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LIDAM Discussion Paper ISBA
2026 / 31

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Inflation impact on multi-cash-flow hybrid life products

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Abstract

The current economic environment is frequently subject to inflationary pressures that severely erode retirees' purchasing power. Consequently, insurers are increasingly expected to offer life annuity products that explicitly incorporate inflation protection. However, pricing these hybrid multi-cash-flow products is challenging because financial markets lack assets that perfectly replicate long-term inflation, thereby leading directly to an incomplete financial market. Since inflation-indexed annuities are also exposed to mortality risk, we need a multi-decomposition-step operator able to formally separate the hedgeable financial strategy from idiosyncratic biometric pooling and systematic risk buffering, while ensuring a market–actuarial-consistent valuation. Moreover, in practice, inflation-indexed annuities are exposed to inflation that fluctuates stochastically, which, in turn, calls for a joint stochastic treatment of nominal interest rates. Driven by the previous requirements, our first contribution is to extend the stochastic-interest-rate three-step method of Belhouari et al. (2025) to incomplete financial markets for multi-cash-flow hybrid life payments. We found that both the hedgeable component and its associated systematic financial risk depend on the hedging strategy. However, we deliberately evaluate the hedgeable component and its associated systematic financial risk such that their sum is invariant to the hedging strategy and equals the best-estimate value. Hence, our goal is to explore how the insurer can allocate the same best-estimate value between these two components depending on the hedging strategy chosen and the assets available in the hedging portfolio. First, we start by relying only on standard nominal zero-coupon bonds, which are readily available in the financial market as traded assets, and highlight that the hedge price does not benefit from the nominal interest rate–inflation correlation. We then examine cross-hedging with an inflation-correlated stock and find that it improves partial hedging and modestly reduces the hedge price, and increases the systematic financial risk premium that can be absorbed by holding additional capital.

Keywords: Inflation risk, hybrid life insurance, multi–cash-flow, actuarial, financial.

1 Introduction

In recent years, the global economy has faced a succession of major shocks, including the COVID-19 pandemic and energy crises arising from geopolitical conflicts. These shocks have resulted in a considerable rise in inflation, increasing public awareness and concerns regarding the preservation of purchasing power. Therefore, the impact of inflation on households' purchasing capacity has become an increasingly relevant economic issue. In response to these challenges, insurers have developed pension products incorporating mechanisms designed to maintain the real value of retirement benefits over time. Among these products, inflation-indexed life annuities represent a natural choice for adjusting retirement benefits in line with inflation and constitute the primary focus of our work. A key motivation for focusing on inflation-indexed life annuities is their ability to provide a direct contractual mechanism for protecting retirement income against inflation. This feature distinguishes them from standard alternative products, such as participating annuities-which offer only an uncertain inflation hedge contingent on stock gains-and variable annuities, which fully transfer the investment risk to the annuitant. Consequently, both leave retirees exposed to inflation risk; see Lazar (2007).

A major hurdle for private insurers seeking to offer inflation-indexed life annuities is the limited availability of appropriate hedging instruments, such as inflation-linked zero-coupon bonds, in financial markets. As evidenced by Jarrow & Yildirim (2003), attempts to artificially complete the market by stripping prices of inflation-linked zero-coupon bonds from coupon-bearing TIPS (U.S market) have proven empirically unreliable over long-term horizons. Therefore, perfect replication of these long-term products is practically impossible, forcing insurers to operate in a financial market that is inherently incomplete regarding inflation risk. This reality dictates a valuation methodology capable of disentangling the hedgeable financial component-replicated via standard nominal zero-coupon bonds-from the unhedgeable residual risks. Given that inflation-indexed annuities are also structurally exposed to systematic biometric risks, a holistic risk decomposition is imperative. The Three-step valuation method-introduced by Deelstra et al. (2020) and advanced by Belhouari et al. (2025) for stochastic interest rates-naturally fits this requirement. Since inflation risk and nominal interest rates are inherently intertwined in practice, we consider their joint stochastic modeling essential for modeling inflation risk in an economically consistent manner. Indeed, Ang et al. (2008) highlight that nominal interest rates are influenced by inflation expectations, making joint affine modeling of nominal interest rates and the inflation rate essential for accurate assessment; see also the seminal contribution of Jarrow & Yildirim (2003). The sign of the inflation–interest rate correlation is regime-dependent: in response to rising inflation, it turns positive during high inflation periods as central banks raise nominal rates to cool consumer demand, but becomes negative during stable macroeconomic regimes when monetary policy lowers nominal rates to stimulate economic growth. Motivated by this context, our primary contribution is to extend the stochastic interest rate Three-step method of Belhouari et al. (2025) to incomplete financial markets with multiple cash-flow payments. We carefully design this extension to ensure that it constitutes a fair valuation, which means that it satisfies both actuarial consistency, as discussed in Barigou et al. (2023), and market consistency, following Pelsser & Stadje (2014). This extension isolates a distinct systematic financial risk, allowing the resulting decomposition to be interpreted as a four-step decomposition. Furthermore, a direct consequence of operating in an incomplete financial market is that the Three-step decomposition is no longer unique; rather, it fundamentally depends on the chosen hedging strategy. We intentionally evaluate the hedge price and the associated systematic financial risk premium such that their sum remains equal to the best-estimate value, independently of the hedging portfolio. Our objective will therefore be to examine, with respect to our selected hedging strategies and available asset portfolios, how this fixed best-estimate value will be allocated between the hedge price and the systematic financial risk premium. We investigate two distinct hedging approaches: Mean-Variance Hedging, applying a squared error penalty from the general convex class proposed by Chen et al. (2020), and Quantile Regression Hedging, based on Barigou et al. (2022). In a recent working paper, Ling et al. (n.d.) made another contribution to the isolation of systematic financial risk from the other three components in incomplete financial markets, but focused just on a valuation approach, in a constant interest rate framework and using the same pricing tool (standard deviation) for all non-hedgeable risks. Our four-step methodology merges valuation with risk management. By pricing each component distinctly, we recognize the very different nature of each component and we assist insurers in launching new business by allocating specific costs to respective risk buckets—explicitly distinguishing the budget required for the hedging portfolio from the capital retained and held as a systematic buffer, and isolate the risk margin loading charged for finite-portfolio idiosyncratic pooling. Consequently, we can explicitly observe how the hedging strategy impacts the price of each component (specifically the new systematic financial risk), identifying which parts are priced under a risk-neutral measure, which under the real-world measure and which use a standard deviation principle justified by pooling arguments. Looking at long term instruments such as indexed lifetime annuities, our valuation framework has been of course developed under stochastic interest rates and stochastic inflation rates, generalizing in incomplete market Belhouari et al. (2025).

Relying exclusively on nominal zero-coupon bonds as hedging instruments restricts the hedgeable component to a purely deterministic cash-matching strategy at each maturity, leading to a higher hedging cost that is insensitive to the nominal interest-inflation rate correlation, as illustrated in this work for an inflation-linked temporary life annuity. Therefore, we investigate whether augmenting the financial market with standard equities, which are naturally imperfectly correlated with inflation in practice, can generate cross-hedging benefits that lower this hedge cost and mitigate the hedging error, with the mitigation effect depending on stock–inflation correlation. The optimal dynamic allocation is derived using a Hamilton-Jacobi-Bellman (HJB) control problem with a quadratic terminal error (i.e., MVH). Given the intensive computational time required to solve the resulting non-linear PDE, we focus on a single inflation-linked pure endowment payment. The results show that cross-hedging indeed reduces the hedge value, although the magnitude of this reduction remains modest. Moreover, we observe an increasing monotonic trend in the systematic financial risk premium with respect to the stock–inflation correlation, as this correlation cannot affect the ICP-linked best-estimate value due to the absence of the stock in the ICP-linked payoff. Exploring these limitations of the cross-hedging mechanism and their impact on pricing constitutes the second contribution of this paper.

The paper proceeds as follows. Section 2 introduces an incomplete financial market with inflation, where only nominal zero-coupon bonds are traded. We extend the Three-step valuation method to this incomplete financial market, multi-cash-flow payment setting, highlighting that the resulting risk decomposition becomes strategy hedging-dependent. We explicitly detail the Mean-Variance and Quantile Regression cases. In Section 3, we illustrate the proposed framework by pricing an inflation-linked temporary life annuity under correlated Vasicek dynamics for both inflation and nominal interest rates. In Section 4, we introduce an inflation-correlated stock into the financial market to investigate its potential for cross-hedging the inflation-linked pure endowment payment, considered as a representative single payment of an inflation-linked temporary life annuity, and its potential for reducing hedging costs. Section 5 provides the numerical results, and Section 6 concludes.

2 Three-step valuation of multi-cash-flow hybrid payments in incomplete financial markets

2.1 Market setup: Inflation-induced incompleteness and zero-coupon bond trading

We consider discrete annual benefits payable at the end of each year $t = 1, \ldots, T$, assuming that the financial market is arbitrage-free but incomplete. Consequently, perfect hedging is generally unattainable, and we face a multiplicity of equivalent risk-neutral probability measures under which the discounted prices of tradable assets are martingales. We study a homogeneous portfolio of n policyholders, all aged x at time 0. The remaining lifetimes of these individuals are modeled by the non-negative random variables $\tau_1, \ldots, \tau_n$ and are assumed to be identically distributed. We maintain the standard assumption that the actuarial and financial risks are independent. Hence, we work on a product probability space $(\Omega, \mathcal{F}, \mathbb{P})$, constructed as $\Omega = \Omega^a \times \Omega^f$ and $\mathbb{P} = \mathbb{P}^a \times \mathbb{P}^f$, explicitly separating the actuarial (a, for actuarial) and financial (f, for financial) universes, along with their respective real-world probability measures. The global flow of information over time is modeled by the filtration $\mathcal{F} = (\mathcal{F}_s)_{0 \le s \le T}$, which is defined as the natural join of the actuarial and financial filtrations, such that $\mathcal{F}_s \triangleq \mathcal{F}_s^a \vee \mathcal{F}_s^f$. Specifically, the underlying sub-filtrations are specified as follows:

- $\mathcal{F}_s^f$ represents the information generated by the financial market up to time s.
- $\mathcal{F}_s^a$ captures the actuarial information, which is further split into two components: the systematic mortality risk filtration, denoted by $(\mathfrak{S}_s)_{0 \le s \le T}$, and the unsystematic mortality risk associated with the n policyholders. Formally, it is given by:
$$\mathcal{F}_s^a \triangleq \mathfrak{S}_s \vee \bigvee_{i=1}^{n} \sigma(\mathbb{1}_{\tau_i \le u} : 0 \le u \le s).$$

We shall use $\mathbb{E}_{\mathcal{F}_s}[.]$ and $\mathbb{V}_{\mathcal{F}_s}[.]$ to denote, respectively, the almost surely defined conditional expectation and conditional variance at time s, namely $\mathbb{E}[. \mid \mathcal{F}_s]$ and $\mathbb{V}[. \mid \mathcal{F}_s]$. We assume the existence of a stochastic mortality intensity process $(\lambda_s)_{0 \le s \le T}$, which is adapted to the systematic filtration $(\mathfrak{S}_s)_{0 \le s \le T}$. The physical survival probability over the period $[0, t]$ for an individual aged x is defined by

$$_tp_x \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[e^{-\int_0^t \lambda_s \mathrm{d}s}\right], \quad t \in \{1, \ldots, T\}. \tag{1}$$

We assume that the lifetimes of n individuals are conditionally independent given the systematic mortality information available at each maturity payment date

$t \in \{1, \ldots, T\}$; that is, for any $s \in [0, t]$

$$\mathbb{P}^a(\tau_1 > s, \tau_2 > s, ..., \tau_n > s|\mathfrak{S}_t) = \prod_{i=1}^{n} \mathbb{P}^a(\tau_i > s|\mathfrak{S}_t). \tag{2}$$

We assume a financial market where the interest rate risk can be perfectly hedged. Specifically, the market provides access to a family of traded risk-free zero-coupon bonds $(P(s,t))_{0 \le s \le t}$ maturing at each payment date $t \in \{1, \ldots, T\}$, whose price dynamics are driven by an underlying stochastic short-rate process $(r_s)_{0 \le s \le T}$. Employing these zero-coupon bonds as traded assets allows avoiding the reinvestment risk inherent to a standard bank account, as they secure a deterministic payoff of 1 at each maturity t instead of a stochastic accumulation $e^{\int_0^t r_u \, du}$. We assume a freely tradable market with deep liquidity, transparency, efficiency, and no transaction costs. We model inflation using the instantaneous rate i_s, representing the continuous growth rate of the general price level. The resulting CPI-U inflation index accumulated between times s and t is defined as

$$I_s^t \triangleq e^{\int_s^t i_u \, du}, \quad 0 \le s \le t, \quad \textit{and } t \in \{1, \ldots, T\}. \tag{3}$$

From an economic perspective, $(I_s^t)_{0 \le s \le t}$ acts as the cumulative revaluation factor, reflecting the realized evolution of the cost of living and effectively linking nominal cash flows to their real value. The macroeconomic inflation factor $(I_s^t)_{0 \le s \le t}$ is assumed to be imperfectly correlated with the nominal stochastic short rate. Hence, while the interest rate risk is fully hedgeable, the financial market is assumed to be incomplete regarding inflation: the macroeconomic inflation index is non-tradable, and no inflation-linked derivatives are available to fully hedge this risk. As a direct consequence of this incompleteness, an equivalent martingale measure for pricing is not unique. Despite market incompleteness yielding multiple risk-neutral measures, the absence of arbitrage opportunities enforces that the observable time-s prices of freely tradable zero-coupon bonds match their discounted expectations under any selected measure $\mathbb{Q}^f$:

$$P(s,t) \triangleq \mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}^f} \left[\frac{P(t,t)}{e^{\int_s^t r_u \mathrm{d}u}} \right], \quad 0 \le s \le t, \quad \textit{and } t \in \{1, \ldots, T\}. \tag{4}$$

Assuming a specific risk-neutral measure $\mathbb{Q}^f$ has been selected; we denote by $\mathcal{I}(s,t)$ the market-implied expected inflation accumulation over the period $[s,t]$, defined as:

$$\mathcal{I}(s,t) = \mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}^f} \left[I_s^t \right], \quad 0 \le s \le t, \quad \textit{and } t \in \{1, \ldots, T\}. \tag{5}$$

$\mathcal{I}(s,t)$ represents a risk-neutral expected value of the accumulated inflation at time t. It serves as a projection estimate rather than a time-s discounted price, forming

the baseline required to evaluate future inflation-contingent cash flows.
We consider here the natural extension of the single-cash-flow hybrid life payoff, presented in Belhouari et al. (2025), to the multi-cash-flow setting, defined as follows

$$H^n = \sum_{t=1}^{T} H_t^n \quad \text{with} \quad H_t^n \triangleq \frac{1}{n} \sum_{j=1}^{n} g_t(\tau_j)\, h_t(I_0^t), \tag{6}$$

where $g_t :]0, \infty[\to [0, \infty[$ and $h_t :]0, \infty[\to [0, \infty[$ are two functions corresponding, respectively, to the actuarial and financial components of the payment at time t. Henceforth, we focus on the associated payoff set, denoted by

$$\mathcal{H} \triangleq \left\{ H^n = \sum_{t=1}^{T} H_t^n \right\}. \tag{7}$$

The subclass of multi-cash-flow payments including inflation risk and without biometric risk is denoted by

$$\mathcal{H}^f \triangleq \left\{ h(I) = \sum_{t=1}^{T} h_t(I_0^t) \right\}, \tag{8}$$

Similarly, the sub-class of multi-cash-flow actuarial payments is indicated by

$$\mathcal{H}^a \triangleq \left\{ g^n = \sum_{t=1}^{T} \frac{1}{n} \sum_{j=1}^{n} g_t(\tau_j) \right\}. \tag{9}$$

We consider a $\{1, T\}-$valuation operator, denoted by $\pi_0^{\mathbb{Q}^f}$, representing the time-0 value of the hybrid life payoff $H^n \in \mathcal{H}$, assumed to satisfy the axioms stated in the following definition.

Definition 1 ($\{1, T\}$**-Valuation operator).**

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure. Let $\pi_0^{\mathbb{Q}^f} : \mathcal{H} \to \mathbb{R}$ be a function representing the time-0 value of the hybrid life payoff H^n defined in (6). We say that $\pi_0^{\mathbb{Q}^f}$ is a valuation operator if it satisfies the following axioms:

- ***Normalization*** $\pi_0^{\mathbb{Q}^f}(0|\mathcal{F}_0) = 0$.
- ***Translation invariance*** *For any $H^n \in \mathcal{H}$ and any cash-flow sequence $(x_t)_{t=1}^T$ such that each x_t is $\mathcal{F}_0$-measurable and payable at time t,*

$$\pi_0^{\mathbb{Q}^f}\left(H^n + \sum_{t=1}^{T} x_t \mid \mathcal{F}_0 \right) = \pi_0^{\mathbb{Q}^f}(H^n \mid \mathcal{F}_0) + \sum_{t=1}^{T} P(0,t) x_t.$$

- ***Positive homogeneity*** $\forall a>0, \forall H^n \in \mathcal{H}$, $\pi_0^{\mathbb{Q}^f}(aH^n|\mathcal{F}_0) = a\pi_0^{\mathbb{Q}^f}(H^n|\mathcal{F}_0)$.
- ***Sub-additivity*** $\forall H^n \in \mathcal{H}, \forall G^n \in \mathcal{H}$, $\pi_0^{\mathbb{Q}^f}(H^n + G^n|\mathcal{F}_0) \leq \pi_0^{\mathbb{Q}^f}(H^n|\mathcal{F}_0) + \pi_0^{\mathbb{Q}^f}(G^n|\mathcal{F}_0)$.

In a complete financial market, the valuation of a financial payoff is uniquely determined by its discounted expectation under the unique equivalent risk-neutral measure. However, in our setting, the financial market is incomplete regarding inflation, so inflation-linked financial payoffs $h(I)$ defined in (8) do not admit a unique valuation. Furthermore, we assumed that the financial market consisted only of traded zero-coupon bonds, without relying on a continuously compounded bank account. Consequently, the only purely hedgeable financial claims are deterministic cash-flow sequences $(x_t)_{t=1}^T$, which are uniquely determined by their corresponding time-0 market value, namely $\sum_{t=1}^T x_t P(0,t)$. If we want to extend this concept to the valuation of hybrid life payoffs $H^n \in \mathcal{H}$, we could define the following financial valuation operator

$$\pi_0^{\mathbb{Q}^f,f}(H^n|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a\times\mathbb{Q}^f}\Big[\sum_{t=1}^T e^{-\int_0^t r_s\,\mathrm{d}s} H_t^n\Big], \tag{10}$$

with $\mathbb{Q}^f$ being an equivalent martingale probability measure.

Remark 1.

The operator $\pi_0^{\mathbb{Q}^f,f}$ defined in (10) correctly prices hedgeable financial payoffs represented in our framework by deterministic cash-flow sequences $(x_t)_{t=1}^T$, since:

$$\pi_0^{\mathbb{Q}^f,f}\left(\sum_{t=1}^T x_t \middle| \mathcal{F}_0\right) = \sum_{t=1}^T x_t\, \mathbb{E}_{\mathcal{F}_0}^{\mathbb{Q}^f}\left[e^{-\int_0^t r_s\,\mathrm{d}s}\right] = \sum_{t=1}^T x_t P(0,t).$$

Furthermore, since the zero-coupon bond prices $P(0,t)$ are observable, this valuation is strictly independent of the chosen equivalent martingale measure $\mathbb{Q}^f$.

Remark 2 (**Market-consistent property**).

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure. Let H^n belong to $\mathcal{H}$. In the setting considered in this paper, the market-consistent property of Belhouari et al. (2025) in the complete financial market case and of Dhaene et al. (2017) in the incomplete financial market case, with respect to $\pi_0^{\overline{\mathbb{Q}^f},f}$, is captured by the translation invariance axiom, namely:

$$\pi_0^{\mathbb{Q}^f}(H^n + \sum_{t=1}^T x_t|\mathcal{F}_0) = \pi_0^{\mathbb{Q}^f}(H^n|\mathcal{F}_0) + \pi_0^{\overline{\mathbb{Q}^f},f}(\sum_{t=1}^T x_t|\mathcal{F}_0),$$

for any equivalent martingale probability measure $\overline{\mathbb{Q}^f}$, and with $(x_t)_{t=1}^T$ a cash-flow sequence such that each x_t is $\mathcal{F}_0$-measurable and payable at time t.

On the actuarial front, we rely on a standard premium principle by adapting the generalized standard deviation premium principle introduced in Belhouari et al. (2025) to the setting of an incomplete financial market with multi-cash-flow hybrid life payoff.

Definition 2 ($\mathbb{Q}^f$**-dependent generalized standard deviation premium principle**).
Let $\mathbb{Q}^f$ be an equivalent martingale probability measure. For any multi-cash-flow hybrid life payoff $H^n \in \mathcal{H}$, the $\mathbb{Q}^f$-dependent generalized standard deviation premium principle is the $\{1, T\}$-valuation operator $\pi_0^{\mathbb{Q}^f,a}$ defined by

$$\pi_0^{\mathbb{Q}^f,a}(H^n|\mathcal{F}_0) \triangleq \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{Q}^f}\left[\sum_{t=1}^{T} e^{-\int_0^t r_s \mathrm{d}s} H_t^n\right] + \sum_{t=1}^{T} \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a \times \mathbb{P}^f}[H_t^n]} \times P(0,t), \quad (11)$$

for $H^n \in \mathcal{H}$.

A $\{1, T\}$-valuation operator $\pi_0^{\mathbb{Q}^f,a}$ is consistent with diversification-based pricing under the standard deviation premium principle:

$$\pi_0^{\mathbb{Q}^f,a}\Big(\sum_{t=1}^{T} \frac{\sum_{j=1}^n g_t(\tau_j)}{n}\Big|\mathcal{F}_0\Big) = \sum_{t=1}^{T} P(0,t) \times \Big(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}\Big[\frac{\sum_{j=1}^n g_t(\tau_j)}{n}\Big] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a}\Big[\frac{\sum_{j=1}^n g_t(\tau_j)}{n}\Big]}\Big).$$

It can be seen that the valuation of purely actuarial claims remains completely independent of the chosen equivalent martingale measure $\mathbb{Q}^f$, as it is determined exclusively by observable zero-coupon bond prices and actuarial pooling.

Definition 3 (**Actuarial-consistent property**).
Let $\mathbb{Q}^f$ be an equivalent martingale probability measure. A $\{1, T\}$-valuation operator $\pi_0^{\mathbb{Q}^f}$ is said to be actuarial-consistent with respect to the $\overline{\mathbb{Q}^f}$-dependent generalized standard deviation premium principle $\pi_0^{\overline{\mathbb{Q}^f},a}$ if and only if, for any purely actuarial multi-cash-flow payoff $g^n \in \mathcal{H}^a$, the following equality holds:

$$\pi_0^{\mathbb{Q}^f}(g^n|\mathcal{F}_0) = \pi_0^{\overline{\mathbb{Q}^f},a}(g^n|\mathcal{F}_0), \quad (12)$$

for any equivalent martingale probability measure $\overline{\mathbb{Q}^f}$.

Definition 4 (**Fair valuation**).

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure. A $\{1, T\}$-valuation operator $\pi_0^{\mathbb{Q}^f}$ is said to be a fair valuation if it satisfies both the market-consistent property relative to the financial valuation operator $\pi_0^{\overline{\mathbb{Q}_1^f}, f}$ and the actuarial-consistent property relative to $\pi_0^{\overline{\mathbb{Q}_2^f}, a}$, for any two equivalent martingale probability measures $\overline{\mathbb{Q}_1^f}$ and $\overline{\mathbb{Q}_2^f}$.

From now on, we focus on the Three-step valuation methodology for hybrid life-contingent payoffs presented in Deelstra et al. (2020), extending it to the setting of an incomplete financial market with multi-cash-flow hybrid payments. Later in the paper, we establish that our resulting extension retains the fair valuation property.

2.2 Three-step decomposition: identifying the hedgeable component via mean-variance hedging

In incomplete financial markets, perfect replication is unattainable, rendering the identification of the deterministic hedgeable component non-unique and strictly dependent on the adopted hedging strategy. By choosing a mean-variance hedging (VH) strategy to determine the hedgeable component [1]—which minimizes the quadratic hedging error in the real-world (see Belhouari et al. (2026))—we proceed in this subsection. Since only a single traded asset is available, hedging in our framework is therefore restricted to static strategies over the intervals $[0, t]$, $t \in \{1, \ldots, T\}$. The Three-step decomposition of Deelstra et al. (2020) extends to incomplete financial markets in a multiple cash-flow and hybrid payment setting by aggregating, at each payment date, the hedgeable, diversifiable, and systematic components, as follows.
Indeed, for any payment date $1 \leq t \leq T$ and individual $1 \leq j \leq n$, the hybrid cash flow can be algebraically decomposed by isolating its physical expectations:

$$g_t(\tau_j) \times h_t(I_0^t) = \underbrace{\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_j)] \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^f}[h_t(I_0^t)]}_{\text{Deterministic hedgeable component}} + \underbrace{\left(g_t(\tau_j) - \mathbb{E}_{\mathcal{F}_0 \vee \mathfrak{G}_t}^{\mathbb{P}^a}[g_t(\tau_j)]\right) \times h_t(I_0^t)}_{\text{Idiosyncratic deviation}}$$
$$+ \underbrace{\left(\mathbb{E}_{\mathcal{F}_0 \vee \mathfrak{G}_t}^{\mathbb{P}^a}[g_t(\tau_j)] \times h_t(I_0^t) - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_j)] \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^f}[h_t(I_0^t)]\right)}_{\text{Systematic deviations}}.$$

[1] Using only nominal zero-coupon bonds means the static hedging portfolio delivers a deterministic amount c_t at time t. The MVH objective is formulated as $\arg\min_{c_t} \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^f}\left[\left(h_t(I_0^t) - c_t\right)^2\right]$, whose well-known solution is $c_t = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^f}[h_t(I_0^t)]$.

Consequently, the total payoff H^n defined in (6) can be rewritten as follows:

$$H^n = {}_nH^1_{VH} + {}_nH^2_{VH} + {}_nH^3_{VH}, \tag{13}$$

where the aggregated components are defined by:

$${}_nH^1_{VH} \triangleq \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)]}{n} = \sum_{t=1}^{T} {}_nH_t^{1-VH}, \tag{14}$$

$${}_nH^2_{VH} \triangleq \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \Big(g_t(\tau_j) - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{S}_t}[g_t(\tau_j)]\Big) h_t(I_0^t)}{n} = \sum_{t=1}^{T} {}_nH_t^{2-VH}, \tag{15}$$

$${}_nH^3_{VH} \triangleq \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \Big(\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{S}_t}[g_t(\tau_j)]h_t(I_0^t) - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)]\Big)}{n} = \sum_{t=1}^{T} {}_nH_t^{3-VH}. \tag{16}$$

The hedgeable part ${}_nH^1_{VH}$ is identified through a static deterministic component that can be funded upfront with zero-coupon bonds. In contrast to the complete market framework—where the financial payoff $h_t(I_0^t)$ is perfectly replicable and maintained as a stochastic factor in the hedgeable step—our incomplete market setting restricts the hedgeable base to purely deterministic cash flows. Consequently, the stochastic payoff $h_t(I_0^t)$ is replaced by its physical expectation $\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)]$ in the hedgeable component. Thus, ${}_nH^1_{VH}$ acts as the sum of purely deterministic expected cash flows under the joint physical probability measure $\mathbb{P}^a \times \mathbb{P}^f$. The diversifiable component ${}_nH^2_{VH}$ remains structurally identical to the complete market case of Deelstra et al. (2020). Since it relies purely on the mutualization of idiosyncratic mortality risks, which are assumed to be independent of the financial market, it is entirely unaffected by market incompleteness. Indeed, since equation (2) states conditional independence of the policyholders' lifetimes, the law of large numbers implies that for each $t \in \{1, \dots, T\}$:

$${}_nH_t^{2-VH}\Big|_{\mathcal{F}_0 \vee \mathcal{F}_t^f \vee \mathfrak{S}_t} \xrightarrow{n \to \infty} 0,$$

and then, the aggregated diversifiable risk converges to zero asymptotically:

$${}_nH^2_{VH}\Big|_{\mathcal{F}_0 \vee \mathcal{F}_T^f \vee \mathfrak{S}_T} \xrightarrow{n \to \infty} \sum_{t=1}^{T} 0 = 0.$$

The residual term ${}_nH^3_{VH}$ captures the unhedgeable systematic risks. Rearranging equation (16), this systematic component splits into two distinct physical shocks:

$$
{}_nH^3_{VH} = \underbrace{\sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \left(\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{G}_t}[g_t(\tau_j)] - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] \right) h_t(I^t_0)}{n}}_{{}_nH^{3,M}\text{: Systematic mortality shock}} + \tag{17}
$$

$$
\underbrace{\sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] \left(h_t(I^t_0) - \mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I^t_0)] \right)}{n}}_{{}_nH^{3,F}_{VH}\text{: Unexpected inflation shock}} .
$$

This formulation highlights how the systematic risk stems simultaneously from deviations in the mortality trend and unpredictable macroeconomic inflation fluctuations. By explicitly isolating these two distinct physical shocks, the model can be naturally interpreted as a four-step decomposition.

2.3 Three-step decomposition: identifying the hedgeable component via quantile hedging

Beyond mean-variance hedging, other strategies can be employed to determine the hedgeable component. By adopting the quantile regression hedging strategy (see Barigou et al. (2022)), an insurer chooses to cover the financial risk up to a confidence level p. Under this strategy, the financial static deterministic base is determined by the value-at-risk $Q^{\mathbb{P}^f}_p[\cdot]$ [2], defined as $Q^{\mathbb{P}^f}_p[h_t(I^t_0)] \triangleq \inf\{c_t \in \mathbb{R} \mid P^{\mathbb{P}^f}(h_t(I^t_0) \leq c_t) \geq p\}$. The three-step decomposition is accordingly modified as follows:

$$
{}_nH^1_{QH} \triangleq \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] Q^{\mathbb{P}^f}_p[h_t(I^t_0)]}{n} = \sum_{t=1}^{T} {}_nH^{1-QH}_t, \tag{18}
$$

$$
{}_nH^2_{QH} \triangleq \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \left(g_t(\tau_j) - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{G}_t}[g_t(\tau_j)] \right) h_t(I^t_0)}{n} = \sum_{t=1}^{T} {}_nH^{2-QH}_t, \tag{19}
$$

$$
{}_nH^3_{QH} \triangleq \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \left(\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{G}_t}[g_t(\tau_j)] h_t(I^t_0) - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] Q^{\mathbb{P}^f}_p[h_t(I^t_0)] \right)}{n} = \sum_{t=1}^{T} {}_nH^{3-QH}_t. \tag{20}
$$

[2] In our deterministic static hedging setting, where only nominal zero-coupon bonds are traded, the quantile regression hedging in Equation (2.8) of Barigou et al. (2022), reduces to the quantile operator $Q^{\mathbb{P}^f}_p[h_t(I^t_0)]$.

This quantile-based decomposition remains structurally coherent and analogous to the mean-variance hedging case. The financial hedgeable risk is now identified via $Q_p^{\mathbb{P}^f}[h_t(I_0^t)]$, for each $t \in \{1, \cdots, T\}$. The diversifiable component is structurally immune to the chosen financial hedging strategy (${}_nH^2_{QH} = {}_nH^2_{VH}$) and therefore still converges to zero asymptotically almost surely. Finally, similarly to the mean-variance approach, the unhedgeable risk ${}_nH^3_{QH}$ splits into a systematic mortality shock and a systematic unexpected inflation shock, as follows

$$
{}_nH^3_{QH} = \underbrace{\sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \Big(\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{G}_t}[g_t(\tau_j)] - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] \Big) h_t(I_0^t)}{n}}_{{}_nH^{3,M}\text{: Systematic mortality shock}} + \underbrace{\sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] \Big(h_t(I_0^t) - Q_p^{\mathbb{P}^f}[h_t(I_0^t)] \Big)}{n}}_{{}_nH^{3,F}_{QH}\text{: Unexpected inflation shock}}. \tag{21}
$$

2.4 Valuation of the Three-step components

To assess the systematic risk component in an incomplete financial market for multi-cash-flow hybrid life products, we extend the Esscher transform operator presented in Belhouari et al. (2025) as follows [3]

$$
\begin{cases}
\pi_0^{\mathbb{Q}^f,n}(H^n|\mathcal{F}_0) \triangleq \displaystyle\sum_{t=1}^{T} \mathbb{E}^{\mathbb{Z}^a \times \mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{-\int_0^t r_s \, \mathrm{d}s} H_t^n \right], \\[2ex]
\left.\dfrac{d\mathbb{Z}^a}{d\mathbb{P}^a}\right|_{\mathcal{F}_t} \triangleq \dfrac{e^{\int_0^t \theta_t^{(1)} \lambda_s \, \mathrm{d}s}}{\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}\left[e^{\int_0^t \theta_t^{(1)} \lambda_s \, \mathrm{d}s} \right]},
\end{cases} \tag{22}
$$

with $\mathbb{Q}^f$ being an equivalent martingale probability measure. In our framework, this Esscher operator acts as a dual risk-loading operator: the measure $\mathbb{Z}^a$ specifically loads the valuation of the systematic mortality risk ${}_nH^{3,M}$, whereas $\mathbb{Q}^f$ loads the valuation of the systematic financial risk ${}_nH^{3,F}$. Continually, the mortality measure $\mathbb{Z}^a$ naturally drops out during the valuation of the pure systematic financial inflation component ${}_nH^{3,F}$, as confirmed later in Lemma 3.

Regardless of the chosen static hedging strategy (e.g., VH or quantile hedging), the evaluation of the resulting three-step decomposition follows a unified framework. Since the hedgeable component ${}_nH^1$ consists of purely deterministic cash

[3] The expression for $\theta_t^{(1)}$ depends on the specified dynamics of stochastic mortality rates and will be given later.

flows, its financial valuation $\pi_0^{\mathbb{Q}^f,f}$ depends exclusively on observable zero-coupon bond prices, rendering it completely independent of the choice of the financial equivalent martingale probability measure $\mathbb{Q}^f$; see Remark 1. Similarly, the diversifiable component ${}_nH^2$ is assessed using the actuarial operator $\pi_0^{\mathbb{Q}^f,a}$, relying solely on the physical probability measure; the best estimate of ${}_nH^2$ is equal to zero (see Lemma 2 later). Consequently, the choice of a specific financial equivalent martingale measure $\mathbb{Q}^f$ is restricted entirely to the Esscher operator $\pi_0^{\mathbb{Q}^f,n}$ for pricing the systematic financial risk ${}_nH^{3,F}$.

Definition 5 (**The** $\{1, T\}$**-Three-step method**).

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure associated with the Esscher operator $\pi_0^{\mathbb{Q}^f,n}$ and let $({}_nH^1,\ {}_nH^2,\ {}_nH^3)$ generically denote the three-step components derived from the chosen static hedging strategy for a hybrid life payoff $H^n \in \mathcal{H}$. The $\{1, T\}$-Three-step valuation method $\pi_0^{\mathbb{Q}^f,3}$ is defined by:

$$\pi_0^{\mathbb{Q}^f,3}(H^n|\mathcal{F}_0) = \pi_0^{\overline{\mathbb{Q}_1^f},f}({}_nH^1|\mathcal{F}_0) + \pi_0^{\overline{\mathbb{Q}_2^f},a}({}_nH^2|\mathcal{F}_0) + \pi_0^{\mathbb{Q}^f,n}({}_nH^3|\mathcal{F}_0), \tag{23}$$

for any two equivalent martingale probability measures $\overline{\mathbb{Q}_1^f}$ and $\overline{\mathbb{Q}_2^f}$.

Theorem 1.

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure associated with the Esscher operator $\pi_0^{\mathbb{Q}^f,n}$. Whether the hedgeable component is identified via mean-variance hedging or quantile regression hedging, the $\{1, T\}$-Three-step method $\pi_0^{\mathbb{Q}^f,3}$ is a fair valuation operator.

Proof. See the Appendix. □

We now establish three lemmas deriving the explicit valuation of each component of the three-step method, to be used in the subsequent numerical application.

Lemma 1.

For any equivalent martingale probability measure $\overline{\mathbb{Q}_1^f}$, the financial valuation of the hedgeable component ${}_nH^1$ is independent of the choice of $\overline{\mathbb{Q}_1^f}$ and is given by:

$$\pi_0^{\overline{\mathbb{Q}_1^f},f}({}_nH^1|\mathcal{F}_0) = \sum_{t=1}^{T} P(0,t) \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_1)] \times c_t, \tag{24}$$

where the deterministic financial base c_t is equal to

$$c_t = \begin{cases} \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^f}[h_t(I_0^t)] & \textit{under mean-variance hedging,} \\ Q_p^{\mathbb{P}^f}[h_t(I_0^t)] & \textit{under quantile hedging.} \end{cases} \tag{25}$$

Proof.

The proof follows directly from (18), (14), and Remark 1. □

Remark 3.

One might note, from Lemma 1, the apparent anomaly that the value of the hedgeable component is blind to the correlation between nominal interest rates and inflation. This occurs naturally because hedging through zero-coupon bonds imposes a deterministic cash-flow structure. However, this hedgeable cost represents only one piece of the total valuation under our multi-step decomposition. The impact of this correlation has instead been captured in the systematic risk valuation component through the best estimate $\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{-\int_0^t r_s\,\mathrm{d}s} h_t(I_0^t)\right]$*, as will be shown in Lemma 3.*

Lemma 2.

Irrespective of the chosen static hedging strategy (e.g., mean-variance or quantile hedging), the actuarial valuation of the diversifiable component ${}_nH^2$ *depends solely on the physical probability measure. For any equivalent martingale measure* $\overline{\mathbb{Q}^f_2}$*, it is explicitly given by:*

$$\pi_0^{\overline{\mathbb{Q}^f_2},a}({}_nH^2|\mathcal{F}_0) = \sum_{t=1}^{T} \frac{\alpha}{2} P(0,t) \times \sqrt{\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)^2]} \times \sqrt{\frac{1}{n}\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}\left[\mathbb{V}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{S}_t}[g_t(\tau_1)]\right]}. \quad (26)$$

Proof.

See the Appendix. □

Lemma 1 and Lemma 2 confirm that evaluating both the hedgeable and diversifiable components relies solely on real-world probability measures (except $P(0,t)$ which are observable).

Lemma 3.

Let $\mathbb{Q}^f$ *be an equivalent martingale probability measure associated with the Esscher operator* $\pi_0^{\mathbb{Q}^f,n}$*. The valuation of the systematic component* ${}_nH^3$ *additively decouples into a systematic mortality risk premium and a systematic inflation risk premium, as follows*

$$\pi_0^{\mathbb{Q}^f,n}({}_nH^3|\mathcal{F}_0) = \pi_0^{\mathbb{Q}^f,n}({}_nH^{3,M}|\mathcal{F}_0) + \pi_0^{\mathbb{Q}^f,n}({}_nH^{3,F}|\mathcal{F}_0), \quad (27)$$

where the respective valuations are given by:

$$\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,M}|\mathcal{F}_0) = \sum_{t=1}^{T} \left(\mathbb{E}^{\mathbb{Z}^a}_{\mathcal{F}_0}[g_t(\tau_1)] - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_1)]\right) \times \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{-\int_0^t r_s\,\mathrm{d}s} h_t(I_0^t)\right], \quad (28)$$

$$\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,F}|\mathcal{F}_0) = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_1)] \times \left(\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{-\int_0^t r_s\,\mathrm{d}s} h_t(I_0^t)\right] - P(0,t)\, c_t\right), \quad (29)$$

with c_t denoting the deterministic financial base (expectation or quantile) defined in Lemma 1.

Proof.

The proof follows directly from the two decompositions of the systematic risk ((17) and (21)), using the fact that $\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0\vee\mathfrak{G}_t}[g_t(\tau_1)] = \mathbb{E}^{\mathbb{Z}^a}_{\mathcal{F}_0\vee\mathfrak{G}_t}[g_t(\tau_1)]$ and the $\mathcal{F}_0$-measurability of $\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)]$, $Q_p^{\mathbb{P}^f}[h_t(I_0^t)]$, and $\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_1)]$. □

Remark 4.

It is worth noting that the sum of the hedge price and the systematic financial risk premium, always returns the exact best-estimate value, completely independent of the hedging portfolio. From (24), we have:

$$\pi_0^{\overline{\mathbb{Q}_1^f},f}({}_nH^1|\mathcal{F}_0) + \pi_0^{\mathbb{Q}^f,n}({}_nH^{3,F}|\mathcal{F}_0) = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_1)] \times \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{-\int_0^t r_s ds} h_t(I_0^t)\right].$$

This invariance allows us to study the pure allocation effect of hedging in the balance sheet. Should one desire the hedging strategy to directly impact the total value of the liability, the systematic financial risk ${}_nH^{3,F}$ could be evaluated using a non-additive risk operator (e.g. Cost-of-Capital approach: 6% of VaR), instead of the additive linear expectation operator driven by the Esscher transform defined in (22).

Remark 5.

In the absence of biometric risk ($g_t(\tau_j) \equiv 1$), the Three-step decomposition naturally collapses into a two-step financial valuation model. Without mortality uncertainty, no pooling risk exists; thus, the diversifiable component is eliminated, and the systematic mortality shock disappears as well. Furthermore, with the elimination of systematic mortality risk, the mortality Esscher-transformed probability measure $\mathbb{Z}^a$ is no longer required. Consequently, the total payoff simplifies exactly to a purely financial deterministic hedgeable sum and a residual systematic unexpected inflation shock:

$$H^n = \underbrace{\sum_{t=1}^{T} c_t}_{\text{Hedgeable financial risk}} + \underbrace{\sum_{t=1}^{T} \left(h_t(I_0^t) - c_t\right)}_{\text{Unhedgeable financial risk}},$$

where $c_t \in \left\{\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)], Q_p^{\mathbb{P}^f}[h_t(I_0^t)]\right\}$. The pricing of this specific subcase can be directly deduced from Lemma 1, Lemma 2, and Lemma 3. Since the variance of the biometric factor in Lemma 2 is zero, the actuarial premium of the diversifiable *component is canceled out. The financial valuation of the hedgeable part strictly reduces to: $\sum_{t=1}^{T} P(0,t)c_t$. The evaluation of the remaining unhedgeable financial risk depends solely on the chosen financial risk-neutral measure $\mathbb{Q}^f$, yielding*

$$\pi_0^{\mathbb{Q}^f,f}({}_nH^3|\mathcal{F}_0) = \sum_{t=1}^{T} \Big(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Q}^f} \big[e^{-\int_0^t r_s ds} h_t(I_0^t) \big] - P(0,t)c_t \Big).$$

This ensures that this generalization of the Three-step method in an incomplete financial market remains perfectly coherent: without biometric risk, the total value is governed solely by the financial upfront static hedge and the market premium assigned to the unhedgeable financial residual risk under a selected $\mathbb{Q}^f$.

3 Inflation-linked temporary life annuity

3.1 Payoff specification

We consider an inflation-linked temporary life annuity designed to preserve the real value of the pensioner's benefits in terms of purchasing power. The nominal payment at each date $t \in \{1, \cdots, T\}$ is contingent upon the policyholder's survival and is indexed to the accumulated CPI inflation index I_0^t. This mechanism guarantees a real income stream, effectively transferring inflation risk from the pensioner to the insurer. The corresponding multi-cash-flow hybrid payoff is given by

$$H^n = K_0 \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} I_0^t \times \mathbb{1}_{\tau_j > t}}{n}, \tag{30}$$

where $K_0 > 0$ represents the base nominal payout, serving as the starting reference value for the inflation indexation.

3.2 Dynamics

To ensure that the accumulated CPI inflation index follows a lognormal distribution, aligned with Jarrow & Yildirim (2003), we specifically adopt the Vasicek dynamics for the instantaneous inflation rate; see also the Vasicek inflation rate model in Fergusson (2020). Under the probability measure $\mathbb{P}^f$, the continuous-time dynamics of both the inflation rate and the nominal interest rate are governed by the following SDE system. A Cholesky decomposition is applied to the covariance structure to express the correlated stochastic shocks as linear combinations of independent standard Brownian motions:

$$di_s = (b_i - a_i i_s)ds + \sigma_i d\widetilde{W}_i(s),$$

$$dr_s = (b_r - a_r r_s)ds + \sigma_r(\rho d\widetilde{W}_i(s) + \sqrt{1-\rho^2} d\widetilde{W}_r(s)),$$

with $0 \leq s \leq T$. For any given maturity $t \in \{1, \ldots, T\}$, the dynamics of the nominal zero-coupon bond $P(s,t)$ over the period $0 \leq s \leq t$ under $\mathbb{P}^f$ are given by:

$$\frac{dP(s,t)}{P(s,t)} = \mu_P(s,t)ds - \sigma_r B(s,t)\, \rho\, d\widetilde{W}_i(s) - \sigma_r B(s,t) \sqrt{1-\rho^2}\, d\widetilde{W}_r(s),$$

where $\mu_P(s,t)$ denotes the stochastic drift of the zero-coupon bond price and $B(s,t) = \frac{1-e^{-a_r(t-s)}}{a_r}$. Here, $\tilde{W}_i(s)$ and $\tilde{W}_r(s)$ denote two mutually independent standard Brownian motions. The constant parameters (a_k, b_k, σ_k) for $k \in \{i, r\}$ govern the mean-reversion speed, the long-term mean, and the volatility of the macroeconomic rates, respectively. The structural dependency between the nominal short rate and the inflation rate is parameterized by the correlation coefficient ρ.

We define the new standard Brownian motions $(W_i(s), W_r(s))$ under an equivalent risk-neutral probability measure $\mathbb{Q}^f$ as:

$$W_i(s) \triangleq \tilde{W}_i(s) - \nu_i s, \tag{31}$$

$$W_r(s) \triangleq \tilde{W}_r(s) - \nu_r s. \tag{32}$$

We assume that the market prices of risk, ν_i and ν_r, are constant to preserve the Vasicek affine structure under any risk-neutral measure. The density process of the Radon-Nikodym derivative of $\mathbb{Q}^f$ with respect to $\mathbb{P}^f$ is explicitly given by:

$$\frac{d\mathbb{Q}^f}{d\mathbb{P}^f} = e^{\nu_i \tilde{W}_i(T) + \nu_r \tilde{W}_r(T) - \frac{1}{2}\left(\nu_i^2 + \nu_r^2\right)T}. \tag{33}$$

To preclude arbitrage opportunities across the entire yield curve, the discounted price of any traded zero-coupon bond must be a martingale under $\mathbb{Q}^f$; see (4). This implies the following equation:

$$\nu_r = \frac{\mu_P(s,t) - r_s}{\sigma_r B(s,t)\sqrt{1-\rho^2}} - \frac{\rho \nu_i}{\sqrt{1-\rho^2}}. \tag{34}$$

Since ν_r is constant, condition (34) dictates that the expected excess return per unit of risk must be identical for all $s \leq t \in \{1, \cdots, T\}$. This leads to imposing a constrained structure on the physical drift $\mu_P(s,t)$ as

$$\mu_P(s,t) = r_s + \sigma_r B(s,t)\Lambda_r,$$

where Λ_r acts as a constant aggregate market premium for interest rate risk. Hence:

$$\nu_r = \frac{\Lambda_r - \rho \nu_i}{\sqrt{1-\rho^2}}. \tag{35}$$

ν_r is uniquely determined with respect to ν_i. In contrast, the pure inflation risk price ν_i remains unconstrained. Thus, any choice of ν_i determines the entire system and yields a valid pricing measure $\mathbb{Q}^f$, highlighting the incompleteness of the financial market. The final risk-neutral dynamics of the nominal interest rate $(r_s)_{0\leq s\leq T}$ and the inflation rate $(i_s)_{0\leq s\leq T}$ under any chosen $\mathbb{Q}^f$ are:

$$di_s = (\theta_i - a_i i_s)ds + \sigma_i dW_i(s),$$

$$dr_s = (\theta_r - a_r r_s)ds + \sigma_r \left(\rho dW_i(s) + \sqrt{1-\rho^2} dW_r(s)\right),$$

with $\theta_i = b_i + \sigma_i \nu_i$, $\theta_r = b_r + \sigma_r \left(\rho \nu_i + \sqrt{1-\rho^2}\nu_r\right)$.
Consequently, the nominal zero-coupon bond price evaluated at time s for a maturity t is given by

$$P(s,t) = \tilde{A}(s,t) e^{-B(s,t) r_s}, \tag{36}$$

where $\tilde{A}(s,t) = e^{\left(\frac{\theta_r}{a_r} - \frac{\sigma_r^2}{2a_r^2}\right)\left(B(s,t)-t+s\right) - \frac{\sigma_r^2}{4a_r} B(s,t)^2}$, for $t \in \{1,\dots,T\}$. Similarly, a risk-neutral expected value of the accumulated CPI inflation index over the period $[s,t]$ is given by

$$\mathcal{I}(s,t) = \tilde{A}_I(s,t) e^{B_I(s,t) i_s}, \tag{37}$$

where $B_I(s,t) = \frac{1-e^{-a_i(t-s)}}{a_i}$ and $\tilde{A}_I(s,t) = e^{\left(-\frac{\theta_i}{a_i} - \frac{\sigma_i^2}{2a_i^2}\right)\left(B_I(s,t)-t+s\right) - \frac{\sigma_i^2}{4a_i} B_I(s,t)^2}$, for $t \in \{1,\dots,T\}$.

Remark 6.

The nominal zero-coupon bond price $P(s,t)$ is uniquely determined. Indeed, substituting the structural constraint (35) into θ_r yields the following expression:

$$\theta_r = b_r + \sigma_r \Lambda_r.$$

Thus, θ_r is a strict constant, fully independent of the unconstrained inflation risk price ν_i. Conversely, the market-implied expected inflation accumulation $\mathcal{I}(s,t)$ remains non-unique, since $\theta_i = b_i + \sigma_i \nu_i$ intrinsically depends on the arbitrary choice of ν_i.

For the actuarial modeling of the instrument, stochastic mortality rates $(\lambda_s)_{s\geq 0}$, are described by an Ornstein–Uhlenbeck intensity process (see, e.g., Luciano & Vigna (2008)). Accordingly, under the probability measure $\mathbb{P}^a$, it follows that

$$d\lambda_s = \mu_m \lambda_s dt + \sigma_m dW_m(s),$$

where $\mu_m > 0$ and $\sigma_m > 0$. Therefore, the survival probability until a payment date $t \in \{1,\dots,T\}$ is given by

$${}_t p_x = e^{-m_{0,t} + \frac{1}{2} s_{0,t}^2},$$

where

$$m_{0,t} = \lambda_0 \frac{e^{\mu_m t} - 1}{\mu_m},$$

and

$$s_{0,t}^2 = \frac{\sigma_m^2}{2\mu_m^3}(1 - e^{\mu_m t})^2 + \frac{\sigma_m^2}{\mu_m^2}(t - \frac{e^{\mu_m t} - 1}{\mu_m}).$$

In accordance with the prudential framework established by Belhouari et al. (2025), the Esscher transform parameter $\theta_t^{(1)}$ driving the systematic mortality loading is defined as, $\theta_t^{(1)} = \frac{-0.16}{s_{0,t}}$.

3.3 Three-step valuation of the Inflation-linked temporary life annuity

The objective of this subsection is to explicitly price the multi-cash-flow inflation-linked annuity using the Three-step method. Leveraging the previously established lemmas, we sequentially derive closed-form pricing formulas for the hedgeable, diversifiable, and systematic risk components.

Proposition 1.

For any equivalent martingale measure $\overline{\mathbb{Q}_1^f}$, the financial valuation of the hedgeable component ${}_nH^1$ for the Inflation-linked temporary life annuity is given by:

$$\pi_0^{\overline{\mathbb{Q}_1^f},f}({}_nH^1|\mathcal{F}_0) = K_0 \sum_{t=1}^{T} P(0,t) \times c_t \times {}_tp_x, \tag{38}$$

where the deterministic financial base c_t admits the following explicit formulations depending on the adopted static hedging strategy:

$$c_t = \begin{cases} e^{m_I(0,t) + \frac{1}{2}v_I(0,t)} & \text{under } MVH, \\ e^{m_I(0,t) + \Phi^{-1}(p)\sqrt{v_I(0,t)}} & \text{under } QH, \end{cases} \tag{39}$$

with $m_I(0,t)$ and $v_I(0,t)$ denoting respectively the mean and variance of the normally distributed integrated process $ln(I_0^t) = \int_0^t i_s ds$ under $\mathbb{P}^f$, given by:

$$m_I(0,t) = B_I(0,t)(i_0 - \frac{b_i}{a_i}) + \frac{b_i}{a_i}t,$$
$$v_I(0,t) = \frac{-\sigma_i^2}{2a_i}B_I(0,t)^2 + \frac{\sigma_i^2}{a_i^2}(t - B_I(0,t)).$$

Proof.

Recalling that $g_t(\tau_1) = \mathbb{1}_{\tau_1>t}$, and computing the mean and variance of the gaussian process $\int_0^t i_s ds$ under $\mathbb{P}^f$, the result follows directly from Lemma 1. $\square$

Proposition 2.

For any equivalent martingale measure $\overline{\mathbb{Q}_2^f}$, the actuarial valuation of the diversifiable component ${}_nH^2$ for the Inflation-linked temporary life annuity is given by:

$$\pi_0^{\overline{\mathbb{Q}_2^f},a}({}_nH^2|\mathcal{F}_0) = K_0 \sum_{t=1}^{T} \frac{\alpha}{2} \times P(0,t) \times \sqrt{e^{2\times(m_I(0,t)+v_I(0,t))}} \times L^t, \tag{40}$$

where the actuarial idiosyncratic risk factor L^t is given by:

$$L^t = \sqrt{\frac{1}{n}({}_tp_x - e^{-2m_{0,t}+2s_{0,t}^2})},$$

and where $m_I(0,t)$ and $v_I(0,t)$ are as previously defined in Proposition 1.

Proof.

Using that $\mathbb{V}_{\mathcal{F}_0\vee\mathfrak{G}_t}^{\mathbb{P}^a}[g_t(\tau_1)] = e^{-\int_0^t \lambda_s \, \mathrm{d}s}(1 - e^{-\int_0^t \lambda_s \, \mathrm{d}s})$, and that $m_I(0,t)$ and $v_I(0,t)$ denote, respectively, the mean and variance of $\int_0^t i_s ds$, the resulting pricing formula follows directly from Lemma 2. $\square$

The term $\sqrt{e^{2\times(m_I(0,t)+v_I(0,t))}}$ measures the volatility of future inflation, reflecting the risk that realized inflation deviates unpredictably from its expected trend.

Proposition 3.

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure associated with the Esscher operator. The valuation of the systematic component ${}_nH^3$ for the Inflation-linked temporary life annuity additively decouples into $\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,M}|\mathcal{F}_0)$ and $\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,F}|\mathcal{F}_0)$, explicitly given by:

$$\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,M}|\mathcal{F}_0) = K_0 \sum_{t=1}^{T} P(0,t) \times \mathcal{I}(0,t) \times A_t \times {}_tp_x \times \left(e^{-s_{0,t}^2\times\theta_t^{(1)}} - 1\right),$$

$$\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,F}|\mathcal{F}_0) = K_0 \sum_{t=1}^{T} {}_tp_x \times P(0,t) \times \left(\mathcal{I}(0,t) \times A_t - c_t\right).$$

where c_t is the deterministic financial base defined in Proposition 1. The market-implied expected inflation value $\mathcal{I}(0,t)$ is given by (37), while the factor A_t is defined as $A_t = e^{-\frac{\rho\sigma_r\sigma_i}{a_i a_r}\times[t-B(0,t)-B_I(0,t)+\frac{1-e^{-(a_r+a_i)t}}{a_r+a_i}]}$.

Proof.

See the Appendix. □

The primary distinction in the systematic valuation price of a life annuity with inflation relative to one without inflation is the inclusion of the term $\mathcal{I}(0,t) \times A_t$. The term $\mathcal{I}(0,t)$ quantifies the baseline cost of incorporating expected inflation. The term A_t represents the additional cost attributable to the correlation between interest rates and inflation: it leads to underpricing compared with the case where inflation and interest rates are independent, under positive correlation, and to overpricing under negative correlation.

4 Impact of a correlated stock as an additional traded asset on an inflation linked pure endowment payment

4.1 Three-step extension with a stochastic hedgeable component: dynamic portfolio rebalancing

To investigate the cross-hedging efficiency of a traded stock-relying solely on its correlation with inflation to hedge the temporary life annuity defined in (30)-and its potential to reduce the initial hedging cost, we extend the financial market by introducing another traded asset $(S_s)_{0\leq s\leq T}$. Due to the substantial computational time required to numerically solve the non-linear partial differential equation (PDE) needed to determine the financial hedging cost (see below), we investigate this impact for a single fixed payment date $t^* \in \{1,\dots,T\}$. Additionally, we focus exclusively on the continuous Mean-variance hedging criterion. Under the physical measure $\mathbb{P}^f$, and using a Cholesky decomposition to capture the correlation structure, the continuous-time dynamics of the stock $(S_s)_{0\leq s\leq T}$ are given by:

$$dS_s = \mu_S S_s ds + \sigma_S S_s \Big(\rho_{iS} d\widetilde{W}_i(s) + C_1 d\widetilde{W}_r(s) + C_2 d\widetilde{W}_S(s)\Big),^{4}$$

with $C_1 = \frac{\rho_{rS}-\rho\rho_{iS}}{\sqrt{1-\rho^2}}$, $C_2 = \sqrt{1-\rho_{iS}^2 - C_1^2}$, and for $s \leq T$. μ_S and σ_S define the expected return and constant volatility of the stock. The structural dependencies are parameterized by ρ_{iS}, representing the correlation between i_s and S_s, and by ρ_{rS}, representing the correlation between r_s and S_s. The introduction of this second traded asset imposes an additional no-arbitrage condition. In addition to

[4]We assume that the stock is not perfectly correlated with inflation, i.e., avoiding $C_1 = 0$, & $C_2 = 0$ and $\rho_{iS} \in \{-1, 1\}$.

the zero-coupon bond constraint derived in equation (35), the discounted stock price must also be a martingale under any equivalent measure $\mathbb{Q}^f$. This yields the following structural constraint for the stock's market price of risk $\nu_S(s)$ [5]:

$$\nu_S(s) = \frac{\mu_S - r_s}{C_2\sigma_S} + \frac{\rho_{iS}\nu_i + C_1\nu_r}{C_2}. \tag{41}$$

Despite the presence of two traded assets ($(P(s,t^*)_{s\leq t^*}$ and $(S_s)_{s\leq t^*}$), they are insufficient to perfectly cover the three independent Brownian shocks $(\tilde{W}_i, \tilde{W}_r, \tilde{W}_S)$. Consequently, the financial market remains strictly incomplete, leaving the pure inflation risk price ν_i unconstrained.

With the inclusion of the stock, we can now dynamically cross-hedge the target inflation risk $I_0^{t^*}$. At any time $s \in [0, t^*]$, we manage a continuously rebalanced, self-financing hedging portfolio whose total wealth process is denoted by X_s. We dynamically select a stochastic scalar control ξ_s, representing the amount [6] allocated to the stock. The remaining wealth $(X_s - \xi_s)$ is inherently allocated to the nominal zero-coupon bond $P(s,t^*)$, yielding the following

$$dX_s = \xi_s \frac{dS_s}{S_s} + (X_s - \xi_s)\frac{dP(s,t^*)}{P(s,t^*)}. \tag{42}$$

The three-step decomposition remains structurally analogous to the case where the nominal zero-coupon bond was the only traded asset in the financial market. The fundamental difference lies in leveraging this newly added traded asset $(S_s)_{s\leq t^*}$: the purely deterministic hedgeable financial base $\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[I_0^{t^*}]$ (MVH case) is now replaced by the stochastic wealth process X_{t^*} generated by the dynamic cross-hedging portfolio. Specifically, targeting the inflation-linked pure endowment payoff at maturity t^*, the hybrid cash flow payable at this date is decomposed as follows:

[5] Because the short rate r_s is stochastic, the specific market price of risk for the stock, $\nu_S(s)$, is necessarily stochastic as well.

[6] We define the control ξ_s as an absolute cash amount rather than the proportion of shares, as the latter would explicitly embed the stock price S_s into this wealth SDE: $dX_s = \xi_s dS_s + (X_s - \xi_s S_s)\frac{dP(s,t^*)}{P(s,t^*)}$; hence introducing an unnecessary state variable (S_s) in the PDE derived later, unnecessarily increasing its dimension.

$$
\begin{cases}
{}_nH^{1-VH}_{t^*} = K_0 \dfrac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[\mathbb{1}_{\tau_j>t^*}] X_{t^*}}{n}, \\[2ex]
{}_nH^{2}_{t^*} = K_0 \dfrac{\sum_{j=1}^{n} \left(\mathbb{1}_{\tau_j>t^*} - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{S}_{t^*}}[\mathbb{1}_{\tau_j>t^*}]\right) I_0^{t^*}}{n}, \\[2ex]
{}_nH^{3-VH}_{t^*} = K_0 \underbrace{\dfrac{\sum_{j=1}^{n} \left(\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0 \vee \mathfrak{S}_{t^*}}[\mathbb{1}_{\tau_j>t^*}] - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[\mathbb{1}_{\tau_j>t^*}]\right) I_0^{t^*}}{n}}_{\text{Systematic mortality shock}} + K_0 \underbrace{\dfrac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[\mathbb{1}_{\tau_j>t^*}] \left(I_0^{t^*} - X_{t^*}\right)}{n}}_{\text{Unhedgeable inflation shock}}.
\end{cases}
\tag{43}
$$

Consequently, evaluating these components at inception ($s = 0$) is strictly analogous to the results established in Propositions (1), (2), and (3). The diversifiable value remains strictly identical, as idiosyncratic mortality risk is fully immune to the addition of another traded stock. However, evaluating the hedgeable and systematic components now relies entirely on the initial cost of the dynamic portfolio: X_0^*. Therefore, pricing this three-step decomposition in the presence of the additional traded asset reduces exclusively to determining the optimal initial capital X_0^*.

Remark 7.

Following the introduction of a stock, for an inflation-linked pure endowment with a single payment at t^*, we note the following:

- *With the inclusion of the traded stock, one can easily verify that the three-step valuation operator remains fair. Indeed, by adapting the proof of Theorem 1 provided in the Appendix, the market-consistency property becomes*

$$\pi_0^{\mathbb{Q}^f,3}(H^n_{t^*} + Y_{t^*}|\mathcal{F}_0) = \pi_0^{\mathbb{Q}^f,3}(H^n_{t^*}|\mathcal{F}_0) + \pi_0^{\overline{\mathbb{Q}^f},f}(Y_{t^*}|\mathcal{F}_0),$$

  for a strictly hedgeable stochastic payoff Y_{t^*} generated by a dynamically rebalanced hedging portfolio over the period $[0, t^*]$, and for any equivalent martingale probability measure $\overline{\mathbb{Q}^f}$. The property is preserved by exploiting in the proof the fact that the mean-variance hedge of Y_{t^*} is perfectly equal to Y_{t^*} itself. Regarding actuarial consistency, it holds exactly as detailed in the Appendix, without requiring any adaptation.

- *Echoing Remark 4, the stability of the best estimate is preserved even with the introduction of the stock, as follows:*

$$
\begin{aligned}
\pi_0^{\overline{\mathbb{Q}^f_1},f}({}_nH^1_{t^*}|\mathcal{F}_0) + \pi_0^{\mathbb{Q}^f,n}({}_nH^{3,F}_{t^*}|\mathcal{F}_0) &= \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[\mathbb{1}_{\tau_j>t^*}]\left(X_0^* + \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{-\int_0^{t^*} r_s ds} I_0^{t^*}\right] - X_0^*\right) \\
&= \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[\mathbb{1}_{\tau_j>t^*}] \times \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{-\int_0^{t^*} r_s ds} I_0^{t^*}\right].
\end{aligned}
$$

4.2 Optimal dynamic allocation via the HJB stochastic control problem

We aim to find the optimal dynamic allocation that minimizes the expected real-world squared hedging error at maturity t^*. The solution to this optimization problem then uniquely determines the required value of X_0^*. This translates into a standard stochastic control problem where the value function $V_s^{X_0}$ at time $s \leq t^*$, representing the minimal expected squared error that is inherently dependent on the initial hedging capital X_0 provided to fund the strategy at inception, is defined as follows:

$$V_s^{X_0} = \inf_{\substack{\xi_u \\ s \leq u \leq t^*}} \mathbb{E}_{\mathcal{F}_s}^{\mathbb{P}^f} \left[\left(I_0^{t^*} - X_{t^*} \right)^2 \right]. \tag{44}$$

It is well known that the value function $V_s^{X_0}$ satisfies the Hamilton-Jacobi-Bellman (HJB) partial differential equation, which, in the context of the dynamics of our actuarial-financial instruments, takes the following form:

$$\partial_s V_s^{X_0} + (b_i - a_i i_s)\partial_i V_s^{X_0} + (b_r - a_r r_s)\partial_r V_s^{X_0} + i_s I_0^s \partial_I V_s^{X_0} + \frac{1}{2}\sigma_i^2 \partial_i^2 V_s^{X_0} + \frac{1}{2}\sigma_r^2 \partial_r^2 V_s^{X_0} + \sigma_i \sigma_r \rho \partial_{ri}^2 V_s^{X_0} + \inf_{\xi_s} \left(JY^{(2)} \xi_s^2 + JY^{(1)} \xi_s + JY^{(0)} \right) = 0 \tag{45}$$

where $JY^{(2)}$, $JY^{(1)}$, and $JY^{(0)}$ are coefficients to be determined. In our specific context, they are explicitly identified as follows:

$$JY^{(2)} = \frac{1}{2}\Sigma_{XX}^{(2)} \partial_X^2 V_s^{X_0}, \tag{46}$$

$$JY^{(1)} = (\mu_S - r_s - \sigma_r B(s,t^*)\Lambda_r)\partial_X V_s^{X_0} + \frac{1}{2}\Sigma_{XX}^{(1)} \partial_X^2 V_s^{X_0} + \Sigma_{Xr}^{(1)} \partial_{Xr}^2 V_s^{X_0} + \Sigma_{Xi}^{(1)} \partial_{Xi}^2 V_s^{X_0}, \tag{47}$$

$$JY^{(0)} = X_s[r_s + \sigma_r B(s,t^*)\Lambda_r]\partial_X V_s^{X_0} + \frac{1}{2}\Sigma_{XX}^{(0)} \partial_X^2 V_s^{X_0} + \Sigma_{Xr}^{(0)} \partial_{Xr}^2 V_s^{X_0} + \Sigma_{Xi}^{(0)} \partial_{Xi}^2 V_s^{X_0}, \tag{48}$$

with $\Sigma_{XX}^{(2)} = \sigma_r^2 B(s,t^*)^2 + \sigma_S^2 + 2\sigma_S\sigma_r B(s,t^*)\rho_{rS}$, $\Sigma_{XX}^{(1)} = -2X_s\sigma_r^2 B(s,t^*)^2 - 2\sigma_S\sigma_r B(s,t^*)\rho_{rS}X_s$, $\Sigma_{XX}^{(0)} = \sigma_r^2 B(s,t^*)^2 X_s^2$, $\Sigma_{Xi}^{(1)} = \sigma_i\sigma_S\rho_{iS} + \sigma_i\sigma_r B(s,t^*)\rho$, $\Sigma_{Xi}^{(0)} = -\sigma_i\sigma_r B(s,t^*)X_s\rho$, $\Sigma_{Xr}^{(1)} = \sigma_r\sigma_S\rho_{rS} + \sigma_r^2 B(s,t^*)$, and $\Sigma_{Xr}^{(0)} = -\sigma_r^2 B(s,t^*)X_s$. (The detailed derivation of the coefficients $JY^{(2)}$, $JY^{(1)}$, and $JY^{(0)}$ is provided in the Appendix.)

Because the value function $V_s^{X_0}$ is strictly convex with respect to the portfolio wealth, the leading coefficient $JY^{(2)}$ is strictly positive. Consequently, the quadratic polynomial inside the minimum operator is strictly convex with respect to the control ξ_s. The minimum is thus analytically reached at the optimal dynamic allocation $\xi_s^* = -\frac{JY^{(1)}}{2JY^{(2)}}$. Substituting this optimal allocation back into the HJB equation eliminates the minimum operator entirely, evaluating the optimized term as

$$\inf_{\xi_s} \left(JY^{(2)}\xi_s^2 + JY^{(1)}\xi_s + JY^{(0)} \right) = JY^{(0)} - \frac{(JY^{(1)})^2}{4JY^{(2)}},$$

thereby yielding the final non-linear PDE.

Ultimately, our core objective is to determine the optimal initial price of the financial hedge X_0^*. By numerically solving[7] this final PDE backward from maturity t^* down to inception $s = 0$, we obtain the expected squared error function $V_0^{X_0}$ evaluated at the current market state. The optimal initial capital X_0^* is then directly extracted as the argument minimizing this initial expected error, namely:

$$X_0^* = \arg\min_{X_0} V_0^{X_0}.$$

5 Numerical results

5.1 Market inputs and calibration

We consider a homogeneous portfolio of $n = 500$ female policyholders, initially aged $x = 65$, who purchase an inflation-linked temporary life annuity to preserve their purchasing power during retirement. The contract provides annual payments over a maximum horizon of $T = 45$ years. To evaluate this contract on a standard per-unit basis, the initial nominal payout is normalized to $K_0 = 1$. The initial instantaneous inflation rate, i_0, is set to 2.47%, reflecting the annualized Belgian inflation rate of 2025[8]. The 1-year Belgian government bond yield (OLO) fluctuates between approximately 2.0% and 3.3% in 2025, with a representative level of 2.8%, based on secondary market data from the National Bank of Belgium. Due to the general unavailability of proprietary derivatives data for full calibration, we adopt a hybrid parameterization approach. The mean-reversion speed and volatility parameters for inflation (a_i, σ_i) are obtained from a maximum-likelihood calibration to historical Belgian inflation data based on the Consumer Price Index (CPI) series from 1953 to 2023, whereas the parameters for the nominal interest rate (a_r, σ_r) are taken from Diez & Korn (2020). To ensure robust macroeconomic consistency, the long-term mean parameters are structurally adjusted. We enforce a strictly positive long-term real interest rate under both the physical and

[7]We solve the PDE using the finite difference method and omit the implementation details.

[8]Data retrieved from https://www.rateinflation.com/inflation-rate/belgium-historical-inflation-rate/.

risk-neutral measures (i.e., $b_r/a_r > b_i/a_i$ and $\theta_r/a_r > \theta_i/a_i$), setting the physical nominal interest rate target to 2.50% against an inflation target of 2.00% [9]. To move from our theoretical incomplete financial market to practical valuation, we fix the unconstrained market price of inflation risk ($\nu_i = 0.15$), thereby selecting a single representative equivalent martingale measure $\mathbb{Q}^f$ for our illustration. The market premium for interest rate risk is set to $\Lambda_r = 0.30$. To ensure a realistic representation of longevity, the parameters of the Ornstein–Uhlenbeck stochastic mortality intensity are calibrated using the IA|BE prospective best-estimate mortality tables. The calibration (using the least-squares method) is based on the projected survival probabilities of a 65-year-old female cohort, with the initial time $t = 0$ corresponding to the year 2025 and the projection horizon extending to 2070. Finally, the stock parameters are fixed at standard market levels ($\mu_S = 6\%$, $\sigma_S = 20\%$). All numerical inputs employed for our implementation are detailed in Table 1, except for the correlation parameters, which are left free for sensitivity analysis.

5.2 Numerical values of a temporary indexed annuity under nominal zero-coupon bond trading

Table 2 reports the decomposition of the time-0 value of the inflation-linked temporary life annuity under a static hedging framework using exclusively zero-coupon bonds, for both Mean-Variance Hedging and Quantile Hedging. The hedgeable component is valued higher under QH than under MVH. This difference is intuitive, as MVH targets the expected inflation index under the physical measure, whereas QH hedges against a predefined extreme scenario (the 95th percentile), requiring a larger initial capital allocation. The systematic mortality risk premium is strictly invariant to the chosen financial hedging strategy. This independence reflects the fundamental structure of the three-step decomposition: because systematic mortality risks are completely orthogonal to the penalization criterion—whether targeting the expected value or a quantile of the inflation index—the latter has no theoretical or numerical impact on the valuation of systematic longevity trends. The systematic inflation risk premium differs markedly between the two hedging strategies: it is positive under MVH but negative under QH. Under MVH, the positive premium compensates the insurer for covering the residual, unhedgeable uncertainty of the inflation index around its expected path. Conversely, under the QH strategy, the insurer has already over-provisioned upfront to cover the

[9] This 2.00% rate is consistent with the long-term inflation target pursued by the European Central Bank (ECB) for the Eurozone (https://www.nbb.be/fr/articles/les-attentes-dinflation-et-la-politique-monetaire).

95th percentile of the inflation distribution through the hedgeable component. Consequently, the systematic inflation premium becomes negative, reflecting an ex-ante discount corresponding to the present value of the expected surplus generated by the over-hedged position. As a benchmark, Table 2 also reports the valuation of a standard, unindexed nominal annuity. For this nominal contract, perfect replication of the financial payoff part is achievable using traded nominal zero-coupon bonds, naturally driving the systematic financial risk premium to zero; only systematic mortality risk requires a premium. Naturally, the overall price of the inflation-linked annuity is higher than that of the standard nominal annuity, capturing the cost of transferring the macroeconomic inflation risk from the pensioner to the insurer. Figure 1 shows a strictly decreasing relationship between the systematic inflation risk premium under MVH hedging and the interest–inflation correlation, ρ. As implied by the three-step decomposition, the valuation of the deterministic hedgeable component is invariant to ρ, hence the correlation affects only the valuation of the best estimate. The more positive the correlation, the more actively the central bank raises nominal interest rates to suppress inflation, systematically lowering the price of the fully indexed best estimate: $\sum_{t=1}^{T} \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_1)] \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{Q}^f}\left[e^{-\int_0^t r_s \, \mathrm{d}s} I_0^t\right]$. Consequently, the associated systematic inflation risk premium decreases. The same comments apply under the Quantile Hedging (QH) framework, as illustrated in Figure 2. Compared to the impact ρ has on the systematic financial risk premiums (as seen in Figure 1 and Figure 2), Figure 3 shows that the systematic mortality risk premium is only marginally affected by the correlation ρ. This weak sensitivity arises because the valuation is primarily driven by the exponential mortality Esscher transform factor, $e^{-s_{0,t}^2 \times \theta_t^{(1)}}$. Consequently, this mortality loading factor dominates the valuation formula, while the correlation ρ contributes only a negligible second-order effect. This finding has a clear interpretation: this risk component is, by definition, a systematic mortality risk, driven by long-term systematic longevity shocks, heavily overshadowing any day-to-day covariance between inflation and nominal interest rate.

5.3 Numerical values when cross-hedging an inflation linked pure endowment payment with a traded stock

We now turn to our numerical analysis considering a setting where the stock is actively traded. Figure 4 illustrates the optimal expected quadratic error at inception as a function of the initial wealth X_0 when hedging a pure financial inflation payment at $t^* = 8$. The strictly parabolic shape of V_0^X is structurally consistent with the quadratic terminal condition, confirming backward numerical stability of the PDE. The global minimum precisely yields the optimal initial capital $X_0^* = 0.8800404$. Furthermore, this deeply centered optimum rigorously validates the absence of spatial truncation effects from our chosen grid boundaries [10]. Table 3 compares the valuation components, revealing that stock integration successfully reduces the hedgeable capital allocation, while simultaneously elevating the systematic financial premium to account for the difference that will be injected to reach our reference best-estimate value. However, hedging with the stock remains strictly advantageous: it actively reduces replication costs, while the higher systematic premium solely acts as a prudential capital buffer. It is preferable to prioritize the reduction of actual upfront liquidity outflows over theoretical reserving margins. The modest numerical hedge-price differences (stock vs. no stock) observed in Table 3 arise because our current payoff is indexed to a purely macroeconomic inflation metric, completely detached from the stock price dynamics. This structural separation inherently limits the effectiveness of cross-hedging. In contrast, the impact would be greater for a mixed-indexation product that dynamically switches between equity returns and inflation. Consider, for example, a recursive equity-linked annuity that provides stock market upside and ensures that the pensioner receives at least the macroeconomic inflation, defined as

$$h_t(I_0^t) = h_{t-1}(I_0^{t-1}) \times \max\left(\frac{1 + r_d(t)}{1+l},\, I_{t-1}^t\right),$$

where l is the insurer's technical interest rate and $r_d(t) = \frac{S_t}{S_{t-1}} - 1$. Because this indexation mechanism directly incorporates the stock price dynamics, hedging with the traded stock would capture more of the underlying equity risk, leading to a larger numerical difference between the two approaches (stock vs. no stock) than in our pure-inflation setting. To quantify the risk-reduction capability of integrating a correlated stock as a cross-hedging asset, Figure 5 plots the optimal initial price of the financial hedge X_0^* against the stock-inflation correlation ρ_{iS}. The declining trend demonstrates that a stronger correlation enhances hedging efficiency. Ultimately, this confirms that introducing a stock to cross-hedge an inflation-linked payment becomes advantageous in financial markets characterized by strong stock-inflation dependencies. We can conclude indirectly from Figure 5, without requiring an additional plot, that the systematic financial risk premium will have an increasing monotonic trend with respect to the stock–inflation correlation, as the best-estimate value $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Q}^f}\left[e^{-\int_0^t r_s \, \mathrm{d}s} h_t(I_0^t)\right]$ is insensitive to this correlation. This provides another limitation of cross-hedging. In contrast, the trend of the systematic financial risk premium could be decreasing if the best-estimate value contributed to the monotonicity with a decreasing effect (as in subsection 5.2), thereby enabling the systematic financial risk premium to also benefit from the level of stock–inflation correlation observed in the market.

[10] $X_{\min} = -1$, $X_{\max} = 3$.

Parameters	Numerical Values
r_0	0.028
a_r	0.4
θ_r	0.0115
σ_r	0.005
b_r	0.01
i_0	0.0247
a_i	3.3434
θ_i	0.077293
σ_i	0.0695
b_i	0.066868
σ_S	0.2
μ_S	0.06
Λ_r	0.3
λ_0	0.007081186
μ_m	0.1015878
σ_m	0.001
α	0.3
K_0	1

Table 1: Numerical inputs.

6 Conclusion

This paper examines inflation-linked annuity products that promise retirees constant CPI-indexed purchasing power. By focusing only on financial payoffs linked to CPI outcomes, we directly face an incomplete financial market, as no traded assets are available to perfectly replicate such payoffs. In private insurance practice, collective pools of inflation-linked annuities are also exposed to systematic longevity risk, in a setting where inflation varies stochastically jointly with nominal interest rates, making the stochastic-interest-rate three-step method of Belhouari et al. (2025) a suitable valuation operator for such hybrid life products. Relying on standard nominal zero-coupon bonds as the available assets in the market, our first objective then was to directly extend this operator to an incomplete financial market for multicash-flow hybrid life payments and preserve market–actuarial consistency. The resulting risk decomposition leads to a non-unique, risk-dependent partial hedging strategy, with an additional and clearly distinguishable new layer of risk attributable to systematic financial risk. More precisely, we identified this strategy-dependence in the hedgeable component and its associated systematic fi-

Valuation component	Mean-Variance Hedging (MVH)	Quantile Hedging (QH)	Standard nominal annuity
Hedgeable part	20.29401	22.70415	15.71529
Diversifiable part	0.08128349	0.08128349	0.05343413
Systematic part: Mortality risk premium	0.4856761	0.4856761	0.2662869
Systematic part: Inflation risk premium	0.8224102	−1.587732	0

Table 2: Three-step valuation of the inflation-linked temporary life annuity using exclusively traded zero-coupon bonds for hedging, with $\rho = 0.5$, and $p = 95\%$, alongside standard nominal annuity valuation.

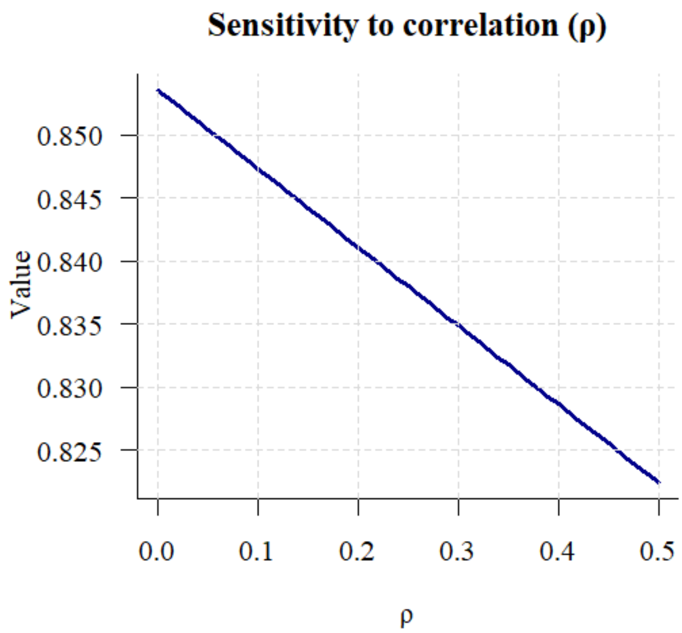

Figure 1: Systematic inflation risk premium of the inflation-linked temporary life annuity under Mean–Variance Hedging (MVH) using only traded zero-coupon bonds, as a function of the interest–inflation correlation ρ.

nancial risk. Accordingly, we keep the ICP-linked best-estimate value fixed and focus on how its allocation between the hedgeable price and its associated systematic financial risk premium evolves with the selected hedging strategy and available asset portfolio. We find that, irrespective of the strategy considered—Mean variance hedging or Quantile regression hedging—the hedge price is insensitive to the nominal interest rate–inflation correlation. This shows that, when the hedging

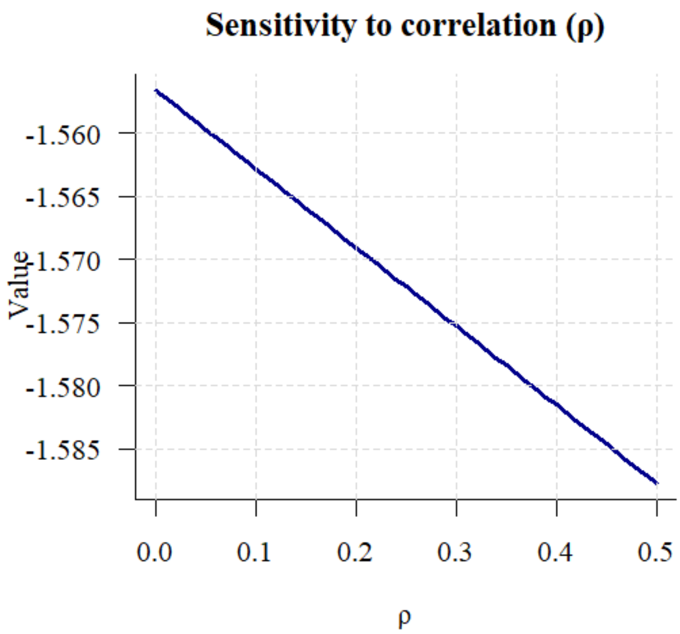

Figure 2: Systematic inflation risk premium of the inflation-linked temporary life annuity under Quantile Hedging (QH) using only traded zero-coupon bonds, as a function of the interest–inflation correlation ρ ($p = 95\%$.)

portfolio contains only standard nominal zero-coupon bonds, our hedging strategy may not be optimal for CPI-linked annuities, as it does not allow the hedge to benefit from the prevailing level of market dependence between nominal interest rates and inflation.

As a further attempt to improve partial replication in this context, where no financial assets directly replicate the CPI index, we investigate whether standard traded stocks exhibiting a fairly strong correlation with inflation can be incorporated, alongside standard nominal zero-coupon bonds, as an additional traded asset, with the potential to provide a more effective hedge of CPI-linked payoffs and consequently reduce the cost of the hedge. Our results indeed show that incorporating the stock reduces the hedge price and, importantly, makes the hedge cost responsive to the stock–inflation correlation: the stronger the correlation, the lower the resulting hedge price. Nevertheless, the magnitude of the net reduction remains modest. This is the first limitation of using the stock as a cross-hedging instrument through its correlation with the CPI-linked payoff; as it is not directly involved in the payoff itself. Introducing the stock leads to a higher systematic financial risk premium, which offsets the reduction in the hedge price and preserves the fixed ICP-linked best-estimate value across the two hedging portfolios, with and without the stock ; this is naturally expected from our study-specific allocation. In addition, we observe an increasing trend in the systematic financial risk premium with respect to the stock–inflation correlation. This represents a second limitation of cross-hedging, as the correlation does not affect the ICP-linked best-estimate value $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Q}^f}\left[e^{-\int_0^t r_s\,\mathrm{d}s} h_t(I_0^t)\right]$, preventing it from contributing a decreasing effect. Notwithstanding this, the additional systematic risk premium is a capital buffer; still, the second stock-based approach is attractive compared to the case without stock, as it reduces hedge cash outflow, which is preferable from a risk management perspective.

Finally, using only nominal zero-coupon bonds to partially hedge ICP-linked products, or, as a further step, cross-hedging them with an inflation-correlated stock, remains insufficient as traded assets in an incomplete inflation market and presents distinct and significant limitations. Hence, we can provide a clear signal that there is a structural and pressing need for governments and central treasuries to deepen the market by facilitating the issuance of long-term Inflation-Linked Bonds (ILBs), thereby providing more appropriate instruments to facilitate insurers in offering inflation-linked retirement products.

Hedging strategy	**Valuation of hedgeable component**	**Valuation of systematic inflation risk premium**
Hedging without stock	0.8589355	0.02023798
Hedging with stock	0.8065153	0.07265827

Table 3: Comparison of the value of the hedgeable component and the systematic inflation risk premium for an inflation-linked pure endowment payment at $t^* = 8$: hedging with versus without the traded stock (MVH, $\rho = 0.5$, $\rho_{rS} = -0.2$, and $\rho_{iS} = 0.5$).

Appendix

Proof of Theorem 1.
It is straightforward to verify that $\pi_0^{\mathbb{Q}^f,3}$ is a valuation operator; hence, we omit the proof. We focus instead on the details of the actuarial-consistency and market-consistency properties under both hedging frameworks.

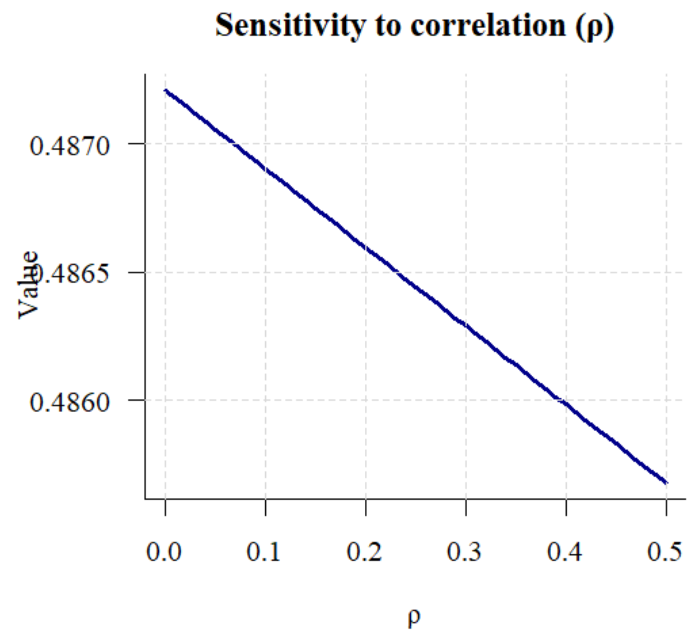

Figure 3: Systematic mortality risk premium of the inflation-linked temporary life annuity as a function of the interest–inflation correlation ρ.

1. Mean-Variance hedging case.
Actuarial-consistent property.
Let $g^n \in \mathcal{H}^a$ be a purely actuarial payoff that does not contain any systematic mortality risk. Consequently, its three-step decomposition does not depend on $(\mathfrak{S}_t)_{t\in\{1,\cdots,T\}}$ and therefore reduces to

$$ {}_nH^1_{VH} = \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]}{n} = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}\Big[\frac{\sum_{j=1}^{n} g_t(\tau_j)}{n}\Big], $$

$$ {}_nH^2_{VH} = \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \Big(g_t(\tau_j) - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]\Big)}{n}, $$

$$ {}_nH^3_{VH} = \sum_{t=1}^{T} \frac{\sum_{j=1}^{n} \Big(\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] - \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]\Big)}{n} = 0. $$

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure associated with the Esscher operator $\pi_0^{\mathbb{Q}^f,n}$. Let $\overline{\mathbb{Q}^f_1}$ and $\overline{\mathbb{Q}^f_2}$ be two arbitrary equivalent martingale prob-

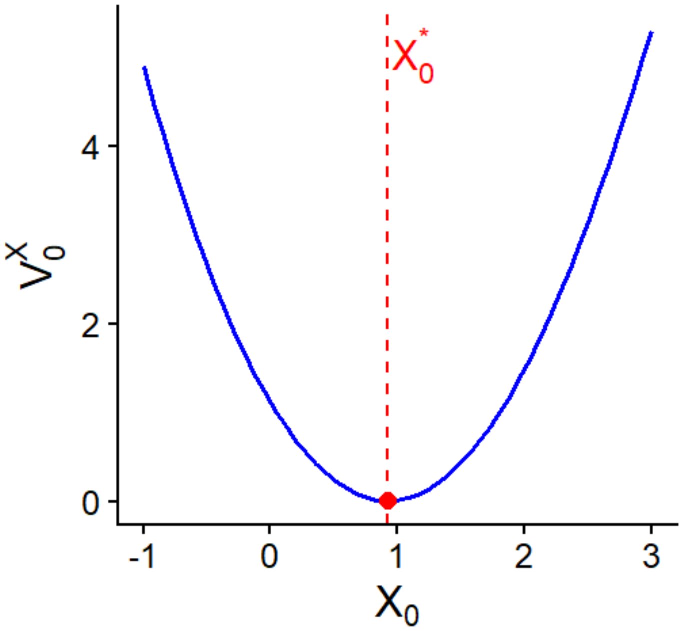

Figure 4: Optimal expected quadratic error at inception as a function of initial wealth X_0 when hedging a pure inflation-linked payment at $t^* = 8$ using an additional correlated traded stock (MVH, $\rho = 0.5$, $\rho_{rS} = -0.2$, and $\rho_{iS} = 0.5$).

ability measures. Applying our three-step valuation, this yields:

$$\pi_0^{\mathbb{Q}^f,3}[g^n] = \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \overline{\mathbb{Q}_1^f}}\left[\sum_{t=1}^{T} e^{-\int_0^t r_s\,\mathrm{d}s} \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[\frac{\sum_{j=1}^n g_t(\tau_j)}{n}\right]\right] + \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a \times \overline{\mathbb{Q}_2^f}}\Bigg[\sum_{t=1}^{T} e^{-\int_0^t r_s \mathrm{d}s} \times$$
$$\frac{\sum_{j=1}^n \Big(g_t(\tau_j) - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_j)]\Big)}{n}\Bigg] + \sum_{t=1}^{T} \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[\frac{\sum_{j=1}^n \Big(g_t(\tau_j) - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_j)]\Big)}{n}\right]} \times P(0,t)$$
$$+ \pi_0^{\mathbb{Q}^f,n}[0].$$

Using the fact that both $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_j)]$ and $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[\frac{\sum_{j=1}^n g_t(\tau_j)}{n}\right]$ are $\mathcal{F}_0$-measurable, together with the independence between the actuarial and financial markets, we

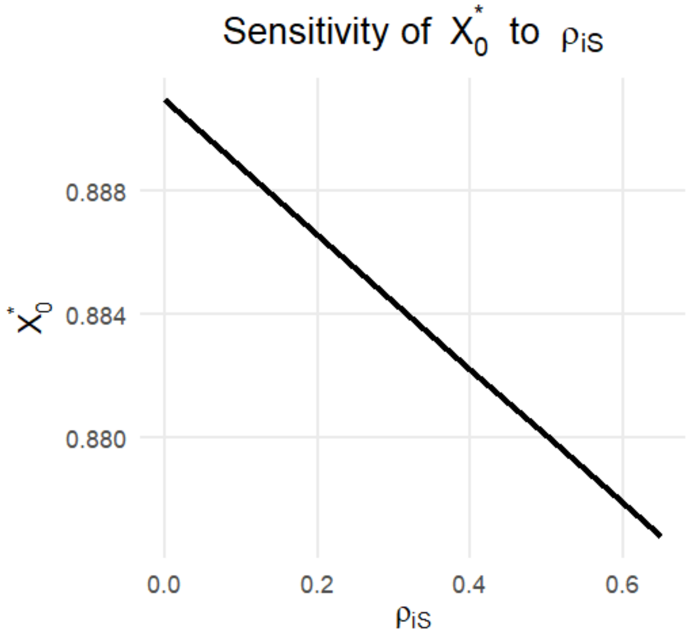

Figure 5: X_0^* as a function of the stock-inflation correlation ρ_{iS} when hedging a pure inflation-linked payment at $t^* = 8$ using an additional correlated traded stock (MVH, $\rho = 0.5$, $\rho_{rS} = -0.2$).

obtain actuarial consistency as follows

$$
\begin{aligned}
\pi_0^{\mathbb{Q}^f,3}[g^n] = &\sum_{t=1}^{T} P(0,t) \times \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[\frac{\sum_{j=1}^n g_t(\tau_j)}{n}\right] + \sum_{t=1}^{T} P(0,t) \times \frac{\sum_{j=1}^n \left(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_j)] - \mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}[g_t(\tau_j)]\right)}{n} + \\
&\sum_{t=1}^{T} \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[\frac{\sum_{j=1}^n \ g_t(\tau_j)}{n}\right]} \times P(0,t) \\
&= \sum_{t=1}^{T} P(0,t) \times \left(\mathbb{E}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[\frac{\sum_{j=1}^n g_t(\tau_j)}{n}\right] + \frac{\alpha}{2}\sqrt{\mathbb{V}_{\mathcal{F}_0}^{\mathbb{P}^a}\left[\frac{\sum_{j=1}^n \ g_t(\tau_j)}{n}\right]}\right) \\
&= \pi_0^{\overline{\mathbb{Q}^f},a}(g^n|\mathcal{F}_0),
\end{aligned}
$$

for any equivalent martingale probability measure $\overline{\mathbb{Q}^f}$.

Market-consistent property.

Let H_n be in $\mathcal{H}$. Let $(x_t)_{t=1}^T$ be a cash-flow sequence such that each x_t is $\mathcal{F}_0$-measurable and payable at time t. The three-step decomposition of $(H^n + \sum_{t=1}^T x_t) = \sum_{t=1}^T \left(x_t + \frac{\sum_{j=1}^n g_t(\tau_j)\, h_t(I_0^t)}{n}\right)$ is given as follows

$$\text{Hedgeable} = \sum_{t=1}^{T} \Big(\frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)]}{n} + \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[x_t] \Big).$$

$$\text{Diversifiable} = \sum_{t=1}^{T} \Bigg[\Big(\frac{\sum_{j=1}^{n} g_t(\tau_j)\, h_t(I_0^t)}{n} + x_t \Big) - \Big(\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0\vee\mathfrak{G}_t\vee\mathcal{F}^f_t} \Big[\frac{\sum_{j=1}^{n} g_t(\tau_j)\, h_t(I_0^t)}{n} + x_t \Big] \Big) \Bigg].$$

$$\text{Systematic} = \sum_{t=1}^{T} \Bigg[\Big(\mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0\vee\mathfrak{G}_t\vee\mathcal{F}^f_t} \Big[\frac{\sum_{j=1}^{n} g_t(\tau_j)\, h_t(I_0^t)}{n} + x_t \Big] \Big) - \Big(\frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]\mathbb{E}^{\mathbb{P}^f}_{\mathcal{F}_0}[h_t(I_0^t)]}{n} + \mathbb{E}^{\mathbb{P}^a\times\mathbb{P}^f}_{\mathcal{F}_0}[x_t] \Big) \Bigg].$$

It can be seen that for each payment date $t \in \{1, \cdots, T\}$, the hedgeable claim x_t naturally cancels out from both the systematic and diversifiable components; thus,

$$\text{Hedgeable} = {}_nH^1_{VH} + \sum_{t=1}^{T} x_t, \quad \text{Diversifiable } = {}_nH^2_{VH}, \quad \text{Systematic} = {}_nH^3_{VH}.$$

Let $\mathbb{Q}^f$ be an equivalent martingale probability measure associated with the Esscher operator $\pi_0^{\mathbb{Q}^f,n}$. Let $\overline{\mathbb{Q}^f_1}$ and $\overline{\mathbb{Q}^f_2}$ be two arbitrary equivalent martingale probability measures. Using the additivity of the $\{1,T\}$-operator $\pi_0^{\overline{\mathbb{Q}^f_1},f}$, market consistency follows:

$$\begin{aligned}
\pi_0^{\mathbb{Q}^f,3}[H^n + \sum_{t=1}^{T} x_t] &= \pi_0^{\overline{\mathbb{Q}^f_1},f}({}_nH^1_{VH}|\mathcal{F}_0) + \pi_0^{\overline{\mathbb{Q}^f_1},f}(\sum_{t=1}^{T} x_t) + \pi_0^{\overline{\mathbb{Q}^f_2},a}({}_nH^2_{VH}|\mathcal{F}_0) + \pi_0^{\mathbb{Q}^f,n}({}_nH^3_{VH}|\mathcal{F}_0) \\
&= \pi_0^{\mathbb{Q}^f,3}[H^n] + \sum_{t=1}^{T} P(0,t)x_t \\
&= \pi_0^{\mathbb{Q}^f,3}[H^n] + \pi_0^{\overline{\mathbb{Q}^f},f}(\sum_{t=1}^{T} x_t),
\end{aligned}$$

for any equivalent martingale probability measure $\overline{\mathbb{Q}^f}$.

2. Quantile hedging case.

The proof relies strictly on the mathematical properties of the quantile operator $Q_p^{\mathbb{P}^f}[\cdot]$, namely translation invariance ($Q_p^{\mathbb{P}^f}[Y + c] = Q_p^{\mathbb{P}^f}[Y] + c$) and positive homogeneity ($Q_p^{\mathbb{P}^f}[aY] = aQ_p^{\mathbb{P}^f}[Y]$ for $a \geq 0$).

Actuarial-consistent property.
Let $g^n \in \mathcal{H}^a$ $(h_t(I_0^t) \equiv 1)$. Since the quantile of a constant is the constant itself $(Q_p^{\mathbb{P}^f}[1] = 1)$, the QH hedgeable base reduces exactly to the MVH proof base:

$$ {}_nH^1_{QH} = \sum_{t=1}^{T} \left(\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] \right) Q_p^{\mathbb{P}^f}[1] = \sum_{t=1}^{T} \left(\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] \right), $$

naturally rendering ${}_nH^3_{QH} = 0$. The diversifiable component remains invariant $({}_nH^2_{QH} = {}_nH^2_{VH})$. Since, the decomposition is therefore structurally identical to the expectation proof-based case, yielding actuarial consistency immediately.
Market-consistent property.
Consider the augmented payoff $H^n + \sum_{t=1}^{T} x_t$. The conditional expectation given the financial filtration of the hybrid cash flow does not affect the deterministic x_t. Subsequently, we apply $Q_p^{\mathbb{P}^f}[\cdot]$ to determine the hedgeable component. Because the averaged actuarial expectation $\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]$ is a non-negative $\mathcal{F}_0$-measurable scalar, we sequentially apply positive homogeneity and translation invariance:

$$ \begin{aligned} \text{Hedgeable} &= \sum_{t=1}^{T} Q_p^{\mathbb{P}^f} \left[\frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)] h_t(I_0^t)}{n} + x_t \right] \\ &= \sum_{t=1}^{T} \left(\frac{\sum_{j=1}^{n} \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]}{n} \times Q_p^{\mathbb{P}^f}[h_t(I_0^t)] + x_t \right) \\ &= {}_nH^1_{QH} + \sum_{t=1}^{T} x_t. \end{aligned} $$

As in the MVH case, x_t perfectly cancels out in the diversifiable and systematic terms. The additivity of the $\{1,T\}$-financial operator $\pi_0^{\overline{\mathbb{Q}^f},f}$ directly yields that,

$$ \pi_0^{\mathbb{Q}^f,3} \left[H^n + \sum_{t=1}^{T} x_t \right] = \pi_0^{\mathbb{Q}^f,3}[H^n] + \pi_0^{\overline{\mathbb{Q}^f},f} \left(\sum_{t=1}^{T} x_t \right), $$

for any equivalent martingale probability measure $\overline{\mathbb{Q}^f}$. Thus, the $\{1,T\}$-Three-step method version based on QH is also a fair valuation.

Details about Lemma 2.
The best estimate value is equal to zero for any equivalent martingale probability measur $\overline{\mathbb{Q}^f_2}$, since, using the independence between the actuarial and financial

markets, we obtain that:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^a\times\overline{\mathbb{Q}_2^f}}_{\mathcal{F}_0}\left[\sum_{t=1}^T e^{-\int_0^t r_s\mathrm{ds}}{}_nH_t^2\right] &= \sum_{t=1}^T \mathbb{E}^{\overline{\mathbb{Q}_2^f}}_{\mathcal{F}_0}[e^{-\int_0^t r_s\mathrm{ds}}h_t(I_0^t)]\times\frac{\sum_{j=1}^n\left(\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]-\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_j)]\right)}{n}\\ &= \sum_{t=1}^T 0\\ &= 0.\end{aligned}$$

For the loading part, by setting, for each $t\in\{1,\cdots,T\}$, $P_t = \frac{\sum_{j=1}^n\left(g_t(\tau_j)-\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0\vee\mathfrak{S}_t}[g_t(\tau_j)]\right)}{n}$, and using the fact that $\mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[P_t]=0$ together with the law of total variance and assumption (2), we obtain the final formula.

Details about Proposition 3.
Starting from Lemma 3 and using that $\mathbb{E}^{\mathbb{Z}^a}_{\mathcal{F}_0}[g_t(\tau_1)] = \mathbb{E}^{\mathbb{P}^a}_{\mathcal{F}_0}[g_t(\tau_1)]\times e^{-s_{0,t}^2\times\theta_t^{(1)}} = {}_tp_x\times e^{-s_{0,t}^2\times\theta_t^{(1)}}$ in the case where $g_t(\tau_1)=\mathbb{1}_{\tau_1>t}$, we obtain the following

$$\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,M}|\mathcal{F}_0) = K_0\sum_{t=1}^T {}_tp_x\times\left(e^{-s_{0,t}^2\times\theta_t^{(1)}}-1\right)\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{\int_0^t(i_s-r_s)\,\mathrm{ds}}\right],$$

$$\pi_0^{\mathbb{Q}^f,n}({}_nH^{3,F}|\mathcal{F}_0) = K_0\sum_{t=1}^T {}_tp_x\times\left(\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{\int_0^t(i_s-r_s)\,\mathrm{ds}}\right]-P(0,t)\,c_t\right).$$

It remains to derive the explicit formula for $\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{\int_0^t(i_s-r_s)\,\mathrm{ds}}\right]$ to complete the pricing formula of the systematic component. Since the expectation of $e^{\int_0^t(i_s-r_s)\,\mathrm{ds}}$ cannot be decomposed into separate expectations involving i_s and r_s, we must derive the joint dynamics of the integral $\int_0^t(i_s-r_s)\mathrm{ds}$ accordingly:

$$\begin{aligned}\int_0^t(i_s-r_s)\mathrm{ds} = &\frac{1-e^{-a_it}}{a_i}(i_0-\frac{\theta_i}{a_i})+\frac{\theta_i}{a_i}t+\int_0^t\frac{\sigma_i}{a_i}(1-e^{-a_i(t-x)})\,dW_i(x)-\\ &\frac{1-e^{-a_rt}}{a_r}(r_0-\frac{\theta_r}{a_r})-\frac{\theta_r}{a_r}t-\rho\int_0^t\frac{\sigma_r}{a_r}(1-e^{-a_r(t-x)})\,dW_i(x)-\\ &\sqrt{1-\rho^2}\int_0^t\frac{\sigma_r}{a_r}(1-e^{-a_r(t-x)})\,dW_r(x),\end{aligned}$$

while computing the variance as follows:

$$\begin{aligned}\mathbb{V}^{\mathbb{Q}^f}_{\mathcal{F}_0}[\int_0^t(i_s-r_s)\mathrm{ds}] = &\mathbb{V}^{\mathbb{Q}^f}_{\mathcal{F}_0}[\int_0^t i_s\mathrm{ds}]+\mathbb{V}^{\mathbb{Q}^f}_{\mathcal{F}_0}[\int_0^t r_s\mathrm{ds}]-\\ &\frac{2\rho\sigma_r\sigma_i}{a_ia_r}\times[t-\frac{1-e^{-a_rt}}{a_r}-\frac{1-e^{-a_it}}{a_i}+\frac{1-e^{-(a_r+a_i)t}}{a_r+a_i}].\end{aligned}$$

Using this development and the expectation of a lognormal distribution, we obtain that

$$\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_0}\left[e^{\int_0^t (i_s - r_s)\,\mathrm{d}s}\right] = P(0,t) \times \mathcal{I}(0,t) \times A_t,$$

with $A_t = e^{-\frac{\rho\sigma_r\sigma_i}{a_i a_r}\times[t - \frac{1-e^{-a_r t}}{a_r} - \frac{1-e^{-a_i t}}{a_i} + \frac{1-e^{-(a_r+a_i)t}}{a_r+a_i}]}$.

Details on the derivation of the coefficients $JY^{(2)}$, $JY^{(1)}$, and $JY^{(0)}$.

The core of solving the Hamilton-Jacobi-Bellman (HJB) equation (45) resides in evaluating the controlled generator $\inf_{\xi_s}\{\mathcal{L}^{\xi_s} V_s^{X_0}\}$. Since the control variable ξ_s exclusively governs the portfolio allocation, it solely affects the drift and diffusion of the wealth process X_s. Therefore, identifying the exact dependence of the HJB equation on ξ_s requires a careful expansion of the instantaneous drift of X_s, its quadratic variation $dX_s dX_s$, and its cross-covariances with the variables r_s and i_s.

Step 1: Expanding the wealth dynamics.

We first simplify the SDE describing the total wealth of the hedging portfolio (eq (42)) by collecting and explicitly separating the terms associated with the three brownian motions $d\widetilde{W}_i(s)$, $d\widetilde{W}_r(s)$, and $d\widetilde{W}_S(s)$:

$$dX_s = \Big[X_s(r_s + \sigma_r B(s,t^*)\Lambda_r) + \xi_s(\mu_S - r_s - \sigma_r B(s,t^*)\Lambda_r)\Big]ds + \Big[\xi_s\sigma_S\rho_{iS} - \sigma_r B(s,t^*)(X_s - \xi_s)\rho\Big]d\widetilde{W}_i(s) + \Big[C_1\sigma_S\xi_s - \sigma_r B(s,t^*)(X_s - \xi_s)\sqrt{1-\rho^2}\Big]d\widetilde{W}_r(s) + \xi_s\sigma_S C_2 d\widetilde{W}_S(s).$$

Step 2: Deriving the instantaneous variances and cross-covariances of X_s.

This, in turn, allows us to derive the different interaction terms involving X_s. Following a series of lengthy calculations, we arrive at

$$dX_s dX_s = \left(\xi_s^2\Sigma^{(2)}_{XX} + \xi_s\Sigma^{(1)}_{XX} + \Sigma^{(0)}_{XX}\right)ds,$$

$$dX_s dr_s = \left(\xi_s\Sigma^{(1)}_{Xr} + \Sigma^{(0)}_{Xr}\right)ds,$$

$$dX_s di_s = \left(\xi_s\Sigma^{(1)}_{Xi} + \Sigma^{(0)}_{Xi}\right)ds.$$

Step 3: Grouping $\mathcal{L}^{\xi_s} V_s^{X_0}$ as a quadratic polynomial in ξ_s.

By rearranging the expression and collecting terms with respect to the powers of

ξ_s, we obtain:

$$\begin{aligned}
\mathcal{L}^{\xi_s} V_s^{X_0} &= \partial_X V_s^{X_0} \Big[X_s(r_s + \sigma_r B(s,t^*)\Lambda_r) + \xi_s(\mu_S - r_s - \sigma_r B(s,t^*)\Lambda_r) \Big] + \\
&\frac{1}{2}\partial_X^2 V_s^{X_0} \left(\xi_s^2 \Sigma_{XX}^{(2)} + \xi_s \Sigma_{XX}^{(1)} + \Sigma_{XX}^{(0)} \right) + \partial_{Xr}^2 V_s^{X_0} \left(\xi_s \Sigma_{Xr}^{(1)} + \Sigma_{Xr}^{(0)} \right) + \\
&\partial_{Xi}^2 V_s^{X_0} \left(\xi_s \Sigma_{Xi}^{(1)} + \Sigma_{Xi}^{(0)} \right) \\
&= \frac{1}{2}\Sigma_{XX}^{(2)} \partial_X^2 V_s^{X_0} \xi_s^2 + \Big[(\mu_S - r_s - \sigma_r B(s,t^*)\Lambda_r) \partial_X V_s^{X_0} + \frac{1}{2}\Sigma_{XX}^{(1)} \partial_X^2 V_s^{X_0} + \\
&\Sigma_{Xr}^{(1)} \partial_{Xr}^2 V_s^{X_0} + \Sigma_{Xi}^{(1)} \partial_{Xi}^2 V_s^{X_0} \Big] \xi_s + \Big[X_s[r_s + \sigma_r B(s,t^*)\Lambda_r] \partial_X V_s^{X_0} + \frac{1}{2}\Sigma_{XX}^{(0)} \partial_X^2 V_s^{X_0} \\
&+ \Sigma_{Xr}^{(0)} \partial_{Xr}^2 V_s^{X_0} + \Sigma_{Xi}^{(0)} \partial_{Xi}^2 V_s^{X_0} \Big] \\
&= JY^{(2)} \xi_s^2 + JY^{(1)} \xi_s + JY^{(0)}.
\end{aligned}$$

This algebraic regrouping perfectly identifies the coefficients $JY^{(2)}$, $JY^{(1)}$, and $JY^{(0)}$.

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