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LMI stability test for multidimensional linear state–space models

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ABSTRACT

Stability is a basic property of dynamical systems. In this paper we analyze the stability of multidimensional systems and present new sufficient conditions for the asymptotic stability in terms of linear matrix inequalities. We treat both the discrete-time and continuous-time cases and also propose variants that require linear matrix inequalities of more moderate size.

1. Introduction

The study of the stability of multidimensional systems has been the subject of much research for several decades. It is important for the analysis of robustness of such systems, and has applications in systems theory but also in engineering areas such as circuit theory, digital filtering and image processing. Multidimensional systems propagate the state in several independent spatial directions. We will restrict ourselves here to causal systems, for which we assume to have a generalized state–space model. It is very important to make sure that the propagation of the state-variable then remains bounded as a function of time, which implies the stability of the model. Stability of multi-dimensional systems has been considered by several authors. Michael G. Strintzis [1] gave the first generalization of stability conditions of multidimensional filters of dimension two or more, using the theory of algebraic functions. This led to progress in important areas such as the stabilization problem, see [2].

One approach to define stability is the extension of what exists in the one-dimensional (1D) case: it uses the standard definition of stability of linear systems, which specifies that the fundamental solution converges to zero for arbitrary

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initial conditions. A second equivalent definition of stability, formulates necessary and sufficient conditions in terms of the roots of a characteristic polynomial. In the one dimensional case there is a third equivalent condition, which is the Lyapunov stability criterion and amounts to the solution of a Linear Matrix Inequality (LMI).

In this paper we look at the extension of this third approach to characterize asymptotic stability of multidimensional systems. Unfortunately, the feasibility of LMIs yields only sufficient conditions in the multidimensional case, but the main advantage of this approach is that LMIs can be solved in polynomial time. The transition of the 1D case to the multidimensional case is done naturally by studying the two-dimensional model first. In the 1970s several extensions were proposed for 2D-systems (see e.g. [3–5]). In [6], Huang gave a stability test for 2D digital recursive filters, using a simplified version of a stability theorem of Shanks and proved that it is equivalent to stability results of Ansell [7]. In [8], Davis pointed out a small problem in Huang's proof and corrected it. Other stability criteria were also introduced by Jury [4] and Siljak [9] for 2D discrete-time and continuous-time systems. These approaches were all based on function theoretic criteria. Instead of this approach, several authors have attempted to use matrix algebraic techniques, such as Lyapunov matrix functions or linear matrix inequalities (LMIs) for testing stability of 2D systems, but only sufficient conditions were found so far. In [10], Anderson et al. constructed a 2D Lyapunov matrix equation that is sufficient for stability but not necessary. Sufficient conditions have also been derived in terms of LMIs by Galkowski et al. [11] who consider the problem of positive real control. In [12], Zou et al. gave sufficient LMI conditions for the internal stability of 2D singular systems, including acceptability and jump modes freeness. More recently, an LMI based stability test was developed in [13]. The aim of this paper is to develop new sufficient algebraic conditions for asymptotic stability of d-dimensional (dD) state–space models. Based on techniques developed in [14], we derive here LMIs for guaranteed asymptotic stability of the considered models.

The three models of conventional state–space 2D discrete time include, the Givone–Roesser Model [15], the Attasi Model [16], and the Fornasini–Marchesini Model [3]. These papers are considered as the precursors of the theory of multidimensional systems. In the 1970*s*, they introduced a description of these systems by linear state models that led to design problems and stability tests.

2. Stability of dD discrete-time systems

We first introduce the notion of bounded-input bounded-output (BIBO) stability for dD discrete-time systems. Consider an input–output system described by the equation

$$y(i_1,\dots,i_d)=\sum_{k_1=0}^{+\infty}\cdots\sum_{k_d=0}^{+\infty}g(i_1-k_1,\dots,i_d-k_d)u(k_1,\dots,k_d) \tag{1}$$

where $u(i_1,\dots,i_d)$, $y(i_1,\dots,i_d)$ are the input and output vectors, respectively, and $g(i_1,\dots,i_d)$ is the impulse response. We assume the system to be causal, which means that $g(i_1,\dots,i_d)=0$ for $i_1<0$ or $i_2<0,\dots$, or $i_d<0$. Following [17], we define the d-dimensional (dD) z-transform for $f(i_1,\dots,i_d)=0$ for all $(i_1,\dots,i_d)\notin\mathbb{Z}_+^d$ as

$$F(z_1,\dots,z_d)=\sum_{i_1+i_2\dots+i_d\geq 0}f(i_1,\dots,i_d)z_1^{i_1}\cdots z_d^{i_d}, \tag{2}$$

where $F(.)$ stands for the z domain representation of the signal $f(.)$. Let us use the following notation

$$\chi_r=\left\{x(i_1,i_2,\dots,i_d)\in\mathbb{R}^d,\ \sum_{\alpha=1}^{d}i_\alpha=r\right\}, \tag{3}$$

introduced in [18] and [19], and

$$\|\chi_r\|=\sup\left\{\|x(i_1,i_2,\dots,i_d)\|,\ \sum_{\alpha=1}^{d}i_\alpha=r\right\}. \tag{4}$$

Then the following theorem gives a necessary and sufficient condition for the BIBO stability of a dD system (see [18]).

Definition 2.1. The system (1) is BIBO stable if and only if for every $\delta_u>0$ there exists an $\delta_y>0$ such that, if $\|u(i_1,\dots,i_d)\|\leq\delta_u$ for all $(i_1,\dots,i_d)$, then $\|y(i_1,\dots,i_d)\|\leq\delta_y$ for all $(i_1,\dots,i_d)$.

Theorem 2.2 (*See [18,19]*)**.** *The dD discrete system (1) is BIBO stable if and only if*

$$\sum_{i_1=0}^{\infty}\dots\sum_{i_d=0}^{\infty}\|g(i_1,\dots,i_d)\|<\infty.$$

Let $D(z_1,\dots,z_d)$ be the least common multiple of the denominators of the entries of a transfer function matrix $G(z_1,\dots,z_d)$. Then

$$G(z_1,\dots,z_d)=\frac{N(z_1,\dots,z_d)}{D(z_1,\dots,z_d)}$$

where $N(z_1,\dots,z_d)$ is a polynomial matrix in $z_1,\dots,z_d$ and

$$D(z_1,\dots,z_d)=\sum_{i_1=0}^{n_1}\dots\sum_{i_d=0}^{n_d}d_{i_1,\dots,i_d}z_1^{i_1}\dots z_d^{i_d}.$$

We assume that $N(z_1,\dots,z_d)$ and $D(z_1,\dots,z_d)$ are coprime, i.e. the only common divisor of $D(z_1,\dots,z_d)$ and all entries of $N(z_1,\dots,z_d)$ is a nonzero constant.

We denote by $\mathbb{R}^{m\times n}$, $(\mathbb{C}^{m\times n})$, the set of real (complex) matrices with m rows and n columns and by $\mathbb{R}^m$, $(\mathbb{C}^m)$, the set of real (complex) vectors. We will use $\mathbb{Z}_+$ for the non-negative integers, $\mathbb{R}_+$ for the non-negative real numbers, and j for the square root of -1.

We consider the general *dD* discrete-time model proposed in [5] as a generalization of the d-dimensional system given in [4],

$$\begin{cases} Ex(i_1+1,i_2+1,\dots,i_d+1)=A_0x(i_1,i_2,\dots,i_d)+\displaystyle\sum_{i=1}^{d}A_ix(i_1+\delta_1^i,\dots,i_d+\delta_d^i)\\ \qquad\qquad\qquad\qquad +\displaystyle\sum_{i=1}^{d}B_iu(i_1+\delta_1^i,i_2+\delta_2^i,\dots,i_d+\delta_d^i)\\ y(i_1+1,i_2+1,\dots,i_d+1)=Cx(i_1+1,i_2+1,\dots,i_d+1).\end{cases}\tag{5}$$

with $\delta_\ell^i=1$ if $i=\ell$ and 0 if $i\neq\ell$ (the Kronecker δ) and where, $x(i_1,i_2,\dots,i_d)\in\mathbb{R}^n$, $u(i_1,i_2,\dots,i_d)\in\mathbb{R}^m$, $y(i_1,i_2,\dots,i_d)\in\mathbb{R}^p$, A_i, B_i, C and E $(i=1,\dots,d)$ are real matrices of appropriate dimensions, and where E is assumed invertible to ensure causality.

By applying the z-transform introduced in [17] to the state–space model (5) with $f(i_1,i_2,\dots,i_d)=0$ for $(i_1,i_2,\dots,i_d)\notin Z_+^n$, and with zero initial conditions $x(0,0,\dots,0)=0$ and with boundary conditions

$$X(0)=\left\{x(0,0,\dots,i_d),x(0,\dots,i_{d-1},0),\dots,x(i_1,0,\dots,0),\ 1\le i_j\le m_j\right\}$$

for $m_j>1$, we obtain the following rational transfer function, realizing this generalized state–space system:

$$G(z_1,z_2,\dots,z_d)=C\left(Ez_1z_2\dots z_d-\sum_{i=1}^{d}A_iz_i-A_0\right)^{-1}\left(\sum_{i=1}^{d}B_iz_i\right)\tag{6}$$

Therefore, the characteristic polynomial describing the poles of the system (5) is given by

$$D(z_1,z_2,\dots,z_d)=\det\left(Ez_1z_2\dots z_d-\sum_{i=1}^{d}A_iz_i-A_0\right)\tag{7}$$

In the following, we introduce the notion of asymptotic stability of dD discrete-time systems. The first definition is the essential definition of asymptotic stability.

Definition 2.3. The dD system (5) is asymptotically stable if the state $x(i_1,i_2,\dots,i_d)$ converges to zero for zero input and every bounded initial conditions, i.e.

$$\lim_{i_1,i_2,\dots,i_d\to\infty}\|x(i_1,i_2,\dots,i_d)\|=0$$

for

$$\begin{array}{l} u(i_1,i_2,\dots,i_d)=0 \text{ for } i_d\in\mathbb{Z}_+\\ \sup_{i_1\in\mathbb{Z}_+}\|x(i_1,0,\dots,0)\|<\infty\\ \sup_{i_2\in\mathbb{Z}_+}\|x(0,i_2,\dots,0)\|<\infty\\ \qquad\vdots\\ \sup_{i_d\in\mathbb{Z}_+}\|x(0,0,\dots,i_d)\|<\infty \end{array}$$

Definition 2.4 ([20]). The *dD* discrete system (5) is asymptotically stable if for $u(i_1,i_2,\dots,i_d)=0$ and finite $\|\chi_0\|$, we have $\|\chi_r\|\longrightarrow 0$ as $r\longrightarrow+\infty$.

We now recall the necessary and sufficient conditions of stability of such systems in terms of the characteristic polynomial. These conditions were derived in [4].

Theorem 2.5. *The dD system (5) is asymptotically stable if and only if* $D(z_1, z_2, \dots, z_d) \neq 0$ *for every* $z = (z_1, z_2, \dots, z_d)$, *such that* $|z_i| \leq 1$, $i = 1, \dots, d$.

We refer to [4] for a detailed proof, but it is informative for the rest of the paper to give the basic ideas of the proof. Let

$$G(z_1, z_2, \dots, z_d) = \frac{N(z_1, z_2, \dots, z_d)}{D(z_1, z_2, \dots, z_d)} = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\dots\sum_{i_d=0}^{\infty} H_{i_1 i_2 \dots i_d} z_1^{i_1} z_2^{i_2} \dots z_d^{i_d}, \tag{8}$$

where the coefficients $H_{i_1 i_2 \dots i_d}$ represent the impulse response of the system (5). The rational function $G(z_1, z_2, \dots, z_d)$ is stable if and only if the impulse response converges to zero for all $z = (z_1, z_2, \dots, z_d)$ in the polydisk $|z_i| \leq 1$ with, $i = 1, \dots, d$. Using the properties of the convergence of such a series, this implies that $G(z_1, z_2, \dots, z_d)$ is analytic for all $z = (z_1, z_2, \dots, z_d)$ in the polydisc $|z_i| \leq 1$, $i = 1, \dots, d$. For a rational function, this also implies that $D(z_1, z_2, \dots, z_d)$ has no roots in the polydisc.

The following equivalent condition was derived in [1] for causal spatial systems. The proof, given in [1], is based on the fact that $D(z_1, z_2, \dots, z_d)$ is polynomial in each z_i, $i = 1, \dots, d$.

The following theorem gives necessary and sufficient conditions for *dD* asymptotic stability in terms of singularities of the system characteristic polynomial.

Theorem 2.6. *The dD system (5) is asymptotically stable if and only if the following polynomials have no roots in the unit circle:*

$$\begin{cases} D(z_1, 1, 1, \dots, 1) \neq 0 & \text{for} \quad |z_1| \leq 1 \\ D(1, z_2, 1, \dots, 1) \neq 0 & \text{for} \quad |z_2| \leq 1 \\ \vdots & \\ D(1, 1, \dots, 1, z_d) \neq 0 & \text{for} \quad |z_d| \leq 1 \end{cases} \tag{9}$$

and

$$D(z_1, z_2, \dots, z_d) \neq 0 \quad \text{for} \quad |z_i| = 1, i = 1, \dots, d. \tag{10}$$

Verifying these conditions turns out to be very complex. In the next section, we derive a much simpler set conditions phrased in terms of LMIs. These can be checked in polynomial time but they are only sufficient conditions.

2.1. Sufficient LMI conditions for stability

Linear matrix inequalities are matrix expressions of the form

$$M(p_1, \dots, p_n) \succ 0,$$

where $M(p_1, \dots, p_n)$ is a square matrix whose elements depend linearly on the parameters p_i, $i = 1, \dots, n$ and where $M \succ 0$ means that the matrix $M(p_1, \dots, p_n)$ is positive definite for particular values of the parameters. The feasible set of such equations can be computed in polynomial time. The following results present sufficient LMI conditions for the asymptotic stability of the system (5).

Theorem 2.7. *The model (5) is asymptotically stable if there exist d real symmetric matrices* $X_1, \dots, X_d$ *and* $d + 2$ *Hermitian matrices* Y_i, $i = 0, \dots, d + 1$ *such that the following LMIs are feasible,*

$$X_i \succ 0, \text{ and } \hat{A}_i^T X_i \hat{A}_i - (E - A_i)^T X_i (E - A_i) \succ 0, \text{ for } i = 1, \dots, d, \tag{11}$$

$$\sum_{i=0}^{d+1} Y_i = 0, \quad V^T V + \text{diag}\{Y_0, Y_1, \dots, Y_d, Y_{d+1}\} \succ 0, \tag{12}$$

where $\hat{A}_i := \sum_{\ell=0, \ell\neq i}^{d} A_\ell$, *and* $V := \begin{bmatrix} -A_0, & -A_1, & \dots & -A_d, & E \end{bmatrix}$.

Proof. The conditions (9) can be reformulated in terms of the characteristic polynomial as follows:

$$\begin{cases} D(z_1, 1, 1, \dots, 1) = \det[z_1(E - A_1) - \hat{A}_1] \neq 0 \text{ for } |z_1| \leq 1 \\ D(1, z_2, 1, \dots, 1) = \det[z_2(E - A_2) - \hat{A}_2] \neq 0 \text{ for } |z_2| \leq 1 \\ \vdots \\ D(1, 1, \dots, 1, z_d) = \det[z_d(E - A_d) - \hat{A}_d] \neq 0 \text{ for } |z_d| \leq 1. \end{cases} \tag{13}$$

These are satisfied if and only if the following LMIs are feasible with real symmetric matrices X_i:

$$\hat{A}_i^T X_i \hat{A}_i - (E - A_i)^T X_i (E - A_i) \succ 0, \text{ and } X_i \succ 0, \text{ for } i = 1, \dots, d. \tag{14}$$

The condition (10) says that for any $\omega_i \in \mathbb{R}$ with $i = 1, 2, \dots, d$, we have

$$\begin{aligned} D(e^{j\omega_1}, e^{j\omega_2}, \dots, e^{j\omega_d}) &= \det[e^{j\omega_1} e^{j\omega_2} \dots e^{j\omega_d} E - A_0 - e^{j\omega_1} A_1 \dots - e^{j\omega_d} A_d] \\ &= \det[V \Pi_D(\omega_1, \dots, \omega_d)] \neq 0 \end{aligned} \tag{15}$$

where

$$\Pi_D(\omega_1, \dots, \omega_d) := \begin{bmatrix} I, & e^{j\omega_1} I, & e^{j\omega_2} I, & \dots & e^{j\omega_d} I, & e^{j\sum_{i=1}^{d} \omega_i} I \end{bmatrix}^T.$$

It is easy to verify that if $\sum_{i=0}^{d+1} Y_i = 0$ and Y_i are Hermitian, then

$$\Pi_D^*(\omega_1, \dots, \omega_d) \operatorname{diag}\{Y_0, Y_1, \dots, Y_d, Y_{d+1}\}\, \Pi_D(\omega_1, \dots, \omega_d) = 0.$$

Therefore we have that

$$\Pi_D^*(\omega_1, \dots, \omega_d) V^T V \Pi_D(\omega_1, \dots, \omega_d) \succ 0 \text{ and } \det[V \Pi_D(\omega_1, \dots, \omega_d)] \neq 0$$

for any set of Hermitian matrices Y_i such that

$$\sum_{i=0}^{d+1} Y_i = 0, \text{ and } V^T V + \operatorname{diag}\{Y_0, Y_1, \dots, Y_d, Y_{d+1}\} \succ 0.$$

This is clearly a sufficient condition for the inequality (10). □

Remark 2.8. Notice that the conditions for the LMIs imply that matrices X_i are real symmetric and Y_i Hermitian matrices. But, since all coefficients E and A_i are real, the feasibility of the LMIs can be constrained to real symmetric matrices Y_i as well.

3. Stability of dD continuous-time systems

We now derive LMI conditions for the stability of dD continuous-time state–space systems of the following form.

Definition 3.1. A dD continuous-time state–space system can be described by the equations,

$$\begin{cases} E \dfrac{\partial^n x(t_1, t_2, \dots, t_d)}{\partial t_1 \partial t_2 \dots \partial t_d} = A_0 x(t_1, t_2, \dots, t_d) + \displaystyle\sum_{i=1}^{d} A_i \dfrac{\partial^i x(t_1, t_2, \dots, t_d)}{\partial t_i} \\ \qquad\qquad + \displaystyle\sum_{i=1}^{n} B_i \dfrac{\partial^i u(t_1, t_2, \dots, t_n)}{\partial t_i} \\ y(t_1, t_2, \dots, t_d) = C x(t_1, t_2, \dots, t_d), \end{cases} \tag{16}$$

where $x(t_1, t_2, \dots, t_d) \in \mathbb{R}^n$ is the state vector, $u(t_1, t_2, \dots, t_d) \in \mathbb{R}^m$ the input vector, $y(t_1, t_2, \dots, t_d) \in \mathbb{R}^p$ the output vector and A_0, A_i, B_i, C and E $(i = 1, 2, \dots, d)$ are real matrices of appropriate dimensions. E is assumed invertible to ensure causality.

First we begin by defining the dD Laplace transform. The multiple Laplace transform relates functions $f(t_1, t_2, \dots, t_d)$ of the d independent real variables $t_1, t_2, \dots, t_d$ to a function $F(s_1, s_2, \dots, s_d)$ of d independent complex variables $s_1, s_2, \dots, s_d$ through the equation

$$\begin{aligned} L[f(t_1, t_2, \dots, t_d)] :=& F(s_1, s_2, \dots, s_d) \\ =& \int_0^{+\infty} \int_0^{+\infty} \cdots \int_0^{+\infty} \Big[e^{-s_d t_d - s_{d-1} t_{d-1} - \dots - s_1 t_1} \times \\ & f(t_1, t_2, \dots, t_d) \Big] dt_1 dt_2 \cdots dt_d \end{aligned} \tag{17}$$

The function defined by (17) is called the multiple Laplace transform of $f(t_1, t_2, \dots, t_d)$, where $F(.)$ stands for the s domain representation of the signal $f(.)$.

By applying the Laplace transform we obtain the dD transfer function,

$$G(s_1, s_2, \dots, s_d) = C \left[E s_1 s_2 \dots s_d - \sum_{i=1}^{d} A_i s_i - A_0 \right]^{-1} \left[\sum_{i=1}^{d} B_i s_i \right]. \tag{18}$$

The characteristic polynomial of (16) is easily seen to be

$$D(s_1, s_2, \dots, s_n) = \det\left[Es_1s_2\dots s_d - \sum_{i=1}^{d} A_is_i - A_0\right] \tag{19}$$

Definition 3.2. The dD continuous-time system (16), is asymptotically stable if and only if the state $x(t_1, t_2, \dots, t_d)$ converges to zero for zero input and every bounded initial conditions, i.e.

$$\lim_{t_1,t_2,\dots,t_d\to\infty} \|x(t_1, t_2, \dots, t_d)\| = 0$$

for

$$\begin{array}{l} u(t_1, t_2, \dots, t_d) = 0 \text{ for } t_i \in \mathbb{R}_+ \\ \sup_{t_1\in\mathbb{R}_+} \|x(t_1, 0, \dots, 0)\| < \infty \\ \sup_{t_2\in\mathbb{R}_+} \|x(0, t_2, \dots, 0)\| < \infty \\ \qquad\qquad \vdots \\ \sup_{t_d\in\mathbb{R}_+} \|x(0, 0, \dots, t_d)\| < \infty \end{array}$$

An algebraic condition for stability of such systems was given in [4].

Theorem 3.3. *The dD continuous-time system (16), is asymptotically stable if and only if $D(s_1, s_2, \dots, s_d) \neq 0$ for all $s = (s_1, s_2, \dots, s_d)$, with $\Re s_i \geq 0$, $i = 1, 2, \dots, d$.*

The equivalent condition which we will use here is based on results derived in [1] and is given below.

Theorem 3.4. *The dD continuous-time system (16), is asymptotically stable if and only if*

$$\begin{cases} D(s_1, 0, \dots, 0) = \det[-A_1s_1 - A_0] \neq 0 \text{ for } \Re s_1 \geq 0, \\ D(0, s_2, \dots, 0) = \det[-A_2s_2 - A_0] \neq 0 \text{ for } \Re s_2 \geq 0, \\ \qquad\qquad \vdots \\ D(0, \dots, 0, s_d) = \det[-A_ds_d - A_0] \neq 0 \text{ for } \Re s_d \geq 0, \end{cases} \tag{20}$$

and

$$D(j\omega_1, j\omega_2, \dots, j\omega_d) \neq 0 \text{ for } \omega_i \in \mathbb{R}, i = 1, \dots, d. \tag{21}$$

Sufficient LMI conditions for the asymptotic stability of *dD* continuous-time systems are now derived.

Theorem 3.5. *The model (16) is asymptotically stable if there are d real symmetric matrices X_i, $i = 1, \dots, d$ such that the following LMIs are feasible,*

$$X_i \succ 0, \text{ and } A_i^TX_iA_0 + A_0^TX_iA_i \succ 0, \text{ for } i = 1, \dots, d \tag{22}$$

and a Hermitian matrix Y with $(d+1)(d+2)/2$ blocks $Y_{i,m}$, $i = 1, \dots, d+1$, $m = i+1, \dots, d+1$ such that

$$V^TV + \begin{bmatrix} 0 & Y_{0,1} & \dots & Y_{0,d} & Y_{0,d+1} \\ Y_{0,1} & 0 & \dots & Y_{1,d} & Y_{1,d+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{0,d} & -Y_{1,d} & \dots & 0 & Y_{d,d+1} \\ -eY_{0,d+1} & eY_{1,d+1} & \dots & eY_{d,d+1} & 0 \end{bmatrix} \succ 0, \tag{23}$$

where $e = (-1)^d$ and $V := \begin{bmatrix} -A_0, & -A_1, & \dots & -A_d, & E \end{bmatrix}$.

Proof. The conditions (23) can be reformulated in terms of the characteristic polynomial as follows:

$$\begin{cases} D(s_1, 0, \dots, 0) = \det[-A_1s_1 - A_0] \neq 0 \text{ for } \Re s_1 \geq 0 \\ D(0, s_2, \dots, 0) = \det[-A_2s_2 - A_0] \neq 0 \text{ for } \Re s_2 \geq 0 \\ \qquad\qquad \vdots \\ D(0, \dots, 0, s_d) = \det[-A_ds - A_0] \neq 0 \text{ for } \Re s_d \geq 0. \end{cases} \tag{24}$$

These are satisfied if and only if the following LMIs are feasible with real symmetric matrices X_i:

$$A_i^TX_iA_0 + A_0^TX_iA_i \succ 0, \text{ and } X_i \succ 0, \text{ for } i = 1, \dots, d. \tag{25}$$

The condition (23) expresses the fact that for any $\omega_i \in \mathbb{R}$ with $i = 1, 2, \ldots, d$, we have

$$\begin{aligned} D(j\omega_1, j\omega_2, \ldots, j\omega_d) &= \det[j\omega_1 j\omega_2 \ldots j\omega_d E - A_0 - j\omega_1 A_1 - j\omega_2 A_2 \ldots - j\omega_d A_d] \\ &= \det[V\Pi_C(\omega_1, \ldots, \omega_d)] \neq 0 \end{aligned} \tag{26}$$

where

$$\Pi_C(\omega_1, \ldots, \omega_d) := \begin{bmatrix} I, & j\omega_1 I, & j\omega_2 I, & \ldots & j\omega_d I, & j^d \prod_{i=1}^{d} \omega_i I \end{bmatrix}^T.$$

It is easy to verify that for *all frequencies* ω_i and for $e := (-1)^d$

$$\Pi_C^*(\omega_1, \ldots, \omega_d) \begin{bmatrix} 0 & Y_{0,1} & \ldots & Y_{0,d} & Y_{0,d+1} \\ Y_{0,1} & 0 & \ldots & Y_{1,d} & Y_{1,d+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{0,d} & -Y_{1,d} & \ldots & 0 & Y_{d,d+1} \\ -eY_{0,d+1} & eY_{1,d+1} & \ldots & eY_{d,d+1} & 0 \end{bmatrix} \Pi_C(\omega_1, \ldots, \omega_d) = 0,$$

provided the middle matrix Y is Hermitian, implying that the pairs of blocks $(Y_{i,j}, Y_{j,i})$ with equal sign are Hermitian, and those with opposite signs are skew-Hermitian. Then it follows that

$$\Pi_C^*(\omega_1, \ldots, \omega_d) V^T V \Pi_C(\omega_1, \ldots, \omega_d) \succ 0 \text{ and } \det[V\Pi_C(\omega_1, \ldots, \omega_d)] \neq 0,$$

for any Hermitian matrix Y with the above structure and with $e := (-1)^d$, such that:

$$V^T V + \begin{bmatrix} 0 & Y_{0,1} & \ldots & Y_{0,d} & Y_{0,d+1} \\ Y_{0,1} & 0 & \ldots & Y_{1,d} & Y_{1,d+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{0,d} & -Y_{1,d} & \ldots & 0 & Y_{d,d+1} \\ -eY_{0,d+1} & eY_{1,d+1} & \ldots & eY_{d,d+1} & 0 \end{bmatrix} \succ 0.$$

This is therefore a sufficient condition for the inequality (23). □

Remark 3.6. It is interesting to point out that there are more degrees of freedom in the LMIs of the continuous-time case than in the LMIs of the discrete-time case.

4. The 2D case

We revisit the discrete-time condition (12) for the case where $d = 2$. We can then simplify the LMI conditions since the matrix Y satisfies

$$\Pi_D^*(\omega_1, \omega_2) Y \Pi_D(\omega_1, \omega_2) = 0 \text{ with } \Pi_D(\omega_1, \omega_2) := \begin{bmatrix} I, & e^{j\omega_1} I, & e^{j\omega_2} I, & e^{j(\omega_1+\omega_2)} I \end{bmatrix}^T$$

One can indeed choose Y to be Hermitian and of the form

$$Y := \begin{bmatrix} Y_1 & Y_5 & Y_6 & 0 \\ Y_5^* & Y_2 & 0 & -Y_6 \\ Y_6^* & 0 & Y_3 & -Y_5 \\ 0 & -Y_6^* & -Y_5^* & Y_4 \end{bmatrix}, \quad Y_1 + Y_2 + Y_3 + Y_4 = 0, \tag{27}$$

implying that the blocks Y_1, Y_2, Y_3, Y_4 are Hermitian.

The equivalent matrix Y for the continuous-time case condition

$$\Pi_C^*(\omega_1, \omega_2) Y \Pi_C(\omega_1, \omega_2) = 0 \text{ with } \Pi_C(\omega_1, \omega_2) := \begin{bmatrix} I & j\omega_1 I & j\omega_2 I & -\omega_1\omega_2 I \end{bmatrix}^T$$

is then also Hermitian but of the form

$$Y := \begin{bmatrix} 0 & Y_1 & Y_2 & Y_5 \\ Y_1 & 0 & Y_6 & Y_3 \\ Y_2 & -Y_6 & 0 & Y_4 \\ -Y_5 & Y_3 & Y_4 & 0 \end{bmatrix} \tag{28}$$

implying that the blocks Y_1, Y_2, Y_3, Y_4 are Hermitian and the blocks Y_5, Y_6 are skew-Hermitian.

Below, we give a number of numerical experiments illustrating the 2D case.

Example 4.1. We first consider the multidimensional discrete-time system (5) for the case $d = 2$ with the following system matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 0.10 & 0.10 & 0.20 \\ 0.02 & 0.01 & 0.25 \\ 0 & 0.30 & 0.20 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.10 & 0.10 & 0.02 \\ 0.01 & 0.10 & 0.25 \\ 0.01 & 0.03 & 0.02 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.10 & 0.10 & 0.02 \\ 0.30 & 0.10 & 0.07 \\ 0.10 & 0.10 & 0.10 \end{bmatrix}.$$

Applying Theorem 2.7, we find that the *LMIs* are feasible and the considered system is asymptotically stable. A solution for the *LMIs* (11), (12) is as follows

$$X_1 = 10^{-11} \begin{bmatrix} 0.0690 & 0.0891 & 0.1016 \\ 0.0891 & 0.1197 & 0.1455 \\ 0.1016 & 0.1455 & 0.1605 \end{bmatrix}, \; X_2 = 10^{-11} \begin{bmatrix} 0.0903 & 0.1188 & 0.1250 \\ 0.1188 & 0.1385 & 0.1583 \\ 0.1250 & 0.1583 & 0.1817 \end{bmatrix}$$

and for the submatrices $Y_i, i = 1, \dots, 6$ of the matrix Y in (27):

$$Y_1 = \begin{bmatrix} 0.2421 & -0.0029 & 0.0314 \\ -0.0029 & 0.2741 & 0.0386 \\ 0.0314 & 0.0386 & 0.2636 \end{bmatrix}, \; Y_2 = \begin{bmatrix} 0.1537 & -0.0613 & -0.0524 \\ -0.0613 & 0.1316 & -0.0342 \\ -0.0524 & -0.0342 & 0.1567 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 0.2100 & -0.0709 & -0.0635 \\ -0.0709 & 0.1009 & -0.0609 \\ -0.0635 & -0.0609 & 0.1071 \end{bmatrix}, \; Y_4 = \begin{bmatrix} -0.6058 & 0.1351 & 0.0845 \\ 0.1351 & -0.5066 & 0.0565 \\ 0.0845 & 0.0565 & -0.5274 \end{bmatrix},$$

$$Y_5 = \begin{bmatrix} -0.0335 & -0.0608 & -0.0251 \\ -0.0608 & -0.0610 & -0.0619 \\ -0.0251 & -0.0619 & -0.0757 \end{bmatrix}, \; Y_6 = \begin{bmatrix} -0.0830 & -0.0409 & -0.0038 \\ -0.0409 & -0.0401 & -0.0662 \\ -0.0038 & -0.0662 & -0.0474 \end{bmatrix}.$$

In the next example, we consider a singular discrete-time model to show the effects of singular systems on our results.

Example 4.2. We consider the multidimensional discrete-time system (5) for $d = 2$ with the following system matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}$$

In [12], this system is shown to be stable and by the use of our method we find that the *LMIs* in Theorem 2.7 are feasible, and a feasible solution is as follows

$$X_1 = X_2 = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.4992 \end{bmatrix},$$

and for the submatrices $Y_i, i = 1, \dots, 6$ of the matrix Y in (27):

$$Y_1 = \begin{bmatrix} 0.26 & 0.0000 \\ 0.0000 & -0.0075 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.2600 & 0.0000 \\ 0.0000 & 0.0025 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 0.2200 & 0.0000 \\ 0.0000 & 0.0025 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} -0.7400 & 0.0000 \\ 0.0000 & 0.0025 \end{bmatrix},$$

$$Y_5 = \begin{bmatrix} 0.1000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, \quad Y_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Example 4.3. We finally consider the multidimensional continuous-time system (16) for $d = 2$ with the following system matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A_0 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}$$

By the use of our method we find that the *LMIs* in Theorem 3.5 are feasible, and a feasible solution is as follows

$$X_1 = \begin{bmatrix} 1.2001 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1.7845 & 0.7637 \\ 0.7637 & 1.7845 \end{bmatrix}$$

and for the submatrices $Y_i, i = 1, \dots, 6$ of the matrix Y in (28):

$$Y_1 = Y_2 = Y_3 = \begin{bmatrix} 1.2001 & 1.2001 \\ 1.2001 & 1.2001 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 1.4001 & 1.2001 \\ 1.2001 & 1.2001 \end{bmatrix}$$

and $Y_5 = Y_6 = 0$.

5. Conclusion

In this paper we studied the stability problem of multidimensional systems and derived new sufficient conditions for the stability of such systems. These conditions can be verified in polynomial time since they are formulated in terms of linear matrix inequalities. We also simplified these conditions further in the case of two-dimensional systems.

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