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Lyapunov Function PDEs to the Stability of Some Complex Balancing Derivative and Compound Networks

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***Abstract*—This paper contributes to extending the validity of Lyapunov function PDEs whose solution is conjectured to be able to behave as a Lyapunov function in stability analysis to more mass-action chemical reaction networks. First, we have proved that the Lyapunov function PDEs method is valid in capturing the asymptotic stability of the networks compounded of a complex balanced network and any species-dependent two-species autocatalytic network if some moderate conditions are included. Then by defining a new class of networks, called complex balanced produced networks, we also show the asymptotic stability of this class of networks, and also to their compound with any species-independent 1-dimensional network and with any species-dependent two-species autocatalytic network under some conditions by using the same method. A notable point is that these classes of networks are non-weakly reversible, of any dimension, and of any deficiency. Finally, we apply our results to some practical biochemical reaction networks including birth-death processes, motifs related networks etc., to illustrate validity.**

***Index Terms*—chemical reaction networks, complex balanced produced networks, mass-action systems, Lyapunov function PDEs, asymptotic stability**

I. Introduction

It has been extensively recognized that chemical reaction networks (CRNs) appear in chemistry, biology and process industries. The study of CRNs, namely, CRN theory, originating from the well-known literature [1], [2], aims at exploring the correlation between the dynamical properties and structural features of networks. In particular, as the emergence of the discipline–systems biology, CRN theory has received considerable attention [3] once again, as a powerful tool to analyze and explain the underlying dynamical behaviors of chemical and biochemical networks from the mathematical point of view. Starting from this practical point, we are encouraged to develop CRN theory, especially the research on dynamical characteristics, which may be stability, oscillation, persistence, etc. Up to now there are still a limited number of results about the stability of mass action systems (MAS, CRN assigned mass-action kinetics). In this paper, we mainly focus our attention to the stability analysis of equilibria, with the help of constructing suitable Lyapunov functions.

Regarding the stability property there are many results [1], [4]–[9] that concentrated on certain CRNs with special structures, such as detailed balancing, complex balancing, among which the deficiency zero theorem [2] is probably the best known. It shows that a class of CRNs equipped with deficiency zero and weakly reversible structure is complex balancing regardless of the specific parameter values, and possess only one equilibrium in every positive stoichiometric compatibility class. More importantly, it has also pointed out that each of these equilibria is asymptotically stable by taking the pseudo-Helmholtz free energy function as the Lyapunov function. Later this work has been improved by [8], known as the deficiency one theorem, which reported the existence and uniqueness of equilibria for a complex balanced MAS within restrictions on deficiency (not necessarily zero). Furthermore, based on these results, global asymptotic stability [10] of equilibria for complex balanced MASs can be achieved when satisfying persistent condition [11]–[13]. Besides, some works [14], [15] focused on the detailed balanced MAS with reversible structure, which is a special case of a complex balanced MAS. Particularly, van der Schaft and his coauthors [15] proposed a compact formulation to depict the network dynamics by using graph theory for this class of MASs, and the stability properties were also achieved.

For the sake of stability analysis of MASs, it is largely accepted to look for proper Lyapunov functions according to their structural properties. Study of [16], [17] has shown that when the considered MAS can be mapped into a complex balanced MAS through linear conjugacy method, it will have the same stability property as the complex balanced MAS. Alradhawi and Angeli [18] established piecewise linear in rates Lyapunov functions for some balanced MASs, and further, they suggested the asymptotic stability property if LaSalle's condition was met. Another Lyapunov function candidate coming from [19], called generalized pseudo-Helmholtz function, served to establish asymptotic stability for a general balanced MAS which is defined on the notions of reconstructions and reverse reconstructions.

Apart from the above results, several attempts have been made to address the stability problem from a microscopic stochastic viewpoint. Anderson [20] put forward the scaling limits of nonequilibrium potential as the Lyapunov function for birth-death systems. Fang and Gao [21] developed par-

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tial differential equations (PDEs) from the chemical master equation for general MASs, termed as Lyapunov function PDEs, whose solution is able to be Lyapunov functions under required conditions. This systematic approach works well on multiple CRNs, including complex balanced CRNs, CRNs of 1-dimensional stoichiometric subspace and a few special cases of higher dimensional stoichiometric subspace.

Nevertheless, those CRNs with more general structures remain an arduous obstacle. Meanwhile a conjecture has been proposed in [21] that says that for any MAS admitting a stable positive equilibrium, Lyapunov function PDEs can produce Lyapunov functions to render the system locally asymptotically stable at the equilibrium when choosing a proper boundary complex set. Recent evidence [22] takes a small further step for this guess, which indicated that the PDEs are also valid for a class of complex balanced produced (CBP) CRNs with non-weakly reversible structure.

Motivated by these results, our paper extends the validity of Lyapunov function PDEs in asymptotic stability analysis to three kinds of networks with high-dimension, arbitrary deficiency as well as non-weakly reversible structure. Inspired by the research on autocatalytic reactions [23], a type of CRNs which play a vital role in the processes of life [24], [25], such as biological metabolism, the initial transcripts of rRNA, etc., we define a CRN compounded of a complex balanced CRN and any two-species autocatalytic CRNs where these subnetworks have shared species, named as Com-ℓts-Autoca CRNs. Then following the CBP CRNs which are generated from complex balanced networks, we further study their compound with any 1-dimensional independent networks and with any two-species autocatalytic CRNs, referred to as CBP-ℓsub1 CRNs and CBP-ℓts-Autoca CRNs, respectively. Compared with [21], [22], the current paper treats these three kinds of newly-defined and more complex networks, and makes the following contributions: (i) a systematic algorithm is proposed to compute CBP mass-action systems; (ii) it has been proved that Lyapunov function PDEs method is valid in capturing the asymptotic stability properties for the MASs mentioned above. And it has been proved that in each positive stoichiometric compatibility class there is a unique positive equilibrium for a CBP MAS, and at most one positive equilibrium for a Com-ℓts-Autoca MAS and CBP-ℓts-Autoca MAS under some moderate conditions; (iii) a dimensionality reduction strategy has been proposed, which aims at obtaining the stability property for a network by decomposing it into a CBP CRN and several 1-dimensional CRNs; (iv) the result greatly supports the conjecture proposed in [21].

The remainder of this paper proceeds as follows. Section II reviews the relevant notations about chemical reaction networks and Lyapunov function PDEs. In section III we show the equilibrium distribution about the Com-ℓts-Autoca MAS and further verify the validity of the Lyapunov function PDEs method for this MAS. Section IV indicates the equilibrium distribution and stability property of a CBP MAS as well as an algorithm that is proposed to compute CBP MASs. Then it shows that the Lyapunov function PDEs method works for the CBP-ℓSub1 MAS and CBP-ℓts-Autoca MAS to produce suitable Lyapunov functions for asymptotic stability analysis in Section V. Finally, Section VI concludes the paper.

Mathematical Notation:

$\mathbb{R}^n, \mathbb{R}^n_{\geq 0}, \mathbb{R}^n_{>0}$	n-dimensional real space, non-negative real space, positive real space, respectively.
$\mathbb{Z}^n_{\geq 0}$	n-dimensional non-negative integer space.
$x^{v_{\cdot i}}$	$x^{v_{\cdot i}} = \prod_{j=1}^d x_j^{v_{ji}}$, where $x \in \mathbb{R}^d, v_{\cdot i} \in \mathbb{Z}^d$ and $0^0 = 1$.
$\frac{x}{y}$	$\frac{x}{y} = (\frac{x_1}{y_1}, \cdots, \frac{x_n}{y_n})$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n_{>0}$.
$\text{Ln}(x)$	$\text{Ln}(x) = (\ln x_1, \cdots, \ln x_n)^\top$, where $x \in \mathbb{R}^n_{>0}$.
$\mathscr{C}^i(\cdot\ ; *)$	the set of ith continuous differentiable functions from "$\cdot$" to "$*$".
$\lvert v_{\cdot i}\rvert$	$\lvert v_{\cdot i}\rvert = \sum_{j=1}^n v_{ji}$.

II. Preliminaries

In this section, we shall provide a basic conceptual framework of CRNs and Lyapunov function PDEs for the understanding of subsequent results.

A. Chemical reaction networks

Consider a network involved with n species $S_1, \cdots, S_n$ and r chemical reactions. The ith $(i = 1, \cdots, r)$ reaction is written as

$$\sum_{j=1}^n v_{ji} S_j \to \sum_{j=1}^n v'_{ji} S_j,$$

where $v_{ji},\ v'_{ji} \in \mathbb{Z}_{\geq 0}$ represent the stoichiometric coefficients of the reactants and the resultants, respectively. Following with [6], here come some elementary definitions related to CRNs.

Definition 1 (CRN): . A CRN consists of three finite sets:

1) a set of species $\mathcal{S} = \{S_1, \cdots, S_n\}$;
2) a set of complexes $\mathcal{C} = \bigcup_{i=1}^r \{v_{\cdot i}, v'_{\cdot i}\}$ with $\text{Card}(\mathcal{C}) = c$, and the jth entry of $v_{\cdot i}$ represents the stoichiometric coefficient of S_j in this complex;
3) a set of reactions $\mathcal{R} = \{v_{\cdot 1} \to v'_{\cdot 1}, \cdots, v_{\cdot r} \to v'_{\cdot r}\}$, which satisfies that $\forall\ v_{\cdot i} \in \mathcal{C}, v_{\cdot i} \to v_{\cdot i} \notin \mathcal{R}$ but $\exists\ v'_{\cdot i}$, s.t. $v_{\cdot i} \to v'_{\cdot i} \in \mathcal{R}$ or $v'_{\cdot i} \to v_{\cdot i} \in \mathcal{R}$.

The triple $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ is often used to represent a CRN.

Definition 2 (stoichiometric subspace): For a CRN $(\mathcal{S}, \mathcal{C}, \mathcal{R})$, the linear subspace $\mathscr{S} \triangleq span\{v'_{\cdot 1} - v_{\cdot 1}, \cdots, v'_{\cdot r} - v_{\cdot r}\}$ is called the stoichiometric subspace of this network, and $dim\mathscr{S}$ represents the dimension of $\mathscr{S}$.

Definition 3 (stoichiometric compatibility class): Let $\mathscr{S}$ be the stoichiometric subspace of a CRN $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ and $x_0 \in \mathbb{R}^n_{\geq 0}$, then the sets $\mathscr{S}(x_0) \triangleq \{x_0 + \xi \mid \xi \in \mathscr{S}\}$, $\bar{\mathscr{S}}^+(x_0) \triangleq \mathscr{S}(x_0) \bigcap \mathbb{R}^n_{\geq 0}$ and $\mathscr{S}^+(x_0) \triangleq \mathscr{S}(x_0) \bigcap \mathbb{R}^n_{>0}$ are called the stoichiometric compatibility class, nonnegative and positive stoichiometric compatibility class of x_0, respectively.

When a CRN follows the mass-action kinetics, the reaction rate of the ith reaction $v_{\cdot i} \to v'_{\cdot i}$ is evaluated by

$$R_i(x) \triangleq k_i x^{v_{\cdot i}} = k_i \prod_{j=1}^d x_j^{v_{ji}}$$

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with $k_i \in \mathbb{R}_{>0}$, $x \in \mathbb{R}^n_{\geq 0}$ representing the rate constant of this reaction and the vector of concentrations x_i of the substance $\mathcal{S}_i$. Let $k = \{k_1, \cdots, k_r\}$ represent the set of reaction rate constants, then we have the definition of MAS.

Definition 4 (MAS): A CRN $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ assigned mass-action kinetics is said to be an MAS, often represented by the quadruple $\mathcal{M} \triangleq (\mathcal{S}, \mathcal{C}, \mathcal{R}, k)$.

The dynamics of an $\mathcal{M}$ that depicts the evolution of concentrations of the species over time is represented as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Gamma R(x), \quad x \in \mathbb{R}^n_{\geq 0}, \tag{1}$$

where $\Gamma \in \mathbb{Z}_{n\times r}$ is the stoichiometric matrix with the ith column given by $\Gamma_{\cdot i} = v'_{\cdot i} - v_{\cdot i}$ called the reaction vector, and $R(x)$ is the vector function of reaction rate defined in $\mathbb{R}^r_{\geq 0}$ with each element $R_i(x) = k_i x^{v_{\cdot i}}$.

Definition 5 (balanced MAS): A point $x^* \in \mathbb{R}^n_{>0}$ is said to be a positive equilibrium in $\mathcal{M}$ if it satisfies $\Gamma R(x^*) = 0$. A MAS that possesses a positive equilibrium is a balanced MAS.

Definition 6 (complex balanced MAS): For an $\mathcal{M}$, if $\exists x^* \in \mathbb{R}^n_{>0}$, s.t.

$$\sum_{\{i|v_{\cdot i}=z\}} k_i (x^*)^{v_{\cdot i}} = \sum_{\{i|v'_{\cdot i}=z\}} k_i (x^*)^{v_{\cdot i}}, \qquad \forall z \in \mathcal{C}, \tag{2}$$

which means the consuming rate equals the producing rate at this state for any complex, then x^* is called a complex balanced equilibrium, and this $\mathcal{M}$ is called a complex balanced MAS.

Definition 7 (reaction vector balanced MAS [26]): For a $\mathcal{M}$, if $\exists x^* \in \mathbb{R}^n_{>0}$, s.t.

$$\sum_{\{i|v'_{\cdot i}-v_{\cdot i}=\eta\}} k_i (x^*)^{v_{\cdot i}} = \sum_{\{i|v'_{\cdot i}-v_{\cdot i}=-\eta\}} k_i (x^*)^{v_{\cdot i}}, \qquad \forall \eta \in \mathbb{R}^n \tag{3}$$

then x^* is a reaction vector balanced equilibrium in $\mathcal{M}$, and the MAS is a reaction vector balanced MAS.

Remark 1: The notion of reaction vector balancing means, for each pair of reaction sets with opposite reaction vectors induced by an MAS, their consumption rate at the equilibrium point is the same. This class of MASs is often seen in some birth-death processes [20] and autocatalytic CRNs (see Definition 8), etc.

We use the following example to illustrate the reaction vector balanced equilibrium.

Example 1: A MAS takes the reaction route like

$$S_2 \xrightarrow{k_1} S_1, \; S_1 + S_2 \xrightarrow{k_2} 2S_2$$
$$2S_1 \xrightarrow{k_3} 2S_2, \; 3S_2 \xrightarrow{k_4} 2S_1 + S_2.$$

There are four reaction vectors $(1, -1)^\top$, $(-1, 1)^\top$, $(-2, 2)^\top$, $(2, -2)^\top$ in the network. We only need to consider two of them, i.e., $\eta = (-1, 1)^\top$ and $\eta = (-2, 2)^\top$, respectively. If a positive concentration vector $x^* = (x_1^*, x_2^*)^\top$ is a reaction vector equilibrium, then for $\eta = (-1, 1)^\top$ it should satisfy $k_2 x_1^* x_2^* = k_1 x_2^*$, i.e., $x_1^* = \frac{k_1}{k_2}$, while for $\eta = (-2, 2)^\top$ there should be $k_3 {x_1^*}^2 = k_4 {x_2^*}^3$, i.e., $x_2^* = \sqrt[3]{\frac{k_3}{k_4}\frac{k_1^2}{k_2^2}}$.

B. Lyapunov function PDEs

For any balanced $\mathcal{M}$, Fang and Gao [21] introduced the Lyapunov function PDEs to analyze the stability of $\mathcal{M}$, whose concrete form are

$$\sum_{i=1}^r k_i x^{v_{\cdot i}} - \sum_{i=1}^r k_i x^{v_{\cdot i}} \exp\left\{(v'_{\cdot i} - v_{\cdot i})^\top \nabla f(x)\right\} = 0, \tag{4}$$

where $x \in \mathbb{R}^n_{>0}$, along with the following boundary condition,

$$\lim_{\substack{x\to\bar{x} \\ x\in(\bar{x}+\mathscr{S})\cap\mathbb{R}^n_{>0}}} \sum_{\{i|v_{\cdot i}\in\mathcal{C}_{\bar{x}}\}} k_i x^{v_{\cdot i}} - \sum_{\{i|v'_{\cdot i}\in\mathcal{C}_{\bar{x}}\}} k_i x^{v_{\cdot i}} \exp\left\{(v'_{\cdot i} - v_{\cdot i})^\top \nabla f(x)\right\} = 0, \tag{5}$$

where $\mathcal{C}_{\bar{x}}$ stands for the complex set induced by any boundary point $\bar{x} \in \partial\mathbb{R}^n_{\geq 0}$. One simple alternative is the naive boundary complex set (see details in [21]), defined as

$$\bar{\mathcal{C}}_{\bar{x}} = \{z \in \mathcal{C} \mid \exists\, \epsilon > 0, \text{ such that } \forall j = 1, \cdots, n, \bar{x}_j \geq \epsilon z_j\}. \tag{6}$$

The PDEs (4) and (5) are developed from the chemical master equation that describes the evolution of a CRN as a stochastic process with the solution as the probability density at each point in time. The solution of PDEs, $f(x)$, is an approximation of the scaling nonequilibrium potential defined by the steady distribution, which (if exists) exhibits the following properties in characterizing the dynamical behaviors of the corresponding MAS.

Property 1 ([21]): Given a balanced $\mathcal{M}$, assume there exists a solution $f(x)$ defined in $\mathscr{C}^1(\mathbb{R}^n_{>0}; \mathbb{R})$ for the Lyapunov function PDE (4) with proper boundary condition induced by $\mathcal{M}$, then $f(x)$ possesses the following two properties:

1) $f(x)$ is dissipative, that is $\dot{f}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}t} \leq 0$ with the equality holding if and only if $\nabla f(x) \perp \mathscr{S}$;
2) if $f(x)$ is defined in $\mathscr{C}^2(\mathbb{R}^n_{>0}; \mathbb{R})$, and $\exists \mathcal{D} \subset \mathbb{R}^n_{>0}$, s.t. $\forall x \in \mathcal{D}$ and $\forall \mu \in \mathscr{S}$, there is
$$\mu^\top \nabla^2 f(x) \mu \geq 0, \tag{7}$$
where the equality holds if and only if $\mu = \mathbb{0}_n$, then $\forall x \in \mathcal{D}$, $\dot{f}(x) = 0$ if and only if x is an equilibrium in $\mathcal{M}$.

A sufficient condition is then given to reach the asymptotic stability of MASs based on the solution of the Lyapunov function PDEs.

Theorem 1 ([21]): Given an $\mathcal{M}$ with an equilibrium $x^* \in \mathbb{R}^n_{>0}$, assume that its Lyapunov function PDE (4) admits a solution $f \in \mathscr{C}^2(\mathbb{R}^n_{>0}; \mathbb{R})$, and there exists a region near x^* such that (7) holds for the whole region. Then for any initial condition in this region but with an initial energy lower than that at any boundary point included in the region, the solution $f(x)$ is an available Lyapunov function to establish the locally asymptotic stability of x^*.

It is thus conjectured [21] that, *"for any MAS that admits a stable positive equilibrium, if the boundary complex set is equipped properly, then the Lyapunov function PDEs induced by this system have a solution qualified as a Lyapunov function* *to suggest that the system is locally asymptotically stable at the equilibrium".*

The conjecture has been proved true in the cases of the following three classes of MASs.

(1) complex balanced MASs whose PDEs admit the well-known pseudo-Helmholtz free energy function

$$G(x)=\sum_{j=1}^{n}\left(x_j^*-x_j-x_j\ln\frac{x_j^*}{x_j}\right),\quad x\in\mathbb{R}^n_{>0} \tag{8}$$

to be a solution that can act as a Lyapunov function.

(2) all MASs of dim$\mathscr{S}=1$ whose PDEs have a solution in the form of

$$f(x)=\int_0^{\gamma(x)}\ln u(y^\dagger(x)+\alpha\omega)\mathrm{d}\alpha \tag{9}$$

as an available Lyapunov function, where $\gamma(x), y^\dagger(x), u$, and ω share the same meanings with Corollary 1.

(3) Com-ℓSub1 MASs with dim$\mathscr{S}\geq 2$ composed of a complex balanced MAS $(\mathcal{S}^{(0)},\mathcal{C}^{(0)},\mathcal{R}^{(0)},k^{(0)})$ and some 1-dimensional MASs $(\mathcal{S}^{(p)},\mathcal{C}^{(p)},\mathcal{R}^{(p)},k^{(p)})$, where $p=1,\cdots,\ell$ and all subnetworks are supposed to be mutually independent according to the species. The Lyapunov function PDEs admit a solution

$$F(x)=G(x^{(0)})+\sum_{p=1}^{\ell}f(x^{(p)}),$$

where $G(x^{(0)})$ and every $f(x^{(p)})$ are defined by (8) and (9), respectively. Clearly, $F(x)$ is a suitable Lyapunov function.

In the current work, we continue to exhibit the validity of the Lyapunov function PDEs to more CRNs. Based on the known solutions above, we try to construct more solutions as well as CRNs with special structures to validate the conjecture.

III. Stability of MASs compounded of a Complex balanced MAS and a few autocatalytic MASs

The asymptotic stability of Com-ℓSub1 MASs has been successfully worked out through the Lyapunov function PDEs method [21]. However, a requirement is that all subnetworks in a Com-ℓSub1 MAS are mutually independent according to species. This restriction makes the solution of the PDEs of a Com-ℓSub1 MAS can be constructed by a combination of solution of the corresponding PDEs of every subnetwork. The case will become complicated if the species among all subnetworks are not independent. We follow this issue in this section by defining a kind of MASs compounded of a complex balanced MAS and a few 1-dimensional autocatalytic MASs where these subnetworks have shared species.

A. Compound MASs of a complex balanced MAS and autocatalytic MASs with shared species

Autocatalytic reactions are ubiquitous in living organisms, like metabolism, DNA replications, etc. Generally speaking, they refer to a class of reactions where the products act as catalysts. There are various expressions concerning this notion [23], [25], [27], and one of them is as follows.

Definition 8 (autocatalystic MAS, a reduced version of [23]): A MAS is said to be an autocatalytic one, labeled by $\mathcal{M}=(\mathcal{S},\mathcal{C},\mathcal{R},k)$, if the following conditions are true

(1) all reactions have a net consumption of one S_i and a net production one S_j, i.e., in the form of

$$S_i+(m-1)S_j\xrightarrow{k_m}mS_j\ ,$$

where $m\geq 1, i,j=1,\cdots,n$;

(2) there is one monomolecular linkage class;

(3) if there is a net consumption of one S_i and a net producing one S_j in a reaction, then $S_i\longrightarrow S_j, S_j\longrightarrow S_i\in\mathcal{R}$, which means mass exchange in both directions only happens in single molecular reactions.

Remark 2: From Definition 8, it is obvious that if an autocatalytic MAS only has two species, then its stoichiometric subspace is 1-dimensional. We name this class of networks two-species autocatalytic ones, which are our main concern in the subsequent investigation.

Consider an $\mathcal{M}$ composed of a complex balanced $\mathcal{M}^{(0)}=(\mathcal{S}^{(0)},\mathcal{C}^{(0)},\mathcal{R}^{(0)},k^{(0)})$, and a number of two-species autocatalytic MASs in the form of

$$S_p+(m_i-1)S_{n_0+p}\xrightarrow{k_i^{(p)}}m_iS_{n_0+p},\quad S_{n_0+p}\xrightarrow{k_{r_p}^{(p)}}S_p\ , \tag{10}$$

labeled by $\mathcal{M}^{(p)}=(\mathcal{S}^{(p)},\mathcal{C}^{(p)},\mathcal{R}^{(p)},k^{(p)})$, $p=1,\cdots,\ell$, $m_i\geq 1$ and $m_1=1$, $i=1,\cdots,r_p-1$. Here, we denote $\mathcal{S}^{(0)}=\{S_1,\cdots,S_{n_0}\}$ and $\mathcal{S}^{(p)}=\{S_p,S_{n_0+p}\}$ with $p=1,\cdots,\ell$. We call the above $\mathcal{M}$ a Com-ℓts-Autoca MAS, which naturally meets (i) $n_0\geq\ell$; (ii) $\forall i,j\in\{1,\cdots,\ell\}$, $\mathcal{S}^{(i)}\bigcap\mathcal{S}^{(j)}=\emptyset$ while $\mathcal{S}^{(0)}\bigcap\mathcal{S}^{(p)}=S_p$, $\forall\{p\}_{p=1}^{\ell}$.

Then we denote $n=n_0+\ell$, $v_{\cdot i}^{(0)^\top}=(v_{\cdot i(0)}^\top,\mathbb{0}_\ell^\top)$, $v_{\cdot i}'^{(0)^\top}=(v_{\cdot i(0)}'^\top,\mathbb{0}_\ell^\top)$, and $v_{\cdot i}^{(p)^\top}=(\mathbb{0}_{p-1}^\top,v_{1i(p)},\mathbb{0}_{n_0-1}^\top,v_{2i(p)},\mathbb{0}_{\ell-p}^\top)$, $v_{\cdot i}'^{(p)^\top}=(\mathbb{0}_{p-1}^\top,v_{1i(p)}',\mathbb{0}_{n_0-1}^\top,v_{2i(p)}',\mathbb{0}_{\ell-p}^\top)$ when $p=1,\cdots,\ell$, where n_p and r_p represent the number of species and reactions, $v_{\cdot i(p)}$ and $v_{\cdot i(p)}'$ represent the reactant complex and the resultant complex of the ith reaction ($i=1,\cdots,r_p$) of every subnetwork ($p=0,\cdots,\ell$) in a Com-ℓts-Autoca MAS, respectively. The dynamics follow

$$\dot{x}=\sum_{p=0}^{\ell}\sum_{i=1}^{r_p}k_i^{(p)}x^{v_{\cdot i}^{(p)}}\left(v_{\cdot i}'^{(p)}-v_{\cdot i}^{(p)}\right). \tag{11}$$

From Definition 5, a concentration vector

$$x^*=(x_1^*,\cdots,x_{n_0}^*,x_{n_0+1}^*,\cdots,x_{n_0+\ell}^*)^\top\in\mathbb{R}^n_{>0} \tag{12}$$

is a positive equilibrium in the Com-ℓts-Autoca $\mathcal{M}$ if $\dot{x}=0$ evaluated at $x=x^*$. For every positive equilibrium in the Com-ℓts-Autoca $\mathcal{M}$, we have the following property.

Property 2: For a Com-ℓts-Autoca $\mathcal{M}$ modelled by (11), a concentration vector $x^*\in\mathbb{R}^n_{>0}$ given by (12) is an equilibrium in $\mathcal{M}$ if and only if $x^{(0)^*}=(x_1^*,\cdots,x_{n_0}^*)^\top$ is a positive equilibrium in $\mathcal{M}^{(0)}$ while $x^{(p)^*}=(x_p^*,x_{n_0+p}^*)^\top$ is a reaction vector balanced equilibrium in $\mathcal{M}^{(p)}$ for $p=1,\cdots,\ell$.

Proof: The detailed proof can be found in appendix A. □

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We characterize the number of positive equilibria of a Com-ℓts-Autoca MAS in each positive stoichiometric compatibility class through the following lemma.

Lemma 1: Given a Com-ℓts-Autoca $\mathcal{M}$, ruled by (11) and admitting an equilibrium $x^* \in \mathbb{R}^n_{>0}$ defined in (12), each positive stoichiometric compatibility class contains at most a positive equilibrium if one of the following conditions holds:

(1) $\forall \{p\}_{p=1}^{\ell}$, for all reactions with indexes $i = 1, \cdots, r_p$ in $\mathcal{M}^{(p)}$, there are $|v_{\cdot i(p)}| \leq 2$, or

(2) $\forall \{p\}_{p=1}^{\ell}$, $\mathcal{M}^{(p)}$ is mass-conserved if there exists some reaction with index z in $\mathcal{M}^{(p)}$ such that $|v_{\cdot z(p)}| > 2$.

Proof: The proof can be caught in appendix A. □

Remark 3: Physically speaking, the first condition in Lemma 1 means every two-species autocatalytic CRN is an at-most-biomolecular CRN (the sum of stoichiometric coefficients of any complex in the network is at most two) while the second condition means those two-species autocatalytic CRNs that are not at-most-biomolecular need to be mass-conserved.

B. Lyapunov function PDEs to the stability of Com-ℓts-Autoca MASs

In this subsection, we capture the asymptotic stability of Com-ℓts-Autoca MASs by using the Lyapunov function PDEs strategy.

From the definition of Com-ℓts-Autoca MASs, it is easy to write out the corresponding Lyapunov function PDEs,

$$\sum_{p=0}^{\ell}\sum_{i=1}^{r_p} k_i^{(p)} x^{v_{\cdot i}^{(p)}} \left(1 - \exp\left\{(v'^{(p)}_{\cdot i} - v^{(p)}_{\cdot i})^\top \nabla f(x)\right\}\right) = 0, \tag{13}$$

and

$$\sum_{p=0}^{\ell}\Bigg(\lim_{\substack{x\to\bar{x} \\ x\in(\bar{x}+\mathscr{S})\cap\mathbb{R}^n_{>0}}} \sum_{\{i|v^{(p)}_{\cdot i}\in\mathcal{C}_{\bar{x}}\}} k_i^{(p)} x^{v^{(p)}_{\cdot i}} - \sum_{\{i|v'^{(p)}_{\cdot i}\in\mathcal{C}_{\bar{x}}\}} k_i^{(p)} x^{v^{(p)}_{\cdot i}} \exp\left\{(v'^{(p)}_{\cdot i} - v^{(p)}_{\cdot i})^\top \nabla f(x)\right\}\Bigg) = 0. \tag{14}$$

Lemma 2: For a Com-ℓts-Autoca $\mathcal{M}$ governed by (11) and possessing an equilibrium x^* defined by (12), the twice differentiable function

$$f(x) = \sum_{i=1}^{n_0}\left(x_i^* - x_i - x_i \ln\frac{x_i^*}{x_i}\right) + \sum_{p=1}^{\ell}\int_{x^*_{n_0+p}}^{x_{n_0+p}} \ln\frac{k^{(p)}_{r_p}\alpha_p}{\sum_{i=1}^{r_p-1}k_i^{(p)}x_p^*\alpha_p^{m_i-1}}d\alpha_p, \tag{15}$$

is a solution of the Lyapunov function PDEs (13) and (14) induced by $\mathcal{M}$ if for every $\mathcal{M}^{(p)}$ ($p = 0, \cdots, \ell$), $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}$ is chosen as the naive boundary complex set defined as (6), where $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}} = \emptyset$ or $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}$ includes at least a reactant complex and a resultant complex.

Proof: The detailed proof is given in appendix A. □

Then we can reach the asymptotic stability of Com-ℓts-Autoca MASs based on the above results.

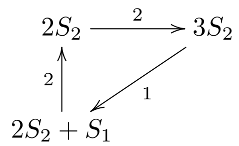
Theorem 2: For a Com-ℓts-Autoca $\mathcal{M}$ described by (11) and admitting an equilibrium $x^* \in \mathbb{R}^n_{>0}$ defined by (12), x^* is locally asymptotically stable

(1) if $\forall \{p\}_{p=1}^{\ell}$, for all reactions with indexes $i = 1, \cdots, r_p$ in $\mathcal{M}^{(p)}$, there are $|v_{\cdot i(p)}| \leq 2$, or

(2) if $\forall \{p\}_{p=1}^{\ell}$, when there exists some reaction with index z in $\mathcal{M}^{(p)}$ such that $|v_{\cdot z(p)}| > 2$, $\mathcal{M}^{(p)}$ is mass-conserved and $\sum_{i=1}^{r_p-1}(2-m_i)k_i^{(p)}x^{*\,m_i-1}_{n_0+p} > 0$.

Proof: The detailed proof can be found in appendix A. □

Example 2: Consider a Com-ℓts-Autoca $\mathcal{M}$ ($\ell = 1$) with dim$\mathscr{S} = 3$ and deficiency 3, as can be seen below, including a complex balanced $\mathcal{M}^{(0)}$ on the upper side and an autocatalytic $\mathcal{M}^{(1)}$ on the downside,



$$S_1 \underset{k_5^{(1)}}{\overset{k_1^{(1)}}{\rightleftharpoons}} S_3,\ S_1 + (m_i - 1)S_3 \xrightarrow{k_i^{(1)}} m_i S_3\ ,$$
$$m_2 = 2, m_3 = 3, m_4 = 4, \tag{16}$$

where $k_1^{(1)} = 8, k_2^{(1)} = 2, k_3^{(1)} = 1, k_4^{(1)} = 1, k_5^{(1)} = 12$. Note that the complex balanced $\mathcal{M}^{(0)}$ possesses an equilibrium $x^{(0)^*} = (1,1)^\top$ and the complexes $v_{\cdot 1(0)} = v'_{\cdot 3(0)} = (1,2)^\top, v'_{\cdot 1(0)} = v_{\cdot 2(0)} = (0,2)^\top, v'_{\cdot 2(0)} = v_{\cdot 3(0)} = (0,3)^\top$. Besides, the autocatalytic $\mathcal{M}^{(1)}$ satisfies (i) $|v_{\cdot 3(1)}| > 2$ and $|v'_{\cdot 4(1)}| > 2$ (ii) reaction vector balancing with an equilibrium $x^{(1)^*} = (1,1)$ in the positive stoichiometric compatibility class constrained by $\{x_1 + x_3 = 2\}$. Its complexes are $v_{\cdot 1(1)} = v'_{\cdot 5(1)} = (1,0)^\top, v_{\cdot 5(1)} = v'_{\cdot 1(1)} = (0,1)^\top, v_{\cdot 2(1)} = (1,1)^\top, v'_{\cdot 2(1)} = (0,2)^\top, v_{\cdot 3(1)} = (1,2)^\top, v'_{\cdot 3(1)} = (0,3)^\top, v_{\cdot 4(1)} = (1,3)^\top, v'_{\cdot 4(1)} = (0,4)^\top$.

From Lemma 2, there is a solution for its Lyapunov function PDEs, which takes the form of

$$f(x) = 2 - x_1 + x_1\ln x_1 - x_2 + x_2\ln x_2 + \int_1^{x_3}\ln\frac{12x_3}{8 + 2x_3 + x_3^2 + x_3^3}dx_3.$$

Then we calculate

$$\nabla^2 f(x) = diag\left(x_1^{-1}, x_2^{-1}, \frac{8 - x_3^2 - 2x_3^3}{x_3(8 + 2x_3 + x_3^2 + x_3^3)}\right),$$

which implies $8 - x_3^2 - 2x_3^3|_{x^*=(1,1,1)} > 0$. Thus, from Theorem 2, $f(x)$ can behave like a Lyapunov function to prove $x^* = (1,1,1)^\top$ is locally asymptotically stable.

IV. Stability of CBP MASs

In this section, we shall define a new class of CRNs based on complex balanced ones, and demonstrate some nice results on stability for them using the Lyapunov function PDEs method.

A. Definition

The concept of reverse reconstruction [19] for an MAS stimulates us to define a wide range of CRNs, which essentially originate from complex balanced MASs. We thus name them CBP CRNs [22].

Definition 9 (CBP MAS): Given a complex balanced $\mathcal{M}$ governed by (1) with an equilibrium $x^* \in \mathbb{R}^n_{>0}$, an $\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{k})$ is called a CBP MAS with respect to $\mathcal{M}$ if for some positive diagonal matrix $D = diag(d_1, \cdots, d_n)$ but not the identity matrix, its species set admits $\tilde{\mathcal{S}} = \mathcal{S}$ while the complexes set $\tilde{\mathcal{C}} = \cup_{i=1}^{\tilde{r}} \{\tilde{v}_{\cdot i}, \tilde{v}'_{\cdot i}\}$ and the reactions set $\tilde{\mathcal{R}} = \cup_{i=1}^{\tilde{r}} \{\tilde{v}_{\cdot i} \xrightarrow{\tilde{k}_i} \tilde{v}'_{\cdot i}\}$ satisfy

(1) $\tilde{r} = r, \tilde{v}_{\cdot i}, \tilde{v}'_{\cdot i} \in \mathbb{Z}^n_{\geq 0}, \tilde{v}_{\cdot i} = v_{\cdot i}, \tilde{v}'_{\cdot i} = v_{\cdot i} + D^{-1}(v'_{\cdot i} - v_{\cdot i})$;

(2) $\tilde{k}_i = k_i \prod_{j=1}^n d_j^{v_{ji}}, \tilde{R}(\tilde{x}) = R(x)$.

Further, the dynamics of $\tilde{\mathcal{M}}$ is expressed by

$$\dot{\tilde{x}} = \tilde{\Gamma}\tilde{R}(\tilde{x}), \quad \tilde{x} \in \mathbb{R}^n_{\geq 0}. \tag{17}$$

Remark 4: Definition 9 suggests that $\tilde{\Gamma} = D^{-1}\Gamma$ and $\tilde{x} = D^{-1}x$, from the latter, i.e., $\tilde{x}^* = D^{-1}x^*$. Moreover, $\tilde{x}^*$ is an equilibrium in $\tilde{\mathcal{M}}$ if and only if x^* is an equilibrium in $\mathcal{M}$.

The one-to-one correspondence between $\tilde{x}^*$ and x^* manifests an important property for CBP MASs, as shown below.

Property 3 ([22]): For any CBP $\tilde{\mathcal{M}}$ generated by a complex balanced $\mathcal{M}$ under a certain matrix D, there is a unique equilibrium in each positive stoichiometric compatibility class.

Remark 5: Substantially, the concept of CBP MAS can be explained by defining a linear transformation, i.e., $\tilde{x} = D^{-1}x$, to bridge the differential equation $\dot{x} = \Gamma R(x)$ to $\dot{\tilde{x}} = \tilde{\Gamma}\tilde{R}(\tilde{x})$. At this point, this notion is consistent with the linear conjugacy concept [16], [17], and a special case of the reconstruction concept [19] from the dynamics. Different from the other two networks, the CBP network has a specific restriction on the structure but for the concepts of linear conjugacy and reconstruction, their structural information is uncertain. The special structure facilitates us to construct a concrete Lyapunov function for the CBP network based on the well-known pseudo-Helmholtz function, and facilitates us to construct the corresponding compound networks, too. Of course, Johnston and Siegel [16] also addressed the stability issue of the linear conjugated systems of a complex balanced MAS, but they got it directly from the definition of linear conjugacy while Lyapunov's Second Theorem is not involved. Throughout their paper, the concrete form of the Lyapunov function for stability analysis is not given. Besides, the CBP networks could stand for practical biochemical networks while the latter twos work as tools and/or even virtual networks.

We use the following example to exhibit that CBP networks are of practical significance.

Example 3: Consider a class of complex balanced MASs like

$$m'S_1 \underset{k_2}{\overset{k_1}{\rightleftharpoons}} mS_2, \qquad m', m \in \mathbb{Z}_{>0}. \tag{18}$$

By taking $D = diag(m', m)$ we get the CBP MAS in the form of

$$m'S_1 \xrightarrow{k_1 m'^{m'}} (m'-1)S_1 + S_2,\ mS_2 \xrightarrow{k_2 m^m} (m-1)S_2 + S_1. \tag{19}$$

Actually, this CBP network can correspond to two types of motifs which have been well studied in [28]. These motifs may be helpful in looking for candidates of biochemical reactions with a small-number effect for possible biological functions. More precisely, when $m' \geq 2, m \geq 2$, (19) belongs to motif K and when $m' = 1$ and $m \geq 2$, the shape of (19) coincides with motif G.

B. An algorithm for producing CBP CRNs

Definition 9 will yield a large class of non-weakly reversible CRNs based on a single complex balanced CRN under various matrices D's. The following algorithm gives a systematic way to generate CBP CRNs from a complex balanced CRN.

Algorithm 1 find all feasible $D = \text{diag}(d_1, \cdots, d_n)$ such that all vectors $\tilde{v}'_{\cdot i} \in \mathbb{Z}^n_{\geq 0}$ for $i = 1, \cdots, r$, and generate $\tilde{v}'_{\cdot i}, \tilde{k}_i$.

Input: $v_{\cdot i}, v'_{\cdot i}, k_i,\ i = 1, \cdots, r$
for $j = 1$ to n **do**
  for $i = 1$ to r **do**
    if $v'_{ji} - v_{ji} < 0$ **then**
      $F_{ji} = \left\{ \frac{v_{ji} - v'_{ji}}{v_{ji} - a}, a = 0, \cdots, v_{ji-1} \right\}$
    else if $v'_{ji} - v_{ji} > 0$ **then**
      $F_{ji} = \left\{ \frac{v_{ji} - v'_{ji}}{v_{ji} - a}, a = 1, \cdots \right\}$
    else
      $F_{ji} = \mathbb{R}_{>0}$
    end if
  end for
  $F_j = \bigcap F_{ji}$
end for
if $F_j = \mathbb{R}_{>0}$ **then**
  $d_j = 1$
end if
$\mathscr{D} = \{\text{diag}(d_1, \cdots, d_n), d_j \in F_j\}$
for D in $\mathscr{D}$ **do**
  for $i = 1$ to r **do**
    $\tilde{v}'_{\cdot i} = v_{\cdot i} + D^{-1}(v'_{\cdot i} - v_{\cdot i}),\ \tilde{k}_i = k_i \prod_{j=1}^n d_j^{v_{ji}}$
  end for
  Output $D, \tilde{v}'_{\cdot i}, \tilde{k}_i, i = 1, \cdots, r$
end for

Proof: The detailed proof of Algorithm 1 can be found in appendix B. □

The following example exhibits how the algorithm works.

Example 4: Consider the following network

$$5S_1 + S_6 \underset{k_2}{\overset{k_1}{\rightleftharpoons}} S_2 \underset{k_4}{\overset{k_3}{\rightleftharpoons}} 3S_3 + S_6$$

$$S_3 + S_7 \underset{k_6}{\overset{k_5}{\rightleftharpoons}} S_4 \underset{k_8}{\overset{k_7}{\rightleftharpoons}} S_5 + S_7$$

If we shut down the two reactions, i.e., $k_4 = k_8 = 0$, this network can be viewed as a subnetwork of the Calvin cycle [29]. Note that this network is complex balanced. In terms of the algorithm, we can compute that $d_1 \in \{\frac{5}{4}, \frac{5}{3}, \frac{5}{2}, 5\}$ while other $d_j = 1$ for $j = 2, \cdots, 7$. Thus it can produce four kinds

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of CBP CRNs, listed as

$$\begin{aligned}
&(1)\quad d_1=\tfrac{5}{4}, && 5S_1+S_6 \xrightarrow{3.05k_1} S_1+S_2,\\
& && 3S_1+S_6 \xleftarrow{k_2} S_2 \underset{k_4}{\overset{k_3}{\rightleftharpoons}} 3S_3+S_6,\\
& && S_3+S_7 \underset{k_6}{\overset{k_5}{\rightleftharpoons}} S_4 \underset{k_8}{\overset{k_7}{\rightleftharpoons}} S_5+S_7;\\
&(2)\quad d_1=\tfrac{5}{3}, && 5S_1+S_6 \xrightarrow{12.86k_1} 2S_1+S_2,\\
& && S_2 \xrightarrow{k_2} 3S_1+S_6, \qquad \cdots;\\
&(3)\quad d_1=\tfrac{5}{2}, && 5S_1+S_6 \xrightarrow{97.66k_1} 3S_1+S_2,\\
& && S_2 \xrightarrow{k_2} 2S_1+S_6, \qquad \cdots;\\
&(4)\quad d_1=5, && 5S_1+S_6 \xrightarrow{3125k_1} 4S_1+S_2,\\
& && S_2 \xrightarrow{k_2} S_1+S_6, \qquad \cdots,
\end{aligned}$$

where we use dots in cases (2), (3), and (4) to represent the remaining reactions that are the same as those reversible reactions emerging in case (1).

In practical application, the problem of how to identify the CBP network is important, which can be addressed by the existing algorithm [30].

C. Lyapunov function PDEs to the stability of CBP MASs

As stated in Remark 5, the asymptotic stability of CBP MASs could be actually addressed through the linear conjugacy or the reconstruction strategy. Hence, the focus should not be on the stability result of CBP MASs itself, but on the alternative way, i.e., the Lyapunov function PDEs, to this result.

For a CBP MAS, defined in Definition 9, its Lyapunov function PDEs are written as

$$\sum_{i=1}^{\tilde{r}} \tilde{k}_i \tilde{x}^{\tilde{v}_{\cdot i}} - \sum_{i=1}^{\tilde{r}} \tilde{k}_i \tilde{x}^{\tilde{v}_{\cdot i}} \exp\left\{(\tilde{v}'_{\cdot i}-\tilde{v}_{\cdot i})^\top \nabla f(\tilde{x})\right\}=0, \quad (20)$$

and

$$\lim_{\substack{\tilde{x}\to\bar{x}\\ \tilde{x}\in(\bar{x}+\tilde{\mathscr{S}})\cap\mathbb{R}^n_{>0}}} \sum_{\{i|\tilde{v}_{\cdot i}\in\tilde{\mathcal{C}}_{\bar{x}}\}} \tilde{k}_i\tilde{x}^{\tilde{v}_{\cdot i}} - \sum_{\{i|\tilde{v}'_{\cdot i}\in\tilde{\mathcal{C}}_{\bar{x}}\}} \tilde{k}_i\tilde{x}^{\tilde{v}_{\cdot i}} \exp\left\{(\tilde{v}'_{\cdot i}-\tilde{v}_{\cdot i})^\top \nabla f(\tilde{x})\right\}=0, \quad (21)$$

We thus have the following stability result for CBP MASs.

Proposition 1 ([22]): For any CBP $\tilde{\mathcal{M}}$ stated in Definition 9, let $\tilde{x}^*\in\mathbb{R}^n_{>0}$ be an equilibrium. Assume $\tilde{\mathcal{C}}_{\bar{x}}$ is selected as the naive boundary complex set, where $\tilde{\mathcal{C}}_{\bar{x}}=\emptyset$ or $\tilde{\mathcal{C}}_{\bar{x}}$ includes both reactant complexes and resultant complexes. Then its induced Lyapunov function PDEs (20) and (21) could produce a solution in the form of

$$\tilde{G}(\tilde{x})=\sum_{j=1}^{n} d_j\left(\tilde{x}_j^*-\tilde{x}_j-\tilde{x}_j\ln\frac{\tilde{x}_j^*}{\tilde{x}_j}\right) \quad (22)$$

as a Lyapunov function to render the local asymptotic stability of $\tilde{x}^*$ with respect to any initial condition in $\tilde{\mathscr{S}}^+(\tilde{x}^*)$ near $\tilde{x}^*$. Furthermore, if the network is persistent, then $\tilde{x}^*$ is globally asymptotically stable with respect to all initial conditions in $\tilde{\mathscr{S}}^+(\tilde{x}^*)$.

We refer to the function $\tilde{G}(\cdot)$ in (22) as the generalized pseudo-Helmholtz function [19].

Example 5: Given a complex balanced CRN as

$$2S_1 \underset{k_2}{\overset{k_1}{\rightleftharpoons}} \emptyset, \quad (23)$$

it possesses a unique positive equilibrium $x^*=\sqrt{\frac{k_1}{k_2}}$. Based on Algorithm 1, we obtain the sole CBP MAS as

$$2S_1 \xrightarrow{4k_1} S_1, \quad \emptyset \xrightarrow{k_2} S_1 \quad (24)$$

under $D=2$. The CBP MAS corresponds to a typical birth-death process, and has a single equilibrium $\tilde{x}^*=\sqrt{\frac{k_1}{4k_2}}$. From Proposition 1, it is straightforward to know that the PDEs (20) and (21) for this CBP MAS admit a solution

$$\tilde{G}(\tilde{x})=2\left(\sqrt{\frac{k_1}{4k_2}}-\tilde{x}-\tilde{x}\ln\sqrt{\frac{k_1}{4k_2}}+\tilde{x}\ln\tilde{x}\right)$$

to behave as a Lyapunov function rendering the local asymptotic stability of $\tilde{x}^*$.

It should be noted that the birth-death processes have been studied well from the viewpoint of microscopic level. Anderson [20] proposed the scaling limit of the non-equilibrium potential as a Lyapunov function to capture asymptotic stability. This example illustrates that the dynamical behavior of some specific biological systems might be analyzed from the viewpoint of CBP MASs.

V. Stability of MASs composed of a CBP MAS and a series of autocatalytic CRNs

In this section, we will (i) consider a class of MASs consisting of a CBP MAS and a few 1-dimensional networks where these subnetworks are independent according to species, and (ii) extend our results on the composition of a CBP MAS and a series of two-species autocatalytic MASs where these subnetworks have shared species.

Like the Com-ℓSub1 MASs, when a CBP MAS and 1-dimensional MASs have mutually independent species sets, there is a straightforward corollary to derive the Lyapunov function for the stability of this CBP-ℓSub1 MAS.

Corollary 1: For a CBP-ℓSub1 $\tilde{\mathcal{M}}$ composed of a CBP MAS $\tilde{\mathcal{M}}^{(0)}$ and ℓ 1-dimensional MASs $\tilde{\mathcal{M}}^{(p)}|_{p=1}^{\ell}$, $\tilde{x}^*\in\mathbb{R}^n_{>0}$ is an equilibrium. Assume that for every $\tilde{\mathcal{M}}^{(p)}$ $(p=0,\cdots,\ell)$, $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}$ is selected as the naive boundary complex set defined as (6), where $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}=\emptyset$ or $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}$ includes at least a reactant complex and a resultant complex. Then its Lyapunov function PDEs admit a solution

$$f(\tilde{x})=\sum_{i=1}^{n_0} d_i\left(\tilde{x}_i^{(0)^*}-\tilde{x}_i^{(0)}-\tilde{x}_i^{(0)}\ln\frac{\tilde{x}_i^{(0)^*}}{\tilde{x}_i^{(0)}}\right) + \sum_{p=1}^{\ell}\int_0^{\gamma_p(\tilde{x}^{(p)})} \ln\tilde{u}^{(p)}(y^\dagger(\tilde{x}^{(p)})+\alpha\omega_p)\mathrm{d}\alpha, \quad (25)$$

where $\tilde{u}^{(p)}$ makes $h_p(\tilde{x}^{(p)}, u^{(p)}) = 0$, and $h_p(\tilde{x}^{(p)}, u^{(p)})$ is defined by

$$h_p(\tilde{x}^{(p)}, u^{(p)}) = \sum_{\{i|\beta_{i(p)}>0\}} (\tilde{k}_i^{(p)} \tilde{x}^{(p)^{\tilde{v}_{\cdot i(p)}}}) \Big(\sum_{j=0}^{\beta_{i(p)}-1} u^{(p)^j} \Big) + \sum_{\{i|\beta_{i(p)}<0\}} (\tilde{k}_i^{(p)} \tilde{x}^{(p)^{\tilde{v}_{\cdot i(p)}}}) \Big(-\sum_{j=\beta_{i(p)}}^{-1} u^{(p)^j} \Big), \tag{26}$$

where $u^{(p)} = \exp\{\omega_p^\top \frac{\partial f}{\partial \tilde{x}^{(p)}}\}$, $\omega_p \in \mathbb{R}^{n_p} \setminus \{\mathbb{0}_{n_p}\}$ represents a set of bases of $\mathscr{S}^{(p)}$, and $\beta_{i(p)} \in \mathbb{Z} \setminus \{0\}$ satisfies $\tilde{v}'_{\cdot i(p)} - \tilde{v}_{\cdot i(p)} = \beta_{i(p)}\omega_p$, $i = 1, \cdots, r_p$, and moreover, $\gamma_p \in \mathscr{C}^2(\mathbb{R}^{n_p}_{>0}; \mathbb{R}_{>0})$ and $y^\dagger \in \mathscr{C}^2(\mathbb{R}^n_{>0}; \mathbb{R}^n_{>0})$ are constrained by $\tilde{x}^{(p)} = y^\dagger(\tilde{x}^{(p)}) + \gamma_p(\tilde{x}^{(p)})\omega_p$ and $\gamma_p(\tilde{x}^{(p)} + \delta\omega_p) = \gamma_p(\tilde{x}^{(p)}) + \delta \ \forall \delta \in \mathbb{R}$, respectively. Further, $\tilde{x}^*$ is locally asymptotically stable, if for every 1-dimensional $\tilde{\mathcal{M}}^{(p)}$ ($p = 1, \cdots, \ell$), there is $\omega_p^\top \frac{\partial}{\partial \tilde{x}^{(p)}} h_p(\tilde{x}^{(p)^*}, 1) < 0$.

Proof: The proof is similar to the Com-ℓsub1 MAS [21] and we omit it here. □

In the next, we replace the complex balanced MAS with the CBP MAS in the Com-ℓts-Autoca MAS. We call this a CBP-ℓts-Autoca MAS, labeled by $\tilde{\mathcal{M}}$, where the CBP MAS and two-species autocatalytic MAS are relabeled by $\tilde{\mathcal{M}}^{(0)}$ and $\tilde{\mathcal{M}}^{(p)}$, respectively.

By taking the same notations as in Com-ℓts-Autoca MASs, we use n_p and r_p to represent the number of species and reactions, and $\tilde{v}_{\cdot i(p)}$ as well as $\tilde{v}'_{\cdot i(p)}$ to represent the reactant complex and the resultant complex of the ith reaction ($i = 1, \cdots, r_p$) of every subnetwork ($p = 0, \cdots, \ell$) in a CBP-ℓts-Autoca MAS, respectively. The dynamics follows the same expression as (11), i.e.,

$$\dot{\tilde{x}} = \sum_{p=0}^{\ell} \sum_{i=1}^{r_p} \tilde{k}_i^{(p)} \tilde{x}^{\tilde{v}_{\cdot i}^{(p)}} \left(\tilde{v}'^{(p)}_{\cdot i} - \tilde{v}^{(p)}_{\cdot i} \right). \tag{27}$$

Similarly, we define the equilibrium points for CBP-ℓts-Autoca MASs as those for Com-ℓts-Autoca MASs and derive the following property.

Property 4: For a CBP-ℓts-Autoca $\tilde{\mathcal{M}}$ modelled by (27), a concentration vector $\tilde{x}^* \in \mathbb{R}^n_{>0}$ is an equilibrium in $\tilde{\mathcal{M}}$ if and only if $\tilde{x}^{(0)^*} = (\tilde{x}_1^*, \cdots, \tilde{x}_{n_0}^*)^\top$ is a positive equilibrium in $\tilde{\mathcal{M}}^{(0)}$ while $\tilde{x}^{(p)^*} = (\tilde{x}_p^*, \tilde{x}_{n_0+p}^*)^\top$ is a reaction vector balanced equilibrium in $\tilde{\mathcal{M}}^{(p)}$ for $p = 1, \cdots, \ell$. In addition, each positive stoichiometric compatibility class of this MAS contains at most a positive equilibrium if it satisfies the same conditions as those in Lemma 1 but with all variables defined for CBP-ℓts-Autoca MASs.

Proof: The proof can be caught in appendix C. □

In the following, we apply the Lyapunov function PDEs method to CBP-ℓts-Autoca MASs for the asymptotic stability analysis.

Lemma 3: For a CBP-ℓts-Autoca $\tilde{\mathcal{M}}$ governed by (27) and possessing an equilibrium $\tilde{x}^*$ defined by Property 4. Let $\bar{\mathcal{C}}^{(p)}_{\tilde{x}^{(p)}}$ be the naive boundary complex set defined as (6), if $\bar{\mathcal{C}}^{(p)}_{\tilde{x}^{(p)}} = \emptyset$ or $\bar{\mathcal{C}}^{(p)}_{\tilde{x}^{(p)}}$ includes at least a reactant complex and a resultant complex, the twice differentiable function

$$f(\tilde{x}) = \sum_{i=1}^{n_0} d_i \Big(\tilde{x}_i^* - \tilde{x}_i - \tilde{x}_i \ln \frac{\tilde{x}_i^*}{\tilde{x}_i} \Big) + \sum_{p=1}^{\ell} \int_{\tilde{x}^*_{n_0+p}}^{\tilde{x}_{n_0+p}} \ln \frac{k_{r_p}^{(p)} \alpha_p}{\sum_{i=1}^{r_p-1} k_i^{(p)} \tilde{x}_p^* \alpha_p^{m_i-1}} d\alpha_p, \tag{28}$$

where $d_p = 1$ for $p = 1, \cdots, \ell$ and $d_p \in F_p$ for $p = \ell + 1, \cdots, n_0$ with F_p calculated by Algorithm 1, is a solution of the Lyapunov function PDEs ((13), (14)-like equations, but with all variables defined as for a CBP-ℓts-Autoca MAS) induced by $\tilde{\mathcal{M}}$.

Proof: The detailed proof is given in appendix C. □

Furthermore, the asymptotic stability of CBP-ℓts-Autoca MASs can be derived.

Theorem 3: For a CBP-ℓts-Autoca $\tilde{\mathcal{M}}$ described by (27) and admitting an equilibrium $\tilde{x}^* \in \mathbb{R}^n_{>0}$ defined by Property 4, $\tilde{x}^*$ is locally asymptotically stable when satisfying the same conditions as those in Theorem 2 but with all variables defined for CBP-ℓts-Autoca MASs.

Proof: The proof can be caught in appendix C. □

The following two examples serve for illustrating the validity of the Lyapunov function PDEs way in CBP-ℓts-Autoca MASs.

Example 6: Consider a CBP-ℓts-Autoca $\tilde{\mathcal{M}}$ ($\ell = 1$) with the reaction route

$$S_3 \underset{k_1^{(1)}}{\overset{k_3^{(1)}}{\rightleftharpoons}} S_1 \xrightarrow{k_1^{(0)}} S_2,$$

$$mS_2 \xrightarrow{k_2^{(0)} m^m} (m-1)S_2 + S_1, \ m \geq 2,$$

$$S_1 + (m'-1)S_3 \xrightarrow{k_2^{(1)}} m'S_3, \ m' \geq 2.$$

This $\tilde{\mathcal{M}}$ is non-weakly reversible, 2-dimensional and of deficiency 2. Note that the present CBP $\tilde{\mathcal{M}}^{(0)}$ (identified by the reaction rate constants with superscript (0)) and the autocatalytic $\tilde{\mathcal{M}}^{(1)}$ correspond to motif G (see example 3) and motif F given in [28], respectively. As a matter of fact, $\tilde{\mathcal{M}}$ can be viewed as a special case of the compound network of motif G and motif F.

Let $m = 2$, $m' = 2$, and $k_1^{(1)} > k_2^{(1)} k_2^{(0)}$, then $\tilde{\mathcal{M}}^{(0)}$ admits a positive equilibrium $(\tilde{x}_1^*, \tilde{x}_2^*) = \Big(k_2^{(0)}, \frac{1}{2}\sqrt{k_1^{(0)}} \Big)$ and has complexes as

$$\tilde{v}_{\cdot 1(0)} = (1,0)^\top, \tilde{v}'_{\cdot 1(0)} = (0,1)^\top, \tilde{v}_{\cdot 2(0)} = (0,2)^\top, \tilde{v}'_{\cdot 2(0)} = (1,1)^\top$$

while $\tilde{\mathcal{M}}^{(1)}$ admits a reaction vector balanced equilibrium

$$(\tilde{x}_1^*, \tilde{x}_3^*) = \Bigg(k_2^{(0)}, \frac{k_3^{(1)} k_2^{(0)}}{k_1^{(1)} - k_2^{(1)} k_2^{(0)}} \Bigg)$$

and possesses complexes as $\tilde{v}_{\cdot 1(1)} = \tilde{v}'_{\cdot 3(1)} = (1,0)^\top$, $\tilde{v}'_{\cdot 1(1)} = \tilde{v}_{\cdot 3(1)} = (0,1)^\top$, $\tilde{v}_{\cdot 2(1)} = (1,1)^\top$, $\tilde{v}'_{\cdot 2(1)} = (0,2)^\top$.

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According to Lemma 3, the corresponding Lyapunov function PDEs for $\tilde{\mathcal{M}}$ admit a solution in the form of

$$f(\tilde{x}) = \tilde{x}_1^* - \tilde{x}_1 - \tilde{x}_1 \ln \frac{\tilde{x}_1^*}{\tilde{x}_1} + 2\left(\tilde{x}_2^* - \tilde{x}_2 - \tilde{x}_2 \ln \frac{\tilde{x}_2^*}{\tilde{x}_2}\right) + \tilde{x}_3 \times \ln(k_3^{(1)}\tilde{x}_3) - \tilde{x}_3^* \ln(k_3^{(1)}\tilde{x}_3^*) - k_2^{(1)^{-1}}\Big[(k_1^{(1)} + k_2^{(1)}\tilde{x}_3)\ln(\tilde{x}_1^* k_1^{(1)} + \tilde{x}_1^* k_2^{(1)}\tilde{x}_3) - (k_1^{(1)} + k_2^{(1)}\tilde{x}_3^*)\ln(\tilde{x}_1^* k_1^{(1)} + \tilde{x}_1^* k_2^{(1)}\tilde{x}_3^*)\Big]$$

Further, $\dot{f}(\tilde{x}) \leq 0$ is guaranteed by Property 1, which also can be verified directly through the following way. The dynamics of the considered MAS is given by

$$\begin{cases} \dot{\tilde{x}}_1 = -k_1^{(0)}\tilde{x}_1 + 4k_2^{(0)}\tilde{x}_2^2 - k_1^{(1)}\tilde{x}_1 - k_2^{(1)}\tilde{x}_1\tilde{x}_3 + k_3^{(1)}\tilde{x}_3, \\ \dot{\tilde{x}}_2 = k_1^{(0)}\tilde{x}_1 - 4k_2^{(0)}\tilde{x}_2^2, \\ \dot{\tilde{x}}_3 = k_1^{(1)}\tilde{x}_1 + k_2^{(1)}\tilde{x}_1\tilde{x}_3 - k_3^{(1)}\tilde{x}_3, \end{cases}$$

Then using $\nabla f(\tilde{x}) = (\ln \frac{\tilde{x}_1}{\tilde{x}_1^*}, 2\ln\frac{\tilde{x}_2}{\tilde{x}_2^*}, \ln \frac{k_3^{(1)}\tilde{x}_3}{k_1^{(1)}\tilde{x}_1^* + k_2^{(1)}\tilde{x}_1^*\tilde{x}_3})^\top$, we have

$$\begin{aligned} \dot{f}(\tilde{x}) &= \nabla f(\tilde{x})^\top \dot{\tilde{x}} \\ &= (4k_2^{(0)}\tilde{x}_2^2 - k_1^{(0)}\tilde{x}_1)(\ln\frac{\tilde{x}_1}{\tilde{x}_1^*} - 2\ln\frac{\tilde{x}_2}{\tilde{x}_2^*}) + (k_3^{(1)}\tilde{x}_3 - \\ &\quad k_1^{(1)}\tilde{x}_1 - k_2^{(1)}\tilde{x}_1\tilde{x}_3)(\ln\frac{\tilde{x}_1}{\tilde{x}_1^*} - \ln\frac{k_3^{(1)}\tilde{x}_3}{k_1^{(1)}\tilde{x}_1^* + k_2^{(1)}\tilde{x}_1^*\tilde{x}_3}) \\ &= (4k_2^{(0)}\tilde{x}_2^2 - k_1^{(0)}\tilde{x}_1)\ln\frac{k_1^{(0)}\tilde{x}_1}{4k_2^{(0)}\tilde{x}_2^2} + (k_3^{(1)}\tilde{x}_3 - k_1^{(1)}\tilde{x}_1 \\ &\quad - k_2^{(1)}\tilde{x}_1\tilde{x}_3)\ln\frac{k_1^{(1)}\tilde{x}_1 + k_2^{(1)}\tilde{x}_1\tilde{x}_3}{k_3^{(1)}\tilde{x}_3} \\ &\leq 0, \end{aligned} \tag{29}$$

where the equality holds only at $-k_1^{(0)}\tilde{x}_1 + 4k_2^{(0)}\tilde{x}_2^2 = 0$ and $-k_1^{(1)}\tilde{x}_1 - k_2^{(1)}\tilde{x}_1\tilde{x}_3 + k_3^{(1)}\tilde{x}_3 = 0$, which implies that $\tilde{x}$ is an equilibrium of the MAS.

In addition, it is not hard to compute the Hessian matrix of $f(\tilde{x})$ to be

$$\nabla^2 f(\tilde{x}) = diag\left(\tilde{x}_1^{-1}, 2\tilde{x}_2^{-1}, \frac{k_1^{(1)}}{\tilde{x}_3(k_1^{(1)} + k_2^{(1)}\tilde{x}_3)}\right),$$

which is obviously strictly convex. From Theorem 3 or Theorem 1, $f(\tilde{x})$ is qualified as a Lyapunov function to suggest the asymptotic stability of $(\tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^*)^\top$.

Example 7: Consider a CBP-ℓts-Autoca $\tilde{\mathcal{M}}$ ($\ell = 1$) with dim$\tilde{\mathscr{S}} = 3$ and deficiency 4, where the upper part is a CBP $\tilde{\mathcal{M}}^{(0)}$ while the lower part is an autocatalytic $\tilde{\mathcal{M}}^{(1)}$,

$$2S_2 + S_1 \xrightarrow{k_1^{(0)}} 2S_2 \xrightarrow{k_2^{(0)}} 4S_2, \quad 3S_2 \xrightarrow{k_3^{(0)}} S_1 + S_2,$$

$$S_1 \underset{k_5^{(1)}}{\overset{k_1^{(1)}}{\rightleftharpoons}} S_3, \; S_1 + (m_i - 1)S_3 \xrightarrow{k_i^{(1)}} m_i S_3 \;,$$

$$m_2 = 2, m_3 = 3, m_4 = 4,$$

where $k_1^{(0)} = \frac{1}{2}, k_1^{(0)} = \frac{1}{2}, k_3^{(0)} = \frac{1}{8}$, and $k_1^{(1)} = 8, k_2^{(1)} = 2, k_3^{(1)} = 1, k_4^{(1)} = 1, k_5^{(1)} = 12$. Note that the CBP $\tilde{\mathcal{M}}^{(0)}$ is generated by the complex balanced MAS given in (16) under $D = \text{diag}(1, \frac{1}{2})$, and possesses an equilibrium $\tilde{x}^{(0)^*} = (1,4)^\top$ while the autocatalytic $\tilde{\mathcal{M}}^{(1)}$ is the same as Example 2.

From Lemma 3, its Lyapunov function PDEs have a solution in the form of

$$f(\tilde{x}) = 3 - \tilde{x}_1 + \tilde{x}_1 \ln \tilde{x}_1 - \frac{1}{2}\tilde{x}_2 - \frac{1}{2}\tilde{x}_2 \ln \frac{4}{\tilde{x}_2} + \int_1^{\tilde{x}_3} \ln \frac{12\tilde{x}_3}{8 + 2\tilde{x}_3 + \tilde{x}_3^2 + \tilde{x}_3^3} d\tilde{x}_3,$$

with the second derivative

$$\nabla^2 f(\tilde{x}) = diag\left(\tilde{x}_1^{-1}, \frac{1}{2}\tilde{x}_2^{-1}, \frac{8 - \tilde{x}_3^2 - 2\tilde{x}_3^3}{\tilde{x}_3(8 + 2\tilde{x}_3 + \tilde{x}_3^2 + \tilde{x}_3^3)}\right).$$

Analogously, we have $8 - \tilde{x}_3^2 - 2\tilde{x}_3^3|_{\tilde{x}^* = (1,4,1)} > 0$. Obviously, $f(\tilde{x})$ can behave like a Lyapunov function to prove this network locally asymptotically stable at the equilibrium $\tilde{x}^* = (1,4,1)^\top$ due to Theorem 3.

Inspired by the CBP-ℓts-Autoca MAS, a method similar to dimensionality reduction is proposed in the following corollary, which shows that if a MAS can be decomposed into a CBP MAS and some 1-dimensional MASs, then the stability of its equilibria can be achieved under certain conditions.

Corollary 2: If an MAS with a positive equilibrium $x^* \in \mathbb{R}^n_{>0}$ can be decomposed into a CBP $\mathcal{M}^{(0)} = (\mathcal{S}^{(0)}, \mathcal{C}^{(0)}, \mathcal{R}^{(0)}, k^{(0)})$ and ℓ independent two-species autocatalytic MASs according to species, labeled by $\mathcal{M}^{(p)} = (\mathcal{S}^{(p)}, \mathcal{C}^{(p)}, \mathcal{R}^{(p)}, k^{(p)})$, $p = 1, \cdots, \ell$, and moreover, $S_p = \mathcal{S}^{(0)} \bigcap \mathcal{S}^{(p)}$ with $\mathcal{S}^{(0)} = \{S_1, \cdots, S_{n_0}\}, \mathcal{S}^{(p)} = \{S_p, S_{n_0+p}\}$, $\ell \leq n_0$ and $n_0 + \ell = n$, then x^* is locally asymptotically stable with the following conditions to be true

(1) for every S_p, $\exists v_{\cdot i(0)} \to v'_{\cdot i(0)} \in \mathcal{M}^{(0)}$ such that $v_{pi(0)} = 1$ while $v'_{pi(0)} = 0$, and $\forall v_{\cdot i(p)} \to v'_{\cdot i(p)} \in \mathcal{M}^{(p)}$ such that $v_{pi(p)}, v'_{pi(p)}$ equal to 0 or 1;
(2) $\sum_{i=1}^{r_p - 1}(2 - m_i)k_i^{(p)} x^{*^{m_i - 1}}_{n_0+p} > 0$.

Proof: Combing the results in Lemma 3 and Theorem 3 about the CBP-ℓts-Autoca MAS, the conclusion follows immediately. □

VI. Conclusions

In this paper, Com-ℓts-Autoca MASs, CBP MASs, CBP-ℓsub1 MASs, and CBP-ℓts-Autoca MASs are consecutively defined from a complex balanced MAS according to some rules, following which an algorithm is proposed to compute CBP MASs systematically. All of these networks can be any dimensional, non-weakly reversible, and of arbitrary deficiency. Moreover, for Com-ℓts-Autoca MASs and CBP-ℓts-Autoca MASs, it has been shown that each positive stoichiometric compatibility class contains at most a positive equilibrium. We use the Lyapunov functions PDEs method to successfully catch the local asymptotic stability of these MASs. The result greatly supports our previous conjecture [21] that the Lyapunov function PDEs of every stable MAS have a solution capable of acting as a Lyapunov function to render the asymptotic stability.

Appendix

In this appendix, we give the detailed proofs to the results presented in Sections III, IV and V.

A. Proofs of results in Section III

Proof of Property 2: Just inserting $x^{(0)^*}$ and $x^{(p)^*}$ ($p = 1, \cdots, \ell$) into the dynamics of the Com-ℓts-Autoca MAS governed by (11), it is easy to know that x^* is a positive equilibrium point of the considered $\mathcal{M}$ distinctly, and vice versa. □

Proof of Lemma 1: The results can be proved with the aid of the dynamics of the considered $\mathcal{M}$, which is specifically represented as

$$\begin{cases} \dot{x}_p = -\sum_{i=1}^{r_p-1} k_i^{(p)} x_p x_{n_0+p}^{m_i-1} + k_{r_p}^{(p)} x_{n_0+p} \\ \qquad + \sum_{i=1}^{r_0} k_i^{(0)} x^{v_{\cdot i}^{(0)}} \left(v_{pi}'^{(0)} - v_{pi}^{(0)} \right), \ p = 1, \cdots, \ell \\ \dot{x}_j = \sum_{i=1}^{r_0} k_i^{(0)} x^{v_{\cdot i}^{(0)}} \left(v_{ji}'^{(0)} - v_{ji}^{(0)} \right), \ j = \ell+1, \cdots, n_0 \\ \dot{x}_{n_0+p} = \sum_{i=1}^{r_p-1} k_i^{(p)} x_p x_{n_0+p}^{m_i-1} - k_{r_p}^{(p)} x_{n_0+p}. \end{cases} \tag{30}$$

It demonstrates that the whole $\mathcal{M}$ is balanced if and only if the involved complex balanced $\mathcal{M}^{(0)}$ and autocatalytic $\mathcal{M}^{(p)}$s are both balanced. Since $\mathcal{M}^{(0)}$ possesses a unique positive equilibrium in every positive stoichiometric compatibility class, then the number of equilibrium in each positive stoichiometric compatibility class induced by the Com-ℓts-Autoca $\mathcal{M}$ can be decided by the remaining $\mathcal{M}^{(p)}$s. Suppose the equilibrium $x^{(0)^*} \in \mathbb{R}^{n_0}_{>0}$ of $\mathcal{M}^{(0)}$ is given, then the remaining dynamic equations turn to be

$$\dot{x}_{n_0+p} = \sum_{i=1}^{r_p-1} k_i^{(p)} x_p^* x_{n_0+p}^{m_i-1} - k_{r_p}^{(p)} x_{n_0+p}.$$

Since $|v_{\cdot i(p)}| = m_i$, we can find when $r_p = 2, |v_{\cdot 1(p)}| = 1$, $\dot{x}_{n_0+p} = 0$ has a unique positive solution. When $r_p = 3, |v_{\cdot 1(p)}| = 1$, $|v_{\cdot 2(p)}| = 2$, it has precisely one positive solution only if $k_3^{(p)} - x_p^* k_2^{(p)} > 0$. Thus $\forall \{p\}_{p=1}^{\ell}$, when $|v_{\cdot i(p)}| \leq 2$, there is at most one equilibrium in $\mathscr{S}^+(x_0)$ for every initial state $x_0 \in \mathbb{R}^n_{>0}$.

Next, let $W \subseteq \{1, \cdots, \ell\}$ represent an index set that satisfies $p \in W$ if there exists some reaction with index z in $\mathcal{M}^{(p)}$ such that $|v_{\cdot z(p)}| > 2$, i.e. $\exists \ m_z > 2$. Then for $\mathcal{M}^{(p)}$ with $p \in W$, there will be at most two positive intersection points when $\sum_{i=1}^{r_p-1} k_i^{(p)} x_p^* x_{n_0+p}^{m_i-1}$ intersects with $k_{r_p}^{(p)} x_{n_0+p}$ at the plane $x_{n_0+p} > 0$. Since such $\mathcal{M}^{(p)}$ is mass-conserved, we know that there is at most one positive equilibrium in its positive stoichiometric compatibility class. Therefore, it indicates that for an $\mathcal{M}$ there is at most an equilibrium in $\mathscr{S}^+(x_0) \bigcap \mathscr{M}$ for any initial condition $x_0 \in \mathbb{R}^n_{>0} \bigcap (\bigcup_{p \in W} \mathscr{M}_p)$, where $\mathscr{M}_p = \{x_p + x_{n_0+p} = M_p, M_p > 0\}$ represents the conservation law that $\mathcal{M}^{(p)}$ follows. □

Proof of Lemma 2: First of all, the corresponding PDE (13) for the Com-ℓts-Autoca $\mathcal{M}$ can be rewritten as

$$\sum_{p=0}^{\ell} \sum_{i=1}^{r_p} k_i^{(p)} x^{(p) v_{\cdot i(p)}} \left(1 - \exp\left\{ (v'_{\cdot i(p)} - v_{\cdot i(p)})^\top \frac{\partial f(x)}{\partial x^{(p)}} \right\} \right) = 0. \tag{31}$$

Then taking

$$\nabla f(x) = \left(\bigotimes_{i=1}^{n_0} \ln \frac{x_i}{x_i^*}, \bigotimes_{p=1}^{\ell} \ln \frac{k_{r_p}^{(p)} x_{n_0+p}}{\sum_{i=1}^{r_p-1} k_i^{(p)} x_p^* x_{n_0+p}^{m_i-1}} \right)^\top$$

into (31), where $\bigotimes$ is the vector concatenation operator, we derive that

$$\sum_{i=1}^{r_0} k_i^{(0)} x^{(0) v_{\cdot i(0)}} \left(1 - \exp\left\{ (v'_{\cdot i(0)} - v_{\cdot i(0)})^\top \ln \frac{x^{(0)}}{x^{(0)^*}} \right\} \right) = 0, \tag{32}$$

where the equality holds on account of (8).

Further, since for $p = 1, \cdots, \ell$, $\zeta = \frac{\partial f(x)}{\partial x^{(p)}} = \left(\ln \frac{x_p}{x_p^*}, \ \ln \frac{k_{r_p}^{(p)} x_{n_0+p}}{\sum_{i=1}^{r_p-1} k_i^{(p)} x_p^* x_{n_0+p}^{m_i-1}} \right)^\top$, then we have

$$\begin{aligned} &\sum_{i=1}^{r_p} k_i^{(p)} x^{(p) v_{\cdot i(p)}} \left(1 - \exp\left\{ (v'_{\cdot i(p)} - v_{\cdot i(p)})^\top \frac{\partial f(x)}{\partial x^{(p)}} \right\} \right) \\ &= \sum_{i=1}^{r_p-1} k_i^{(p)} x_p x_{n_0+p}^{m_i-1} \left(1 - \exp\{(-1,1)\zeta\} \right) + k_{r_p}^{(p)} x_{n_0+p} \\ &\quad \times \left(1 - \exp\{(1,-1)\zeta\} \right) \\ &= \sum_{i=1}^{r_p-1} k_i^{(p)} x_p x_{n_0+p}^{m_i-1} \left(1 - \frac{k_{r_p}^{(p)} x_{n_0+p}}{\sum_{i=1}^{r_p-1} k_i^{(p)} x_p x_{n_0+p}^{m_i-1}} \right) + k_{r_p}^{(p)} \times \\ &\quad x_{n_0+p} \left(1 - \frac{\sum_{i=1}^{r_p-1} k_i^{(p)} x_p x_{n_0+p}^{m_i-1}}{k_{r_p}^{(p)} x_{n_0+p}} \right) \\ &= 0, \end{aligned} \tag{33}$$

Therefore, by summing (33) from $p = 1$ to ℓ and (32) we get (13).

Then we prove that $f(x)$ satisfies the boundary condition. Let $\bar{x}$ be any boundary point of any positive stoichiometric compatibility class of a Com-ℓts-Autoca MAS, denoted by $\bar{x} \in \partial_{(\bar{x}+\mathscr{S}) \bigcap \mathbb{R}^n_{>0}} ((\bar{x} + \mathscr{S}) \bigcap \mathbb{R}^n_{>0} \neq \emptyset)$, then there exists an index set $I_{\bar{x}} \subseteq \{0, \cdots, \ell\}$ satisfies $\bar{x}^{(p)} \in \partial_{(\bar{x}^{(p)}+\mathscr{S}^{(p)}) \bigcap \mathbb{R}^{n_p}_{>0}}$ if $p \in I_{\bar{x}}$. Then the boundary condition of the MAS can be rewritten as

$$\begin{aligned} &\sum_{p \in I_{\bar{x}}} \lim_{\substack{x^{(p)} \to \bar{x}^{(p)} \\ x^{(p)} \in (\bar{x}^{(p)} + \mathscr{S}^{(p)}) \bigcap \mathbb{R}^{n_p}_{>0}}} \left(\sum_{\{i | v_{\cdot i(p)} \in \bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}\}} \theta_1(i,p) - \right. \\ &\left. \sum_{\{i | v'_{\cdot i(p)} \in \bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}\}} \theta_2(i,p) \right) + \sum_{p \notin I_{\bar{x}}} \sum_{i=1}^{r_p} \Big(\theta_1(i,p) - \theta_2(i,p) \Big) = 0, \end{aligned} \tag{34}$$

where $\theta_1(i,p) = k_i^{(p)} x^{(p) v_{\cdot i(p)}}$, $\theta_2(i,p) = k_i^{(p)} x^{(p) v_{\cdot i(p)}} \exp\left\{ (v'_{\cdot i(p)} - v_{\cdot i(p)})^\top \frac{\partial f(x)}{\partial x^{(p)}} \right\}$.

When $\bar{\mathcal{C}}_{\bar{x}} = \emptyset$, it's obvious that Eq. (34) is true. We focus on the case that $\bar{\mathcal{C}}_{\bar{x}} \neq \emptyset$. Since the boundary condition of complex balanced MAS is already derived in [21], we have $\forall x^{(0)} \in (\bar{x}^{(0)} + \mathscr{S}^{(0)}) \bigcap \mathbb{R}^{n_0}_{>0}$, when $x^{(0)} \to \bar{x}^{(0)}$, there is

$$\sum_{\{i | v_{\cdot i(0)} \in \bar{\mathcal{C}}^{(0)}_{\bar{x}^{(0)}}\}} \theta_1(i,0) - \sum_{\{i | v'_{\cdot i(0)} \in \bar{\mathcal{C}}^{(0)}_{\bar{x}^{(0)}}\}} \theta_2(i,0) = 0. \tag{35}$$

Consider the boundary condition of the autocatalytic $\mathcal{M}^{(p)}$, $p=1,\cdots,\ell$, there are two types of naive boundary complex sets.

(i) $\bar{x}^{(p)}=(x_p,0)$ with $x_p>0$, which corresponds to $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}=\left\{v_{\cdot 1(p)},v'_{\cdot r_p(p)}\right\}$, then for any $x^{(p)}\in(\bar{x}^{(p)}+\mathscr{S}^{(p)})\bigcap\mathbb{R}^2_{>0}$, we obtain that

$$\lim_{x^{(p)}\to\bar{x}^{(p)}}\left(\sum_{\{i|v_{\cdot i(p)}\in\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}\}}\theta_1(i,l)-\sum_{\{i|v'_{\cdot i(p)}\in\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}\}}\theta_2(i,l)\right)$$
$$=\lim_{x^{(p)}\to\bar{x}^{(p)}}\left(k_1x_p-k_{r_p}x_{n_0+p}\exp\{(1,-1)\zeta\}\right)$$
$$=\lim_{x^{(p)}\to\bar{x}^{(p)}}\left(k_1x_p-\sum_{i=1}^{r_p-1}k_ix^{m_i-1}_{n_0+p}x_p\right)$$
$$=0.$$

(ii) $\bar{x}^{(p)}=(0,x_{n_0+p})$ with $x_{n_0+p}>0$, then $\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}=\left\{v_{\cdot r_p(p)},v'_{\cdot i(p)},i=1,\cdots,r_p-1\right\}$. Therefore $\forall x^{(p)}\in(\bar{x}^{(p)}+\mathscr{S}^{(p)})\bigcap\mathbb{R}^2_{>0}$, we get

$$\lim_{x^{(p)}\to\bar{x}^{(p)}}\left(\sum_{\{i|v_{\cdot i(p)}\in\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}\}}\theta_1(i,l)-\sum_{\{i|v'_{\cdot i(p)}\in\bar{\mathcal{C}}^{(p)}_{\bar{x}^{(p)}}\}}\theta_2(i,l)\right)$$
$$=\lim_{x^{(p)}\to\bar{x}^{(p)}}\left(k_{r_p}x_{n_0+p}-\sum_{i=1}^{r_p-1}k_ix_px^{m_i-1}_{n_0+p}\exp\{(-1,1)\zeta\}\right)$$
$$=\lim_{x^{(p)}\to\bar{x}^{(p)}}k_{r_p}x_{n_0+p}-k_{r_p}x_{n_0+p}$$
$$=0.$$

Combing the above two cases with (35) we can find that (34) equals to zero since the second part of (34) is also zero according to (31). □

Proof of Theorem 2: Firstly, we compute the second derivative of $f(x)$ as

$$\nabla^2f(x)=\text{diag}\left(\bigotimes_{i=1}^{n_0}x_i^{-1}\bigotimes_{p=1}^{\ell}\frac{\sum_{i=1}^{r_p-1}(2-m_i)k_i^{(p)}x^{m_i-1}_{n_0+p}}{\sum_{i=1}^{r_p-1}k_i^{(p)}x^{m_i}_{n_0+p}}\right) \tag{36}$$

Clearly, $\forall\{p\}_{p=1}^{\ell}$, when $|v_{\cdot i(p)}|\leq 2$, which means $m_i\leq 2$, $f(x)$ is strictly convex for $\nabla^2f(x)>0$ in $\mathscr{S}^+(x_0)$ with respect to any initial value x_0 near x^*. Associated with Theorem 1, x^* is locally asymptotically stable.

Further, we continue to prove the second result. The continuity of the function $\sum_{i=1}^{r_p-1}(2-m_i)k_i^{(p)}x^{m_i-1}_{n_0+p}$ with respect to x_{n_0+p} implies that, there exists a neighborhood of $x^*_{n_0+p}$ for $p=1,\cdots,\ell$, denoted by $\mathcal{N}(x^*_{n_0+p})$, such that $\forall x_{n_0+p}\in\mathcal{N}(x^*_{n_0+p})$, it holds

$$\sum_{i=1}^{r_p-1}(2-m_i)k_i^{(p)}x^{m_i-1}_{n_0+p}>0.$$

So, $\forall x\in\left(\mathbb{R}^{n_0}_{>0}\bigotimes_{p=1}^{\ell}\mathcal{N}(x^*_{n_0+p})\right)\bigcap\mathscr{S}^+(x^*)\bigcap(\bigcup_{p\in W}\mathscr{M}_p)$, where $\mathscr{M}_p,W$ share the same meanings as the ones given in the proof of Lemma 1, we have $\nabla^2f(x)>0$. In the end, Theorem 1 tells us that x^* is locally asymptotically stable. □

B. Proof of Algorithm 1 in Section IV

First we show the algorithm contains all feasible Ds. From $v'_{\cdot i}=v_{\cdot i}+D^{-1}(v'_{\cdot i}-v_{\cdot i}),i=1,\cdots,r$, in Definition 8 (CBP MAS), we have $v'_{ji}=v_{ji}+d_j^{-1}(v'_{ji}-v_{ji}),j=1,\cdots,n$. When the reaction index i is fixed, we denote the feasible region of d_j by F_{ji}. Since $v'_{\cdot i}$ is a nonnegative integer (i.e., $v'_{\cdot i}\in\mathbb{Z}^n_{\geq 0}$), it is easy to get

$$F_{ji}=\begin{cases}\left\{\frac{v_{ji}-v'_{ji}}{v_{ji}-a},a=0,\cdots,v_{ji-1}\right\}, & v'_{ji}-v_{ji}<0,\\ \left\{\frac{v_{ji}-v'_{ji}}{v_{ji}-a},a=1,\cdots\right\}, & v'_{ji}-v_{ji}>0,\\ \mathbb{R}_{>0}, & v'_{ji}-v_{ji}=0.\end{cases} \tag{37}$$

Thus we can get all possible d_j with $d_j\in F_j=\bigcap_{i=1}^r F_{ji},j=1,\cdots,n$. Let $\mathscr{D}=\{\text{diag}(d_1,\cdots,d_n),d_j\in F_j\}$ represent the set of all Ds. Then we prove the set is finite. Due to the fact that the original complex balanced MAS is weakly reversible, then for each j, there must exist some reaction index k such that $v'_{jk}-v_{jk}\leq 0$. Thus from (37) we can know that the number of elements in F_j is finite or $F_j=\mathbb{R}_{>0}$. Note that in the latter case, the value of d_j does not affect the structure of the network because $v'_{ji}=v_{ji}$. Therefore, we stipulate that $d_j=1$ when $F_j=\mathbb{R}_{>0}$ in Algorithm 1. Finally, every set F_j is finite, which implies the number of D is finite. □

C. Proofs of results in Section V

Proof of Property 4: Replacing the complex balanced $\mathcal{M}^{(0)}$ with the CBP MAS in (30), it is easy to find that the CBP-ℓts-Autoca MAS is balanced only if the CBP MAS and autocatalytic MASs are both balanced, which states the first result. Next, according to Property 3 that a CBP MAS possesses a unique positive equilibrium in every positive stoichiometric compatibility class, then the number of equilibrium in each positive stoichiometric compatibility class of a CBP-ℓts-Autoca $\tilde{\mathcal{M}}$ is determined by the autocatalytic MASs. The remaining proof follows Property 2. □

Proof of Lemma 3: We consider the first Lyapunov function PDE induced by the CBP-ℓts-Autoca $\tilde{\mathcal{M}}$, which is

$$\sum_{p=0}^{\ell}\sum_{i=1}^{r_p}\tilde{k}_i^{(p)}\tilde{x}^{(p)\tilde{v}_{\cdot i(p)}}\left(1-\exp\left\{(\tilde{v}'_{\cdot i(p)}-\tilde{v}_{\cdot i(p)})^\top\frac{\partial f(\tilde{x})}{\partial\tilde{x}^{(p)}}\right\}\right)$$
$$=0. \tag{38}$$

Then taking

$$\nabla f(\tilde{x})=\left(\bigotimes_{i=1}^{n_0}d_i\ln\frac{\tilde{x}_i}{\tilde{x}_i^*},\bigotimes_{p=1}^{\ell}\ln\frac{\tilde{k}_{r_p}^{(p)}\tilde{x}_{n_0+p}}{\sum_{i=1}^{r_p-1}\tilde{k}_i^{(p)}\tilde{x}_p^*\tilde{x}^{m_i-1}_{n_0+p}}\right)^\top$$

into (38), we get

$$\sum_{i=1}^{r_0}\tilde{k}_i^{(0)}\tilde{x}^{(0)\tilde{v}_{\cdot i(0)}}\left(1-\exp\left\{(\tilde{v}'_{\cdot i(0)}-\tilde{v}_{\cdot i(0)})^\top\ln\frac{\tilde{x}^{(0)}}{\tilde{x}^{(0)*}}\right\}\right)=0, \tag{39}$$

where the equality holds according to Proposition 1.

Since $d_p = 1$, for $p = 1, \cdots, \ell$, we have $\frac{\partial f(\tilde{x})}{\partial \tilde{x}^{(p)}} = \left(\ln \frac{\tilde{x}_p}{\tilde{x}_p^*}, \ \bigotimes_{p=1}^{\ell} \ln \frac{\tilde{k}_{r_p}^{(p)} \tilde{x}_{n_0+p}}{\sum_{i=1}^{r_p-1} \tilde{k}_i^{(p)} \tilde{x}_p^* \tilde{x}_{n_0+p}^{m_i-1}} \right)^\top$. Then following the result of (33), we know $f(\tilde{x})$ satisfies the PDE of every autocatalytic $\tilde{\mathcal{M}}^{(p)}$, thus (38) holds. The remaining proof of the boundary condition is similar to the second part of Lemma 2. □

Proof of Theorem 3: The second derivative of $f(\tilde{x})$ is calculated as

$$\nabla^2 f(\tilde{x}) = \operatorname{diag}\left(\bigotimes_{i=1}^{n_0} d_i \tilde{x}_i^{-1} \bigotimes_{p=1}^{\ell} \frac{\sum_{i=1}^{r_p-1} (2 - m_i) \tilde{k}_i^{(p)} \tilde{x}_{n_0+p}^{m_i-1}}{\sum_{i=1}^{r_p-1} \tilde{k}_i^{(p)} \tilde{x}_{n_0+p}^{m_i}} \right). \tag{40}$$

The remaining proof is similar to Theorem 2. □

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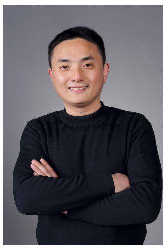

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