transitive. Therefore, there exists a color, say 1, such that F_1 is not transitive on $\{2, ..., d\}$ (otherwise F would be 2-transitive). Moreover, without loss of generality, we can suppose that F_1 does not contain any element mapping 2 to 3.

Consider a vertex $v \in \mathcal{T}$ and starting from this vertex we want to construct a particular geodesic ray $[v, \eta)$, with $\eta \in \partial\mathcal{T}$. We enumerate its vertices by $\{v_i\}_{i\geq 0}$, such that $v_0 = v$. We construct the geodesic ray $[v, \eta)$ by choosing the vertices $\{v_i\}_{i\geq 0}$ such that the successive colors of the edges along the path $[v, \eta)$ are:

$$(12)(13)(12)^2(13)(12)^3(13)\cdots(12)^m(13)(12)^{m+1}(13)\cdots,$$

where $(12)^m$ denotes the geodesic segment of length $2m$ colored successively with the pair of ordered colors $(1, 2)$. Notice that the choice of the vertices $\{v_i\}_{i\geq 0}$ with the above prescribed coloration is unique. Moreover, it determines a unique endpoint $\eta \in \partial\mathcal{T}$ and thus, a unique geodesic ray $[v, \eta)$.

By hypothesis, $\mathbb{G}_\eta$ contains at least one hyperbolic element. Take one of those and denote it by b; consider that its translation length is $2N$ and that its attracting endpoint is η. Notice there exists a vertex v_b on the geodesic ray $[v, \eta)$ such that for every $v \in [v_b, \eta)$ we have that $b(v) \in [v_b, \eta)$. Moreover, if we travel deep enough on the geodesic ray $[v_b, \eta)$ we find a geodesic segment whose coloring is the following string $(12)^N(13)(12)^N$. Let x be the midpoint of this geodesic segment. The two edges in $[v_b, \eta)$ emanating from x have colors 1 and 3, where the two edges in $[v_b, \eta)$ emanating from $b(x)$ have colors 1 and 2. We obtained that b sends the pair of ordered colors $(1, 2)$ into the pair of ordered colors $(1, 3)$. This contradicts that F_1 does not contain any element mapping 2 to 3. The proposition stands proven. $\square$

Proposition 2.2.12. *Let F be transitive and generated by its points stabilizers. Then $\mathbb{G}$ acts transitively on the boundary $\partial\mathcal{T}$ if and only if every hyperbolic element of $\mathbb{G}$ is a power of a hyperbolic element of translation length two.*

Proof. The implication '⇒' follows by applying Propositions 2.2.7 and 2.2.9.

Consider the reverse implication '⇐'. Suppose moreover that $\mathbb{G}$ is not transitive on the boundary $\partial\mathcal{T}$. This is equivalent to F not being 2-transitive. Therefore, there exists a color, say 1, such that F_1 is not transitive on $\{2, \cdots, d\}$. Moreover, without loss of generality, we can assume that F does not contain any element sending the pair of ordered colors $(1, 2)$ to the pair of ordered colors $(1, 3)$.

Construct in $\mathcal{T}$ a bi-infinite geodesic line colored using only the array of ordered colors $(1, 2, 1, 3)$ and denote it by ℓ. As F is transitive and generated by its points stabilizers, the group $\mathbb{G}$ is edge-transitive on $\mathcal{T}$; therefore, $\mathbb{G}$ acts co-compactly on $\mathcal{T}$. Apply then Proposition 3.2.9 to the geodesic line ℓ. There exists thus a hyperbolic element $h \in \mathbb{G}$ such that its translation axis intersects ℓ into a geodesic segment of a big length. By hypothesis h is a power of a hyperbolic element of translation length 2. This implies that there exists an element in $\mathbb{G}$ such that the pair of ordered colors $(1, 2)$ is sent into the pair of ordered colors $(1, 3)$. This gives an element of F sending $(1, 2)$ to $(1, 3)$ which is a contradiction. The conclusion follows. □

2.2.4 Towards the Howe–Moore property

Recall the following definition.

Definition 2.2.13. Let G be a locally compact group and let $\alpha = \{g_n\}_{n>0}$ be a sequence in G. Corresponding to α we define the set:

$$U_\alpha^+ := \{g \in G \mid \lim_{n\to\infty} g_n^{-1} g g_n = e\}.$$

Notice that U_α^+ is a subgroup of G. It is called the **contraction group** corresponding to α. Because it does not need to be closed in general, we denote by N_α^+ the closed subgroup $\overline{U_\alpha^+}$. In the same way, but using $g_n g g_n^{-1}$ we define U_α^- and N_α^-.

When $g_n = a^n$, for every $n > 0$ and for $a \in G$, we simply denote U_α^+ by U_a^+.

For example, the next easy lemma describes the contraction group of a sequence $\alpha = \{g_n\}_{n>0} \subset \mathrm{Aut}(T)$ that enjoys particular properties.

Lemma 2.2.14. *Let T be a locally finite tree and $\alpha = \{g_n\}_{n>0} \subset \mathrm{Aut}(T)$. Assume there exist $x \in T$ and $\xi \in \partial T$ with the property that $g_n(x) \to \xi$ when $n \to \infty$, the limit being considered with respect to the cone topology on $T \cup \partial T$. Then every element $g \in \mathrm{Aut}(T)$ that fixes pointwise the half-tree of T, which emanates from the vertex x and which contains ξ in its boundary, is an element of U_α^+.*

Proof. Let $g \in \mathrm{Aut}(T)$ that fixes pointwise the half-tree $T_1 \subset T$, which emanates from the vertex x and which contains ξ in its boundary ∂T_1. We have to prove that $\lim\limits_{n\to\infty} g_n^{-1} g g_n = e$. This is equivalent to showing that for every $r > 0$, there exists $N_r > 0$ such that $g_n^{-1} g g_n(B(x,r)) = B(x,r)$ pointwise, for every $n > N_r$, where $B(x,r) \subset T$ is the ball centered in x and of radius r. The latter condition is translated as $g(B(g_n(x),r)) = B(g_n(x),r)$ pointwise, for every $n > N_r$. This equality is true by hypothesis, as $g_n(x) \to \xi$ and g fixes pointwise the half-tree T_1. The conclusion follows. $\square$

Notice that the sequence $\{g_n = a^n\}_{n>0}$, where $a \in \mathrm{Aut}(T)$ is a hyperbolic element, satisfies the hypotheses of Lemma 2.2.14; the point $\xi \in \partial T$ is the attracting endpoint of a and x is a vertex of the translation axis of a. In this particular case, one can prove more when considering the group $\mathbb{G}$. The next lemma emphasizes the importance of contractions groups.

Lemma 2.2.15. *Let F be primitive but not cyclic of prime order. Let a be a hyperbolic element in $\mathbb{G}$. Then $\mathbb{G} = \langle U_a^+, U_a^- \rangle$.*

In particular, for every unitary representation $(\pi, \mathcal{H})$ of $\mathbb{G}$, without non-zero $\mathbb{G}$-invariant vectors and where $\mathcal{H}$ a separable complex Hilbert space, all matrix coefficients of $(\pi, \mathcal{H})$ corresponding to the sequence $\{a^n\}_{k\geq 0}$ are C_0.

Proof. Denote the translation axis of the hyperbolic element a by ℓ and its repelling and attracting unique endpoints by ξ_- and respectively, by ξ_+.

Let g be an element in U_a^+, which by definition has the property $\lim\limits_{n\to\infty} a^{-n} g a^n = e$. One deduces that $g(\xi_+) = \xi_+$ and that g fixes at least one vertex of $\mathcal{T}$. Therefore $g \in \mathbb{G}^0_{\xi_+} := \{h \in \mathbb{G} \mid h(\xi_+) = \xi_+ \text{ and } h \text{ is not hyperbolic}\}$. We conclude that $U_a^+ \leq \mathbb{G}^0_{\xi_+}$. In the same way we have that $U_a^- \leq \mathbb{G}^0_{\xi_-}$.

Moreover, we claim that $\mathbb{G}^0_{\xi_+} \leq \langle U_a^+, U_a^- \rangle$. Indeed, let $g \in \mathbb{G}^0_{\xi_+}$ and consider a vertex $x \in \ell$ fixed by g. The geodesic ray $[x, \xi_+) \subset \ell$ is thus pointwise fixed by g. Let e be an edge in the interior of $[x, \xi_+)$. By Tits' independence property we obtain that $g = g_1 h_1$, where g_1 fixes pointwise the half-tree emanating from the edge e and containing the endpoint ξ_+ and h_1 fixes pointwise the other half-tree emanating from e, which moreover contains the endpoint ξ_-. By Lemma 2.2.14 we have that $h_1 \in U_a^-$ and $g_1 \in U_a^+$. In particular $\mathbb{G}_e \leq \langle U_a^+, U_a^- \rangle$ and our claim follows.

To conclude, we obtain that $\mathbb{G}^0_{\xi_+}, \mathbb{G}^0_{\xi_-} \leq \langle U_a^+, U_a^- \rangle$ and $\langle U_a^+, U_a^- \rangle$ is an open subgroup of $\mathbb{G}$, because it contains the open subgroup $\mathbb{G}_e$. But $\langle U_a^+, U_a^- \rangle$ is not compact as otherwise it would be contained in a vertex stabilizer by [CDM11, Lemma 2.6], which is impossible (see also Lemma 2.2.17). Thus $\langle U_a^+, U_a^- \rangle$ is an open, non-compact subgroup of $\mathbb{G}$. By Proposition 2.2.10 we must have that $\langle U_a^+, U_a^- \rangle = \mathbb{G}$.

The last part of the lemma follows by applying Lemma 2.1.11 and Lemma 2.1.17. $\square$

Lemma 2.2.15 motivates the following remark.

Remark 2.2.16. Let F be primitive but not $2-$transitive and not cyclic of prime order. Let $(\pi, \mathcal{H})$ be a unitary representation of $\mathbb{G}$ without non-zero $\mathbb{G}$-invariant vectors and where $\mathcal{H}$ is a separable complex Hilbert space. Consider $v \neq 0 \in \mathcal{H}$ and let $\mathbb{G}_v := \{g \in \mathbb{G} \mid \pi(g)v = v\}$. We record the following two possibilities for the closed subgroup $\mathbb{G}_v$ of $\mathbb{G}$:

i) $\mathbb{G}_v$ is compact;

ii) $\mathbb{G}_v$ is not compact. In this case, as the unitary representation $(\pi, \mathcal{H})$ is without non-zero $\mathbb{G}$-invariant vectors, by Lemma 2.2.15 we obtain that $\mathbb{G}_v$ does not contain hyperbolic elements. Applying the result of Tits [Tit70, Prop. 3.4] we conclude that $\mathbb{G}_v$ is a closed subgroup of a stabilizer of an endpoint $\xi \in \partial \mathcal{T}$.

Remark 2.2.16 is completed by the following lemma and proposition.

Lemma 2.2.17. *Let F be primitive but not cyclic of prime order. Then for every $\xi \in \partial \mathcal{T}$ the subgroup $\mathbb{G}_\xi^0$ is not compact, where $\mathbb{G}_\xi^0 := \{g \in \mathbb{G} \mid g(\xi) = \xi$ and g is not hyperbolic$\}$.*

Proof. Suppose there is $\xi \in \partial \mathcal{T}$ such that $\mathbb{G}_\xi^0$ is compact. Using [CDM11, Lemma 2.6] there exists a vertex $x \in \mathcal{T}$ such that $\mathbb{G}_\xi^0 \leq \mathbb{G}_x$. Take the geodesic ray $[x, \xi)$ and let e be an edge in $[x, \xi)$ with $\mathrm{dist}_{\mathcal{T}}(e, x) > 1$. Using Tits' independence property of $\mathbb{G}$ we claim that for every $g \in \mathbb{G}_e$, $g(e') = e'$, where e' is an edge contained in $[x, \xi)$ such that $d(e', x) \leq d(e, x) - 1$. Indeed, we have $\mathbb{G}_e = \mathbb{G}_{\mathcal{T}_1} \mathbb{G}_{\mathcal{T}_2}$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are the two infinite half sub-trees of $\mathcal{T}$ emanating from the endpoints of e. Therefore, if $g \in \mathbb{G}_{\mathcal{T}_1}$ then $g(e') = e'$, where $\mathcal{T}_1$ is the half-tree not containing the end ξ. If $g \in \mathbb{G}_{\mathcal{T}_2}$ then $g \in \mathbb{G}_\xi \leq \mathbb{G}_v$ and thus $g(e') = e'$.

We conclude that $\mathbb{G}_e \subset \mathbb{G}_{e'}$, for any two edges e, e' of $[x, \xi)$ with $\mathrm{dist}_{\mathcal{T}}(e, x) > 1$ and $d(e', x) \leq d(e, x) - 1$. In particular, if we take e' such that $d(e', x) = d(e, x) - 1$ then we obtain $F_{i(e)} \leq F_{i(e')}$, where by F_j, with $j \in \{1, ..., d\}$, we denote the stabilizer of the color j in the permutation subgroup F.

Now, as the geodesic ray $[x, \xi)$ is d-colored, there are at least two different colors on this geodesic ray, say $k \neq j$, for which $F_j = F_k$.

Choose an edge e_1 on the axis $[x, \xi)$ which is colored j. Consider the unique bi-infinite geodesic line ℓ containing e_1 and colored only with j and k, alternatively. Now $\mathbb{G}_\ell := \{g \in \mathbb{G} \mid g$ stabilizes $\ell\}$ contains $\mathbb{G}_{e_1}$. This is because if the color j is fixed then also is the color k, as $F_j = F_k$, and vice versa. Therefore, $\mathbb{G}_\ell$ is an open subgroup in $\mathbb{G}$.

But $\mathbb{G}_\ell$ is not compact because it contains a hyperbolic element of translation length 2, i.e., whose translation axis is ℓ. Indeed, observe that the group $U(\mathrm{id})$ is a subgroup of $U(F)$ and it preserves the fixed coloring i of the tree. In particular, we can construct in $U(\mathrm{id})$ a hyperbolic element h of translation length 2 whose translation axis is ℓ. Moreover, this hyperbolic element h belongs to $\mathrm{Aut}(\mathcal{T})^+$. As F is primitive, apply [Ama03, Prop. 57]. We obtain that $U(F)^+ = U(F) \cap \mathrm{Aut}(\mathcal{T})^+$ and so $h \in \mathbb{G}_\ell \leq U(F)^+$. We conclude that $\mathbb{G}_\ell$ is a non-compact, open, proper subgroup of $\mathbb{G}$ contradicting Proposition 2.2.10. The conclusion follows. $\square$

Remark 2.2.18. Recall from Section 1.2 that a locally finite tree T is endowed with the cone topology which turns $T \cup \partial T$ into a compact topological space. Let $x \in T$ and take a sequence $\{g_n\}_{n>0} \subset \mathrm{Aut}(T)$ such that $g_n \to \infty$. Therefore $\mathrm{dist}_T(x, g_n(x)) \to \infty$. By compactness, we can extract a subsequence $\{g_{n_k}\}_{n_k}$ such that $\{g_{n_k}(x)\}_{n_k}$ converges with respect to the cone topology to an endpoint $\xi \in \partial T$.

The next proposition is a strengthening of the Mautner's phenomenon when Tits' independence property is fulfilled and is stated in the general framework of groups acting on locally finite trees.

Proposition 2.2.19. *Let T be a locally finite tree. Let $G \leq \mathrm{Aut}(T)$ be a closed, non-compact subgroup with Tits' independence property and $(\pi, \mathcal{H})$ be a unitary representation of G. Assume there exists a sequence $\{g_n\}_{n>0} \subset G$ and a vector $v \in \mathcal{H}$ such that $g_n \to \infty$ and $\{\pi(g_n)(v)\}_{n>0}$ weakly converges to a non-zero vector $v_0 \in \mathcal{H}$.*

Then for every $\xi \in \partial T$, with the property that there exists $x \in T$ and a subsequence $\{g_{n_k}\}_{n_k>0}$ such that $g_{n_k}(x) \to \xi$, we have that $N^+_{\{g_{n_k}\}_{n_k}} \leq G^0_\xi \leq G_{v_0}$.

Moreover, if G_{v_0} is non-compact and does not contain hyperbolic elements then $G_{v_0} = G^0_\xi$, for a point $\xi \in \partial T$.

Proof. First of all, by Mautner's phenomenon [BM00, Ch. III, Thm. 1.4] (see Section 2.1.2) the vector v_0 is $N^+_{\{g_n\}_n}$-invariant. So $N^+_{\{g_n\}_n} < G_{v_0}$.

Consider now a vertex $x \in T$. As $g_n \to \infty$, by Remark 2.2.18 we extract a subsequence $\{g_{n_k}\}_{n_k>0} \subset \{g_n\}_{n>0}$ such that $g_{n_k}(x) \to \xi$ in the cone topology of $T \cup \partial T$.

For such an $\{g_{n_k}\}_{n_k>0}$ and ξ, we claim that $N^+_{\{g_{n_k}\}_{n_k}} < G^0_\xi$. As G^0_ξ is a closed subgroup, it is enough to show that $U^+_{\{g_{n_k}\}_{n_k}} < G^0_\xi$. Indeed, let $g \in U^+_{\{g_{n_k}\}_{n_k}}$. By definition we have $\lim\limits_{n\to\infty} g^{-1}_{n_k} g g_{n_k} = e$. As $g_{n_k}(x) \to \xi$ in the cone topology, we deduce that g fixes the vertices $g_{n_k}(x)$ and so also the endpoint ξ. We conclude that $g \in G^0_\xi$.

Moreover, we claim that $G^0_\xi \leq G_{v_0}$.

Indeed, let $g \in G^0_\xi$ and we want to prove that $g \in G_{v_0}$. Let $[x_0, \xi)$ be a geodesic ray fixed by g and we index its vertices by x_j. Take an edge $[x_j, x_{j+1}]$, with $j \geq 0$ and we have that $g \in G_{[x_j,x_{j+1}]} := \{h \in G \mid h[x_j, x_{j+1}] = [x_j, x_{j+1}]\}$. By Tits' independence property we have that $g = t_j h_j$, where $t_j \in G_{T_{j+1}}$, $h_j \in G_{T_j}$, T_{j+1} is the half-tree emanating from the vertex x_{j+1} and containing the end ξ, and T_j is the half-tree emanating from the vertex x_j and not containing the end ξ. As g and t_j are stabilizing the end ξ we deduce that $h_j \in G^0_\xi$.

In addition, as t_j fixes pointwise the half-tree T_{j+1} containing the endpoint ξ and $g_{n_k}(x) \to \xi$, by Lemma 2.2.14 we have that $\lim_{n\to\infty} g^{-1}_{n_k} t_j g_{n_k} = e$. Thus $t_j \in N^+_{\{g_{n_k}\}_{n_k}} < G_{v_0}$.

Remark that $h_j \to e$ when $j \to \infty$. Therefore $gh_k^{-1} = t_k \to g$. Because G_{v_0} is a closed subgroup and $t_k \in G_{v_0}$ we deduce that $g \in G_{v_0}$. The claim follows.

Let us now prove the last assertion of the proposition by supposing that G_{v_0} is non-compact and does not contain hyperbolic elements. By the result of Tits [Tit70, Prop. 3.4] we have that G_{v_0} is contained in the stabilizer G^0_η of an endpoint $\eta \in \partial T$. As G_{v_0} is non-compact, choose a sequence $\{h_m\}_{m>0} \subset G_{v_0}$ such that $h_m \to \infty$. Remark that $\{\pi(h_m)(v_0)\}_{m>0}$ weakly converge to v_0 and for any vertex $x \in T$ we have that $h_m(x) \to \eta$, with respect to the cone topology of $T \cup \partial T$. Applying the first part of the proposition we have that $G^0_\eta \leq G_{v_0}$ and the conclusion follows. $\square$

By Remarks 2.2.16 and 2.2.18 we obtain:

Corollary 2.2.20. *Let F be primitive but not 2-transitive and not cyclic of prime order. Let $(\pi, \mathcal{H})$ be a unitary representation of $\mathbb{G}$ without non-zero $\mathbb{G}$-invariant vectors and where $\mathcal{H}$ is a separable complex Hilbert space. Consider $v \in \mathcal{H}$ such that $\mathbb{G}_v$ is not compact. Then $\mathbb{G}_v = \mathbb{G}^0_\xi$ for a point $\xi \in \partial\mathcal{T}$.*

Using the results obtained until now, the next theorem provides an equivalent characterization for the group $\mathbb{G}$ to have the Howe–Moore property. We stress here that in general, for locally compact groups an 'if and only if' condition for the Howe–Moore property is difficult to be achieved.

Theorem 2.2.21. *Let F be primitive but not cyclic of prime order. Then $\mathbb{G}$ has the Howe–Moore property if and only if for every unitary representation $(\pi, \mathcal{H})$ of $\mathbb{G}$, without non-zero $\mathbb{G}$-invariant vectors, and for every $v \neq 0 \in \mathcal{H}$ the closed subgroup $\mathbb{G}_v$ is compact.*

Proof. Remark that, by [CdCL+11, Prop. 3.2] and Proposition 2.2.10, the primitivity of F is a necessary condition for the Howe–Moore property. Also, by Remark 2.1.7 and Lemma 2.1.6, it is enough to consider only unitary representations over separable Hilbert spaces.

Moreover, the implication '$\mathbb{G}$ has Howe–Moore property implies $\mathbb{G}_v$ is compact' is easy and holds in general. Indeed, if $\mathbb{G}_v$ is non-compact then there exists a sequence $\{g_n\}_n \subset \mathbb{G}_v$ with $g_n \to \infty$. Therefore, the corresponding matrix coefficients $\langle \pi(g_n)(v), v\rangle = 1$ which is a contradiction.

Consider now the reverse implication.

Suppose there exists a unitary representation $(\pi, \mathcal{H})$ of $\mathbb{G}$, without non-zero $\mathbb{G}$-invariant vectors and where $\mathcal{H}$ is a separable Hilbert space, two non-zero vectors $v, w \in \mathcal{H}$ and a sequence $\{g_n\}_{n>0} \subset \mathbb{G}$ such that $g_n \to \infty$ and $|\langle \pi(g_n v, w\rangle| \nrightarrow 0$. Restricting to a subsequence and without loss of generality we can consider also that $\{\pi(g_n)(v)\}_{n>0}$ weakly converges to $v_0 \neq 0 \in \mathcal{H}$.

Take a vertex $x \in \mathcal{T}$. As $g_n \to \infty$, by Remark 2.2.18 we extract a subsequence $\{g_{n_k}\}_{n_k>0} \subset \{g_n\}_{n>0}$ such that $g_{n_k}(x) \to \xi$ in the cone topology of $T \cup \partial T$. Applying Proposition 2.2.19 we obtain that $\mathbb{G}_\xi^0 \leq \mathbb{G}_{v_0}$. By Lemma 2.2.17 we conclude that $\mathbb{G}_{v_0}$ is not compact, which contradicts our hypotheses. The theorem stands proven. $\square$

Chapter 3

Gelfand pairs and strong transitivity for Euclidean buildings

3.1 Introduction

A main general problem in any field of Mathematics is the 'classification' of various central objects related to that field. In most of the cases this attempt gives rise to new theories and very hard questions.

Such an example is the well-known problem of understanding general groups by their representations. In particular, for several natural classes of non-discrete, topologically simple, locally compact groups one could envision to obtain a complete classification of all their irreducible unitary representations. Towards this classification, many machineries and techniques were developed for both families of simple locally compact groups: the connected and respectively, the totally disconnected ones. Until now, all those attempts are far from being enough to give a complete picture of the theory, except for some particular cases, for example in the case of the semi-simple real Lie group $\mathrm{SL}_2(\mathbf{R})$ or for the full group of automorphisms of a d-regular tree T_d, namely, $\mathrm{Aut}(T_d)$, which is a totally disconnected locally compact group. For the former, one can consult the book of Serge Lang [Lan75]. For the latter example, a good overview of the theory of unitary representations is given in [Ol′77] and [FTN91], which is completed by a more recent general study, the

Amann's PhD thesis [Ama03]. In both of the cases, for $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{Aut}(T_d)$, the set of all irreducible unitary representations is divided in the following main classes: the unitary principal series, the unitary complementary series and the discrete series representations. To obtain all their irreducible unitary representations we add the trivial representation and two more inequivalent irreducible unitary representations.

For a general connected semi-simple real Lie group, the complete classification of all its irreducible unitary representations is still unknown, giving rise to an intense field of research even in the recent years (see of example [SRV98], [Vog00] or [BW00]). Equally, for the general case of non-discrete, locally compact groups acting on trees, the theory of unitary representations is undeveloped to a large extent. The only cases where we have a complete classification are the closed, non-compact subgroups of $\mathrm{Aut}(T_d)$ acting transitively on the vertices and on the boundary of T_d and enjoying Tits' independence property (see [FTN91] and [Ama03] where the results of [Ol′77] are used). An example of group which does not belong to the above mention family of groups is the p-adic Lie group $\mathrm{SL}(2, \mathbf{Q}_p)$; it does not satisfy Tits' independence property.

Regarding the unitary principal and complementary series of $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{Aut}(T_d)$ we record another common feature. In both cases, those two series of irreducible unitary representations are completely determined using a general theory: the theory of Gelfand pairs combined with the computation of all the eigenfunctions of a so-called 'Laplace operator'.

More precisely:

Definition 3.1.1. Let G be a locally compact group and $K \leq G$ be a compact subgroup. We denote by $C_c^K(G)$ the space of continuous, compactly supported functions $\phi : G \to \mathbf{C}$ that are K**-bi-invariant**, i.e., functions that satisfy the equality $\phi(kgk') = \phi(g)$ for every $g \in G$ and all $k, k' \in K$. We view the $\mathbf{C}$-vector space $C_c^K(G)$ as an algebra whose multiplication is given by the convolution product

$$\phi * \psi \colon x \mapsto \int_G \phi(xg)\psi(g^{-1})d\mu(g).$$

Moreover, $C_c^K(G)$ is called the **Hecke algebra** corresponding to $K \leq G$ and (G, K) forms a **Gelfand pair** if the convolution algebra $C_c^K(G)$ is commutative. A good introduction to the theory of

Gelfand pairs for general locally compact groups can be consulted in [vD09].

The following two families provide prototypical examples of locally compact groups that admit Gelfand pairs. In the first place, for the class of connected locally compact groups, we have that all semi-simple non-compact real Lie groups, with finite center, together with their maximal compact subgroup form a Gelfand pair. In the second place, for the totally disconnected locally compact groups, the only known examples of Gelfand pairs are coming from groups acting by automorphisms on Euclidean buildings (e.g., semi-simple p-adic Lie groups or groups acting by automorphisms and with further properties on a d-regular tree). More precisely, in [Léc10] and [APVM13], they have been obtained that, among groups acting strongly transitively on a locally finite thick general building of arbitrary type, one has a Gelfand pair only if the building is Euclidean and the compact subgroup is the stabilizer of a special vertex, the latter assertion following from [Mat77, Corollary 4.2.2]. For completion, we mention that the abstract parabolic Hecke algebras of a Coxteter group studied in [APVM13] are canonically isomorphic to the corresponding Hecke algebras $C_c^{G_x}(G)$, introduced in Definition 3.1.1, where G is a locally compact group acting continuously and properly by type-preserving automorphisms and strongly transitively (see Def. 1.1.7) on a locally finite thick Euclidean building Δ and $G_x \leq G$ is the stabilizer of a vertex x of Δ. For these isomorphisms the reader can consult [Mat77, pages 136–137], where it is concluded that the Hecke algebra $C_c^B(G)$ is canonically isomorphic to the abstract Hecke algebra '$K(W, q)$', where $B \leq G$ denotes the minimal parabolic subgroup, the stabilizer in G of a chamber. To deduce from there that the above mentioned algebras are isomorphic, the reader must notice that $C_c^{G_x}(G)$ are sub-algebras of $C_c^B(G)$.

In fact, by the result presented in the this section (see [CC]), in the Euclidean case the hypothesis of strong transitivity cannot be relaxed:

Theorem 3.1.2. *Let G be a locally compact group acting continuously and properly by type-preserving automorphisms on a locally finite thick Euclidean building Δ. Then the following are equivalent:*

(i) G acts strongly transitively on Δ;

(ii) *G acts strongly transitively on the spherical building at infinity $\partial\Delta$;*

(iii) *G acts cocompactly on Δ, and there exists a compact subgroup K of G such that (G, K) is a Gelfand pair.*

Combining the results of [Léc10] and [APVM13] with Theorem 3.1.2, we conclude the following important fact.

Corollary 3.1.3. *A locally compact group G acting continuously, properly and cocompactly by type-preserving automorphisms on a locally finite thick Euclidean building Δ, admits a Gelfand pair only when G acts strongly transitively on Δ. In this case, its only Gelfand pairs are of the form (G, G_x), where x is a special vertex of Δ.*

Proof. If G admits a Gelfand pair, by Theorem 3.1.2, we conclude that G must acts strongly transitively on Δ. Therefore, by [Léc10] and [APVM13], the only Gelfand pairs in this case are of the form (G, G_x), where x is a special vertex of Δ. □

For the convenience of the reader, in the particular case of a closed, non-compact group G acting co-compactly, by type-preserving automorphisms on a locally finite thick tree, we include below (see Theorem 3.1.9) the idea of the proof of the main Theorem 3.1.2. For the case of trees, Theorem 3.1.2, which was mentioned without proof in the book [FTN91, page 48], asserts that a group G, enjoying the above mentioned conditions, admits a Gelfand pair (G, K), where K is the stabilizer of a vertex of the tree, if and only if G acts 2-transitively on the boundary of the tree. The latter assertion is equivalent also to the transitivity of G on the set of all pairs (ℓ, e), where ℓ is a bi-infinite geodesic line in the tree and which moreover contains the edge e.

A basic criterion to prove the Gelfand property, used for example in the case semi-simple non-compact real Lie group which are with finite center, or for closed, non-compact groups acting by type-preserving automorphisms and strongly transitively on a locally finite thick tree, is given by the following result:

Proposition 3.1.4 (See Proposition 6.1.3 in [vD09])**.** *Let G be a locally compact group and K a compact subgroup of G. Assume there exists a continuous involutive automorphism σ of G such that $\sigma(g) \in Kg^{-1}K$ for all $g \in G$. Then (G, K) is a Gelfand pair.*

Another important fact about Gelfand pairs is the following:

Proposition 3.1.5 (See Proposition 6.3.1 in [vD09]). *Let G be a locally compact group and K a compact subgroup of it. The pair (G, K) is Gelfand if and only if for every irreducible unitary representation π of G on a Hilbert space $\mathcal{H}$ the dimension of the subspace of K-fixed vectors is at most one.*

Before stating an important consequence of Proposition 3.1.5, which is in fact the main motivation for studying Gelfand pairs, let us introduce two more notions.

Definition 3.1.6. Let G be a locally compact group and let $f : G \to \mathbf{C}$ a continuous function. We say that f is **positive-definite** if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} f(g_j^{-1} g_i) \geq 0$$

for all $g_1, ..., g_n$ in G and $c_1, ..., c_n$ in $\mathbf{C}$.

Definition 3.1.7 (Proposition–Definition). Let G be a locally compact group and (G, K) be a Gelfand pair. Let f be a K-bi-invariant, continuous function on G. The function f is **spherical** if and only if:

i) $f(e) = 1$,

ii) for every $F \in C_c^K(G)$ there is a complex number $\lambda(F)$ such that $F * f = \lambda(F) f$.

Corollary 3.1.8 (See Corollary 6.3.3 in [vD09]). *Let (G, K) be a Gelfand pair. The positive-definite spherical functions on G correspond one-to-one to the equivalence classes of irreducible unitary representations of G having a one-dimensional K-fixed vector space.*

Therefore, by Corollary 3.1.8, if we want to classify the representations in the spherical unitary dual of a locally compact group G admitting a Gelfand pair (G, K), it is enough to compute all positive-definite spherical functions on G and to try to classify these ones. By Proposition–Definition 3.1.7, this could be achieved if one computes the common eigenfunctions of the elements in the Hecke algebra $C_c^K(G)$ corresponding to $K < G$. This is why in the literature, the elements of $C_c^K(G)$ are generally called Hecke operators or

Laplacians. However, for a general Gelfand pair (G, K), the positive-definite spherical functions on G cannot be explicitly computed, but in some very particular cases: for example when $G = \mathrm{SO}(n, 1)$, with $n \geq 2$, or for our favorite groups acting on d-regular trees, namely, for closed, non-compact subgroups of $\mathrm{Aut}(T_d)$ acting transitively on the vertices and on the boundary of T_d, without necessarily enjoying Tits' independence property (see [FTN91, Chap.II]). For the former example of $(\mathrm{SO}(n, 1), \mathrm{SO}(n))$, all positive-definite spherical functions are computed via the eigenfunctions of the usual Laplace–Beltrami operator defined on the the corresponding symmetric space $\mathrm{SO}(n, 1)/\mathrm{SO}(n)$ (see for example [vD09, Section 7.5]). Equally, for the latter class of example of groups that act on d-regular trees and that enjoy the above mentioned properties, all positive-definite spherical functions are computed using the combinatorial Laplace operator defined on the corresponding 'symmetric space' which is the d-regular tree (see [FTN91, Chap.II]). In both cases, we obtain a complete and explicit description of the unitary principal and complementary series of those groups. For general connected or totally disconnected locally compact groups the theory is difficult to a large extent. Primely, this is because of the fact that for a general semi-simple real Lie group the eigenfunctions of the Laplace–Beltrami operator of the corresponding symmetric space are very hard to compute. Secondly, for most locally finite thick Euclidean buildings, which are the analog of the symmetric spaces of semi-simple real Lie groups, the combinatorial Laplace operator does not admit sufficiently 'good' properties.

As we promised, we include here the main ideas of the proof of Theorem 3.1.2 by treating the particular case of a locally finite thick tree.

Theorem 3.1.9. *Let G be a closed, non-compact group acting by type-preserving automorphisms on a locally finite thick tree T. Then the following are equivalent:*

(i) G acts strongly transitively on T (i.e., for every two pairs (ℓ_1, e_1) and (ℓ_2, e_2), consisting of a bi-infinite geodesic line ℓ_i in T which contains the edge e_i, there exists $g \in G$ such that $g(\ell_1) = \ell_2$ and $g(e_1) = e_2$);

(ii) G acts 2-transitively on ∂T;

(iii) *G acts co-compactly on T, and (G, K) is a Gelfand pair, where K is the stabilizer of a vertex of T.*

Proof. The implication (i) implies (ii) is immediate.

The converse implication (ii) $\Rightarrow$ (i) is well-known (see [FTN91] and [BM00a, Lemma 3.1.1]). For the convenience of the reader we give here a proof. Let (ℓ_1, e_1) and (ℓ_2, e_2) be two pairs, consisting of a bi-infinite geodesic line ℓ_i in T which contains moreover the edge e_i. By the 2-transitivity of G on ∂T, there exists $g \in G$ such that $g(\ell_1) = \ell_2$, as any two points of ∂T determine a unique bi-infinite geodesic line in the tree T. It remains to show that for a bi-infinite geodesic line $\ell \subset T$, the stabilizer of ℓ in G, namely, $\mathrm{Stab}_G(\ell)$, acts transitively on the set of all edges of ℓ. To obtain this it is enough to prove the following claim: for any two adjacent edges $e_1, e_2 \subset \ell$ there exists $g \in \mathrm{Stab}_G(\ell)$ such that $g(e_1) = e_2$ and $g(e_2) = e_1$. Indeed, denote by v the common vertex of e_1, e_2 and by $\xi_-, \xi_+ \in \partial T$ the two ends of ℓ, where the vertex of e_1 which is different from v is pointing towards ξ_-. As the tree is thick, there exists $\eta \in \partial T$ such that the geodesic ray (v, η) is disjoint from ℓ. By the 2-transitivity of G on ∂T there exists $g_1 \in G$ with $g_1(\eta) = \eta$ and $g_1(\xi_-) = \xi_+$. In particular, as g_1 cannot be a hyperbolic element, we have that $g_1(v) = v$ and g_1 fixes pointwise the geodesic ray $[v, \eta)$. We obtain that $g_1(e_1) = e_2$. Then, there exists $g_2 \in G$ such that $g_2(\xi_+) = \xi_+$ and $g_2(g_1(\xi_+)) = \xi_-$. Notice that, as above, $g_2([v, \xi_+)) = [v, \xi_+)$ pointwise and we have that $g_2(g_1(e_2)) = e_1$. We obtained that $g_2 g_1(v) = v$, $g_2 g_1(\xi_-) = g_2(\xi_+) = \xi_+$ and $g_2 g_1(\xi_+) = \xi_-$. We have found $g := g_2 g_1 \in \mathrm{Stab}_G(\ell)$ with $g(e_1) = e_2$. Our claim stands proven and the conclusion follows.

The implication (ii) $\Rightarrow$ (iii) is proved in [FTN91, Lemma 4.1] and uses Proposition 3.1.4 where σ is the identity on G.

Let us prove the converse implication: (iii) $\Rightarrow$ (ii). We mention that this implication is announced without proof in [FTN91, page 48]. Moreover, this is proved in full generality in this Chapter (see [CC]), namely, for any locally compact group acting continuously and properly, by type-preserving automorphisms on a locally finite thick Euclidean building. For the particular case of locally finite thick trees, the proof can easily be visualized by 'playing' on the tree T.

Let v be a vertex in T such that (G, K) is a Gelfand pair, where K is the stabilizer of v in G. Suppose first that G fixes some point

$\xi \in \partial T$. It then follows from [CM13, Theorem M] that G is not unimodular. This is a contradiction, since any locally compact group endowed with a Gelfand pair is unimodular by [vD09, Proposition 6.1.2]. Therefore, as G acts co-compactly on T and does not fix any point in ∂T, by [Tit70] there exists a hyperbolic element $a \in G$. Denote the translation axis of a by ℓ_a and let its repelling endpoint and the attracting endpoint be denoted by ξ_- and respectively, by ξ_+. Notice that v is not necessarily contained in ℓ_a, but for the simplicity of the exposition and because here we only intend to give the intuition behind the proof of Theorem 3.1.2, we assume that v is a vertex of ℓ_a.

Suppose now that G is not 2-transitive on ∂T. Then, by Theorem 3.3.11 (iv), whose proof works exactly the same as there for the particular case of a locally finite thick tree, we obtain that K has an infinite number of endpoint-orbits in ∂T. By our assumption that $v \in \ell_a$, we have that $K\ell_a$ does not 'cover' all the tree T and moreover we have that there exists $\eta \in \partial T \setminus \{K\xi_- \cup K\xi_+\}$. Notice that, if we consider the ball $B(v,r) \subset T$ around the vertex v and where r is a very big radius, we find a vertex $y := \partial B(v,r) \cap [v,\eta)$ such that every geodesic ray with base point in (y,η) does not belong to $\{K\xi_- \cup K\xi_+\}$. More precisely, we consider the set

$$U_{y,\eta} := \{[y,\eta_1) \mid \eta_1 \in \partial T \text{ and } (y,\eta_1) \cap (y,\eta) \neq \emptyset\},$$

where $y \in (v,\eta)$ is taken such that $\{K\xi_- \cup K\xi_+\} \cap U_{y,\eta} = \emptyset$. With respect to the cone topology on $T \cup \partial T$, the set $U_{y,\eta}$ is an open neighborhood of $[y,\eta)$. Fix a vertex $y \in (v,\eta)$ with the above property. In fact, for such a choice, notice that

$$\{K\xi_- \cup K\xi_+\} \cap KU_{y,\eta} = \emptyset.$$

In addition, remark that the open neighborhoods

$$U_{x_\pm,\xi_\pm} := \{[x_\pm,\eta_1) \mid \eta_1 \in \partial T \text{ and } (x,\eta_1) \cap (y,\xi_\pm) \neq \emptyset\},$$

where $x_\pm := \partial B(v,r) \cap [v,\xi_\pm)$, for the same radius r chosen to obtain y, have the property

$$\{KU_{x_+,\xi_+} \cup KU_{x_-,\xi_-}\} \cap KU_{y,\eta} = \emptyset. \tag{3.1}$$

This can easily be verified because we are living in a tree.

Remember that G acts co-compactly on T; therefore, intuitively, there must exist another hyperbolic element $b \in G$ having a geodesic ray of its translation axis entirely contained in $U_{y,\eta}$. Consider that the attracting endpoint of b, denoted by η_+, belongs to $U_{y,\eta}$. Therefore, $U_{y,\eta}$ is also an open neighborhood of η_+, with resect to the cone topology on $T \cup \partial T$. Notice that, in general, the vertex v does not belong necessarily to the translation axis of b. But, to simplify the exposition, we consider that v is contained in the translation axis of b.

As, by assumption, v is a vertex of the translation axis of the hyperbolic element a, as well as of the translation axis of the hyperbolic element b, we consider big positive powers n, m of a and respectively, of b, such that $a^n(v) \in U_{x_+,\xi_+} \setminus \{x_+\}$ and $b^m(v) \in U_{y,\eta} \setminus \{y\}$.

We claim that, for such chosen powers n and m, the element

$$a^n b^m \notin K b^m K a^n K. \tag{3.2}$$

To prove the claim, it is enough to verify that $a^n b^m(v) \notin K b^m K a^n(v)$. Indeed, by the choice of U_{x_+,ξ_+}, the powers n, m and the fact that we are living in a tree, we easily have that $a^n b^m(v) \in U_{x_+,\xi_+}$.

Let us evaluate the set $K b^m K a^n(v)$. We have that $K a^n(v) \subset K U_{x_+,\xi_+}$. Then $b^m K a^n(v) \subset U_{y,\eta}$. This is because b^m translates the segment $[k a^n(v), v]$ into the segment $[b^m k a^n(v), b^m(v)]$, for every $k \in K$, and that $b^m(v) \neq y \in U_{y,\eta}$, by assumptions. In addition, we obtain that $K b^m K a^n(v) \subset K U_{y,\eta}$. Using now the equation (3.1), our claim that $a^n b^m(v) \notin K b^m K a^n(v)$ follows.

To end the proof, we really copy-paste the same proof of the original Theorem 3.1.2.

Set

$$\phi = \mathbf{1}_{K a^n K} \qquad \text{and} \qquad \psi = \mathbf{1}_{K b^m K}.$$

Clearly the maps ϕ, ψ are both locally constant, hence continuous, and K-bi-invariant. Thus $\phi, \psi \in C_c^K(G)$ and we claim that $\phi * \psi \neq \psi * \phi$.

Indeed, we first observe that

$$\begin{array}{rcl} \phi * \psi(a^n b^m) & = & \int_G \phi(a^n b^m g)\psi(g^{-1}) d\mu(g) \\ & \geq & \int_{b^{-m}K} \phi(a^n b^m g)\psi(g^{-1}) d\mu(g) \\ & = & \int_K \phi(a^n k)\psi(k^{-1} b^m) d\mu(k) \\ & = & \mu(K) \\ & > & 0, \end{array}$$

where μ is a Haar measure on G.

On the other hand, consider

$$\psi * \phi(a^n b^m) = \int_G \psi(a^n b^m g)\phi(g^{-1}) d\mu(g).$$

For the integrand to be nonzero, we need that the variable g satisfies both $g^{-1} \in Ka^nK$ and $a^nb^mg \in Kb^mK$. For such an element $g \in G$, we may find $k_1, k_2, k_3, k_4 \in K$ such that $g^{-1} = k_1a^nk_2$ and $a^nb^mg = k_3b^mk_4$. This yields

$$a^nb^m = k_3b^mk_4g^{-1} = k_3b^mk_4k_1a^nk_2 \in Kb^mKa^nK,$$

which contradicts our choice of a^n and b^m. Therefore, there is no choice of g making the integrand nonzero in the integral defining $\psi * \phi(a^nb^m)$. This implies that $\psi * \phi(a^nb^m) = 0$. The claim stands proven, thereby confirming that the convolution algebra $C_c^K(G)$ is not commutative.

□

For the general case of a locally finite thick Euclidean building Δ, in order to prove Theorem 3.1.2, a more elaborate proof is needed. First, notice that the implication from (i) to (ii) of Theorem 3.1.2 is easy and well-known to hold in general for abstract groups acting on (not necessarily locally finite) thick Euclidean buildings. For the converse implication '(ii) $\Rightarrow$ (i)', the major part of our proof, which is given in Section 3.3 below, also holds without assuming that G is locally compact nor that Δ is locally finite. This hypothesis is only used in the final stage of the proof. However, in Remark 3.3.10 below, we describe a possible strategy towards proving this equivalence in full generality. In the particular case of Δ being a tree, the equivalence between (i) and (ii) is due to Burger–Mozes [BM00a, Lemma 3.1.1].

The fact that (i) implies (iii), in Theorem 3.1.2, is well-known (see [Mat77, Corollary 4.2.2]). The remaining implication from (iii) to (i) is established in Section 3.4 and requires the introduction of several new ingredients: further characterizations of the equivalence of (i) and (ii) given by Theorem 3.3.11, as well as the existence in G of automorphisms with a peculiar dynamics, which are called **strongly regular hyperbolic elements**. We define those as hyperbolic automorphisms of Δ, whose translation axes are contained in a unique apartment, and cross every wall of that apartment. They play a similar role as the regular semi-simple elements in the case of algebraic groups, or as the hyperbolic isometries in the case of tree automorphism groups. The following existence result, established in Section 3.2, is of independent interest.

Theorem 3.1.10. *Let G be a group acting cocompactly by automorphisms on a locally finite Euclidean building Δ. Then G contains a strongly regular hyperbolic element.*

The particular case when G is discrete can be deduced from Ballmann–Brin [BB95], where they use a Poincaré recurrence argument on the geodesic flow. Our proof is different and is inspired by an argument due to E. Swenson [Swe99].

In our considerations, the relevance of strongly regular hyperbolic elements comes from their peculiar dynamical properties, which show some resemblance with the dynamics of hyperbolic isometries of rank one symmetric spaces (or more generally, of Gromov hyperbolic CAT(0) spaces):

Proposition 3.1.11. *Let Δ be a Euclidean building and $a \in \mathrm{Aut}(\Delta)$ be a type-preserving strongly regular element with unique translation apartment A. Then, for any point ξ in the visual boundary $\partial\Delta$, the limit $\lim_{n\to\infty} a^n(\xi)$ exists (in the cone topology) and belongs to ∂A. In particular, the fixed-point-set of a in $\partial\Delta$ is ∂A.*

A slightly more precise statement will be established in Proposition 3.2.10 below.

The remaining of this chapter is organized as follows. Section 3.2 is devoted to strongly regular hyperbolic automorphisms. Various properties are described, and the proofs of Theorem 3.1.10 and Proposition 3.1.11 are completed. Section 3.3 is devoted to strong transitivity. The goal there is to establish Theorem 3.3.11, which contains the equivalence between (i) and (ii) from Theorem 3.1.2, as well as some additional characterizations of independent interest. A good deal of Section 3.3 holds in the general context of abstract groups acting on thick Euclidean buildings. Finally, Section 3.4 is devoted to Gelfand pairs; this is where the proof of Theorem 3.1.2 is completed.

3.2 Strongly regular hyperbolic automorphisms

Throughout this chapter, we view Euclidean buildings both as CAT(0) spaces and as simplicial complexes. We refer to [AB08] and [BH99] for the main concepts and basic properties, which will be used freely in the rest of this chapter.

The goal of this section is to study the basic properties of strongly regular hyperbolic automorphisms of Euclidean buildings and to prove Theorem 3.1.10.

3.2.1 Regular and strongly regular isometries

The following definition is inspired by the notion of **R**-hyper-regular elements, appearing in Prasad–Raghunathan's paper [PR72], or more recently in [BL93], where they are called *h*-regular elements.

Definition 3.2.1. Let X be a CAT(0) space, γ be a hyperbolic isometry of X and let

$$\mathrm{Min}(\gamma) := \{x \in X \mid d(x, \gamma(x)) = |\gamma|\},$$

where $|\gamma|$ denotes the translation length of γ. We say that γ is a **regular hyperbolic isometry** of X if $\mathrm{Min}(\gamma)$ is at bounded Hausdorff distance from a maximal flat of X.

It follows that the visual boundary of $\mathrm{Min}(\gamma)$ is a round sphere in ∂X. In view of [Lee00, Prop. 2.1], this implies that $\mathrm{Min}(\gamma)$ contains a γ-invariant flat of X, which therefore lies at a bounded Hausdorff distance from the corresponding maximal flat. Notice that in a general CAT(0) space, maximal flats need not exist. Moreover, two different maximal flats need not have the same dimension.

In the case when X is a Riemannian symmetric space or a Bruhat–Tits building, the full isometry group $\mathrm{Is}(X)$ is known, is very big and it acts strongly transitively on the spherical building at infinity of X. Using the known properties of $\mathrm{Is}(X)$ in those cases, it is easy to construct hyperbolic isometries $\gamma \in \mathrm{Is}(X)$ such that $\mathrm{Min}(\gamma)$ is a maximal flat. In particular, such elements are regular hyperbolic isometries in the sense above. Moreover, if X is a proper CAT(0) space and γ is a rank one element (in the sense of Ballmann [Bal95]), then γ is also regular hyperbolic; in that case the relevant maximal flat is a rank one geodesic line of X (see [Bal95, Chapter 3]).

In the present section we focus on the case where X is a Euclidean building. In this case the maximal flats coincide with the apartments. Therefore, if γ is a regular hyperbolic isometry, there is a unique apartment A contained in $\mathrm{Min}(\gamma)$. However, even if $A = \mathrm{Min}(\gamma)$, it could be that the γ-axes (which are pairwise parallel) are **singular**, i.e., are parallel to some wall of A. The case where this does not happen is central to our purposes and we give it a special name.

Definition 3.2.2. Let Δ be a Euclidean building and $\gamma \in \mathrm{Aut}(\Delta)$. We say that γ is **strongly regular hyperbolic** if γ is regular hyperbolic and if one (and hence all) of its translation axes crosses all the walls of the unique apartment contained in $\mathrm{Min}(\gamma)$.

We record the following easy characterization.

Lemma 3.2.3. *Let Δ be a Euclidean building and $\gamma \in \mathrm{Aut}(\Delta)$ be a type-preserving automorphism. Then γ is strongly regular hyperbolic if and only if γ is a hyperbolic isometry and the two endpoints of one (and hence all) of its translation axes lie in the interior of two opposite chambers of the spherical building at infinity.*

In particular, if γ is strongly regular hyperbolic, then $\mathrm{Min}(\gamma)$ is an apartment.

The apartment $\mathrm{Min}(\gamma)$ will henceforth be called the **translation apartment** of γ; it is uniquely determined.

Proof of Lemma 3.2.3. The 'only if' part is clear. Conversely, if γ is hyperbolic and if the endpoints of some γ-axis, say ℓ, lie respectively in the interior of two opposite chambers $c_+, c_- \in \mathrm{Ch}(\partial\Delta)$, then the same holds for all γ-axes. Consider now the unique apartment A of Δ whose boundary contains c_+ and c_-. Since γ is type-preserving and it fixes the two endpoints of its translation axes, it follows that γ fixes c_- and c_+ pointwise. Therefore, we conclude that the apartment A is invariant under γ. In particular, we obtain that γ acts trivially on ∂A; in other words γ acts by translations on A. This proves that $A \subseteq \mathrm{Min}(\gamma)$. To prove the converse inclusion, we remark that any translation axis ℓ' of γ has the same endpoints as ℓ, and must thus be entirely contained in A. Thus $A = \mathrm{Min}(\gamma)$. Since the endpoints of ℓ lie in the interior of the chambers c_-, c_+, the line ℓ cannot be parallel to any wall of A. Therefore, it meets every such wall. This confirms that γ is strongly regular hyperbolic. $\square$

Lemma 3.2.3 motivates the following.

Definition 3.2.4. Let Δ be a Euclidean building. A geodesic line ℓ is called **strongly regular** if its endpoints lie in the interior of two opposite chambers of the spherical building at infinity.

Thus, in this case, the line ℓ is contained in a unique apartment A and ℓ crosses every wall of A. Lemma 3.2.3 shows that a type-preserving hyperbolic automorphism is strongly regular if and only if its translation axes are strongly regular.

Notice that strongly regular geodesic lines are defined by means of an asymptotic property, namely a property of its endpoints at infinity. An easy but crucial observation is the existence of a local criterion to recognize strongly regular lines:

Lemma 3.2.5. *Let Δ be a Euclidean building and ℓ be a geodesic line. If ℓ contains two special vertices which are contained, respectively, in the interiors of two opposite sectors in a common apartment of Δ, then ℓ is strongly regular. Conversely, if ℓ is strongly regular and contains three special vertices, then at least two of them must be contained in the interiors of two opposite sectors of a common apartment.*

Proof. Let $v, v' \in \ell$ be special vertices of the same type respectively contained in the interiors of two opposite sectors s, s' of some apartment A. Then A contains the geodesic segment $[v, v']$, which cannot be parallel to any wall of A. In particular, there exists at least one chamber c which is contained in the intersection of all apartments containing both v and v'; in particular c is contained in A. Let now B be an apartment containing ℓ and consider the retraction $\rho_{c,A}$ onto A based at c. As there is an isomorphism of B onto A, which fixes the intersection $A \cap B$ pointwise, we obtain that $\rho_{c,A}$ maps ℓ to a geodesic line ℓ_A of A containing $[v, v']$. By hypothesis, the line ℓ_A is strongly regular. Since the restriction of $\rho_{c,A}$ to B induces an isomorphism $B \to A$, we infer that ℓ is also strongly regular, as desired.

The converse statement is straightforward. $\square$

3.2.2 Existence of strongly regular elements

The first step is the special case of thin buildings.

Lemma 3.2.6. *Let (W, S) be a Euclidean Coxeter system and A be the associated Coxeter complex. Then W contains strongly regular hyperbolic elements.*

Proof. Recall that the translation subgroup of W acts simply transitively on the set of special vertices of a given type of the Coxeter complex A. Let v and v' be two special vertices respectively contained in the interiors of two opposite sectors of A. Then the unique geodesic line ℓ through v and v' is strongly regular and the unique translation t mapping v to v' is an element of W having ℓ as a translation axis. Thus t is strongly regular hyperbolic by Lemma 3.2.3. $\square$

The following basic construction, valid in general CAT(0) spaces, is a key step in proving the existence of strongly regular hyperbolic automorphisms in groups acting on a locally finite Euclidean building Δ. In what follows, given a proper CAT(0) space X, we denote its visual boundary by ∂X and endow the space $X \cup \partial X$ with the **cone topology**, which is compact (see [BH99, Chap.II.8] for the basic definitions). The following lemma is technical, but its proof relies on a standard compactness argument.

Lemma 3.2.7. *Let X be a proper $\mathrm{CAT}(0)$ space, $G < \mathrm{Is}(X)$ be any group of isometries and $\rho\colon \mathbf{R} \to X$ be a geodesic map. Assume there is an increasing sequence $\{t_n\}_{n\geq 0}$ of positive real numbers tending to infinity such that $\sup_n d(\rho(t_n), \rho(t_{n+1})) < \infty$ and the set $\{\rho(t_n)\}_{n\geq 0}$ falls into finitely many G-orbits, each of which is moreover discrete.*

Then there exist a sequence $\{g_n\}_{n\geq 0}$ in G and an increasing sequence $\{f(n)\}_{n\geq 0}$ of positive integers such that for all $m \geq 0$ and $r \in [-m, m]$ the sequence $\{g_n \circ \rho(t_{f(n)} + r)\}_{n\geq m}$ is constant. Furthermore, the limit map

$$\rho'\colon \mathbf{R} \to \Delta : r \mapsto \lim_{n\to\infty} g_n \circ \rho(t_{f(n)} + r)$$

is geodesic. In particular, for $n \geq m$, the intersection $g_n(\rho(\mathbf{R})) \cap \rho'(\mathbf{R})$ is a geodesic segment of length $\geq 2m$.

If in addition X is a locally finite Euclidean building and $\rho(\mathbf{R})$ is strongly regular, then $\rho'(\mathbf{R})$ is strongly regular as well.

Proof. Set $x_n = \rho(t_n)$ and $C = \sup_n d(x_n, x_{n+1})$. Let also $V = \bigcup_n G(x_n)$ denote the union of the G-orbits of the points x_n. Since the sequence $\{x_n\}_{n\geq 0}$ falls into finitely many G-orbits, we may find a subsequence $\{x_{f(n)}\}_{n\geq 0}$ and a sequence $\{g_n\}_{n\geq 0} \subset G$ such that the sequence $\{g_n(x_{f(n)})\}_{n\geq 0}$ is constant. We set $v_0 = g_n(x_{f(n)}) \in V$.

Since X is proper and V is discrete by hypothesis, any ball around v_0 contains finitely many points of V. On the other hand, for all $n \geq m \geq 0$, the geodesic segment $g_n \circ \rho([t_{f(n)} - m, t_{f(n)} + m])$, which is contained in the ball of radius $2m$ around v_0, contains at least $\lfloor \frac{2m}{C} \rfloor$ points of V. Therefore, we may adjust the sequence $\{f(n)\}_n$ to ensure that for all $m \geq 0$ the set of geodesic segments $\{g_n \circ \rho([t_{f(n)} - m, t_{f(n)} + m]) \ : \ n \geq m\}$ is exactly one segment. Thus, by construction, the map $\rho' \colon r \mapsto \lim\limits_{n\to\infty} g_n \circ \rho(t_n + r)$ is well defined and therefore $\rho'(\mathbf{R})$ is the limit, when $n \to \infty$, of the sequence of geodesic lines $g_n(\rho(\mathbf{R}))$, which are converging uniformly on compact sets. Thus $\rho'(\mathbf{R})$ is a also a geodesic line. That $\rho'(\mathbf{R})$ is strongly regular in the case when X is a locally finite Euclidean building and $\rho(\mathbf{R})$ is strongly regular follows from Lemma 3.2.5. □

We will also need the following subsidiary fact.

Lemma 3.2.8. *Let X be a* CAT(0) *space and $\rho\colon \mathbf{R} \to X$ be a geodesic line. Let moreover h be an isometry. Suppose there exist $t < t' \in \mathbf{R}$ and $c > 0$ such that*

$$h \circ \rho(t + \varepsilon) = \rho(t' + \varepsilon)$$

for all $\varepsilon \in [-c, c]$. Then h is a hyperbolic element admitting a translation axis containing the segment $[\rho(t), \rho(t')]$.

Proof. Set $x = \rho(t)$ and $y = \rho(t')$, so $h(x) = y$. By hypothesis, the Alexandrov angle $\angle_y(x, h(y)) = \angle_y(x, h^2(x))$ is equal to π. It follows that the three points x, y and $h(y) = h^2(x)$ are collinear and so the geodesic segment $[x, h^2(x)]$ contains the point y. The same argument used inductively shows that all points of the sequence $\{h^n(x)\}_{n\in\mathbf{Z}}$ are contained in a common geodesic line, which we denote by ℓ. Therefore, by this construction the geodesic line ℓ is preserved by h. Moreover, $x, y \in \ell$ and by hypothesis $h(x) = y \neq x$. Thus, h is a hyperbolic isometry having ℓ as a translation axis. □

The following result is a technical relative of Theorem 3.1.10, valid for more general CAT(0) spaces. We state it separately for the sake of future references. Notice that groups acting isometrically on CAT(0) spaces with discrete orbits are plentiful: they include (not necessarily discrete) groups whose action preserves a locally finite simplicial (or cellular) decomposition of the space, or totally disconnected groups acting minimally on a space with the geodesic extension property (as a consequence of [CM09, Th. 6.1]).

Proposition 3.2.9. *Let X be a proper* CAT(0) *space, $G < \mathrm{Is}(X)$ be any group of isometries and $\rho\colon \mathbf{R} \to X$ be a geodesic map. Assume that there is an increasing sequence $\{t_n\}_{n\geq 0}$ of positive real numbers tending to infinity such that $\sup_n d(\rho(t_n), \rho(t_{n+1})) < \infty$ and that the set $\{\rho(t_n)\}_{n\geq 0}$ falls into finitely many G-orbits, each of which is moreover discrete.*

Then there is an increasing sequence $\{f(n)\}_n$ of positive integers such that, for all $n > m > 0$, there is a hyperbolic isometry $h_{m,n} \in G$ which has a translation axis containing the geodesic segment $[\rho(t_{f(m)}), \rho(t_{f(n)})]$.

If in addition X is a locally finite Euclidean building and the geodesic line $\rho(\mathbf{R})$ is strongly regular, then $h_{m,n}$ is a strongly regular hyperbolic automorphism.

Proof. Let $\{g_n\}_{n\geq 0} \subset G$ and $\{f(n)\}_{n\geq 0} \subset \mathbf{N}$ be the sequences afforded by Lemma 3.2.7. Fix $m > 0$. Then by Lemma 3.2.7 we know that for all $r \in [-m, m]$, the sequence $\{g_n \circ \rho(t_{f(n)} + r)\}_{n\geq m}$ is constant. In particular, for all $n > m$, the element $h_{m,n} = g_m^{-1} g_n \in G$ has the property that

$$h_{m,n} \circ \rho(t_{f(n)} + r) = \rho(t_{f(m)} + r)$$

for all $r \in [-m, m]$. Therefore, Lemma 3.2.8 ensures that $h_{m,n}$ is hyperbolic and has a translation axis passing through $\rho(t_{f(m)})$ and $\rho(t_{f(n)})$. In the case when X is a locally finite Euclidean building and the geodesic line $\rho(\mathbf{R})$ is strongly regular, this axis must be therefore strongly regular by Lemma 3.2.5. In view of Lemma 3.2.3 we conclude that $h_{m,n}$ is strongly regular hyperbolic as well. $\square$

We are now able to complete the proof of Theorem 3.1.10 from the introduction.

Proof of Theorem 3.1.10. We first observe that the subgroup of $\mathrm{Aut}(\Delta)$ consisting of type-preserving automorphisms is of finite index, see [AB08, Prop. A.14]. In particular, G has a finite index subgroup acting by type-preserving automorphisms, whose action remains therefore cocompact. There is thus no loss of generality in assuming that the G-action is type-preserving.

Let A be an apartment in Δ. By Lemma 3.2.6, the Weyl group of Δ acting on A contains strongly regular hyperbolic elements. Since any point and so also any vertex of A belongs to an axis of a fixed such strongly regular element, it follows that there exists a geodesic map $\rho\colon \mathbf{R} \to A$ and some $C > 0$ such that $\ell = \rho(\mathbf{R})$ is strongly regular and that the sequence $\{\rho(nC)\}_{n\geq 0}$ are special vertices of the same type. The set of all special vertices is clearly discrete. Since the G-action is cocompact by hypothesis, it follows that G has finitely many orbits of special vertices. Therefore, the desired conclusion now follows from Proposition 3.2.9. □

3.2.3 Dynamics of strongly regular elements

Hyperbolic isometries of CAT(-1) spaces (and more generally of Gromov hyperbolic metric spaces) enjoy remarkable dynamical properties: they have a unique attracting point and a unique repelling point in the visual boundary and any other point of the boundary is contracted by the positive powers of the isometry towards the attracting fixed point. It turns out that this property is shared by rank one isometries of CAT(0) spaces. However, it cannot be expected that such a peculiar dynamical behavior extends to all regular hyperbolic isometries of CAT(0) spaces, since a hyperbolic isometry g acts trivially on the visual boundary of $\mathrm{Min}(g)$, which is generally not reduced to a pair of boundary points.

We shall however see that the dynamics of a strongly regular automorphism of a Euclidean building has a contracting property which is reminiscent from (but weaker than) the hyperbolic case mentioned above.

In the following statement, the symbol $\rho_{A,c}$ denotes the retraction onto an apartment A and based at the ideal chamber $c \in \mathrm{Ch}(\partial A)$, as it is introduced in Section 3.3.1 below. The following is a strengthening of Proposition 3.1.11 from the introduction.

Proposition 3.2.10. *Let Δ be a Euclidean building and $a \in \mathrm{Aut}(\Delta)$ be a type-preserving strongly regular hyperbolic element with unique* *translation apartment A. Let $c_- \in \mathrm{Ch}(\partial A)$ be the unique chamber containing the repelling fixed point of a.*

Then, for any $\xi \in \partial\Delta$, the limit $\lim_{n\to\infty} a^n(\xi)$ exists (in the cone topology) and coincides with $\rho_{A,c_-}(\xi) \in \partial A$. In particular, the fixed-point-set of a in $\partial\Delta$ is ∂A.

Proof. Let $\xi \in \partial\Delta$ and consider an apartment B whose boundary contains ξ and c_-. Let Q be a sector pointing to c_- and contained in the intersection $A \cap B$. Let moreover $p \in Q$ be a base point contained in the interior of Q. Since B is convex, the geodesic ray $[p, \xi)$ is entirely contained in B. If $\xi \in c_-$, then ξ is fixed by a and the desired conclusion follows. We assume henceforth that $\xi \notin c_-$. In particular, the ray $[p, \xi)$ is not entirely contained in Q and hence there is some $q \in Q$ such that $[p, \xi) \cap Q = [p, q]$.

Since p lies in the interior of Q, it follows that the segment $[p, q]$ is of positive length. Therefore, in the apartment A, the geodesic segment $[p, q]$ can be extended uniquely to a geodesic ray emanating from p; in other words, there is a unique boundary point $\eta \in \partial A$ such that the segment $[p, q]$ is contained in the ray $[p, \eta)$.

We claim that $\lim_{n\to\infty} a^n(\xi) = \eta$. We refer to Figure 3.1 for a graphical illustration of the argument.

Since a is strongly regular, we have $a^n(Q) \supset Q$ for all $n > 0$. Moreover $[a^n(p), a^n(q)] = [a^n(p), a^n(\xi)) \cap a^n(Q)$. Since a^n acts on A as a Euclidean translation, it follows that $[a^n(p), a^n(q)] = a^n([p, q])$ is parallel to $[p, q]$ (in the Euclidean sense). On the other hand, the apartment $a^n(B)$ contains both the ray $[p, a^n(\xi))$ and the ray $[a^n(p), a^n(\xi))$, which are parallel since they have the same endpoint. It follows that $[p, a^n(\xi))$ contains $[p, q]$. We next observe that, since a is strongly regular, for any boundary wall M of Q, the distance between M and $a^n(M)$ grows linearly with n. This implies that the length of the geodesic segment $[p, a^n(\xi)) \cap a^n(Q)$ tends to infinity with n. In particular so does the length of $[p, a^n(\xi)) \cap A$.

We have thus shown that the geodesic ray $[p, a^n(\xi))$ contains $[p, q]$ for all $n \geq 0$ and also a subsegment of A whose length tends to infinity with n. By the definition of η, this implies that $[p, a^n(\xi)) \cap [p, \eta)$ is a segment whose length also tends to infinity with n. In particular we have $\lim_{n\to\infty} a^n(\xi) = \eta$ and the claim stands proven.

Now, let $\xi' \in \partial B$ be another boundary point of B. Since B is a Euclidean flat, the CAT(1) distance between ξ and ξ' coincides with

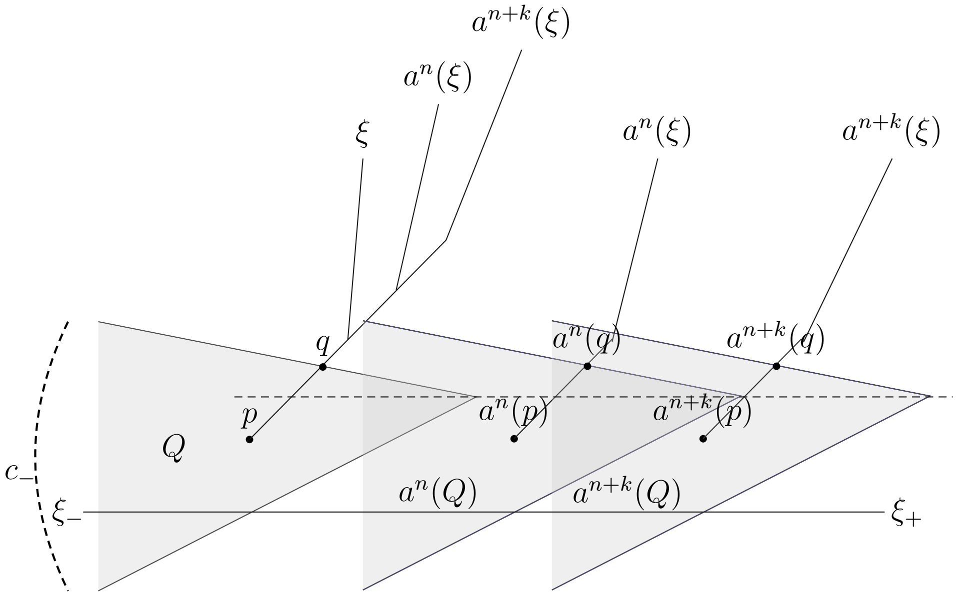

Figure 3.1: The line (ξ_-, ξ_+) is a translation axis of the strongly regular element a; c_- is the chamber in ∂A containing ξ_-; Q is a sector in A corresponding to c_-; the line between p and ξ is the geodesic ray $[p, \xi)$.

the angle at p between the rays $[p, \xi)$ and $[p, \xi')$. The claim above implies that this angle also coincides with the CAT(1) distance between $\lim_{n\to\infty} a^n(\xi)$ and $\lim_{n\to\infty} a^n(\xi')$. Therefore, the restriction to ∂B of the map

$$\partial\Delta \to \partial A : \xi \mapsto \lim_{n\to\infty} a^n(\xi)$$

is an isometry of ∂B onto ∂A fixing c_- pointwise. By the definition of the retraction, this implies that $\lim_{n\to\infty} a^n(\xi) = \rho_{A,c_-}(\xi)$. $\square$

Moreover, Proposition 3.2.10 shows that the boundary of the translation apartment (without the 'repelling' ideal chamber c_-) of a strongly regular hyperbolic element behaves like an attracting locus for the orbits at infinity. The following result highlights an additional 'attracting property', with respect to the interior of the building, of the translation apartment of a strongly regular hyperbolic element: geodesic segments joining a point to its image under large powers of the strongly regular element must pass through that apartment.

Proposition 3.2.11. *Let Δ be a Euclidean building and $a \in \mathrm{Aut}(\Delta)$ be a type-preserving strongly regular hyperbolic element with unique translation apartment A. Let also S be a finite set of special vertices of Δ and let $x_0 \in S$.*

Then for every $T > 0$, there exists $N \geq 1$ such that for all $n \geq N$ and $x \in S$, the geodesic segment $[x_0, a^n(x)]$ contains a subsegment of length greater than T entirely contained in A.

Proof. Let $x \in S$. We will find a constant N_x such that for all $n \geq N_x$, the geodesic segment $[x_0, a^n(x)]$ contains a subsegment of length greater than T entirely contained in A. Once this has been done, the constant $N = \max_{x \in S} N_x$ satisfies the desired conclusions.

Let ξ_-, ξ_+ denote the two endpoints of the strongly regular translation axis of a. By Lemma 3.2.3, these endpoints ξ_- and ξ_+ lie respectively in the interior of some chambers at infinity, say c_-, c_+. In order to find the constant N_x, we distinguish several cases.

Assume first that $x \in A$. Let then A_{x_0} be an apartment in Δ with the property that $x_0 \in A_{x_0}$ and c_+ is an ideal chamber in $\mathrm{Ch}(\partial A_{x_0})$. As c_+ is a common ideal chamber of $\mathrm{Ch}(\partial A_{x_0})$ and $\mathrm{Ch}(\partial A)$, the apartments A and A_{x_0} share a common sector corresponding to c_+. Denote it by Q_{x_0}. Remark that Q_{x_0} is a subsector of the sector, in A_{x_0}, with base point x_0 and ideal chamber c_+. Now because $x \in A$ and $\lim_{n \to \infty} a^n(y) = \xi_+$, there exists $N_x > 1$ such that for every $n > N_x$ we have $a^n(x) \in Q_{x_0}$. Therefore the geodesic segment from x_0 to $a^n(x)$ passes through the apartment A, for every $n > N_x$. Moreover, as n increases, so does the length of the intersection $[x_0, a^n(x)] \cap A$ which is contained in Q_{x_0}.

Assume next that $x_0 \in A$. Permuting x_0 and x and replacing a by a^{-1}, we are reduced to the case that has already been treated.

Consider now the remaining case of x_0 and $x \notin A$. Let A_{x_0} and A_x be two apartments in Δ such that $x_0 \in A_{x_0}$, c_+ is a chamber of ∂A_{x_0}, $x \in A_x$ and c_- is a chamber of ∂A_x. Denote by Q_{x_0} (resp., Q_x) a common sector of A and A_{x_0} (resp., of A and A_x) corresponding to c_+ (resp., to c_-). As above, remark that Q_{x_0} (resp., Q_x) is a subsector of the sector in A_{x_0} (resp., A_x) with base point x_0 (resp., x) and ideal chamber c_+ (resp., c_-). Let moreover z be the base point of the sector Q_{x_0}. We can choose also Q_x to be a sector, with base point y, such that x is contained in the interior of the opposite sector to Q_x, with base point y, of the apartment A_x (see Figure

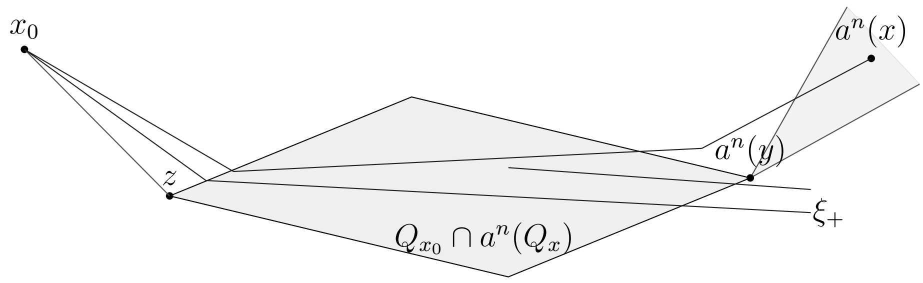

Figure 3.2: The line between x_0 and ξ_+ is the geodesic ray $[x_0, \xi_+)$; the line passing through $a^n(y)$ is the translation of y by powers of a; the line between x_0 and $a^n(x)$ is the geodesic segment $[x_0, a^n(x)]$;

3.2).

Since $\lim_{n\to\infty} a^n(y) = \xi_+$, there exists $N_1 > 1$ such that $a^n(y) \in Q_{x_0}$ and $z \in a^n(Q_x)$, for every $n > N_1$. But every sector is convex. Therefore, so is any intersection of two sectors. Thus, for every $n > N_1$, the intersection $Q_{x_0} \cap a^n(Q_x)$ is a non-empty convex subset of A and so also of $a^n(A_x)$ and A_{x_0}. As $n \to \infty$ the volume of $Q_{x_0} \cap a^n(Q_x)$ grows strictly with n. Moreover, $Q_{x_0} \cap a^n(Q_x)$ converges uniformly on compact sets to the sector Q_{x_0}.

For every $n > N_1$ we have $z \in a^n(Q_x) \subset a^n(A_x)$. Let Q_n denote the sector in $a^n(A_x)$ with base point z and corresponding to the opposite ideal chamber of c_- in $a^n(A_x)$. This sector Q_n contains as a subsector the sector with base point $a^n(y)$ and opposite to $a^n(Q_x)$. Therefore Q_n contains the vertex $a^n(x)$.

Let $n > N_1$ and consider the sector with base point x_0 and corresponding to the opposite ideal chamber of c_- in $a^n(A_x)$. This sector contains as a subsector the sector Q_n and so also the vertex $a^n(x)$ and the intersection $Q_{x_0} \cap a^n(Q_x)$. Since $\lim_{n\to\infty} a^n(x) = \xi_+$, there exists $N_x > N_1$ such that the geodesic segment from x_0 to $a^n(x)$ is passing through the convex set $Q_{x_0} \cap a^n(Q_x) \subset A$, for every $n > N_x$. Since $Q_{x_0} \cap a^n(Q_x)$ tends uniformly to the sector Q_{x_0}, we infer that, given $T > 0$, we may enlarge N_x in such a way that for every $n \geq N_x$ the unique geodesic segment from x_0 to $a^n(x)$ contains a subsegment of length greater than T entirely contained in $Q_{x_0} \cap a^n(Q_x)$ and so in A. The proof is complete. $\square$

3.3 Euclidean buildings and strong transitivity

Let Δ be a (possible spherical) building and G be a group acting by automorphisms on Δ. We say the G-action is **strongly transitive** if for any two pairs (A_1, c_1) and (A_2, c_2) consisting of an apartment A_i and a chamber $c_i \in \mathrm{Ch}(A_i)$, there exists $g \in G$ such that $g(A_1) = A_2$ and $g(c_1) = c_2$.

The goal of this section is to establish the equivalence between (i) and (ii) from Theorem 3.1.2. We shall moreover provide several other characterizations in Theorem 3.3.11 below. Let us first notice that the implication from (i) to (ii) of Theorem 3.1.2 is well-known (see [Gar97, Section 17.1]) and holds in full generality. In fact, in the first part of our discussion towards proving the converse implication, we leave the realm of locally compact groups and locally finite Euclidean buildings, and consider the broader framework of abstract groups acting on Euclidean buildings that are not supposed to be locally finite.

3.3.1 A geometric 'unipotent' radical

Lemma 3.3.1. *Let Δ be a Euclidean building and $G \leq \mathrm{Aut}(\Delta)$ be any group of type-preserving automorphisms. Let $c \in \mathrm{Ch}(\partial\Delta)$ be a chamber at infinity. Then the set*

$$G_c^0 := \{g \in G_c \mid g \text{ fixes some point of } \Delta\}$$

is a normal subgroup of the stabilizer $G_c = \{g \in G \mid g(c) = c\}$.

Proof. Let $g, h \in G_c^0$ and $x, y \in \Delta$ be points fixed by g, h respectively. Since g (resp., h) also fixes the chamber c and is type-preserving, it must fix pointwise an entire sector Q_x (resp., Q_y) emanating from x (resp., y) and pointing to c. Any two sectors in a Euclidean building having the same chamber at infinity contain a common subsector. Thus $Q_x \cap Q_y$ is a non-empty subset of Δ, which is clearly fixed pointwise by $\langle g, h\rangle$. This implies that the product gh belongs to G_c^0. Thus G_c^0 is a subgroup. It is clear that G_c^0 is invariant under conjugation by all elements of G_c. $\square$

There is another interpretation of the subgroup G_c^0 which will play an important role in our considerations. In order to describe it, we first need to recall the notion of retractions from a chamber at infinity.

Given an apartment A in a Euclidean building Δ and a chamber at infinity c contained in ∂A, there is a map

$$\rho_{A,c}\colon \Delta \to A,$$

called the **retraction** on A based at c, characterised by the following properties: the restriction of $\rho_{A,c}$ to A is the identity on A and the restriction of $\rho_{A,c}$ to any apartment B whose boundary contains c, induces an isomorphism of B onto A. We refer to [AB08, §11.7] for more details.

Lemma 3.3.2. *Let Δ be a Euclidean building and $G \leq \mathrm{Aut}(\Delta)$ be any group of type-preserving automorphisms. Let $c \in \mathrm{Ch}(\partial\Delta)$ be a chamber at infinity and A be an apartment whose boundary contains c. Then for any $g \in G_c$, the map*

$$\beta_c(g)\colon A \to A : x \mapsto \rho_{A,c}(g(x))$$

is an automorphism of the apartment A, acting as a (possibly trivial) translation. Moreover the map

$$\beta_c\colon G_c \to \mathrm{Aut}(A)$$

is a group homomorphism whose kernel coincides with G_c^0.

Proof. Let $g \in G_c$. Then $g(A)$ is an apartment whose boundary contains c. Therefore, the restriction of $\rho_{A,c}$ to $g(A)$ is an isomorphism from $g(A)$ to A which fixes c. Hence $\beta_c(g)$ is indeed an automorphism of A which fixes c. Since G is type-preserving $\beta_c(g)$ fixes all chambers in ∂A and acts thus as a translation on A.

Recall that the translation subgroup of $\mathrm{Aut}(A)$ acts freely on the set of special vertices of A. Therefore, the translation $\beta_c(g)$ is uniquely determined by its action on a given special vertex. In particular, in order to show that β_c is a homomorphism, it suffices to find, given $g, h \in G_c$, some special vertex v of A such that $\beta_c(gh)(v) = \beta_c(g)\beta_c(h)(v)$. Since A and $h(A)$ have the chamber at infinity c in common, they also contain a common sector associated to c. Now, for any vertex v deep enough in that sector, we have

$h(v) \in A$. Therefore $\rho_{A,c}(h(v)) = h(v)$ and hence

$$\begin{array}{rcl} \beta_c(g)\beta_c(h)(v) & = & \beta_c(g)(\rho_{A,c}(h(v))) \\ & = & \beta_c(g)(h(v)) \\ & = & \rho_{A,c}(gh(v)) \\ & = & \beta_c(gh)(v). \end{array}$$

This confirms that β_c is indeed a homomorphism.

The fact that $G_c^0 \leq \mathrm{Ker}(\beta_c)$ is clear from the definition. Conversely, given any element $g \in \mathrm{Ker}(\beta_c)$, the element g must fix pointwise the intersection $A \cap g(A)$, and hence $g \in G_c^0$. Thus $G_c^0 = \mathrm{Ker}(\beta_c)$. □

3.3.2 Trees in Euclidean buildings

The following property of Euclidean buildings is well-known and useful.

Lemma 3.3.3. *Let Δ be a thick (respectively, locally finite) Euclidean building of dimension n. Let $\sigma, \sigma' \subset \partial\Delta$ be a pair of opposite panels at infinity. We denote by $P(\sigma, \sigma')$ the union of all apartments of Δ whose boundaries contain σ and σ'. Then $P(\sigma, \sigma')$ is a closed convex subset of Δ, which splits canonically as a product*

$$P(\sigma, \sigma') \cong T \times \mathbf{R}^{n-1},$$

where T is a thick (respectively, locally finite) tree whose ends are canonically in one-to-one correspondence with the elements from the set $\mathrm{Ch}(\sigma)$ of all ideal chambers having σ as a panel. Under this isomorphism, the walls of Δ contained in $P(\sigma, \sigma')$ correspond to the subsets of the form $\{v\} \times \mathbf{R}^{n-1}$ with v a vertex of T.

Proof. See [Ron89, Chapter 10, §2]. □

3.3.3 Stabilizers of pairs of opposite panels

The following result is also well-known; we provide a proof for the reader's convenience.

Lemma 3.3.4. *Let Z be a thick spherical building and $G \leq \mathrm{Aut}(Z)$ be a strongly transitive group of type-preserving automorphisms. Then, for any pair of opposite panels σ, σ', the stabilizer $G_{\sigma,\sigma'}$ is 2-transitive on the set of chambers $\mathrm{Ch}(\sigma)$.*

Proof. Let $c \in \mathrm{Ch}(\sigma)$ and $x, y \in \mathrm{Ch}(\sigma)$ be two chambers different from c. Let $c' = \mathrm{proj}_{\sigma'}(c)$. Then x and y are both opposite c'. Therefore there is $g \in G_{c'}$ mapping x to y. Since G is type-preserving, it follows that g fixes σ' (because it fixes c') and hence σ (because it is the unique panel of x, respectively y, which is opposite σ'). Thus $g \in G_{\sigma,\sigma'}$. Moreover, g fixes c' and hence also $c = \mathrm{proj}_\sigma(c')$.

Thus, for any triple c, x, y of distinct chambers in $\mathrm{Ch}(\sigma)$, we have found an element $g \in G_{\sigma,\sigma'}$ fixing c and mapping x to y. The 2-transitivity of $G_{\sigma,\sigma'}$ on $\mathrm{Ch}(\sigma)$ follows. □

3.3.4 A criterion for strong transitivity

Lemma 3.3.5. *Let Δ be a thick Euclidean building and $G \leq \mathrm{Aut}(\Delta)$ be a group of type-preserving automorphisms. Then the following conditions are equivalent.*

(i) For each chamber $c \in \mathrm{Ch}(\partial\Delta)$, the group G_c^0 is transitive on the set of chambers at infinity opposite c.

(ii) For each chamber $c \in \mathrm{Ch}(\partial\Delta)$ and each panel π which is opposite a panel of c, the group $G_{c,\pi}^0 = G_c^0 \cap G_\pi$ is transitive on the set $\mathrm{Ch}(\pi) \setminus \mathrm{proj}_\pi(c)$.

(iii) G is strongly transitive on Δ.

Proof. (i) $\Rightarrow$ (ii): The hypothesis implies that G is strongly transitive on the spherical building $\partial\Delta$. In particular G is transitive on the set of apartments of Δ.

Let $c \in \mathrm{Ch}(\partial\Delta)$, σ be a panel of c and σ' be a panel which is opposite σ. Let also x and y be two chambers in $\mathrm{Ch}(\sigma')$ that are both different from the projection $\mathrm{proj}_{\sigma'}(c)$. In particular x and y are both opposite c. Therefore, there exists $g \in G_c^0$ mapping x to y. Notice that g fixes σ since G is type-preserving. Since σ' is the unique panel of x (resp., y) which is opposite σ, it follows that g fixes σ' as well. Therefore $g \in G_{c,\sigma'}^0$. Thus (ii) holds.

(ii) $\Rightarrow$ (iii): We start with a basic observation. Let A' and A'' be two apartments of Δ such that the intersection $A' \cap A''$ is a half-apartment. We claim that there is some $u \in G$ mapping A' to A'' and fixing $A' \cap A''$ pointwise.

Indeed, let $c \in \mathrm{Ch}(\partial(A' \cap A''))$ be an ideal chamber having a panel on the boundary of $\partial(A' \cap A'')$. The chamber c has a unique opposite chamber c' (resp., c'') in $\partial A'$ (resp., $\partial A''$). Notice that c has some panel σ on the boundary of the wall $A' \cap A''$; similarly the chambers c' and c'' share a common panel σ' which is opposite σ. By hypothesis, there is some $u \in G^0_{c,\sigma'}$ mapping c' to c''. It follows that u maps A' to A''. Moreover, since u fixes pointwise an entire sector of A', it follows that u fixes pointwise the intersection $A' \cap A''$, which is a half-apartment. The claim stands proven.

The claim implies similar statements for half-apartments at infinity. It follows in turn that the stabilizer G_c of every chamber at infinity is transitive on the set of chambers opposite c. In particular G acts transitively on the collection of all apartments. Therefore, what remains to show is that the stabilizer of each individual apartment acts transitively on the set of chambers of that apartment.

Let thus A be an apartment of Δ and M be a wall of A. Let also H and H' be the two half-apartments of A determined by M. Since Δ is thick, there exists a half-apartment H'' such that $H \cup H''$ and $H' \cup H''$ are both apartments of Δ. By the claim above, we can find an element $u \in G$ fixing H pointwise and mapping H' to H''. Similarly, there are elements $v, w \in G$ fixing H' pointwise and such that $v(H'') = H$ and $w(H) = u^{-1}(H')$. Now we set $r = vuw$. By construction r fixes pointwise the wall M. Moreover we have

$$r(H) = vuu^{-1}(H') = v(H') = H'$$

and

$$r(H') = vu(H') = v(H'') = H,$$

so that r swaps H and H'. It follows that r stabilizes the apartment A and acts on A as the reflection through the wall M. Since this holds for an arbitrary wall of A, it follows that $\mathrm{Stab}_G(A)$ contains elements that realize every reflection of A. In particular $\mathrm{Stab}_G(A)$ is transitive on the chambers of A, as desired.

(iii) $\Rightarrow$ (i): Let x and x' be two chambers opposite $c \in \mathrm{Ch}(\partial\Delta)$. Let moreover A and A' be the two apartments of Δ determined by the pairs $\{c, x\}$ and $\{c, x'\}$. They must share a common sector s pointing to c. In particular they contain a common chamber $C \in \mathrm{Ch}(\Delta)$. By hypothesis there exists $g \in G$ mapping A to A' and fixing C. Since G is type-preserving, it follows that g fixes the intersection $A \cap A'$ pointwise. In particular g fixes s pointwise, and hence preserves c. Therefore $g \in G_c$. Since moreover g fixes C we have in fact $g \in G_c^0$, as desired. $\square$

Lemma 3.3.5 already implies that the implication of (ii) to (i) of Theorem 3.1.2 holds in the special case of trees:

Corollary 3.3.6. *Let T be a thick tree and $G \leq \mathrm{Aut}(T)$ be an automorphism subgroup. If G is 2-transitive on $T(\infty)$, then G is strongly transitive on T.*

Proof. The group G must contain some hyperbolic isometry, otherwise it would fix a point in T or $T(\infty)$, which is absurd. By the 2-transitivity on the set of ends, it follows that G contains hyperbolic isometries fixing any given pair of ends of T. Given $c \in T(\infty)$, let β_c be the homomorphism from Lemma 3.3.2. Remark that for the case of the tree one does not need the type-preserving condition in the hypothesis of Lemma 3.3.2. Also remark that the image of the homomorphism β_c is a cyclic group. Therefore, the restriction of β_c to the cyclic subgroup of G_c generated by a hyperbolic isometry t of minimal non-zero translation length is onto. Denoting the other fixed end of t by c', we deduce from Lemma 3.3.2 that $G_c = G_{c,c'}.G_c^0$. Since G_c is transitive on the ends of T different from c, the same holds for G_c^0. The desired conclusion now follows from Lemma 3.3.5. $\square$

3.3.5 Apartments have cocompact stabilizers

Let Δ be a thick Euclidean building and $G \leq \mathrm{Aut}(\Delta)$ be a group of type-preserving automorphisms. Remark that if G is strongly transitive on $\partial\Delta$, then it is transitive on the set of apartments of Δ. Therefore, under the latter assumption, G is strongly transitive on Δ if and only if the stabilizer of any apartment is transitive on the set of chambers of that apartment. Our next goal is to establish the following approximation of this transitivity property.

Proposition 3.3.7. *Let Δ be a thick Euclidean building and $G \leq \mathrm{Aut}(\Delta)$ be a group of type-preserving automorphisms. Assume that G is strongly transitive on $\partial\Delta$. Then for any apartment A of Δ, the stabilizer $\mathrm{Stab}_G(A)$ acts cocompactly on A (i.e., with finitely many orbits of chambers).*

Proof. In view of Corollary 3.3.6, this holds clearly if Δ is a tree. We assume henceforth that Δ is higher-dimensional.

We start with a preliminary observation. Let H, H' be two complementary half-apartments of A. We claim that there is some $g \in \mathrm{Stab}_G(A)$ which swaps H and H'; in particular g stabilizes the common boundary wall $\partial H = \partial H'$.

In order to prove the claim, choose a pair of opposite panels σ, σ' lying on the boundary at infinity of the wall ∂H. Notice that it makes sense to consider panels at infinity since Δ is not a tree, and thus, the spherical building $\partial\Delta$ has positive rank. By Lemma 3.3.4, the stabilizer $G_{\sigma,\sigma'}$ acts 2-transitively on $\mathrm{Ch}(\sigma)$. We now invoke Lemma 3.3.3 which provides a canonical isomorphism $P(\sigma, \sigma') \cong T \times \mathbf{R}^{n-1}$, where $n = \dim(\Delta)$ and T is a thick tree. The set $\mathrm{Ch}(\sigma)$ being in one-one correspondence with $T(\infty)$, we infer that $G_{\sigma,\sigma'}$ is 2-transitive on $T(\infty)$, and hence strongly transitive on T by Corollary 3.3.6. Therefore, if v (resp., D) denotes the vertex (resp., geodesic line) of T corresponding to the wall ∂H (resp., the apartment A), we can find some $g \in \mathrm{Stab}_G(A)$ stabilizing D and acting on it as the symmetry through v. It follows that g stabilizes A, the wall $\partial H = \partial H'$ and swaps the two half-apartments H and H'. This proves the claim. (We warn the reader that g might however act non-trivially on the wall ∂H, so that it is not clear a priori that g is the reflection through that wall.)

Let $V \subseteq A$ be a minimal Euclidean subspace invariant under $\mathrm{Stab}_G(A)$. Because of the above paragraph $\mathrm{Stab}_G(A)$ is not fixing any point in A. Therefore V is not a point. Since $\mathrm{Stab}_G(A)$ acts on A as a discrete group of Euclidean isometries, its action on V is cocompact on V. Therefore, it suffices to show that $V = A$. Now, the boundary $V(\infty)$ is invariant under $\mathrm{Stab}_G(A)$. Since G is strongly transitive on $\partial\Delta$, it follows that if $V(\infty) \neq \partial A$, then $V(\infty)$ is contained in some wall at infinity. Equivalently, V is contained in some wall of A, say M. Let M' be a wall parallel to, but distinct from M. By the claim above, we find an element $g \in \mathrm{Stab}_G(A)$ stabilizing M' and swapping the half-apartments determined by it. In particular $g(M) \cap M$ is empty, which contradicts the fact that g stabilizes V. This completes the proof. $\square$

3.3.6 Geometric Levi decomposition

A geometric analogue of the Levi decomposition of parabolic subgroups of semi-simple groups has been established in [CM13, Theorem J] in the context of isometry groups of proper CAT(0) spaces. The following result can also be viewed as such a Levi decomposition. Note however that it concerns buildings that are not assumed to be locally finite, so it cannot be proved by invoking [loc. cit.].

Lemma 3.3.8. *Let Δ be a thick Euclidean building and $G \leq \mathrm{Aut}(\Delta)$ be a group of type-preserving automorphisms. Assume that G is strongly transitive on $\partial\Delta$.*

Then for any pair c, c' of opposite chambers at infinity, the group $G_{c,c'}.G_c^0$ is of finite index in G_c. Moreover G_c^0 has finitely many orbits on the set of chambers opposite c, and each of these orbits is invariant under $G_{c,c'}$.

Proof. Let c, c' be a pair of opposite chambers at infinity, and let A be the unique apartment of Δ that they determine. Since $\mathrm{Stab}_G(A)$ acts cocompactly on A by Proposition 3.3.7, so does the translation subgroup of $\mathrm{Stab}_G(A)$, which coincides with $G_{c,c'}$.

Let now $\beta_c \colon G_c \to \mathrm{Aut}(A)$ be the homomorphism from Lemma 3.3.2. Then for any $g \in G_{c,c'}$, we have $\beta_c(g) = g|_A$. We have just observed that $G_{c,c'}$ acts cocompactly on A, so that $\beta_c(G_{c,c'})$ is of finite index in $\mathrm{Aut}(A)$. In particular, the group $H = G_c^0.G_{c,c'} = \beta_c^{-1}(\beta_c(G_{c,c'}))$ is of finite index in G_c. In particular it acts with finitely many orbits on the set $\mathrm{Opp}(c)$ consisting of all chambers opposite c, since G_c is transitive on $\mathrm{Opp}(c)$.

We next claim that every G_c^0-orbit is invariant under $G_{c,c'}$. Indeed, given $d \in \mathrm{Opp}(c)$, there is some $x \in G_c$ with $d = x(c')$. Noticing that the quotient G_c/G_c^0 is abelian by Lemma 3.3.2, and hence that H is normal in G_c, we find

$$\begin{array}{rcl} G_{c,c'}(G_c^0(d)) & = & H(x(c')) \\ & = & x(H(c')) \\ & = & x(G_c^0.G_{c,c'}(c')) \\ & = & xG_c^0(c') \\ & = & G_c^0(x(c')) \\ & = & G_c^0(d). \end{array}$$

Thus, the G_c^0-orbit of d is indeed left invariant by $G_{c,c'}$, as claimed.

It follows that the H-orbits on $\mathrm{Opp}(c)$ coincide with the G_c^0-orbits, and hence, that there are only finitely many of these. $\square$

In view of Lemma 3.3.5, the final step in proving that G is strongly transitive on Δ consists in showing that the index of $G_{c,c'}.G_c^0$ in G_c is actually equal to one, since this means that G_c^0 is transitive on the chambers opposite c. It is only for this last step that we need to assume Δ to be locally finite and G to be a closed subgroup of Δ. The desired equality $G_c = G_{c,c'}.G_c^0$ can then be deduce from the above results by invoking [CM13, Theorem J]; alternatively, at this point it is also possible to provide a direct argument.

Lemma 3.3.9. *Let Δ be a thick Euclidean building and $G \leq \mathrm{Aut}(\Delta)$ be a group of type-preserving automorphisms. Assume that G is strongly transitive on $\partial\Delta$. Assume in addition that Δ is locally finite and that G is closed.*

Then G is strongly transitive on Δ.

First Proof of 3.3.9 . We use the criterion from Lemma 3.3.5. Let thus $c \in \mathrm{Ch}(\partial\Delta)$, let σ be a panel of c and σ' be a panel opposite σ.

Since G is strongly transitive on $\partial\Delta$, we know from Lemma 3.3.4 that $G_{\sigma,\sigma'}$ is 2-transitive on $\mathrm{Ch}(\sigma)$. Moreover, Lemma 3.3.8 implies that $G^0_{c,\sigma'}$ has finitely many orbits on $\mathrm{Ch}(\sigma') \setminus \mathrm{proj}_{\sigma'}(c)$, all of which are invariant under $G_{c,c'}$.

Recall from Lemma 3.3.3 that $G_{\sigma,\sigma'}$ preserves a convex subset $P(\sigma, \sigma') \subset \Delta$ which is canonically isomorphic to a product of a tree T with a flat factor F, and that the set of ends of T is canonically in one-to-one correspondence with $\mathrm{Ch}(\sigma)$. We infer that $G^0_{c,\sigma'}$ fixes an end of T and has finitely many orbits on the other ends. Since $G_{\sigma,\sigma'}$ is doubly transitive on the set of ends of T, it follows that some element of $G_{c,c'}$ must act as a hyperbolic isometry on T.

Now we use the fact that $G^0_{c,\sigma'}$ is closed in G, and hence it acts properly on $P(\sigma, \sigma')$. Moreover, by definition, its induced action on the flat factor F is trivial. It follows that $G^0_{c,\sigma'}$ acts properly on the tree T. Since $G_c^0.G_{c,c'}$ has finite index in G_c, one has that $G^0_{c,\sigma'}.G_{c,c'}$ is an open subgroup of $G_{c,\sigma'}$. From here we deduce that the $G^0_{c,\sigma'}$-orbit of the end of T corresponding to c' is open. On the other hand, any other orbit of the cyclic group generated by a hyperbolic element in $G_{c,c'}$ has the ends c and c' as limit points. This implies that $G^0_{c,\sigma'}$ is transitive on the set of ends of T different from

c. Equivalently, the group $G^0_{c,\sigma'}$ is transitive on $\mathrm{Ch}(\sigma') \setminus \mathrm{proj}_{\sigma'}(c)$. The desired conclusion follows from Lemma 3.3.5. $\square$

Alternative proof of Lemma 3.3.9. Let $c \in \mathrm{Ch}(\partial\Delta)$, $c' \in \mathrm{Opp}(c)$ and denote by A the unique apartment in Δ whose boundary contains c and c'.

As G is strongly transitive on $\partial\Delta$ we have that G acts transitively on the set of all apartments of Δ. Therefore by Proposition 3.3.7 we conclude that G acts cocompactly on Δ. Thus, by Theorem 3.1.10, G contains a strongly regular hyperbolic element. Invoking the transitivity of G on the set of apartments of Δ we conclude that every apartment in Δ admits a strongly regular hyperbolic element in G.

By the assumptions of the lemma we have that G^0_c is an open subgroup of G_c. Thus every $G^0_c G_{c,c'}$-orbit in $\mathrm{Opp}(c)$ is closed in the cone topology on $\partial\Delta$. Applying Proposition 3.2.10 to a strongly regular element with unique translation apartment A we obtain that c' is an accumulation point for every $G^0_c G_{c,c'}$-orbit in $\mathrm{Opp}(c)$. By closedness we conclude that there is only one such orbit, thus $G_c = G^0_c G_{c,c'}$ and therefore G^0_c is transitive on $\mathrm{Opp}(c)$. The desired conclusion follows from Lemma 3.3.5. $\square$

Remark 3.3.10. We have already pointed out that the hypothesis that Δ to be locally finite and G a closed subgroup of $\mathrm{Aut}(\Delta)$ was only used at the very last stage of the proof, namely in Lemma 3.3.9. Without those extra hypotheses, one could also consider the subgroup H of $G_{\sigma,\sigma'}$ defined by $H = \langle G^0_{c,\sigma'} \mid c \in \mathrm{Ch}(\sigma)\rangle$. By definition, this is a normal subgroup of $G_{\sigma,\sigma'}$ which acts trivially on the flat factor F of $P(\sigma,\sigma')$. As in the proof above, we see that for each $c \in \mathrm{Ch}(\sigma)$ the group $G^0_{c,\sigma'}$ has finitely many orbits of ends on the tree T. Since $G_{\sigma,\sigma'}$ is doubly transitive on the set of ends ∂T, it follows that H is transitive on ∂T and has finitely many orbits on $\partial T \times \partial T$. This leads us to the following question: *given a thick tree T (not necessarily locally finite), a group $G \leq \mathrm{Aut}(T)$ acting doubly transitively on ∂T and a normal subgroup H of G acting with finitely many orbits on $\partial T \times \partial T$, is it true that H must itself be doubly transitive on ∂T?*

A positive answer to this question would imply that Lemma 3.3.9 holds without the assumptions on local finiteness of Δ or closedness of G. In other words, the equivalence between (i) and (ii) from Theorem 3.1.2 would hold in full generality.

3.3.7 Characterizing strong transitivity in the locally finite case

We finally record various characterizations of strong transitivity for locally compact groups acting on locally finite thick Euclidean buildings.

Theorem 3.3.11. *Let G be a locally compact group acting continuously and properly by type-preserving automorphisms on a locally finite thick Euclidean building Δ. Then the following are equivalent:*

(i) G is strongly transitive on Δ;

(ii) G is strongly transitive on $\partial\Delta$;

(iii) G has no fixed point in $\partial\Delta$ and for some chamber $c \in \mathrm{Ch}(\partial\Delta)$, the stabilizer G_c acts cocompactly on Δ;

(iv) G acts cocompactly on Δ, without a fixed point in $\partial\Delta$, and there exists a compact open subgroup $K \leq G$ having finitely many chamber-orbits in $\partial\Delta$.

Proof. (i) $\Rightarrow$ (ii) is well-known, see [Gar97, Section 17.1].

(ii) $\Rightarrow$ (i) is contained in Lemma 3.3.9.

(i) $\Rightarrow$ (iii): Since G is strongly transitive on $\partial\Delta$, it is clear that G cannot fix any point of $\partial\Delta$. Moreover, by strong transitivity, the stabilizer G_c of a chamber at infinity c is transitive on the chambers opposite c, and hence, also on the collection of apartments whose boundaries contain c. Now, any chamber x in Δ is contained in such an apartment. In other words, if we fix an apartment A whose boundary contains c, then A contains a representative of every G_c-orbit of chambers. Now our hypothesis ensures that $\mathrm{Stab}_G(A)$ acts cocompactly on A. This implies that the translation subgroup of $\mathrm{Stab}_G(A)$, which is of finite index, also acts cocompactly on A. This translation subgroup acts trivially on ∂A, and it is thus contained in G_c. It follows that G_c acts cocompactly on Δ.

(iii) $\Rightarrow$ (ii): By hypothesis G acts cocompactly on Δ and without a fixed point in $\partial\Delta$. Let $c \in \mathrm{Ch}(\partial A)$ be such that G_c acts cocompactly on Δ. Then G_c is transitive on the set of chambers opposite c by [CM09, Proposition 7.1]. Since G does not fix any point of $\partial\Delta$, it must contain an element g mapping c to a chamber d sharing no simplex with c. Let now A be an apartment whose boundary contains both c and d. By the same arguments as in the proof of Lemma 3.3.5, we see that every reflection associated to a wall separating c from d can be realised by an element of $\mathrm{Stab}_G(\partial A)$. The set of those reflections is easily seen to be a parabolic subgroup of the spherical Weyl group. Since c and d do not share any simplex, they are not contained in a common proper subresidue; therefore the parabolic subgroup in question must be the whole Weyl group. This implies that $\mathrm{Stab}_G(\partial A)$ is transitive on the chambers of ∂A. It follows that G is transitive on $\mathrm{Ch}(\partial\Delta)$. Therefore, for all chambers $c' \in \mathrm{Ch}(\partial\Delta)$ the stabilizer $G_{c'}$ is transitive on the chambers opposite c'. From here we deduce that G acts transitively on the set of all apartments in Δ. Thus the strong transitivity of G on $\partial\Delta$ follows.

(i) $\Rightarrow$ (iv): If G is strongly transitive on Δ and K denotes the stabilizer of a special vertex, then K is transitive on the set of chambers at infinity: this follows from the KAK-decomposition of G, whose existence follows from the Bruhat decomposition of G with respect to the affine BN-pair. The desired implication follows.

(iv) $\Rightarrow$ (iii): Let $K \leq G$ be a compact open subgroup having finitely many orbits of chambers at infinity and let $c \in \mathrm{Ch}(\partial\Delta)$.

As the action of G on $\partial\Delta$ is continuous (here we endow $\partial\Delta$ with the cone topology induced from $\Delta \cup \partial\Delta$), every G-orbit on $\mathrm{Ch}(\partial\Delta)$ is a finite union of K-orbits (as there are only finitely many K-orbits), which by continuity are also compact in the cone topology. To prove the desired conclusion, it is enough to show that for every chamber at infinity $c \in \mathrm{Ch}(\partial\Delta)$, G/G_c is compact in the quotient topology.

Take a class hG_c in G/G_c. As $G(c) = K(g_1(c)) \sqcup ... \sqcup K(g_m(c))$, which is a finite union of K-orbits by hypothesis, there exists $i \in \{1, ..., m\}$ such that $h(c) \in K(g_i(c))$. So we have $h \in Kg_iG_c$ and thus $G/G_c \subset Kg_1G_c \sqcup ... \sqcup Kg_mG_c$. Moreover $G/G_c = (\cup_{i=1}^m Kg_i)\, G_c$. As $\cup_{i=1}^m Kg_i$ is compact in G, it follows that the projection from G of $\cup_{i=1}^m Kg_i$ on G/G_c is onto, thereby showing that G/G_c is also compact. The conclusion follows. $\square$

3.4 The proof of the main Theorem

The goal of this section is to prove the remaining implications from Theorem 3.1.2. This will be achieved at the end of the section. Moreover, the technical core of the proof relies on the following lemma. For the Gelfand pairs see Definition 3.1.1.

Lemma 3.4.1. *Let G be a locally compact group acting continuously and properly by type-preserving automorphisms on a locally finite thick Euclidean building Δ. Suppose that the G-action is cocompact, without a fixed point in $\partial\Delta$ and that it is not strongly transitive.*

Then, given the stabilizer $K \leq G$ of a vertex of Δ, there exist two strongly regular hyperbolic elements $\alpha, \beta \in G$ such that $\alpha\beta \notin K\beta K\alpha K$.

Proof. Let $x_0 \in \Delta$ be a vertex and set $K := G_{x_0}$. By Theorem 3.1.10, the group G contains a strongly regular hyperbolic element a. We denote by A its unique translation apartment.

Furthermore, by Theorem 3.3.11, the group K has infinitely many orbits of chambers at infinity. Since ∂A contains only finitely many chambers, we may thus find some chamber c which does not belong to the K-orbit of any chamber in ∂A.

We endow the set $\Delta \cup \partial\Delta$ with the cone topology (see [BH99, Chap.II.8] for the definitions and properties), which turns $\Delta \cup \partial\Delta$ into a compact Hausdorff space. Thus $\Delta \cup \partial\Delta$ becomes a normal space as well. Moreover, each chamber at infinity is a compact subset of $\partial\Delta$; so is the visual boundary of every apartment. Since the G-action on $\Delta \cup \partial\Delta$ is continuous and since K is compact, we infer that $K\partial A$ is compact. As $K\partial A$ is a union of chambers, all of which are different form c, it follows that the interior of c is disjoint from $K\partial A$.

Let A' be an apartment containing x_0 and such that $c \in \mathrm{Ch}(\partial A')$. In view of Lemma 3.2.6, we may thus find a strongly regular geodesic line $\rho\colon \mathbf{R} \to A'$ such that $\rho(0) = x_0$, $\{\rho(nC)\}_{n\geq 0}$ is a set of vertices of the same type, for some constant $C > 0$, and $\rho(\infty) := \lim\limits_{t\to\infty} \rho(t)$ lies in the interior of c.

We consider now the standard open neighborhoods of $\rho(\infty)$, in the cone topology, which are of the form

$$U(\rho, r, \epsilon) = \{z \in \Delta \cup \partial\Delta \mid d(z, x_0) > r \text{ and } d(p_r(z), \rho(r)) < \epsilon\},$$

with $r, \epsilon > 0$, and where $p_r(z)$ denotes the unique point on the geodesic from x_0 to z at distance r from x_0 (see [BH99, Chap.II.8]). Since $\rho(\infty) \notin KA \cup K\partial A$, which is a compact, thus closed set in the cone topology, there exist r big enough and ϵ small enough such that for every $x \in KA \setminus \overline{B}(x_0, r)$ we have $d(p_r(x), \rho(r)) \geq \epsilon$. Fix some r and ϵ with this property. We have that $(K\partial A \cup KA \setminus \overline{B}(x_0, r)) \cap U(\rho, r, \epsilon) = \varnothing$.

We now apply Proposition 3.2.9 to the geodesic line ρ. This ensures that we may find a strongly regular hyperbolic element $b \in G$ admitting a translation axis which shares a segment of arbitrarily large length with the geodesic ray $\rho(\mathbf{R}_+)$. Denoting by η_+, η_- respectively the attracting and repelling endpoints of such a translation axis of b, it follows that, upon replacing b by b^{-1}, the geodesic ray $[x_0, \eta_+)$ shares a geodesic segment of arbitrarily large length with the geodesic ray $\rho(\mathbf{R}_+)$, which does not necessarily pass through the vertex x_0. To conclude, we find a strongly regular hyperbolic element $b \in G$ such that $\eta_+ \in U(\rho, r, \epsilon)$.

The elements α and β that we are looking for will be defined as suitable high powers of a and b respectively. We now proceed to find those powers.

Let $\rho' \colon \mathbf{R}_+ \to \Delta$ be the geodesic ray such that $\rho'(0) = x_0$ and $\rho'(\infty) = \eta_+$. Since $\eta_+ \in U(\rho, r, \epsilon)$, we may find $r', \epsilon' > 0$ such that

$$U := U(\rho', r', \epsilon') \subset U(\rho, r, \epsilon).$$

Since for each $k \in K$ and $\xi \in \partial A$ we have that $k\xi \notin U(\rho, r, \epsilon)$, there exists $r_{k\xi}$ and $\epsilon_{k\xi}$ such that

$$U([x_0, k\xi), r_{k\xi}, \epsilon_{k\xi}) \cap U = \varnothing.$$

The collection $\{U([x_0, k\xi), r_{k\xi}, \frac{1}{2}\epsilon_{k\xi})\}_{k\in K, \xi \in \partial A}$ forms an open covering of the compact set $K\partial A$, from which we may extract a finite subcovering, corresponding to $k_1, \ldots, k_n \in K$ and $\xi_1, \ldots, \xi_n \in \partial A$. Set

$$r'' = \max\{r_{k_1\xi_1}, \ldots, r_{k_n\xi_n}\} \qquad \text{and} \qquad \epsilon'' = \frac{1}{2}\min\{\epsilon_{k_1\xi_1}, \ldots, \epsilon_{k_n\xi_n}\}.$$

Then by the triangle inequality, for each $k \in K$ and $\xi \in \partial A$, there is some $i \in \{1, \ldots, n\}$ such that

$$U_{k\xi} = U([x_0, k\xi), r'', \epsilon'') \subset U([x_0, k_i\xi_i), r_{k_i\xi_i}, \epsilon_{k_i\xi_i}).$$

In particular, $U_{k\xi}$ is an open neighborhood of $k\xi$ which is furthermore disjoint from U. Since K fixes x_0, we observe that $U_{k\xi} = kU_\xi$ for all $\xi \in \partial A$. This implies that $U \cap KU_\xi = \varnothing$ for all $\xi \in \partial A$.

By Proposition 3.2.10, the sequence $\{a^m(\eta_+)\}_{m\geq 0}$ converges to some $\gamma \in \partial A$. In particular, for all $m > 0$ sufficiently large, we have $a^m(\eta_+) \in U_\gamma$. We fix such an m. We may thus find r''' and ϵ''' with

$$U' := U([x_0, \eta_+), r''', \epsilon''') \subset U \cap a^{-m}(U_\gamma).$$

We deduce that $a^m(U') \subset U_\gamma$ and hence $ka^m(U') \subset kU_\gamma = U_{k\gamma}$ for all $k \in K$. Since $U \cap KU_\gamma = \varnothing$ we have

$$ka^m(U') \cap U = \varnothing, \text{ for every } k \in K. \tag{3.3}$$

Recall that η_+ is the attracting endpoint of a translation axis of b. Therefore, for each $y \in \Delta$, the sequence $\{b^n(y)\}_{n\geq 0}$ converges to η_+. In particular, for each finite set of points $F \subset \Delta$, we may find a sufficiently large n such that $b^n(F) \subset U'$. We apply this argument to the set $F = x_0 \cup Ka^m(x_0)$, which is indeed finite since Δ is locally finite. In this way we obtain a suitable large n.

Thus, for this n found above we have:

$$b^n(x_0) \in U' \qquad \text{and} \qquad b^n ka^m(x_0) \in U' \subset U$$

for each $k \in K$. On the other hand, by (3.3), we also have

$$k'a^m b^n(x_0) \notin U$$

for all $k' \in K$. Therefore $Ka^m b^n K \cap Kb^n Ka^m K = \varnothing$, since otherwise there would exist $k, k' \in K$ such that $b^n ka^m(x_0) = k'a^m b^n(x_0)$. Finally, setting $\alpha = a^m$ and $\beta = b^n$, we obtain $\alpha\beta \notin K\beta K\alpha K$. □

We are now able to complete the proof of the main theorem.

End of proof of Theorem 3.1.2. (i) ⇔ (ii) is covered by Theorem 3.3.11.

(i) ⇒ (iii) follows from [Mat77, Corollary 4.2.2], where it is proved that (G, G_x) is a Gelfand pair, where x is a special vertex of Δ.

(iii) ⇒ (i)

Assume G acts cocompactly on Δ and there exists a compact subgroup K of G such that (G, K) a Gelfand pair. As K is a compact subgroup of G, by the well-known Fixed Point Theorem, there exists a vertex x_0 of Δ such that $K \leq G_{x_0}$. In particular, one notices that (G, G_{x_0}) becomes a Gelfand pair. Without loss of generality, we can consider in what follows that $K = G_{x_0}$.

By contraposition, suppose the G-action is *not* strongly transitive on Δ. We want to prove that (G, K) is *not* a Gelfand pair.

Assume first that G fixes some point $\xi \in \partial\Delta$. It then follows from [CM13, Theorem M] that G is not unimodular. Thus we are done in this case, since any locally compact group endowed with a Gelfand pair is unimodular by [vD09, Proposition 6.1.2].

We assume henceforth that G does not fix any point in $\partial\Delta$. By Lemma 3.4.1, we may find two strongly regular hyperbolic elements $\alpha, \beta \in G$ such that $\alpha\beta \notin K\beta K\alpha K$. Set

$$\phi = \mathbf{1}_{K\alpha K} \qquad \text{and} \qquad \psi = \mathbf{1}_{K\beta K}.$$

Clearly the maps ϕ, ψ are both locally constant, hence continuous, and K-bi-invariant. Thus $\phi, \psi \in C_c^K(G)$ and we claim that $\phi * \psi \neq \psi * \phi$.

Indeed, we first observe that

$$\begin{array}{rcl} \phi * \psi(\alpha\beta) & = & \int_G \phi(\alpha\beta g)\psi(g^{-1})d\mu(g) \\ & \geq & \int_{\beta^{-1}K} \phi(\alpha\beta g)\psi(g^{-1})d\mu(g) \\ & = & \int_K \phi(\alpha k)\psi(k^{-1}\beta)d\mu(k) \\ & = & \mu(K) \\ & > & 0, \end{array}$$

where μ is a Haar measure on G.

On the other hand, consider

$$\psi * \phi(\alpha\beta) = \int_G \psi(\alpha\beta g)\phi(g^{-1})d\mu(g).$$

For the integrand to be nonzero, we need that the variable g satisfies both $g^{-1} \in K\alpha K$ and $\alpha\beta g \in K\beta K$. For such an element $g \in G$, we may find $k_1, k_2, k_3, k_4 \in K$ such that $g^{-1} = k_1\alpha k_2$ and $\alpha\beta g = k_3\beta k_4$. This yields

$$\alpha\beta = k_3\beta k_4 g^{-1} = k_3\beta k_4 k_1 \alpha k_2 \in K\beta K\alpha K,$$

which contradicts our choice of α and β. Therefore, there is no choice of g making the integrand nonzero in the integral defining $\psi * \phi(\alpha\beta)$. This implies that $\psi * \phi(\alpha\beta) = 0$. The claim stands proven, thereby confirming that the convolution algebra $C_c^K(G)$ is not commutative. $\square$

Chapter 4

The flat closing problem for buildings

4.1 Introduction

In 1972, Prasad and Raghunathan proved the following result (see [PR72, Corollary 2.9, Lemma 1.15]): Let G be a semi-simple real Lie group of rank r (which may admit compact factors). Let $\Gamma < G$ be a lattice. Then Γ contains an abelian subgroup of rank r.

To obtain this result, they use in the first place the existence in Γ of a so-called $\mathbf{R}$-hyper-regular element (see [PR72, Definition 1.1 and Theorem 2.5]). Being also an $\mathbf{R}$-regular element (see [PR72, Remark 1.2],[Ste65]), an $\mathbf{R}$-hyper-regular $g \in G$ inherits the property that its G-centralizer $\mathrm{Cent}_G(g)$ contains a unique maximal $\mathbf{R}$-split torus of G of dimension the $\mathbf{R}$-rank of G (see [Ste65] or [PR72]). Using this, the last step of the Prasad–Raghunathan strategy is to show that $\mathrm{Cent}_G(g)/\mathrm{Cent}_\Gamma(g)$ is compact if g is an $\mathbf{R}$-hyper-regular element (see [PR72, proof of Lemma 1.15]). This is obtained in the following way. If Γ is a uniform lattice then one can use a Selberg's lemma (see [PR72, Lemma 1.10]). If not, then [PR72, Theorem 1.14] gives the desired result.

The above result of Prasad and Raghunathan can be related, in the setting of CAT(0) spaces, to the following question of Gromov (also known as the **flat closing problem**) (see [Gro93, Section 6.B_3]). From another point of view, the flat closing problem is a converse to the Flat Torus Theorem.

First, let us recall the following basic notions from the setting of

CAT(0) spaces.

Definition 4.1.1. Let X be a proper CAT(0) space and G be a not necessarily discrete locally compact group acting continuously and properly by isometries on X. By a **geometric flat** $F \subset X$ of dimension d (or a **d-flat**) we mean a closed convex subset of X which is isometric to the Euclidean d-space and where $d \geq 2$. Moreover, we say that a subset $Y \subset X$ is **periodic** if $\mathrm{Stab}_G(Y)$ acts co-compactly on Y.

When γ is a hyperbolic isometry of X we denote

$$\mathrm{Min}(\gamma) := \{x \in X \mid \mathrm{dist}_X(x, \gamma(x)) = |\gamma|\},$$

where $|\gamma| := \inf_{x \in X} \mathrm{dist}_X(x, \gamma(x))$ denotes the translation length of γ.

Problem 4.1.2. *(The flat closing problem) Let X be a proper* CAT(0) *space and Γ be a discrete group acting continuously, properly and co-compactly by isometries on X. Does the existence of a d-flat $F \subset X$ imply that Γ contains a copy of $\mathbf{Z}^d$?*

Remark 4.1.3. In the hypotheses of Problem 4.1.2, note that if F is a periodic d-flat then indeed, by Bieberbach Theorem, $\mathrm{Stab}_\Gamma(F)$ contains a copy of $\mathbf{Z}^d$ and therefore Γ contains a subgroup which is virtually $\mathbf{Z}^d$. Moreover, if we replace Γ with a not necessarily discrete locally compact group G acting continuously, properly and co-compactly by isometries on X and F is a periodic d-flat with respect to the G-action then G contains a compact-by-$\mathbf{Z}^d$ subgroup.

To attack Gromov's Problem 4.1.2, the natural strategy would be to construct periodic flats or more generally periodic subspaces of the form $Y = F \times C \subset X$, where F is a d-flat and C is a compact set. This could be done by showing that Γ contains a hyperbolic element γ which is 'regular' (see Definition 3.2.1) and then to consider $Y := \mathrm{Min}(\gamma) = F \times C$, where F is a flat and C is a compact set. To conclude from here the flat closing problem it would be enough to show that the centralizer of γ in Γ, namely, $\mathrm{Cent}_\Gamma(\gamma) < \mathrm{Stab}_\Gamma(Y)$, acts co-compactly on Y and that $\mathrm{Stab}_{\mathrm{Cent}_\Gamma(\gamma)}(F) \leq \mathrm{Stab}_\Gamma(F)$ has co-compact action on F. Thus, $\mathrm{Stab}_\Gamma(F)$ would contain a copy of $\mathbf{Z}^d$ and the Remark 4.1.3 is applied.

Notice that the above strategy is analogous to the one used in the Prasad–Raghunathan result, where the **R**-split torus is replaced with a flat. Moreover, this strategy, including the existence of 'regular' elements, is successfully implemented for example in [CZ], where it is proved that any discrete group acting properly and co-compactly on a decomposable locally compact CAT(0) space, which admits in addition the geodesic extension property (i.e., every geodesic segment is contained in a bi-infinite geodesic line) contains virtually $\mathbf{Z}^d$, where d is the number of indecomposable de Rham factors.

This chapter proposes to answer further Gromov's Problem 4.1.2 in the case when G is a not necessarily discrete locally compact group acting continuously, properly and co-compactly by type-preserving automorphisms on a locally finite general building. We stress here that we cannot apply directly the result of [CZ]: Firstly, because in the usual Davis realization, buildings are CAT(0) spaces where in general the geodesic extension property is not fulfilled. Secondly, the group G considered in this chapter is not necessarily discrete. Still, the strategy is the same and uses the existence of 'strongly regular hyperbolic automorphisms' (see Definition 3.2.2) acting on locally finite Euclidean buildings. We obtain:

Theorem 4.1.4 (See Theorem 4.3.4)**.** *Let Δ be a locally finite building of not-finite type (W, S), but with S being finite. Let G be a not necessarily discrete, closed, type-preserving subgroup of* $\mathrm{Aut}(\Delta)$*, with co-compact action on Δ. Then G contains a compact-by-$\mathbf{Z}^d$ subgroup, where d is the dimension of a maximal flat of Δ.*

4.2 Preliminaries

From Section 3.2, recall the definitions of regular and strongly regular hyperbolic automorphisms acting on locally finite Euclidean buildings, as they are used further in this chapter. For some notations see also Definition 4.1.1 from the Introduction. Moreover, recall the following proposition.

Proposition 4.2.1 (See Proposition 3.2.9)**.** *Let X be a proper* CAT(0) *space, $G \leq \mathrm{Is}(X)$ be any subgroup of isometries and $\rho\colon \mathbf{R} \to X$ be a geodesic map. Assume there is an increasing sequence $\{t_n\}_{n\geq 0}$ of positive real numbers tending to infinity such that*

$$\sup_n d(\rho(t_n), \rho(t_{n+1})) < \infty$$

and that the set $\{\rho(t_n)\}_{n\geq 0}$ falls into finitely many G-orbits, each of which is moreover discrete.

Then there is an increasing sequence $\{f(n)\}_n$ of positive integers such that, for all $n > m > 0$, there is a hyperbolic isometry $h_{m,n} \in G$ which has a translation axis containing the geodesic segment $[\rho(t_{f(m)}), \rho(t_{f(n)})]$.

Moreover, if X is a locally finite Euclidean building and the geodesic line $\rho(\mathbf{R})$ is strongly regular, then $h_{m,n}$ is a strongly regular hyperbolic automorphism. Still, if the geodesic line $\rho(\mathbf{R})$ is the translation axis of a rank-one hyperbolic element of $\mathrm{Is}(X)$, then for each fixed $m > 0$ there exists an $N_m > 0$ such that for every $n > N_m$ the isometry $h_{m,n}$ is a rank-one element.

Proof. We give here a proof only for the very last assertion of the proposition, this not being part of Proposition 3.2.9. Moreover, for the definition of a rank-one element see [CF10, Section 2].

Let m be fixed. Suppose the contrary, namely, for every $k > 0$ there exists $n_k \geq k$ such that h_{m,n_k} is not a rank-one element. This means that the translation axis of $h_{m,n}$ containing the geodesic segment $[\rho(t_{f(m)}), \rho(t_{f(n)})]$ is contained in the boundary of a flat half-plane. Therefore, as $k \to \infty$ we obtain that the geodesic ray $[\rho(t_{f(m)}), \rho(\infty))$ is contained in the boundary of a flat half-plane as well. As the space X is proper and the geodesic line $\rho(\mathbf{R})$ is the translation axis of a rank-one hyperbolic element, we obtain a contradiction with the fact that the diameter of the projection on $\rho(\mathbf{R})$ of every closed metric ball in X, which is moreover disjoint from $\rho(\mathbf{R})$, must be bounded above by a fixed constant. The conclusion follows. $\square$

4.3 The proof of the main theorem

Before proceeding to the proof of Theorem 4.1.4, we recall some general facts about buildings and we fix some notations. Let Δ be a locally finite general building of type (W, S), with S finite. Fix from now on a chamber $\mathbf{c}$ in $\mathrm{Ch}(\Delta)$. Let $W = W_1 \times W_2 \times \cdots \times W_k$ be the direct product decomposition of W in irreducible Coxeter systems (W_i, S_i). Thus $S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$ is a disjoint union. Denote by Δ_i the W_i-residue in Δ containing the fixed chamber $\mathbf{c}$. From [Ron89, Theorem 3.10] we have that $\Delta \cong \Delta_1 \times \cdots \times \Delta_k$. For what follows, we use the following notations:

$$\mathbf{Eucl} = \{i \in \{1, \cdots, k\} \mid (W_i, S_i) \text{ is Euclidean}\},$$

$$\mathbf{Sph} = \{i \in \{1, \cdots, k\} \mid (W_i, S_i) \text{ is finite}\},$$

$$\mathbf{nSph} = \{1, \cdots, k\} \backslash \mathbf{Sph} \quad \text{and} \quad \mathbf{nEsph} = \{1, \cdots, k\} \backslash (\mathbf{Eucl} \cup \mathbf{Sph}) .$$

Accordingly, we use the notations $\Delta_{\mathbf{A}} := \prod_{i \in \mathbf{A}} \Delta_i$ and $W_{\mathbf{A}} := \prod_{i \in \mathbf{A}} W_i$, where $\mathbf{A}$ is one of the sets **Eucl**, **Sph**, **nSph** or **nEsph**. Moreover, for every $i \in \{1, \cdots, k\}$ we denote by $\mathrm{pr}_i : \Delta \to \Delta_i$ the projection map on Δ_i and by abuse of notation we write $\mathrm{pr}_i(\gamma)$ to represent the element $\gamma \in \mathrm{Aut}(\Delta)$ acting on the factor Δ_i.

The first step towards the main theorem is given by the next proposition which uses an argument of [HK05, Lemma 3.1.2].

Proposition 4.3.1. *Let Δ be a locally finite building of not-finite type (W, S), but with S being finite. Let G be a not necessarily discrete, type-preserving subgroup of $\mathrm{Aut}(\Delta)$ acting co-compactly on Δ and let $\mathcal{R}$ be any residue in Δ containing the chamber $\mathbf{c}$. Then $\mathrm{Stab}_G(\mathcal{R})$ acts co-compactly on $\mathcal{R}$.*

Proof. Let us denote by (W', S') the type of the residue $\mathcal{R}$, where $W' \leq W$ and $S' \subset S$. Take $K \subset \Delta$ to be a compact fundamental domain corresponding to the action of G and containing the chamber $\mathbf{c}$. Let $(g_i)_{i \in I}$ be a subset of G such that $\mathcal{R} \subset \bigcup_{i \in I} g_i(K)$. Because $\mathcal{R}$ is a residue containing $\mathbf{c}$ and G is type-preserving, for every $i \in I$, $g_i^{-1}(\mathcal{R})$ is a residue of the same type as $\mathcal{R}$, containing the chamber $g_i^{-1}(\mathbf{c})$ and intersecting K.

As Δ is locally finite, the set K has a finite number of chambers. Therefore, K intersects a finite number of (W', S')-type residues of Δ. We conclude that there is a finite number of left cosets of the form $\mathrm{Stab}_G(\mathcal{R}) g_i$, with $i \in I$. Denote by $\{g_1, \cdots, g_t\} \subset \{g_i\}_{i \in I}$ the finite set of representatives of these left cosets. We obtain that $\mathcal{R}$ is covered by $\bigcup_{j \in \{1, \cdots, t\}} \mathrm{Stab}_G(\mathcal{R}) g_j(K)$.

Because $g_j(K)$ is compact, let K' be a compact in Δ such that $\bigcup_{j=1}^{t} g_j(K) \subset K'$. Thus $\mathrm{Stab}_G(\mathcal{R}) K'$ covers $\mathcal{R}$ and the conclusion follows.

$\square$

The second step is to find a hyperbolic element in $\mathrm{Stab}_G(\Delta_{\mathbf{nSph}})$. This is given by the following proposition.

Proposition 4.3.2. *Let Δ be a locally finite building of not-finite type (W, S), but with S being finite. Let G be a not necessarily discrete, type-preserving subgroup of $\mathrm{Aut}(\Delta)$ acting co-compactly on Δ. Then there exists a hyperbolic element $\gamma \in \mathrm{Stab}_G(\Delta_{\mathbf{nSph}})$ such that $\mathrm{pr}_i(\gamma)$ is a strongly regular hyperbolic element if $i \in \mathbf{Eucl}$ and a rank-one isometry if $i \in \mathbf{nEsph}$.*

Proof. First, remark that G acts co-compactly on $\Delta_{\mathbf{nSph}}$. Take $i \in \mathbf{Eucl}$. By Lemma 3.2.6, let $\mathtt{l}_i$ be a strongly regular geodesic line contained in some apartment of Δ_i, constructed using a strongly regular hyperbolic element γ_i of W_i. The line $\mathtt{l}_i$ is thus contained in a unique apartment of Δ_i. Denote by $\{v_{i,j}\}_{j\in\mathbf{Z}}$ a bi-infinite sequence of points in $\mathtt{l}_i$ such that $\gamma_i(v_{i,j}) = v_{i,j+1}$, for every $j \in \mathbf{Z}$. For example, the points $\{v_{i,j}\}_{j\in\mathbf{Z}} \subset \mathtt{l}_i$ can be taken to be special vertices of Δ_i, of the same type.

For $i \in \mathbf{nEsph}$ we have a similar construction. Following [CF10, Proposition 4.5], denote by $\mathtt{r}_i$ a rank-one geodesic line given by a rank-one hyperbolic element $h_i \in W_i$. Denote by $\{t_{i,j}\}_{j\in\mathbf{Z}}$ a bi-infinite sequence of points of $\mathtt{r}_i$ such that $h_i(t_{i,j}) = t_{i,j+1}$, for every $j \in \mathbf{Z}$.

Because $\prod\limits_{i\in\mathbf{Eucl}} \mathtt{l}_i \times \prod\limits_{i\in\mathbf{nEsph}} \mathtt{r}_i$ is a flat of $\Delta_{\mathbf{nSph}}$, we consider in its interior the infinite geodesic line determined by the following sequence of points

$$\{o_j := \prod_{i\in\mathbf{Eucl}} v_{i,j} \times \prod_{i\in\mathbf{nEsph}} t_{i,j}\}_{j\in\mathbf{Z}} \subset \Delta_{\mathbf{nSph}}\,.$$

Denote the resulting geodesic line by $\mathtt{L}$ and observe that, by defining $h := \prod\limits_{i\in\mathbf{Eucl}} \gamma_i \times \prod\limits_{i\in\mathbf{nEsph}} \gamma_i$, we have that $h \in W_{\mathbf{nSph}}$ and $h(o_j) = o_{j+1}$, for every $j \in \mathbf{Z}$.

We are now ready to proceed in finding the desired hyperbolic element in G. Apply Proposition 4.2.1 to our geodesic line $\mathtt{L}$, to the sequence of points $(o_j)_{j\in\mathbf{N}}$ and to the $\mathrm{Stab}_G(\Delta_{\mathbf{nSph}})$-action on $\Delta_{\mathbf{nSph}}$ (which is co-compact). As we are working with a locally finite building, all hypotheses of Proposition 4.2.1 are fulfilled. We obtain thus a sequence $\{f(n)\}_{n\geq 0}$ and a sequence of hyperbolic elements $\{\gamma_{m,n}\}_{0<m<n} \subset \mathrm{Stab}_G(\Delta_{\mathbf{nSph}})$ such that every $\gamma_{m,n}$ has a

translation axis containing the geodesic segment $[o_{f(m)}, o_{f(n)}]$. By this construction we obtain that $\mathrm{pr}_i(\gamma_{m,n}(o_{f(m)})) = \mathrm{pr}_i(o_{f(n)})$, for every $i \in \mathbf{nSph}$ and every element $\gamma_{m,n}$. Applying again Proposition 4.2.1, there exists a hyperbolic element $\gamma_{m,n}$ such that $\mathrm{pr}_i(\gamma_{m,n})$ is a strongly regular hyperbolic element if $i \in \mathbf{Eucl}$ and a rank-one isometry if $i \in \mathbf{nEsph}$. The conclusion follows. □

Remark 4.3.3. Let F be a maximal d-flat of a locally finite general building Δ. By [Cap09, Proposition 3.1] there exists a residue $\mathcal{R} \subset \Delta$ of type $(W_{\mathcal{R}}, S_{\mathcal{R}})$ such that $d = \sum\limits_{i \in \mathbf{Eucl}_{\mathcal{R}}} n_i + |\mathbf{nEsph}_{\mathcal{R}}|$, where $\mathbf{Eucl}_{\mathcal{R}}$ and $\mathbf{nEsph}_{\mathcal{R}}$ correspond to the residue $\mathcal{R}$.

Therefore, by Proposition 4.3.1, to conclude the flat closing problem in the case of a locally finite general non-spherical building it is enough to prove the following theorem.

Theorem 4.3.4. *Let Δ be a locally finite building of not-finite type (W, S), with S being finite, and G a closed, not necessarily discrete, type-preserving subgroup of* $\mathrm{Aut}(\Delta)$ *with co-compact action. Then G contains a compact-by- $\mathbf{Z}^d$ subgroup, where $d := \sum\limits_{i \in \mathbf{Eucl}} n_i + |\mathbf{nEsph}|$.*

Proof. Let $\gamma \in \mathrm{Stab}_G(\Delta_{\mathbf{nSph}})$ be a hyperbolic element given by Proposition 4.3.2. We have that

$$\mathrm{Min}(\gamma) = \prod_{i \in \mathbf{Eucl}} \mathbf{R}^{n_i} \times \prod_{j \in \mathbf{nEsph}} (\mathbf{R} \times C_j),$$

where n_i is the Euclidean dimension of the corresponding building Δ_i and C_j is a compact, convex subset of the corresponding building Δ_j. Let $F := \prod_{i \in \mathbf{Eucl}} \mathbf{R}^{n_i} \times \prod_{i \in \mathbf{nEsph}} \mathbf{R} \subset \mathrm{Min}(\gamma)$. By [Rua01, Theorem 3.2], whose proof works also for not necessarily discrete groups, we have that $\mathrm{Cent}_G(\gamma)$ stabilizes and acts co-compactly on $\mathrm{Min}(\gamma)$. As $\prod_{i \in \mathbf{nEsph}} C_i$ is compact, using the argument of [HK05, Lemma 3.1.2], recalled in the proof of Proposition 4.3.1, and the fact that we are working with CAT(0) cellular complexes we obtain that $\mathrm{Stab}_{\mathrm{Cent}_G(\gamma)}(F)$ acts co-compactly on F. In particular, we obtain that the action of $\mathrm{Stab}_G(F)$ on F is properly discontinuously and co-comapct. Therefore, the group $\mathrm{Stab}_G(F)/\mathrm{Fix}_G(F)$ is virtually isomorphic with $\mathbf{Z}^d$, where $d := \sum\limits_{i \in \mathbf{Eucl}} n_i + |\mathbf{nEsph}|$. Since G is closed, the group $\mathrm{Fix}_G(F)$ is compact. Therefore, the group G

contains a compact-by-$\mathbf{Z}^d$ subgroup. This concludes the proof of the theorem. $\square$

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