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Hedging and Pricing Longevity Risks: Actuarial Challenges and Securitization Solutions

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To my parents

Acknowledgments

A thesis is not just about research and science. Behind a thesis, there are joy and ambition, but also tears and despair...

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Chapter 1

Introduction

1.1 Modelling and managing longevity risk

Different demographic studies have shown that human life expectancy has significantly increased worldwide over the past few decades. According to the Office for National Statistics (ONS), the global average life expectancy increased by 5.5 years between 2000 and 2016, which is the fastest increase since the 1960s. The main explanations for this trend are the significant increase in infants' longevity that characterized the early 1900s and the advances in medicine and medical technology (the use of antibiotics and vaccines etc.). In the European Union, life expectancy at birth has also increased by 2.81% over the last 10 years.

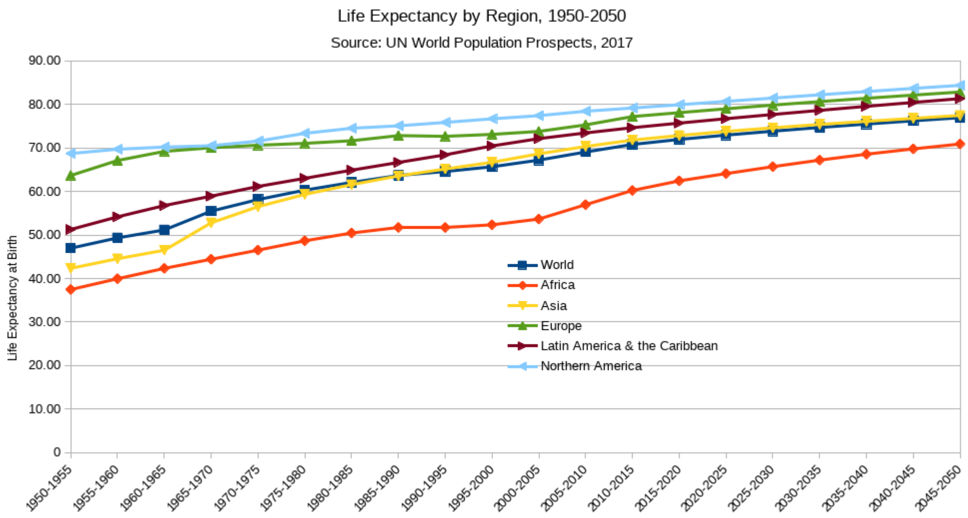

Figure 1.1: Life expectancy at birth in the world

This increase in longevity is impacting most developed countries, and a study made by the Belgian Statistical Office (Statbel), shows that Belgium is not an isolated case as we can see in the following figure:

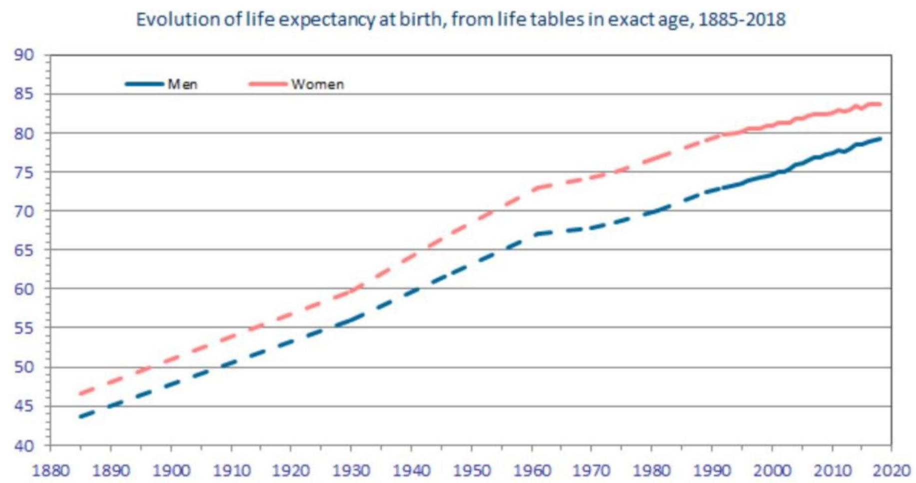

Figure 1.2: Life expectancy at birth in Belgium, Statbel

Clearly, mortality is improving from year to year, and this trend will likely continue increasing in the future years.

Although a higher longevity is a sign of progress, it is also one of the major challenges of the twenty-first century and a considerable financial concern for most life insurers, annuity providers and governments. Consequently, even if this longevity risk has not been a priority for academics due to its long-term aspect, they have now become more interested in the issue.

Currently, companies are attempting to address the problem and develop new innovative ideas to cover this longevity risk, especially with the Solvency II directives. The first challenge is to correctly model the evolution of longevity to obtain accurate future predictions. The regulator requires the integration of mortality risk analysis into stochastic valuation models, and we know from the literature that mortality can be more accurately measured by utilizing stochastic models (see Cairns et al. [6]), given that such models are better able to capture the uncertainty inherent in the problem. Several stochastic mortality models have been proposed by researchers. The leading statistical model used for mortality in the existing literature is the discrete time model developed by Lee and Carter [14] (time series technique).

The choice of this particular model was recommended by two U.S. Social Security Technical Advisory Panels. An alternative modelling technique is based on the use of continuous-time stochastic processes for describing the force of mortality, and Milevsky and Promislow [22] were the first to propose a stochastic force of mortality model. Since then, several other stochastic models have been proposed, using the analogy between mortality and interest rate (Dahl [9], Biffis [5], Denuit and Devolder [10], Luciano and Vigna [21], Schrager [23]). Some of the suggested models were inspired by the natural analogy between mortality intensities, and interest rates or default rates in credit risk. In particular, by using affine processes for intensity of mortality, the survival function for an individual can be given in a closed form, which is extremely useful for pricing mortality/longevity-linked securities (exactly like zero coupon bonds in finance).

Because the longevity risk affects the pricing of life products sold by insurers, it is important to distinguish between two fundamentally different sources of risk: unsystematic and systematic mortality risk. The unsystematic risk refers to the risk associated with the random development of an insurance portfolio: even if mortality rates are known, the actual number of deaths remains uncertain. But according to the law of large numbers, this risk can be alleviated by increasing the portfolio, and if the insurance portfolio becomes infinitely large, the unsystematic mortality risk is eliminated. Thus, the unsystematic mortality risk is diversifiable. The systematic risk represents the uncertainty linked to the global future mortality. That is, it concerns the whole population and this cannot be diversified. There are only a few options available to hedge this systematic risk (Cairns et al [7] and Li and Haberman [16]). One is reinsurance, in which the risk is transferred to a reinsurer who is willing to take it on. However, this option has some serious limitations: on the one hand, the cost of this customized protection is high, and on the other hand, the capacity of reinsurers is limited because longevity risk cannot be diversified. Consequently, only a few reinsurers accept taking on longevity risk. The second option is to transfer this longevity risk to the financial market through securitization. In recent years, some longevity derivatives have arisen. These products are based on mortality/longevity rates, similar to those traded in the financial market. Blake and Burrows [3] proposed the use of survivor bonds to transfer the longevity risk to the capital markets. Subsequently, many authors have focused on longevity derivatives. The following table reports some of the longevity-linked products suggested in the literature:

Longevity products	**Mechanism**
Longevity bond (Blake and Burrows [3], Dowd [11])	Coupons are based on survivor index (on the percentage of the reference population who are still alive).
Longevity swap (Dowd [11], Dowd et al. [12])	Consists on exchanging a serie of future cash flows corresponding to the realized survival rate (based on the reference population who are still alive) of a given population, in return for fixed survival rates agreed at the inception of the contract.
Survivor option (Dowd [11])	The call holder receives a payoff equals to $Max({}_T\hat{p}_x - {}_Tp_x, 0)$ at maturity, and the the pull holder receives a payoff equals to $Max({}_Tp_x - {}_T\hat{p}_x, 0)$ where ${}_T\hat{q}_x$ is the survival fixed rate and ${}_Tp_x$ is the realized survival rate.
q-forward (Coughlan et al. [8])	is an agreement to exchange at a future date T (the maturity of the contract), an amount equal to the realized mortality rate of a given population cohort (floating leg), in return for a fixed survival rate agreed at the inception of the contract (fixed rate payment).
Mortality option (Cairns et al. [7])	The payoffs are similar to those of the Survivir option, with the survival rate being replaced by the realized mortality rate
S-forward (LLMA [20])	This contract is similar to the q-forward derivative, but the realized mortality rate is replaced by the realized survival rate.
Longevity nominal choosing swaption (Bensusan et al. [1])	It is a tool for transferring interest rate risk from insurers to banks: the insurer exchanges a fixed interest rate for a floating interest rate, according to its expectations on the evolution of its policyholders' future mortality and on interest rates.

Table 1.1: Some mortality/longevity-linked securities proposed in literature

Even if these derivatives can be constructed in theory, their practical implementation turns out to be problematic. In fact, just a handful of longevity/mortality-linked securities have been announced or launched in the capital market. In the following we give some of these products:

- **Longevity bonds:** two types of longevity bonds have been issued in the market: coupon-based bonds that pay decreasing coupons based on a cohort survival index (EIB/BNP bonds), and principal at risk longevity bonds that pay coupons, and whose principal depends on an index measuring the difference between two mortality rates (Kortis/Swiss Re)[4].

- **Mortality bonds**: inspired by Cat Bonds, bonds issued so that insurers and reinsurers can be protected against natural disasters. If the loss occurs under predefined conditions, investors lose some or all of the return and even the nominal value of the bond. This way insurers partially protect themselves against the risk of a sudden increase in mortality by sharing it with other investors. Mortality Bonds work in the same way: the principal amount of the bond is linked to a mortality rate, and if the mortality rate exceeds the expected threshold, investors will lose part or all of their investment. The first mortality bonds were released by Swiss Re in 2003 (VITA1), and then other similar bonds (VITA2 and VITA3) were launched later (2004 and 2007)

- **Longevity swaps:** longevity swaps can mainly take two forms, depending on whether they provide a custom-made cover or a standardized index-based cover [2]. In 2008 JP Morgan contracts a customized swap with Canada life, and a standardized swap with Lucida.

- **Longevity options:** call option with underlying indexed to a longevity product. Deutsch Bank has launched in 2013 a 10-year maturity out-of-the-money option based on a 5-year cohort of men and women aged between 55 and 79 (population of England, Wales and the Netherlands). It is an OTC product that allows its holders to make gains if the actual longevity rates are higher than those of the underlying asset.

Unlike the classical financial derivatives, these securities were not sufficiently attractive for investors, mainly due to different reasons depending on the type of the security. For instance, the standardized products are based on transparent indexes of a population that differ from those utilized by the insurer. Indeed, unlike professional reinsurers, financial investors are not experts in insurance risks and consequently need public indexes to avoid any manipulation. Therefore, these standardized products leave the insurer uncovered against the basis risk, which can be very high. Customized securities eliminate this problem since the longevity risk is completely covered, however, this full protection is usually expensive.

Moreover, pricing could be the main reason these linked securities have failed to survive. At present, the longevity market remains immature and incomplete, with an evident lack of liquidity. As a result, we lack data for pricing purposes, which leads to the absence of mark-to-market prices for the longevity derivatives. Different pricing methods have been proposed in the literature, and the majority of these techniques were inspired by the classical pricing methods used in the financial market, such as risk-neutral, Sharpe and Wang approaches.

However, these methods require market data to calibrate the risk parameter (market price of risk, Sharpe ratio and Wang parameter).

In addition, a good pricing method should be consistent with the directives of Solvency II, as these products are primarily used by insurers or reinsurers. The main concern of Solvency II is to protect policyholders, and keep the insurer solvent by requiring the hold of a solvency capital requirement (SCR) based on the risks that the insurer faces. This SCR is defined as the value-at-risk (VaR) of the variation of the own funds over a one-year period, subject to a confidence level of 99.5%. According to the regulator, insurance liabilities are computed as the sum of a best estimate plus a risk margin (RM). This latter is determined by the cost of capital approach based on future SCRs.

1.2 Purpose and contribution of the thesis

The main purpose of this thesis is to propose a longevity hedging and pricing framework in line with the directives of Solvency II, in a stochastic continuous-time environment. In each chapter we focus on a specific issue.

As mentioned previously, the first challenge is to properly describe the evolution of longevity and provide accurate predictions. Luciano and Vigna [21] have studied the appropriateness of affine processes, such as the Ornstein-Uhlenbeck, Feller and Vasiçek processes, in modelling mortality intensities. Their approach focused on analyzing and comparing the fitting of the survival curve provided by these models on historical data. The authors report that non-mean reverting affine processes are more accurate for modelling the force of mortality. These results have been based on reversion models with a fixed target, which is not in line with the increasing trend of mortality intensity. Our contribution in this thesis is to re-examine this comparison by considering fixed-target mean-reverting models and mean-reverting models with a moving target. More specifically, we study in **Chapter 2** the impact of adding a time-dependent long-term mean element to the Ornstein-Uhlenbeck and Feller processes. Using different calibration and forecasting tests, we show that adding a time-dependent long-term exponential target to the selected processes seems to be more suitable to describe mortality of individuals.

Chapter 3 is devoted to elucidating the pricing framework of S-forward and survival swap (S-swap) contracts. Some authors have been inspired by the pricing methods used in finance such as Sharpe, risk-neutral and Wang approaches. However, due to the illiquidity of the longevity market, the longevity risk premium cannot be observed, and therefore the calibration of these longevity parameters could be a source of discussion. Levantesi and Menzietti [15] have provided an approach that makes it possible to avoid this problem: they have proposed in a discrete-time environment a pricing method for S-forward contracts inspired by the directive of Solvency II. That is, the price is computed as the sum of the best estimate and the RM, which is determined by the cost of capital approach based on the SCR.

This pricing method is also based on one parameter: the cost of capital percentage. But in this case, this parameter does not need to be estimated because of its regulatory value in Solvency II (6% at present). Nevertheless, their framework is not entirely consistent with the one-year view of Solvency II since the VaR is computed over a multi-year horizon. In our framework, we propose a pricing approach more in line with Solvency II and based on successive one-year SCR using stochastic affine processes for mortality (the Hull and White and the CIR extended models selected in the previous Chapter). We also study the consistency between the Cost of Capital approach and three important classical pricing methods used in finance: risk-neutral, Wang transform and Sharpe ratio.

Due to the eventual mismatch between the reference population and the portfolio to be hedged, these proposed longevity securities allow only for immunizing the portfolio against a portion of the systemic longevity risk, leaving the insurer exposed to the basis risk. In the longevity pricing literature, the general tendency was to ignore this basis risk, but this mismatch can be highly important due to different factors (Li et al. [17]). Recently, some few works focusing on this risk have been published (for instance Haberman et al. [13] and Villegas et al. [24]). These frameworks were based on the spread between a two-population mortality model for assessing demographic basis risk.

In **Chapter 4** we present different possibilities for describing the mortality intensity of the insurer's portfolio by modifying the trend, the volatility, or both in the force of mortality process, under the Hull and White model. Our second important contribution consists in evaluating the basis risk through a survival exchange forward (S-exchange) instrument, for which we provide fair prices using the Cost of Capital approach and the three classical pricing approaches considered in Chapter 3. Moreover, we provide a comparison between different hedging options depending on the insurer's risk aversion.

Up to this point of the thesis, we have considered the case of an insurer using one S-forward contract per cohort to hedge its longevity risk. Namely, if we assume that the insurer holds a portfolio of policyholders belonging to n cohorts, n individual S-forwards contracts related to these n cohorts should be purchased. This same insurer can also cover the longevity risk by entering one global S-forward for the n generations, if such a product is available. However, in this second case the eventual correlation between cohorts' mortality should be taken into account. To do so, we need to develop a multi-population model that allows for the incorporation of dependence factors.

Some multi-population models that have been proposed in literature are based on the Lee-Carter model, to which an additional common factor between the multiple populations is added (see for instance the models presented by Li and Lee [19], Haberman et al. [13], and Li et al. [18]). Our first contribution in this respect is presented in **Chapter 5**, where we propose a multi-generation approach based on the Hull and White model. The idea consists in incorporating risk factors, which allows for the introduction of the inter-generational correlation between generations with different levels. We have also provided the global S-forward prices in closed form under the Cost of Capital, risk-neutral and Sharpe approaches for different correlation levels. In addition, to measure the impact of including this dependence, we have compared the S-forward prices both with and without taking into account the correlation between generations.

1.3 Published material

In this thesis, we have replicated the text of each previously submitted or published paper to facilitate an independent reading of the different chapters. **Chapter 2** has been published in the European Actuarial Journal (EAJ 2020), **Chapter 3** has been published in Risks (2019), and **Chapter 5** has been published in Risks (2020). **Chapter 4** have been submitted.

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Chapter 2

Mean reversion in stochastic mortality: why and how ?[1]

Abstract

Life insurance companies use stochastic models to forecast mortality. According to the literature, non-mean reversion models are more suitable for mortality modelling than mean reversion models with a fixed long-term target. In this paper, we adopt stochastic affine processes for the force of mortality and study the impact of adding a time-dependent long-term mean reversion level to two non-mean-reverting processes. We calibrate the models to different generations of the Belgian population and assess these models' abilities to predict mortality using different statistical methodologies. The backtest shows that the survival curves provided by the mean-reverting processes are closer to reality. Thus, we conclude that incorporating a time-dependent target into these considered models improves their performance significantly.

2.1 Introduction

For life insurance companies and pension funds, mortality risk has been traditionally estimated using a simplified approach to reality, based on deterministic life tables projected from historical mortality rates and added safety margins. While this traditional method may be prudent for risk management, it does not account for uncertainty related to mortality evolution, since a deterministic mortality scenario is applied to risk management and pricing. Ultimately, this approach may lead to a flawed evaluation of life insurance products. Moreover, under the Solvency II Framework Directive, solvency capital must be held to avoid the risk of ruin. This capital is related to the risk of deviation of future mortality based on life expectancy and, therefore, should use stochastic projections. Thus, stochastic models for mortality are needed to measure the trend of improvement in mortality over the next few decades, along with associated uncertainty in the projections. By modelling mortality as a stochastic process, reserves can expand to include uncertainty associated with future development of mortality intensity. This

[1]Zeddouk, F. and Devolder, P. Mean reversion in stochastic mortality: why and how? (2020), European Actuarial Journal, 10, 499–525.

method should allow for a more accurate assessment of future commitments that account for the systematic risk of mortality. Moreover, the development of mortality and longevity-related securities requires the use of stochastic models to accurately price financial instruments according to demographic risks.

To better predict longevity risk, research and development efforts have recently focused on stochastic mortality models. The Lee-Carter model [12], was the first model to consider increased life expectancy trends in age mortality dynamics and has been used for stochastic forecasts of the U.S. Social Security system and other aspects of the U.S. federal budget (see Congressional Budget Office of the United States [2]). Even though data are typically reported at discrete intervals (e.g. annually), other models have considered the development of mortality rates in continuous time. Since the 2000s, with requirements of Solvency II for the assessment of mortality risk, both insurance companies and academic researchers have become increasingly interested in continuous-time stochastic mortality models, and various continuous-time stochastic mortality models have been proposed in the literature. Our framework focuses on mortality and modelling the survival function of individuals with one-factor short-rate models which are used in finance to model changes in the term structure of interest rates. The similarities between mortality and interest rate dynamics were exploited by Milevsky and Promislow [14], Cairns et al.[3], Biffis [2], Dahl [6], Schrager [16], and Denuit and Devolder [7] to model the force of mortality, using tools and techniques developed in interest rate modelling.

In this context, the eventual mean-reverting property of the model is essential. The analogy between interest rates and the force of mortality models motivates the issue of mean reversion models. The mean reversion concept is highly intuitive and combines long-term convergence to an equilibrium level (the trend of the process), with a stochastic noise (the diffusion part of the model). Milevsky and Promislow [14] were the first to propose a stochastic force of mortality model. Several stochastic affine models facilitate mean-reverting characteristics; however, according to Cairns et al. [4], affine stochastic models should not incorporate mean-reverting elements. Moreover, Luciano and Vigna [13] state that non-mean-reverting affine processes are better for modelling the force of mortality. These conclusions were based on fixed-target reversion models (e.g., the Vasiçek model).

We revisit this comparison by considering fixed-target mean-reverting models as well as moving-target mean-reverting models. Not surprisingly, a fixed-time target is inconvenient in representing the evolution of time on mortality intensity, which increases with age. Thus, examining a mean-reverting model with an increasing target is more relevant, especially with an exponential target consistent with the classical Gompertz model. Hence, we explore the impact of adding a time-dependent long-term mean element to the Ornstein-Uhlenbeck (OU) and Feller processes. In this paper, we show that introducing a moving target in the mean reversion process strongly improves the performance of these models when compared to non-mean-reverting models.

[2]Congressional Budget Office Of The United States, "Long-Term Budgetary Pressures and Policy Options." (1998))

This paper is organised as follows. Section 2.2 offers a brief review of the longevity risk, and presents the considered mean and non-mean reversion processes. Section 2.3 provides an in-depth comparison between these models using statistical methodologies, including calibration, mortality trend estimation, and backtesting. Section 2.4 presents a multicohort vision, and Section 2.5 concludes this paper.

2.2 Models presentation

We model mortality using a single explanatory variable: the intensity of mortality μ_x at age x defined by:

$$\mu_x = \lim_{\triangle \to 0} \frac{\mathbb{P}[x < \bar{T} \leq x + \triangle \mid \bar{T} > x]}{\triangle}, \tag{2.1}$$

where $\overline{T}$ is a random variable representing the future lifespan of an individual.
Within our framework, we model the intensity of mortality $\mu_x(t)$ as a continuous stochastic process, defined on the real probability space $(\Omega, F, \mathbb{P})$ and adapted to its natural filtration $(F_t)_{0 \leq t \leq T}$. Thus, μ_{x+t} is written as: $\mu_x(t, \omega)$, with ω an element of the probability space $(\Omega, F, \mathbb{P})$. Therefore, the starting point of our modelling describes the dynamics of the mortality through the specification of a stochastic differential equation (SDE). Afterwards, we focus on the survival process, which represents the index at time t of an individual initially aged x, and alive at time t to survive $T - t$ years more. For one individual, the survival process is defined as follows:

$$S(x, t, T, \omega) = e^{-\int_t^T \mu_x(u,\omega) du}. \tag{2.2}$$

At time t, the survival probability is obtained by accounting for the expectation of the survival process (2.2), i.e.:

$${}_{T-t}p_{x+t} = E_{\mathbb{P}}[S(x, t, T, \omega) \mid \mathcal{F}_t]. \tag{2.3}$$

Several stochastic mortality models have been proposed in the literature. According to Luciano and Vigna [13], constant mean reversion models are not suitable to describe the mortality of individuals; thus, non-mean reversion models work better.
While we aim to study two mean reversion models, we first incorporate a moving mean-reverting target, inspired by the Gompertz model, and compare the performances between a mean reversion model with a constant mean level and a non-mean reversion model. Therefore, we consider the following five affine stochastic processes:

- Ornstein-Uhlenbeck (OU)
- Feller
- Vasiçek
- Hull & White (HW)
- Cox-Ingersoll-Ross extended (CIR)

For simplicity, we write $\mu_x(t, \omega) = \mu_x(t)$ to represent the intensity of mortality of an individual having the age $x + t$ at time t, and $w(t, \omega) = w(t)$, a standard Brownian motion.

The advantage of considering affine models for modelling $\mu_x(t)$ is the ease of finding an explicit expression for the survival probability (2.3). Therefore, for affine processes under technical conditions (see Duffie et al. [8]), the properties of the classical stochastic process allow to write:

$$\begin{aligned} {}_{T-t}p_{x+t} &= E_{\mathbb{P}}[S(x,t,T,\omega) \mid \mathcal{F}_t] \\ &= e^{\alpha(t,T)-\beta(t,T)\mu_x(t)}, \end{aligned} \tag{2.4}$$

where $\alpha(.)$ and $\beta(.)$ are two deterministic functions and solutions of the Riccati generalized ordinary differential equations that can be determined numerically, and in some cases analytically (see for instance Duffie et al. [9]).

2.2.1 Non-mean reversion models

Ornstein-Uhlenbeck model

The force of mortality is assumed to satisfy the SDE:

$$d\mu_x(t) = a\mu_x(t)dt + \sigma dw(t), \tag{2.5}$$

with $a > 0$, $\sigma > 0$ and $w(t)$ a standard Brownian motion.
As the OU model is Gaussian, we can easily find the expression of the survival probability by determining its distribution. We can also use the affine framework that gives the following system of ODEs for $\alpha(t,T)$ and $\beta(t,T)$ (see for example [13]):

$$\begin{cases} \frac{d\alpha(t,T)}{dt} = \frac{1}{2}\sigma^2\beta(t,T)^2 \\ \frac{d\beta(t,T)}{dt} = a\beta(t,T) - 1 \end{cases} \tag{2.6}$$

with boundary conditions $\alpha(T,T) = \beta(T,T) = 0$.
By solving the system (2.6) we find:

$$\begin{cases} \alpha(t,T) = \frac{\sigma^2(T-t)}{2a^2} + \frac{\sigma^2}{4\sigma^3}e^{2a(T-t)} - \frac{\sigma^2}{a^3}e^{a(\text{T-t})} + \frac{3\sigma^2}{4a^3} \\ \beta(t,T) = \frac{1}{a}(1 - e^{a(T-t)}). \end{cases} \tag{2.7}$$

The solution $\mu_x(t)$ of the SDE (2.5) is:
$(0 \leq s \leq t \leq T)$

$$\mu_x(t) = \mu_x(s)e^{a(t-s)} + \sigma \int_s^t e^{a(t-u)}dw(u). \tag{2.8}$$

In this model, the force of mortality has always a strictly positive probability of becoming negative. This probability is given by:

$$\mathbb{P}[\mu_x(t) < 0 \mid \mathcal{F}_s] = \Phi\left(-\frac{\mu_x(s)e^{a(t-s)}}{\sigma\sqrt{\frac{e^{2a(t-s)}-1}{2a}}}\right) = \Phi(\zeta(a,\sigma)), \tag{2.9}$$

where $\Phi(.)$ is the distribution function of a standard normal random variable.

The function $\zeta(a,\sigma)$ is an increasing function of σ and a decreasing function of a. With this model, we can obtain negative values of $\mu_x(t)$, which is a disadvantage, but in practice, this probability tends to be very small, since the values of σ and a are respectively small and high enough. This is confirmed by calibration results in our simulation.

Feller model

The SDE of the Feller process describing the force of mortality dynamics is given by:

$$d\mu_x(t) = a\mu_x(t)dt + \sigma\sqrt{\mu_x(t)}dw(t), \tag{2.10}$$

with $a > 0$, $\sigma > 0$ and $w(t)$ a standard Brownian motion.
Compared to the OU process, the Feller process avoids negative values for the force of mortality but can reach zero with a positive probability.
In a Feller setting, $\alpha(.)$ and $\beta(.)$ are determined using the Feynman-Kac representation theorem [11].
The survival probability (2.4) is the solution of the following differential equation (PDE):

$$\begin{cases} \frac{\partial p}{\partial t} + a\mu_x\frac{\partial p}{\partial \mu_x} + \frac{1}{2}\sigma^2\mu_x\frac{\partial^2 p}{\partial \mu_x^2} - \mu_x p = 0 \\ {}_0p_{x+t} = 1. \end{cases} \tag{2.11}$$

By substituting (2.4) in the PDE (2.11), we obtain a system of Riccati's ODE that $\alpha(t,T)$ and $\beta(t,T)$ must verify:

$$\begin{cases} \frac{d\alpha(t,T)}{dt} = 0 \\ \frac{d\beta(t,T)}{ds} = 1 - a\beta(t,T) - \frac{1}{2}\sigma^2\beta^2(t,T) \\ \alpha(T,T) = \beta(T,T) = 0. \end{cases} \tag{2.12}$$

The unique solution of system (2.12) is given by:

$$\begin{cases} \alpha(t,T) = 0 \\ \beta(t,T) = \frac{1-e^{(b(T-t))}}{c+d\cdot e^{(b(T-t))}} \\ \beta(T,T) = 0, \end{cases} \tag{2.13}$$

with $b = \sqrt{a^2 + 2\sigma^2}$, $c = \frac{b+a}{2}$ and $d = \frac{b-a}{2}$.

2.2.2 Mean reversion models

Vasiçek model

The Vasiçek model of mortality is given by (see for instance [13]):

$$d\mu_x(t) = (a - b\mu_x(t))dt + \sigma dw(t), \tag{2.14}$$

with $a > 0$, $b > 0$, $\sigma > 0$, $w(t)$ a standard Brownian motion.
The drift of this model presents a mean reversion to a constant target $\frac{a}{b}$, realised with a speed b.

As the Vasiçek model is affine, the survival probability takes the form:

$$_{T-t}p_{x+t} = e^{\alpha(t,T)-\beta(t,T)\mu_x(t)}, \tag{2.15}$$

$\alpha(t,T)$ and $\beta(t,T)$ are given by (see for instance [13]):

$$\begin{cases} \alpha(t,T) & = (\frac{a}{b} - \frac{\sigma^2}{2b^2})(\frac{1}{b}(1-\exp(-b(T-t)) - T + t) \\ & \quad -\frac{\sigma^2}{4b^3}(1-\exp(-b(T-t))^2) \\ & \\ \beta(t,T) & = \frac{1}{b}(1-e^{-b(T-t)}). \end{cases} \tag{2.16}$$

Luciano and Vigna [13] considered the OU, Feller, and Vasiçek models and demonstrated that non-mean-reverting models outperform mean-reverting models. This finding leads us to question the idea of mean reversion. Nevertheless, the poor performance of the Vasiçek model is not surprising, not due to its mean-reverting property, but rather because of its fixed-target $\frac{a}{b}$. Indeed, contrary to interest rates models, the force of mortality has, in essence, an increasing trend with age and thus with time in equation (2.14). Therefore, it seems essential to recheck the performance of the mean-reverting models with an increasing target. A natural candidate in this perspective is to use as increasing target an exponential form, in line with the classical Gompertz model.

HW model

In order to assess the impact of using a moving target instead of a fixed target, we consider the Hull and White model [10], a time-dependent extension of the Vasiçek model, where the force of mortality tends to return to a moving-target $\frac{\xi(t)}{b}$ with restoring force b. The expression of the model is as follows:

$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma dw(t), \tag{2.17}$$

with $b > 0$, $\sigma > 0$, $w(t)$ a standard Brownian motion, and $\xi(t)$ a deterministic function. The form of the target is naturally given by an exponential function, with the same kind of evolution as the trend of the non-mean-reverting models:

$$\xi(t) = Ae^{Bt}, \tag{2.18}$$

with A and B two strictly positive constants. $\xi(t)$ corresponds to the classical function of Gompertz, which expresses that the force of mortality increases exponentially with age, where $\frac{A}{b}$ is the baseline mortality at age x, and B is the senescent component.[3]

[3] The Gompertz model is an exponential process given by:

$$\begin{aligned} \mu_x(t) &= A' e^{B(x+t)} \\ &= A' e^{Bx} e^{Bt} \\ &= \mu_x(0) e^{Bt}. \end{aligned}$$

The Hull and White model is then given by:

$$d\mu_x(t) = b(\frac{A}{b}e^{Bt} - \mu_x(t))dt + \sigma dw(t). \tag{2.19}$$

If we consider that $\frac{A}{b} = \mu_x(0)$, this equation becomes:

$$d\mu_x(t) = b(\mu_x(0)e^{Bt} - \mu_x(t))dt + \sigma dw(t). \tag{2.20}$$

Since the Hull and White model is Gaussian, we can easily find the expression of the survival probability by determining its distribution. We can also use the affine framework that gives the following system of ODEs for $\alpha(t,T)$ and $\beta(t,T)$:

$$\begin{cases} \frac{d\alpha(t,T)}{dt} = \beta(t,T)\xi(t) - \frac{1}{2}\sigma^2\beta^2 \\ \frac{d\beta(t,T)}{dt} = \beta(t,T)b - 1, \end{cases} \tag{2.21}$$

with the boundary conditions $\alpha(T,T) = \beta(T,T) = 0$.
$\alpha(t,T)$ and $\beta(t,T)$ are given by:

$$\begin{cases} \alpha(t,T) & = \frac{A}{b}[e^{-bT}\frac{e^{(B+b)T}-e^{(B+b)t}}{B+b} - \frac{e^{BT}-e^{Bt}}{B}] - \frac{\sigma^2}{2b^2}[\frac{1}{b}(1-e^{-b(T-t)}) - T + t] \\ & \quad -\frac{\sigma^2}{4b^3}(1-e^{-b(T-t)})^2 \\ & \\ \beta(t,T) & = \frac{1}{b}(1-e^{-b(T-t)}). \end{cases} \tag{2.22}$$

By short calculations we find that the force of mortality in the HW model is given by: $(0 \le s \le t \le T)$

$$\mu_x(t) = \mu_x(s)e^{-b(t-s)} + \frac{A}{b+B}(e^{Bt} - e^{Bs-b(t-s)}) + \sigma e^{-bt}\int_s^t e^{bu}dw(u), \tag{2.23}$$

and, conditionally on $\mathcal{F}_s$, normally distributed with the following expectation and variance:

$$\begin{cases} E_{\mathbb{P}}[\mu_x(t) \mid \mathcal{F}_s] = \mu_x(s)e^{-b(t-s)} + \frac{A}{b+B}(e^{Bt} - e^{Bs-b(t-s)}) \\ \\ Var_{\mathbb{P}}[\mu_x(t) \mid \mathcal{F}_s] = \frac{\sigma^2}{2b}(1-e^{-2b(t-s)}). \end{cases} \tag{2.24}$$

As in the OU model, for each time t, the intensity of mortality can be negative with a positive probability:

$$\Phi\left(-\frac{\mu_x(s)e^{-b(t-s)} + \frac{A}{b+B}(e^{(Bt} - e^{Bs-b(t-s)})}{\sqrt{\frac{\sigma^2}{2b}(1-e^{-2b(t-s)})}} \right) = \Phi(\psi(A,B,b,\sigma)), \tag{2.25}$$

which is usually negligible in practice.

Cox-Ingersoll-Ross (CIR) extended model

The CIR model was developed by Cox et al. [5] and is known as square root process. We consider here an extension of the CIR by considering a time-dependent long-term mean. The CIR extended process for the force of mortality is characterised by:

$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dw(t), \tag{2.26}$$

with $b > 0$, $\sigma > 0$, $w(t)$ a standard Brownian motion.
To compare the two models, we choose the same expression for $\xi(t)$ as for the model of Hull and White (Gompertz model).

Unlike the model of Hull and White, this process has the advantage of ensuring the positivity of the intensity of mortality, and due to the presence of the square root, the intensity process can never be negative. The intensity process can reach zero, but in practice, this probability is negligible. In our framework, we prevent the intensity process from reaching 0 by imposing the following sufficient condition (e.g., Mishura and Posashkova [15]): $A > \frac{\sigma^2}{2}$.
Nevertheless, this improvement has a price in the sense that the intensity of mortality cannot be handled analytically.
The CIR model is affine but not Gaussian, (2.26) is a complicated stochastic process with a distribution difficult to evaluate. We use the Feynman-Kac representation to determine the survival function. By short calculations we get:

$$\begin{cases} \frac{d\alpha(t,T)}{dt} = \xi(t)\beta(t,T) \\ \frac{d\beta(t,T)}{dt} = b\beta(t,T) + \frac{1}{2}\sigma^2\beta^2(t,T) - 1, \end{cases} \tag{2.27}$$

with the boundary conditions $\alpha(T,T) = \beta(T,T) = 0$, and $\gamma = \frac{(b^2+2\sigma^2)^{\frac{1}{2}}}{2}$.
$\alpha(t,T)$ and $\beta(t,T)$ are given by:

$$\begin{cases} \alpha(t,T) = 2A\int_t^T \frac{e^{Bs}(e^{\gamma(T-s)} - e^{-\gamma(T-s)})}{(2\gamma+b)e^{\gamma(T-s)} + (2\gamma-b)e^{-\gamma(T-s)}} ds \\ \\ \beta(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t) + \frac{1}{2}b\sinh(\gamma(T-t))}. \end{cases} \tag{2.28}$$

2.3 Models comparison

In this section, we evaluate the effectiveness of the aforementioned models to predict mortality using statistical methodologies. We calibrate the models on historical and projected mortality data. Next, we compare the mortality trend for the different models. Then, we assess the quality of these calibrations by performing a backtest. Finally, we use the models to predict mortality.

2.3.1 Calibration

We aim to calibrate our models on different historical survival processes, corresponding to four different cohorts of individuals, born in 1900, 1915, 1965, and 1970, respectively, and initially aged 50 years old. Hence, we calibrate our models on mortality data of the Belgian population (unisex data) considering the following assumptions:

• We consider the 1900 and 1915 generations, we use historical data from the Human Mortality Database[4], we have unisex annual death rates for ages 0 to 110, from the years 1841 to 2015;
• We consider the 1965 and 1970 generations, we use projected data from the IABE projected generational mortality table[5] starting from the year 2015 to 2060.
• Initial age of observation $x_0 = 50$.
• $\mu_{50}(0) = -\ln(s_{50})$, where s_{50} is the one-year survival probability for an individual initially aged $x_0 = 50$.

We have chosen these particular generations and $x_0 = 50$ as initial age of observation to have approximately the same number of observations (50 for historical data and 45 for projected data).
Let $(s_1, s_2, ...s_n)$ be the annual survival probabilities assumed to be functions of both the age and the calendar year extracted from both life tables. We denote by ${}_t\hat{p}_{50}$ the probability that an individual aged $x_0 = 50$ reaches age $x_0 + t$. ${}_t\hat{p}_{50}$ is computed as follows:

$$ {}_t\hat{p}_{50} = \prod_{i=50}^{50+t-1} s_i. \tag{2.29} $$

Let ${}_{t_j}p_x(\theta)$ be the function to calibrate and defined by equation (2.3), where θ is the vector of the model's parameters. To estimate θ, we use Least Square Estimation (LSE). While there are many ways to calibrate each of these models, different calibration techniques yield different results. For instance, we could estimate the parameters using the maximum likelihood (MLE) or method of moments techniques. However, for this study, the LSE method was the easiest to implement. We minimise the least square errors on the survival probabilities computed between the observed survival probabilities (around which we want to calibrate our models) and the predicted mortality by the stochastic model:

$$ R = \min_{\theta} \sum_{j=0}^{n} ({}_jp_x(\theta) - {}_j\hat{p}_x)^2. \tag{2.30} $$

For the Least Square minimisation, it is necessary to specify the initial parameters where the R routine starts searching the minimum of (2.30). To solve this non-linear system, we use the generalised Newton-Raphson method provided by R. Since the final result is highly sensitive to these starting values, we generate various initial parameters randomly and calculate some optimisation steps. The values with minimal residual are taken as start parameters for the final LS optimisation, which supplies the final parameters. We use this type of algorithm as well for all other calibration issues described in the following sections.

[4] Available at www.mortality.org
[5] Mortality projection for the Belgian population (2015), available at https://www.iabe.be

Non-mean reversion models

We calibrate the OU and Feller models to the data using the above described algorithm. We find the optimal parameters reported in Table 2.1.

	1900	**1915**	**1965**	**1970**
$\mu_{50}(0)$	0.007909195	0.005820909	0.002850792	0.002600332
OU-error	0.005885259	0.003403252	0.055996630	0.018659380
a-OU	0.080696570	0.082023980	0.078277481	0.074116615
σ-OU	0.000000129	0.000000130	0.000000177	0.000000036
FELL-error	0.005885342	0.003403291	0.055996670	0.018659451
a-FELL	0.080687850	0.082019520	0.078272931	0.074114700
σ-FELL	0.000001527	0.000010657	0.000001104	0.000011621

Table 2.1: Optimal parameter values for the survival function in the non mean-reverting models.

The values of the parameters are quite coherent in both models, and parameter a has a decreasing trend. The values of σ are very small which confirms that the volatility of the mortality is much lower as compared to the financial market. Also, the residuals are more important for younger generations.
Figures 2.1 and 2.2 present calibration results graphically. As predicted by Table 2.1 above, we see that the fit is acceptable for older generations and poor for younger generations:

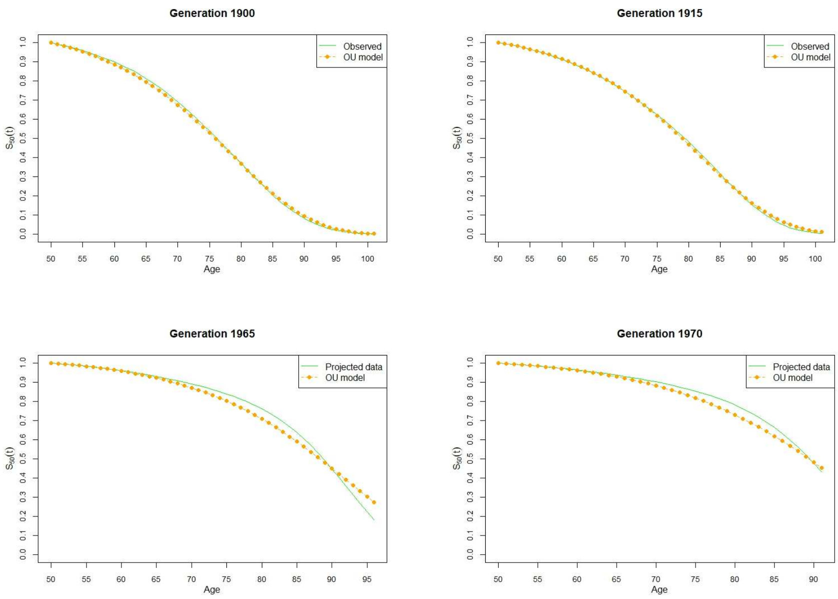


Figure 2.1: Survival functions, 1900,1915,1965 and 1970 generation for OU model.

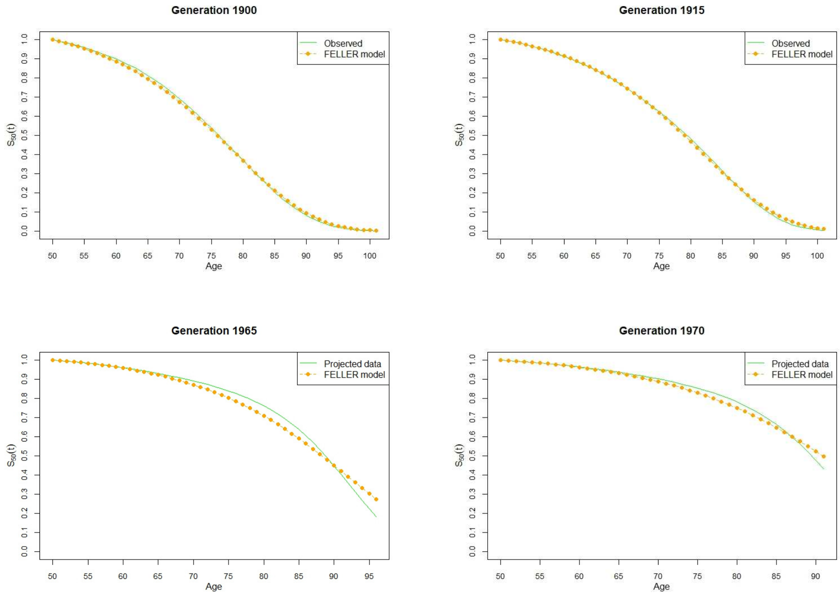


Figure 2.2: Survival functions, 1900,1915,1965 and 1970 generation for Feller model.

Mean reversion models

Table 2.2 reports the optimal calibration parameters for Vasiçek, HW and CIR models:

	1900	**1915**	**1965**	**1970**
$\mu_{50}(0)$	0.007909195	0.005820909	0.002850792	0.002600332
Vasi-error	0.355228820	0.379264950	0.345576900	0.145937100
Vasi-a	0.002379388	0.001945061	0.000852597	0.000997105
Vasi-b	0.023799120	0.020618021	0.009913990	0.055305637
Vasi-σ	0.000027800	0.000009621	0.000307905	0.000652782
Vasi-a/b	0.009977980	0.094337871	0.085999420	0.018029101
HW-error	0.000049506	0.002591606	0.006926344	0.003449854
HW-A	0.004333549	0.004015638	0.000239767	0.000307570
HW-B	0.084061184	0.084626391	0.109395638	0.100627916
HW-b	0.472769826	0.650482593	0.104155542	0.138550587
HW-σ	0.030988143	0.000444231	0.000408286	0.000519610
HW-A/b	0.009166298	0.006173321	0.002302015	0.002219915
CIR-error	0.000116668	0.002568589	0.007464623	0.004277590
CIR-A	0.005041583	0.003145545	0.0002742925	0.000608089
CIR-B	0.088362257	0.084695177	0.1086040555	0.097400314
CIR-b	0.665089137	0.491036196	0.1276709251	0.329482866
CIR-σ	0.021254297	0.036289101	0.0191874838	0.021601300
CIR-A/b	0.007580312	0.006405933	0.0021484340	0.001845586

Table 2.2: Optimal parameter values for the survival function in the mean-reverting models.

The residuals of the Vasiçek model are very large for the four generations (10^{-1}), and low for HW and CIR models as compared with the OU and Feller non-mean reversion models above. We also notice that the volatility of the Vasiçek model (Vasi-σ) is very low as compared with the other two models.

As expected, the starting point of the target strongly decreases with the generations; however, less intuitively, B seems more important for the younger generations, which could be explained by the fact that the target is much lower for the younger generations. Figures 2.3, 2.4 and 2.5 present the calibration results graphically. The fit is very poor for the Vasiçek model and very satisfying for the HW and CIR models, as we cannot distinguish the two survival functions.

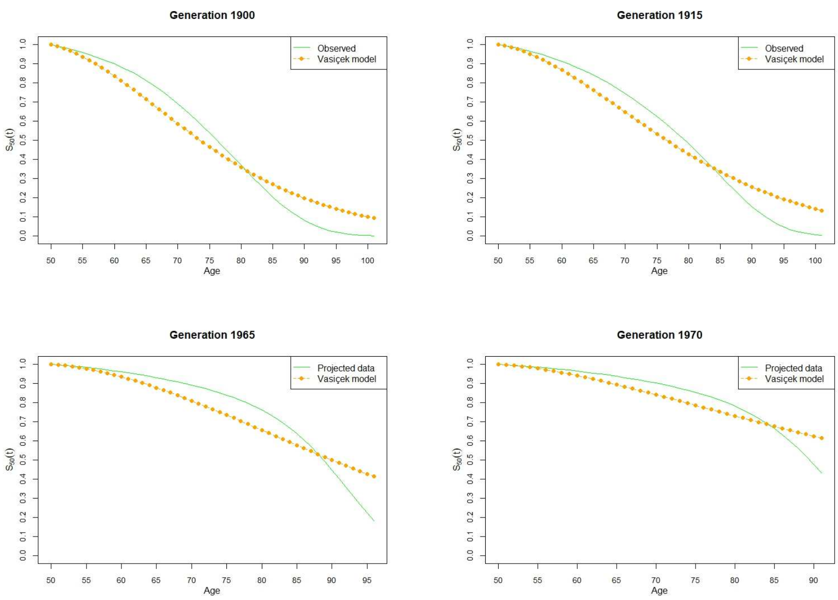


Figure 2.3: Survival functions, 1900,1915,1965 and 1970 generation for Vasiçek model.

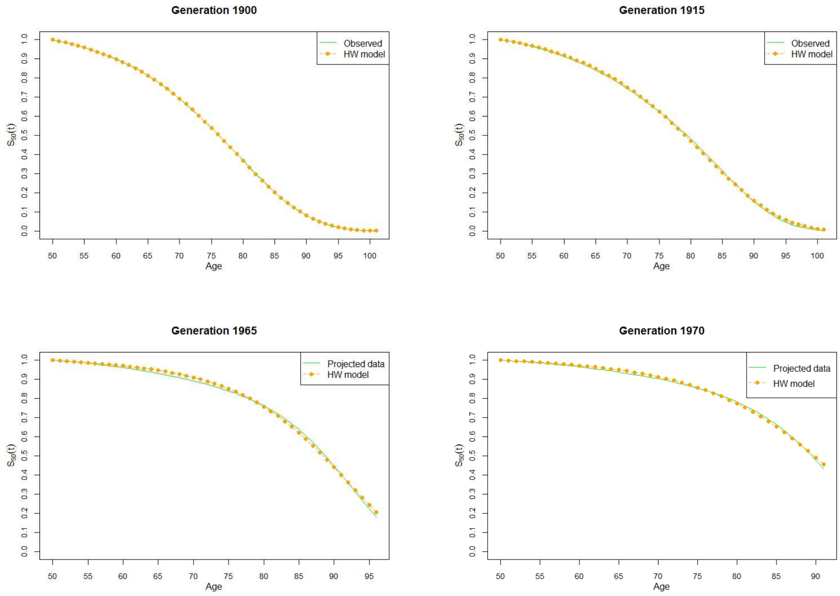


Figure 2.4: Survival functions, 1900,1915,1965 and 1970 generation for HW model.

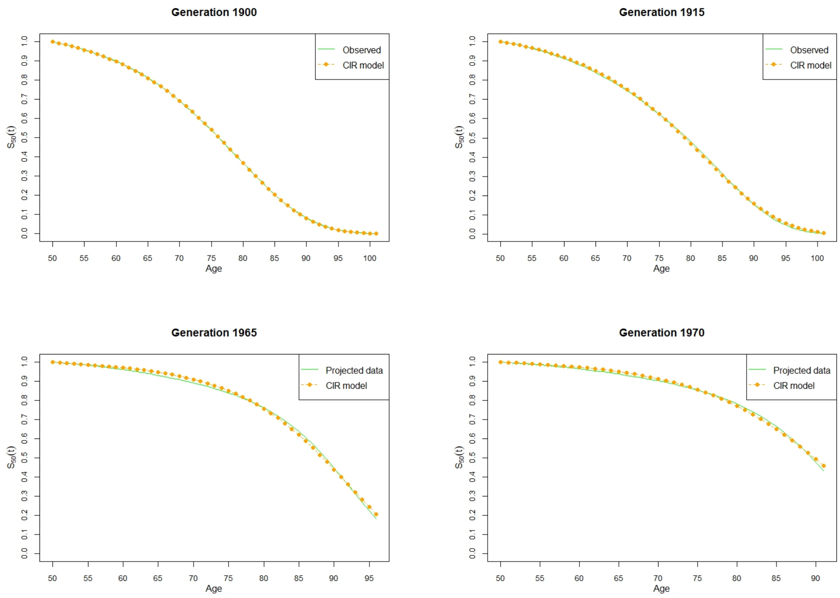


Figure 2.5: Survival functions, 1900,1915,1965 and 1970 generation for CIR model.

Calibration analysis

To conclude, we plot the calibration errors (i.e,. the differences between the observed survival probabilities and their theoretical counterparts for all generations). Table 2.3 represents a summary of the errors for each model and each generation:

	1900	**1915**	**1965**	**1970**
OU	0.005885259	0.003403252	0.055996630	0.018659380
Feller	0.005885342	0.003403291	0.055996670	0.018659451
Vasiçek	0.355228820	0.379264950	0.345576900	0.145937100
HW	0.000049506	0.002591606	0.006926344	0.003449854
CIR	0.000116668	0.002568589	0.007464623	0.004277590

Table 2.3: Calibration errors, all models.

The four graphs in figure 2.6 show that the Vasiçek fitting error is very large, which confirms the conclusions of Luciano and Vigna [13] against mean reversion models; thus, the Vasiçek model is not appropriate to describe mortality. In general, the OU and Feller models give similar acceptable errors. The HW and CIR models can fit the observed data with a smaller error. Therefore, we conclude that adding a variable long-term mean level following a Gompertz law leads to a significant improvement in the fit, and the mean reversion models perform better than non-mean reversion models, with the condition of using a variable target.

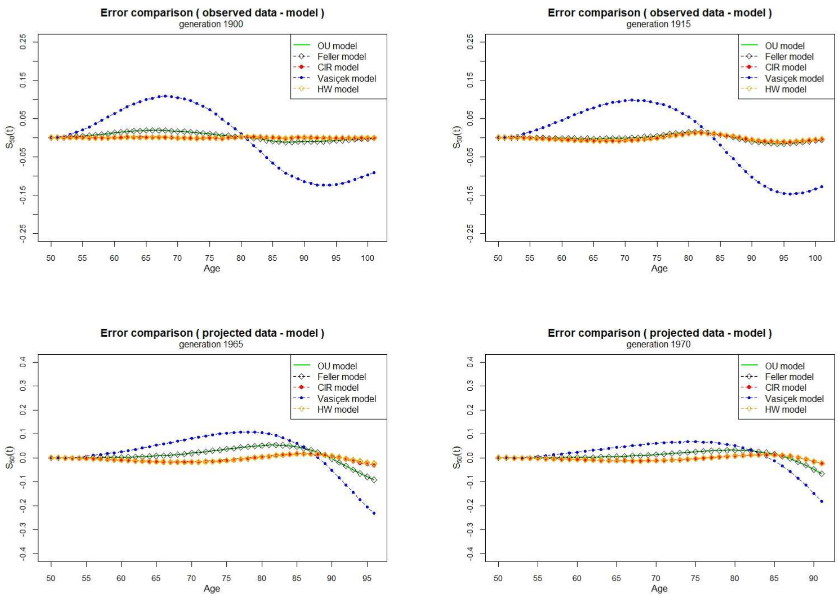


Figure 2.6: Calibration errors: Vasiçek, OU, Feller, HW and CIR, different generations.

To compare the quality of each model relative to each of the other models, we follow the routine used by Apicella et al. [1]: we compute the Bayesian Information Criterion (BIC). This model selection method allows selecting the model that provides a good fit for the data without losing generality from overfitting the data. This model selection method identifies the model with the best balance of goodness of fit between data and complexity. The BIC is given by:

$$BIC = n \ln(\frac{R}{n}) + k \ln(n), \tag{2.31}$$

where n is the number of observations, k is the number of parameters and R the residual found with the LSE method.
In Table 2.4 we report the BIC criteria for all models and different generations:

	1900	**1915**	**1965**	**1970**
BIC^{OU}	-444.5423	-471.9283	-293.3974	-342.8498
BIC^{Feller}	-444.5416	-471.9277	-293.3973	-342.8497
$BIC^{Vasiçek}$	-235.6147	-232.3411	-207.6941	-246.4859
BIC^{HW}	-675.6233	-477.7269	-379.8322	-411.1972
BIC^{CIR}	-632.7620	-478.1730	-424.8321	-452.6713

Table 2.4: BIC values for all models and different generations.

The two non-mean reversion models, OU and Feller, provide almost identical BIC values for all generations, which is not surprising because both models have the same number of parameters and give close residuals. BIC values of the HW and CIR models are also close for the same reason. We clearly see the gap between the BIC of the two mean reversion models and non-mean reversion models. For all generations and in all the cases, the non-mean reversion models have the highest BIC values as compared with the HW and CIR models. The Vasiçek model is the model with the highest BIC criteria and should be eliminated. Therefore, we keep the Hull and White and CIR models for reference, as well as the OU and Feller models for further testing.

2.3.2 Best estimate of future mortality

To look at the mortality trend, we compare the best estimate of $\mu_x(t)$ for different generations in order to illustrate the longevity pattern. We provide the mean of $\mu_{50}(t)$ for the generations 1900, 1915, 1965 and 1975, from age 50 until age 95, using the optimal parameters found in subsection 2.3.1.

Non-mean reversion models

Figure 2.7 reports the mean of $\mu_{50}(t)$ for the four generations for the OU and Feller models:

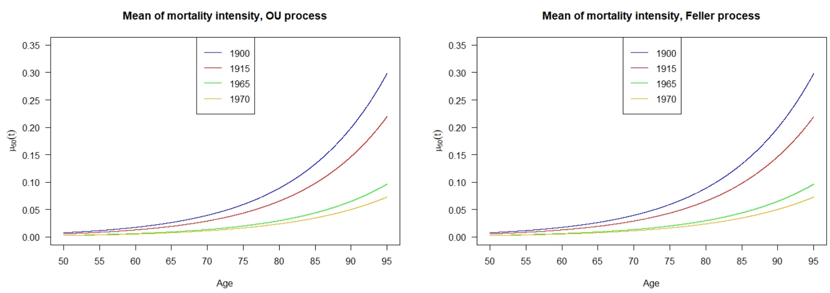


Figure 2.7: Mean of $\mu_{50}(t)$ for generation 1900, 1915, 1965 and 1970, OU and Feller models.

Mean reversion models

Figure 2.8 reports the mean of $\mu_{50}(t)$ for the four cohorts for the HW and CIR models:

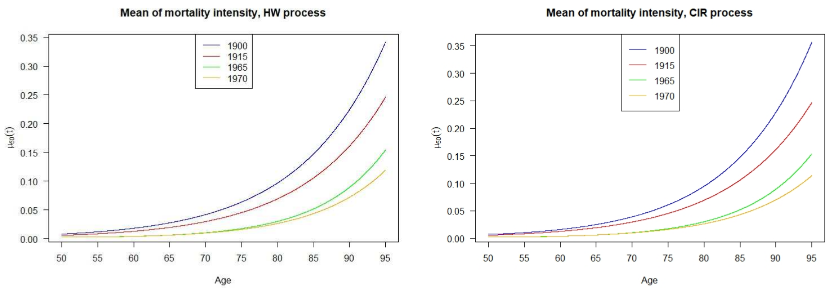


Figure 2.8: Mean of $\mu_{50}(t)$ for generation 1900, 1915, 1965 and 1970, HW and CIR models.

For all models, the older the generation, the higher the mean of $\mu_x(t)$; however we observe that for HW and CIR models, the mean of $\mu_x(t)$ appears to increase more rapidly with age.

2.3.3 Backtesting

In this section, we check if, after the calibration of our models on a portion of the historical survival curve, they can extend it correctly. Hence, we perform backtesting. Using historical data of the generation 1900, we consider:

- $x_0 = 50$,the initial age of observation ;
- $N = 50$, the total amount of historical data;
- N_0, the portion of historical data we use for calibration;
- $\delta = \frac{N_0}{N}$, the ratio of data used to total data.

The backtest consists on varying δ from 20% to 100% to see if the survival curves provided by the models meet reality.

Non-mean reversion models

Ornstein-Uhlenbeck model

We calibrate the models to the portion of the observed survival function; then, we stock the optimal parameters and the global error, which refers to the difference between the curve projected by the model and the historical one. For the OU model, we obtain the following results:

N_0	Age	δ	Global error	a	σ
10	50-60	20%	0.779266417	0.057915586	0.000001304
20	50-70	40%	0.076667020	0.073191902	0.000000237
30	50-80	60%	0.009913895	0.078843759	0.000000006
40	50-90	80%	0.005931298	0.080496310	0.000001238
50	50-100	100%	0.005885259	0.080696570	0.000000129

Table 2.5: Backtesting OU model, generation 1900.

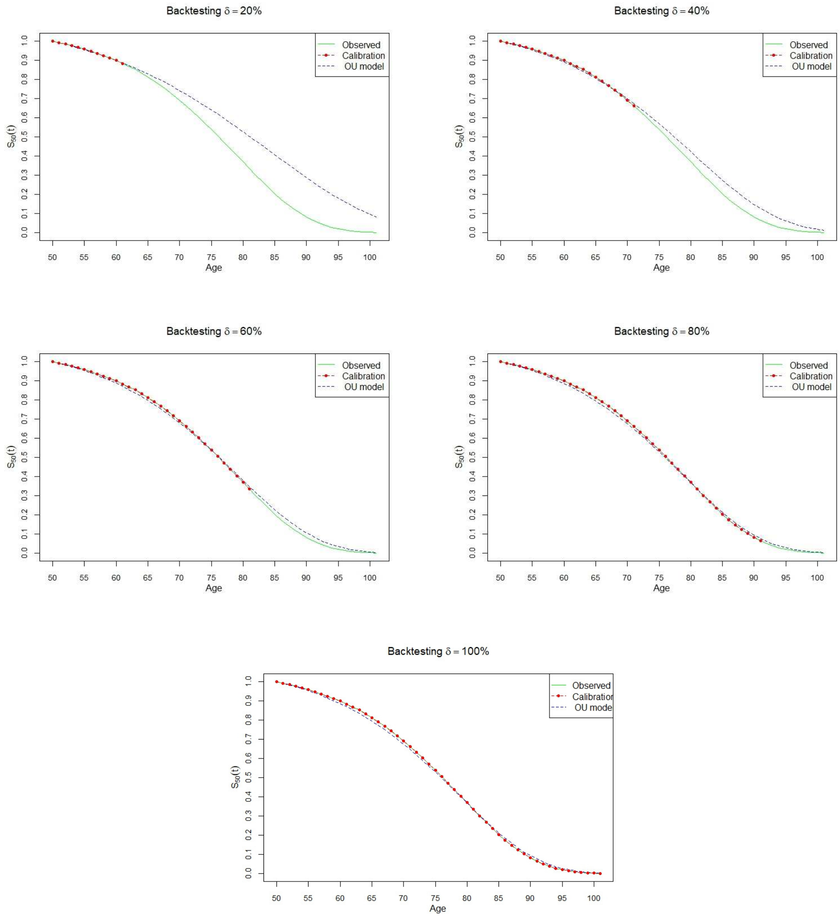


Figure 2.9: Evolution of the fitting quality of the OU model according to the portion of observed data considered.

Feller model

N_0	Age	δ	Global error	a	σ
10	50-60	20%	0.779101200	0.057919067	0.000203210
20	50-70	40%	0.076726301	0.073188955	0.000038991
30	50-80	60%	0.009921382	0.078842051	0.000010750
40	50-90	80%	0.005931992	0.080494890	0.000024904
50	50-100	100%	0.005885342	0.080687850	0.000001527

Table 2.6: Backtesting Feller model, generation 1900.

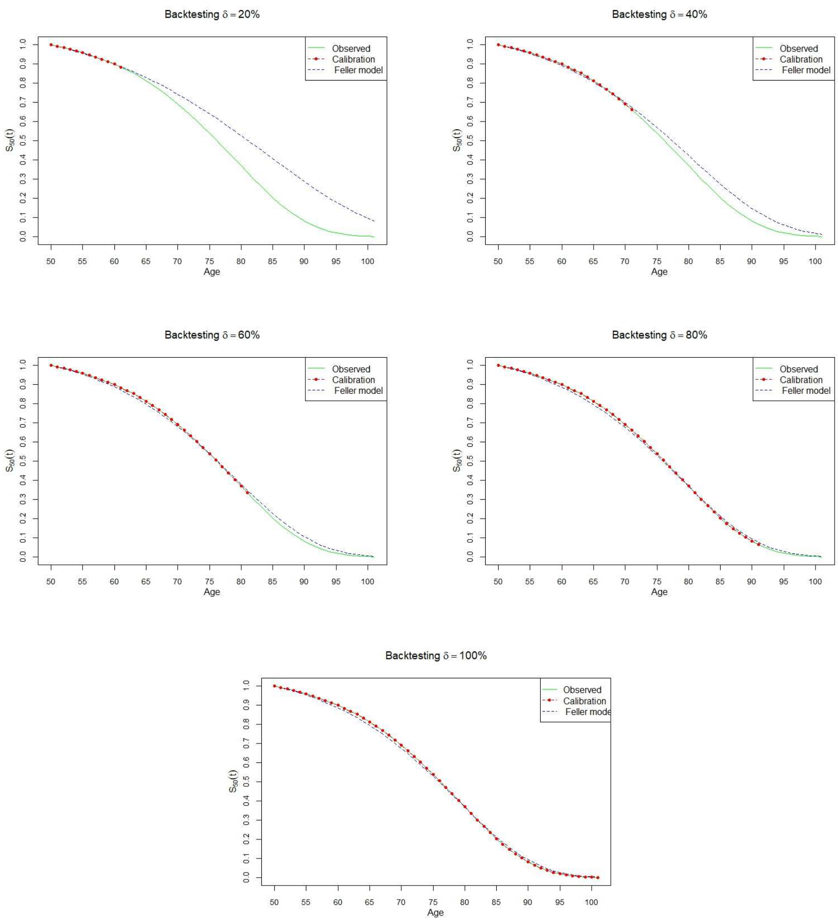


Figure 2.10: Evolution of the fitting quality of the Feller model according to the portion of observed data considered.

Both models OU and Feller, need an important volume ($\delta \geq 80\%$) of information to correctly complete the historical curve. Below this value, both models overestimate the survival function.

Mean reversion models

HW model

N_0	Age	δ	Global error	A	B	b	σ
10	50-60	20%	0.003464492	0.002733086	0.088094733	0.334805279	0.001641703
20	50-70	40%	0.000594809	0.003718993	0.085592503	0.400761822	0.027073209
30	50-80	60%	0.000245888	0.006525992	0.081024406	0.690103707	0.053059593
40	50-90	80%	0.000051370	0.004631080	0.083927720	0.509788920	0.033306650
50	50-100	100%	0.000049506	0.004333549	0.084061184	0.472769826	0.030988143

Table 2.7: Backtesting HW model, generation 1900.

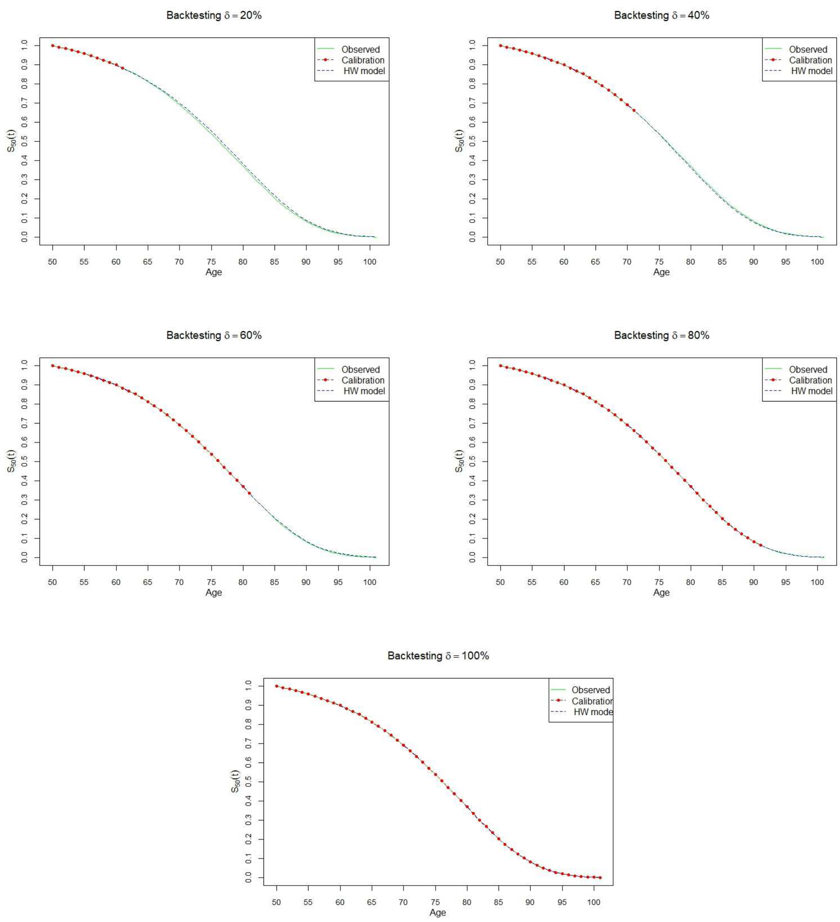


Figure 2.11: Evolution of the fitting quality of the HW model according to the portion of observed data considered.

CIR model

For the CIR model we obtain the following results:

N_0	Age	δ	Global error	A	B	b	σ
10	50-60	20%	0.009092467	0.003205452	0.086116506	0.397389283	0.051463759
20	50-70	40%	0.003353861	0.003299728	0.092763104	0.431962054	0.023858361
30	50-80	60%	0.000321363	0.004015533	0.089673479	0.518644014	0.077440960
40	50-90	80%	0.000139469	0.003613061	0.088649272	0.450780512	0.068252594
50	50-100	100%	0.000116668	0.005041583	0.088362257	0.665089137	0.021254297

Table 2.8: Backtesting CIR model, generation 1900.

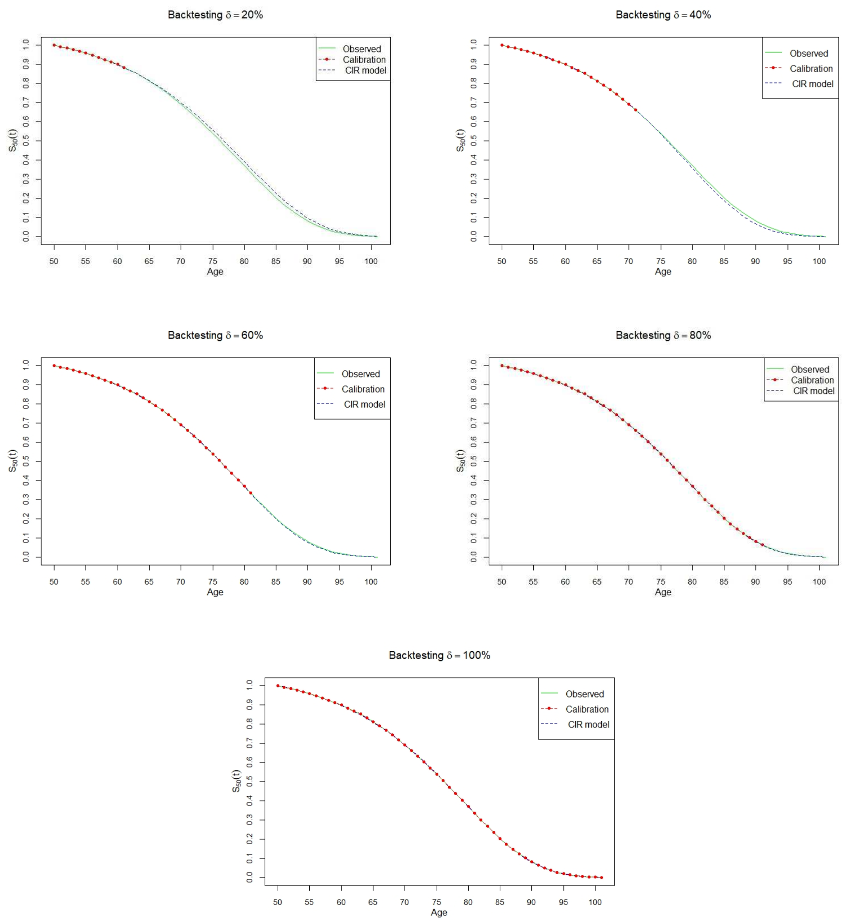


Figure 2.12: Evolution of the fitting quality of the CIR model according to the portion of observed data considered.

In the backtesting strategy, the HW and CIR models perform better than OU and Feller model; the fit is very satisfying when $\delta \geq 40\%$ for both models when compared with non-mean reversion models that require more data to fit the historical survival function correctly[6].

2.4 Multicohort vision

In this section, we plot the mortality surface to represent the survival functions of different cohorts simultaneously. To have enough values, we calibrate the models to projected data.

2.4.1 Mortality surface

Let us define $S^f(x_0, 65, z)$, the survival forward probability between age 65 and age z for an individual aged x_0 at time 0 (2015 in our example), and assumed to be alive at age 65. We define this probability as the ratio between the two corresponding survival probabilities:

$$S^f(x_0, 65, z) = \frac{{}_{z-x_0}p_{x_0}}{{}_{65-x_0}p_{x_0}}. \tag{2.32}$$

This definition is similar to the concept of forward rates in the interest rate theory. Taking into account (2.2), (2.3) and (2.4), this probability is explicitly given by:

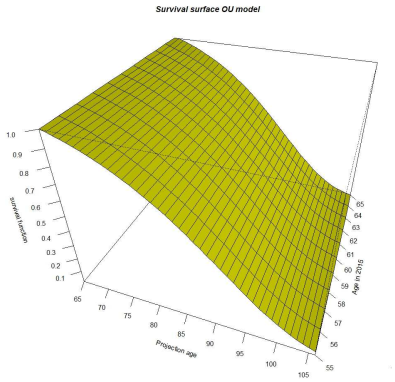
$$\begin{aligned} S^f(x_0, 65, z) &= S^f(x_0, z) && (2.33)\\ &= \frac{E_{\mathbb{P}}(S(x_0, 0, \boldsymbol{z} - x_0))}{E_{\mathbb{P}}(S(x_0, 0, 65 - x_0))} && (2.34)\\ &= \frac{e^{\alpha(0,z-x_0)-\beta(0,z-x_0)\mu_{x_0}(0)}}{e^{\alpha(0,65-x_0)-\beta(0,65-x_0)\mu_{x_0}(0)}}. && (2.35) \end{aligned}$$

To plot the mortality surface, we follow these steps:

- We consider 11 cohorts of individuals initially aged from 55 to 65 years in 2015;
- We calibrate the models on projected data for the different cohorts (from the age x_0 until the end of the projected life table);
- We consider that all individuals survive until the age of 65;
- We use the models to project the survival functions for the 11 cohorts from the age of 65 until the age of 105;
- We plot the mortality surface in three dimensions.

[6]This result allows to consider using only a portion of historical data for calibration, especially when projected data is not available

Non-mean reversion models:



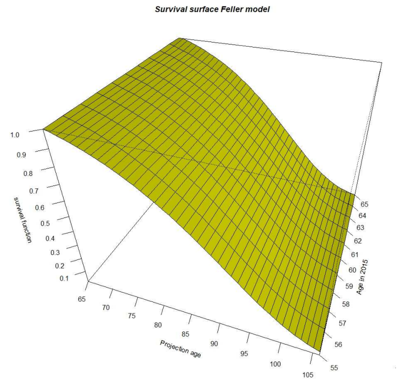


Figure 2.13: Forward survival surface for individuals aged from 55 to 65 in 2015, projected from the age 65 to 105, OU and Feller models.

Tables 2.9 and 2.10 present the annual survival probabilities given by the OU and Feller models, which are used to plot the three-dimensional figure above:

Projection age z	$S^f(55,z)$	$S^f(58,z)$	$S^f(61,z)$	$S^f(64,z)$
70	0.94270999	0.94086362	0.94000700	0.94073165
75	0.86324605	0.85867284	0.85584838	0.85594137
80	0.75692340	0.74867934	0.74240724	0.73964739
85	0.62208501	0.60956896	0.59843903	0.59017254
90	0.46416890	0.44786853	0.43160361	0.41629487
95	0.29981629	0.28209663	0.26297026	0.24269607
100	0.15614449	0.14104335	0.12407613	0.10538469
105	0.05896950	0.04987813	0.03973097	0.02901933

Table 2.9: Predicted forward survival probabilities with OU model, different cohorts.

Projection age z	$S^f(55,z)$	$S^f(58,z)$	$S^f(61,z)$	$S^f(64,z)$
70	0.94270779	0.94059967	0.93981972	0.94066064
75	0.86323974	0.85786186	0.85519102	0.85559036
80	0.75691057	0.74695390	0.74089639	0.73871205
85	0.62206344	0.60658881	0.59570571	0.58833943
90	0.46413823	0.44358153	0.42757920	0.41348398
95	0.29978022	0.27708191	0.25826448	0.23939465
100	0.15611136	0.13656292	0.12000066	0.10264470
105	0.05894795	0.04713311	0.03741897	0.02763406

Table 2.10: Predicted forward survival probabilities with Feller model, different cohorts.

Mean reversion models:

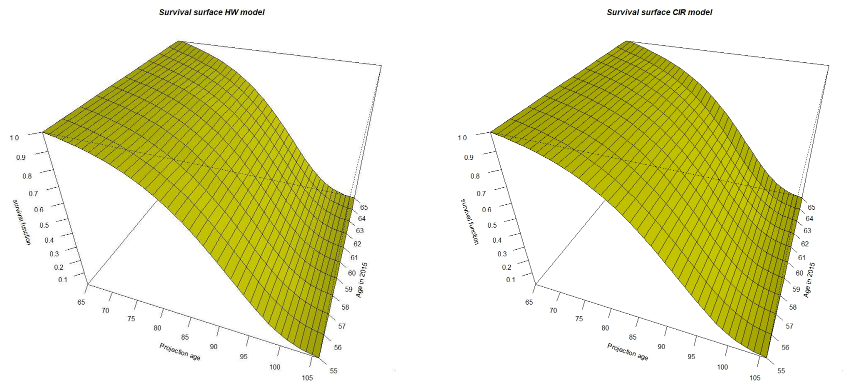


Figure 2.14: Forward survival surface for individuals aged from 55 to 65 in 2015, projected from age 65 to 105, HW and CIR models.

Tables 2.11 and 2.12 present the annual survival probabilities given by the HW and CIR models, which are used to plot the three-dimensional figure above:

Projection age z	$S^f(55, z)$	$S^f(58, z)$	$S^f(61, z)$	$S^f(64, z)$
70	0.962835443	0.960325946	0.957746038	0.950056248
75	0.904339900	0.899711357	0.897777534	0.884516037
80	0.811135375	0.803912194	0.801521846	0.784853820
85	0.669693860	0.659217546	0.652808569	0.635412171
90	0.477145956	0.463747219	0.451074084	0.435174359
95	0.261767768	0.248404861	0.233137192	0.220321818
100	0.090383387	0.081991900	0.072303009	0.064728824
105	0.013744286	0.011454980	0.009139222	0.007139386

Table 2.11: Predicted forward survival probabilities with HW model, different cohorts.

Projection age z	$S^f(55, z)$	$S^f(58, z)$	$S^f(61, z)$	$S^f(64, z)$
70	0.96289906	0.960181543	0.957353056	0.949143243
75	0.90423742	0.899746640	0.893606531	0.882913764
80	0.81075621	0.804358436	0.793837357	0.783807917
85	0.66912780	0.659968649	0.645760840	0.635573943
90	0.47668261	0.464236350	0.450115603	0.435762844
95	0.26168739	0.248033054	0.239427926	0.219823304
100	0.09058580	0.081125847	0.079370316	0.063408897
105	0.01386676	0.011058342	0.011505930	0.006615424

Table 2.12: Predicted forward survival probabilities with CIR model, different cohorts.

A comparison between Tables 2.9, 2.10 and Tables 2.11, 2.12 shows that HW and CIR models are once again more efficient: the forward survival probabilities provided by HW and CIR models, are more in line with the improvement of longevity for younger generations.

2.4.2 Parameter smoothing

Next, we examine the evolution trend of the HW and CIR models' parameters. We focus on smoothing B, $\frac{A}{b}$ and σ following these three steps:

1. We use projected data for calibration and consider individuals aged from 55 to 85 years old in 2015.
2. We draw the data points related to each parameter and each age;
3. We use smooth non-linear functions to approximate these data points.

We find the following results:

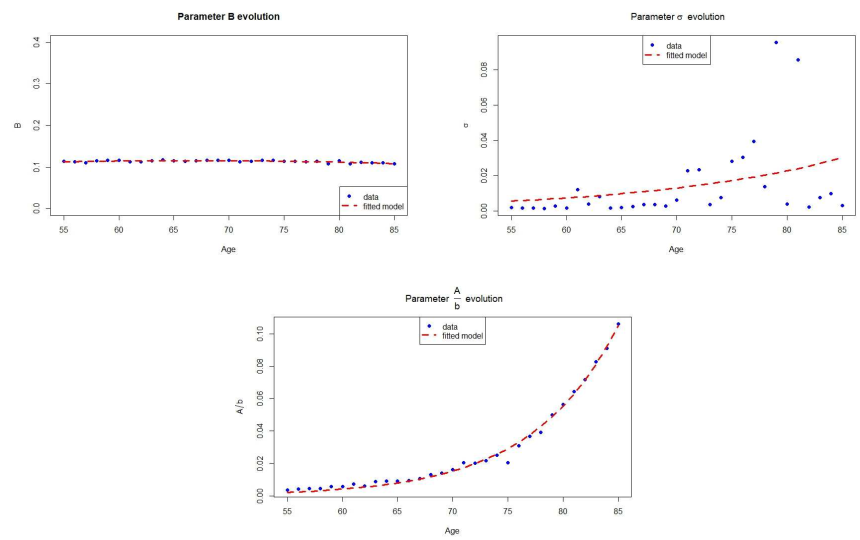


Figure 2.15: Non-linear regression applied to the different parameters, HW model.

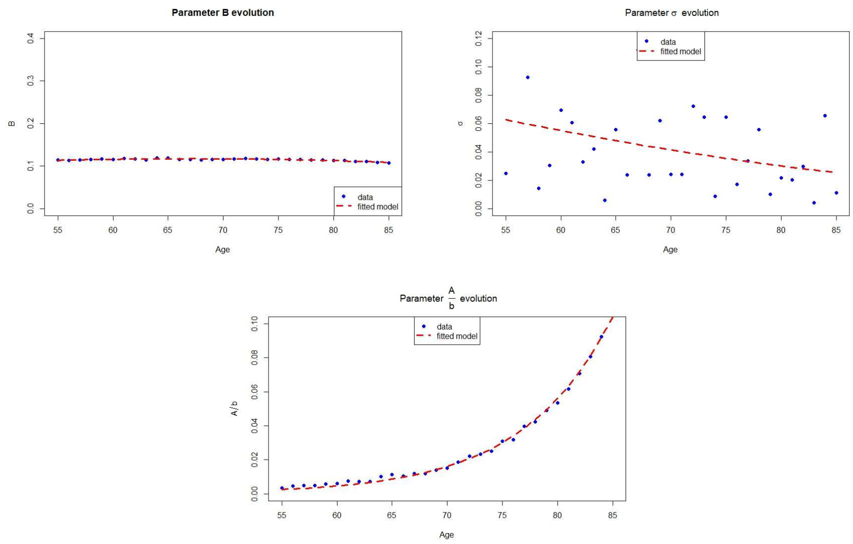


Figure 2.16: Non-linear regression applied to the different parameters, CIR model.

From these figures, we see that parameter B is very stable[7]. The initial level of the target $\frac{A}{b}$ increases exponentially with age, which is also in line with the Gompertz assumption.

2.5 Conclusion

In this paper, we described the stochastic evolution of mortality using affine processes. We considered five kinds of stochastic processes that differ in their drift and stochastic part, and their survival probabilities were provided in closed form and studied analytically. The intensity processes were calibrated to the Belgian population, using historical data for old generations and projected mortality tables for younger generations. For each model, we performed different calibration tests and assessed their ability to forecast mortality. As a result, we showed that adding a time-dependent long-term mean to the selected processes seems to be more appropriate to describe the death intensity of individuals, satisfying most of the reasonable criteria for a good mortality model. Moreover, we proposed procedures for mortality forecasting and mortality trend assessment, which describe the future evolution of mortality within a single cohort and between different cohorts, respectively. Furthermore, we voluntarily confined ourselves to one-factor, non-jump basic models in order to focus on our research question concerning the eventual mean reversion mortality property. We conclude that while the mean reversion aspect is not suitable for mortality when using a fixed target, it is entirely appropriate when using a variable target linked to the increase in mortality with age. For further research, an interesting study would be to consider different functions for the floating target and assess the performance of the mean-reverting models.

[7]This stability confirms the adequacy of Gompertz model.

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Chapter 3

Pricing of longevity derivatives and Cost of Capital[1]

Abstract

Annuities providers become more and more exposed to longevity risk due to the increase in life expectancy. To hedge this risk, new longevity derivatives have been proposed (longevity bonds, q-forwards, S-swaps...). Although academic researchers, policy makers and practitioners have talked about it for years, longevity-linked securities are not widely traded in financial markets, due in particular to the pricing difficulty. In this paper, we compare different existing pricing methods and propose a Cost of Capital approach. Our method is designed to be more consistent with Solvency II requirement (longevity risk assessment is based on a one-year time horizon). The price of longevity risk is determined for a S-forward and a S-swap but can be used to price other longevity-linked securities. We also compare this Cost of Capital method with some classical pricing approaches. The Hull and White and CIR extended models are used to represent the evolution of mortality over time. We use data for Belgian population to derive prices for the proposed longevity linked securities based on the different methods.

3.1 Introduction

Significant improvements in longevity have been experienced in most developed countries. For annuity providers, longevity risk, i.e., the risk that future mortality trends differ from those anticipated, constitutes an important risk factor.

Under the new European regulatory environment for the insurance industry, Solvency II, it becomes a requirement for insurers to measure and evaluate longevity risks (Levantesi and Menzietti [17]). As a consequence, the level of the capital required for longevity is increasing (EIOPA 2011 [14]), and this creates the need for some longevity risk management solutions.

In the past, annuity providers were only able to transfer the longevity risk to reinsurer who can offer a standardized, or a customized hedging with no residual basis

[1] Zeddouk, F. and Devolder, P. Pricing Longevity Derivatives and Cost of Capital (2019), Risks, 7(2),1–29

risk, but this risk transfer arrangement has become increasingly expensive and present a large counterparty credit risk. In addition, the capacity of reinsurers is limited because longevity risk cannot be diversified, and as a consequence, just few reinsurers accept taking longevity risk (Blake [3]). Hence, finding other alternatives to transfer the longevity risk, notably securitization, became a necessity (Cox et al. [10]; Li et al. [18]). Another de-risking approach for a pension provider are the pension buy-in and the pension buy-out (D'Amato et al. [11]).

In recent years, some longevity derivatives arose. These products are based on mortality/longevity rates, similar to those existing in the financial market (Blake et al. [4])
The two main longevity-linked securities that were proposed in the literature are longevity bonds (Blake and Burrows [5]), and Survival swaps (Dowd et al. [13]).
Longevity swaps are considered better because their transaction costs are lower, are more flexible and could be customized to suit individual circumstances. Moreover, they do not require the existence of a liquid market. Therefore, longevity swaps appear to be the most relevant derivative to consider for hedging longevity risk.

Like the interest rate swaps, the survival swaps can be broken down into a collection of more simple derivatives: the Survival-forwards (Coughlan et al. [9]).
These derivatives have attracted many researchers and practitioners who have published many papers on this subject, but because of the pricing difficulties, and the fact that these products remove only the longevity risk, they are still not widely traded in the financial market (Lin et al. [19]).
Some authors have introduced a new methodology inspired by Solvency II: the pricing of longevity derivatives using the Cost of Capital approach. A version of this method has been proposed by Levantesi and Menzietti [17] in a discrete time model.

In this paper, we generalize this approach in two directions: first we use a risk measurement more in line with Solvency II and based on successive one-year time horizons; secondly we model longevity risk with continuous-time processes, as they offer analytical tractability and can be used for the pricing of financial instruments. We also study the consistency between this Cost of Capital approach (COC) and three important classical pricing methods used in finance: risk-neutral pricing, Wang transform and Sharpe ratio.

This paper is organized as follows: in Section 3.2, we give the general pricing framework for the S-forward with the classical the COC approach, then the same framework is provided in Section 3.3 for the S-swaps. In Section 3.4, we translate the theoretical framework into a numerical illustration, enabling the analysis of the consistency between the proposed and the existing classical pricing approaches. We show that none of these methods is completely consistent with the COC approach. Finally, in Section 3.5 we conclude.

3.2 Survival-forwards

As an interest-rate swap is essentially a portfolio of forward-rate agreement (Hull [15]), the survival swap can be broken down into a collection of a more simple derivative:

the Survival-forward. Therefore, we will begin our study by focusing on them.

3.2.1 Definition and structure

A Survival forward (or S-forward) is an agreement between two counterparties to exchange at a future date T (the maturity of the contract), an amount equal to the realized survival rate of a given population cohort (floating leg), in return for a fixed survival rate agreed at the inception of the contract (fixed rate payment).

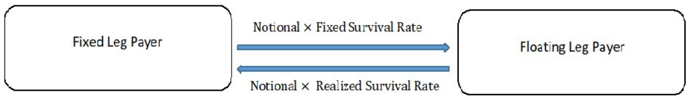

Figure 3.1: S-forward

For easiness of representation, we consider the notional amount equal to one monetary unit. The payoff of the S-forward is then given by:

$$Payoff(T) = {}_Tp_x - {}_T\hat{p}_x, \tag{3.1}$$

where ${}_Tp_x$ is the realized survival rate ($\mathcal{F}_T$ measurable), and ${}_T\hat{p}_x$ is a fixed probability of an individual aged x at time 0 to be alive at age $x+T$ ($\mathcal{F}_0$ measurable).

3.2.2 Assumptions

In order to price the S-forward and S-swap, we will need to make some assumptions. For the whole paper we will consider the following:

- The longevity is stochastic through one explanatory variable, the mortality intensity denoted by $\mu_x(t,\omega)$, which represents the force of mortality for an individual aged $x+t$ at time t.
- For this mortality intensity, we consider two affine stochastic time-continuous models (Luciano and Vigna [21]):
  1. Hull & White model (HW):

$$d\mu_x(t) = (\xi(t) - \mu_x(t))dt + \sigma dw(t). \tag{3.2}$$

  2. Cox-Ingersoll-Ross (CIR) extended model:

$$d\mu_x(t) = (\xi(t) - \mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dw(t). \tag{3.3}$$

with $\xi(t) = Ae^{Bt}$ and A, B, b, σ positive numbers, $w(t)$ a standard Brownian motion under the real world probability measure $\mathbb{P}$.
The CIR model guarantees the positivity of the force of mortality.

- The spot interest rate $r(t,\omega)$ is stochastic and is solution of an SDE of the form:

$$dr(t,\omega) = h(t,r)dt + z(t,r)d\bar{w}(t,\omega), \tag{3.4}$$

  where h and z are two given functions.

- w and $\bar{w}$ are independent standard Brownian motions under the real probability measure $\mathbb{P}$.

- We denote by $V(t,T)$ the price at a given time t of an S-forward with maturity T.

- The survival index at time t of an individual initially aged x, alive at time t and surviving $T-t$ years more is given by the following process:

$$I(x+t, T-t) = e^{-\int_t^T \mu_x(u,\omega)du}. \tag{3.5}$$

- ${}_tp_x^{obs} = e^{-\int_0^t \mu_x(s)ds}$ is the observed survival function ($\mathcal{F}_t$ measurable).

- The two models being affine, the expectation of their survival index is directly given by (Dowd et al. [13]):

$$E_{\mathbb{P}}(I(x+t, T-t) \mid \mathcal{F}_t) = e^{\alpha_{\mathbb{P}}(t,T) - \beta_{\mathbb{P}}(t,T)\mu_x(t)}. \tag{3.6}$$

  1. In the HW model we have:

$$\begin{cases} \alpha_{\mathbb{P}}^{HW}(t,T) & = \frac{A}{b}[e^{-bT}\frac{e^{(B+b)T}-e^{(B+b)t}}{B+b} - \frac{e^{BT}-e^{Bt}}{B}] - \frac{\sigma^2}{2b^2}[\frac{1}{b}(1-e^{-b(T-t)}) - T + t] \\ & \quad -\frac{\sigma^2}{4b^3}(1-e^{-b(T-t)})^2 \\ \beta_{\mathbb{P}}^{HW}(t,T) & = \frac{1}{b}(1-e^{-b(T-t)}). \end{cases} \tag{3.7}$$

  2. In the CIR extended model we have:

$$\begin{cases} \alpha_{\mathbb{P}}^{CIR}(t,T) = 2A\int_t^T \frac{e^{Bs}(e^{\gamma(T-s)}-e^{-\gamma(T-s)})}{(2\gamma+b)e^{\gamma(T-s)}+(2\gamma-b)e^{-\gamma(T-s)}}ds \\ \beta_{\mathbb{P}}^{CIR}(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)+\frac{1}{2}b\sinh(\gamma(T-t)),} \end{cases} \tag{3.8}$$

  with $\gamma = \frac{(b^2+2\sigma^2)^{\frac{1}{2}}}{2}$.

- $P(t,T) = E_{\mathbb{Q}}(e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t)$ is the price at time t of a zero coupon bond with maturity T under the risk-neutral probability measure $\mathbb{Q}$.

- There is no basis risk: the insurer's reference population is the same as in the longevity linked security.

- There is no counterparty default risk.

3.2.3 S-forward pricing by Cost of Capital

There is an ongoing debate and many authors have addressed the question of how to price longevity-linked securities, but the calibration of the market price of longevity risk remains an open question as " in one hand, there is a quest for actuarial or economic methods of how to derive prices "conceptually" and, on the other hand, there is the problem of how to derive prices given a certain methodology, i.e. what data to use for calibration purposes "(Bauer et al. [2]).
Some authors have suggested that, following the directive of Solvency II, the price of longevity linked securities could be associated with the Solvency Capital Requirement (SCR) to cover unexpected losses (Börger [6]).

Laventesi and Menzietti [17] have proposed to price a S-forward using the risk margin implicit within the calculation of the technical provisions as defined by Solvency II.
The overall idea is based on linking the price of longevity-linked securities with the capital that the insurer should hold to cover unexpected losses: according to Solvency II, insurance liabilities that cannot be hedged are evaluated as the sum of a best estimate and a risk margin (RM), which is the market value of the uncertainty on insurance liabilities. The RM is determined by the cost of capital approach based on the SCR, the capital required to cover with 99.5% probability the unexpected losses on a one-year time horizon.
In other words, by entering an S-forward, the insurer can lower its exposition to the longevity risk and in consequence, the SCR for longevity risk can be mitigated, or even reduced to zero if the longevity risk is completely covered, and so does the corresponding RM. Hence, the price of an S-forward should not be higher than the cost of holding the SCR, otherwise insurers will not be interested in buying it. In fact, this approach provides an estimation of the maximum price that the insurer would be willing to pay to transfer longevity risk.
Let us consider a T-years S-forward contract.

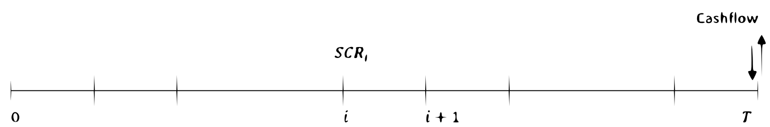

Figure 3.2: T-years S-forward contract

Under this approach, the price of the S-forward at time t is given by:

$$V_{COC}(t,T) = BE_t^{\mathbb{P}} + RM_t, \tag{3.9}$$

where $BE_t^{\mathbb{P}}$ is the best estimate, equal to the discounted expectation of the payoff of the S-forward at time t under the real-world measure, and RM_t the risk margin.
At time 0, we obtain:

$$V_{COC}(0,T) = BE_0^{\mathbb{P}} + RM_0, \tag{3.10}$$

where $BE_0^{\mathbb{P}}$ is as follows:

$$BE_0^{\mathbb{P}} = P(0,T)(E_{\mathbb{P}}(I(x,T) - {}_T\hat{p}_x). \tag{3.11}$$

The risk margin is defined as the present value of future returns on the successive SCRs:

$$RM_0 = C\% \sum_{i=0}^{T-1} SCR_i \, P(0, i+1), \tag{3.12}$$

where SCR_i is the solvency capital Requirement to cover with 99.5% probability the unexpected losses for year i, and C is the Cost of Capital rate (6% in Solvency II).
These future SCR_i are random variables. In order to compute the initial risk margin RM_0, we need to use their estimation at time 0 denoted by: $\hat{SCR}_i = SCR_i \mid_0$.
The risk margin at time 0 becomes then:

$$RM_0 = C\% \sum_{i=0}^{T-1} \hat{SCR}_i \, P(0, i+1). \tag{3.13}$$

In order to determine the price of an S-forward, we will need to calculate the estimation the future solvency capitals. Solvency II provides different methods to calculate these capitals. We will consider the VaR method, that measures the potential loss in value of a risky asset or liability over a defined time horizon for a given confidence level, say 99.5% (Solvency II standard).

Levantesi and Menzietti's specification

A first approach developed by Levantesi and Menzietti [17] is based on the VaR of the payoff at 99,5% from the moment of calculation until maturity, as represented by the following figure:

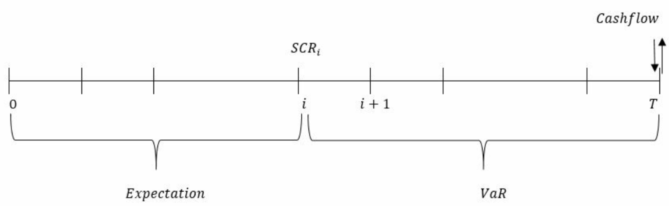

Figure 3.3: Computation of the successive SCR_i in the Levantesi and Menzietti's specification

Explicitely, the SCR_i are given by:

$$\begin{aligned}
SCR_i &= \mathrm{VaR}_{99,5\%}(P(i,T) Payoff_T{}^{\mathrm{VaR}_{99,5\%}} - BE_i^{\mathbb{P}}) & (3.14)\\
&= P(i,T)\, \mathrm{VaR}_{99,5\%}(I(x,T) - {}_T\hat{p}_x) - P(i,T)(E_{\mathbb{P}}(\, I(x,T) - {}_T\hat{p}_x)) & (3.15)\\
&= \mathrm{VaR}_{99,5\%}(P(i,T)(I(x,T) - {}_T\hat{p}_x)) - P(i,T)(E_{\mathbb{P}}(I(x,T) - {}_T\hat{p}_x)) & (3.16)\\
&= P(i,T)(I(x,i)[\mathrm{VaR}_{99,5\%}(I(x+i, T-i)) - E_{\mathbb{P}}(\, I(x,T))] & (3.17)\\
SCR_i \mid_0 &= P(i,T)[E_{\mathbb{P}}(I(x,i))\, \mathrm{VaR}_{99,5\%}(I(x+i, T-i) - E_{\mathbb{P}}(\, I(x,T))]. & (3.18)
\end{aligned}$$

From expression (3.18), we can see that the Value-at-Risk is measured on the whole interval $[i, T]$.

The price of the S-forward is then equal to:

$$\begin{aligned} V_{COC}(0,T) &= P(0,T)(E_{\mathbb{P}}(I(x,T)) - {}_T\hat{p}_x) \\ &+ 6\% \sum_{i=0}^{T-1} P(i,T)P(0,i+1)\big[E_{\mathbb{P}}((I(x,i))\,\mathrm{VaR}_{99,5\%}(I(x+i,T-i)) - E_{\mathbb{P}}(I(x,T))\big]. \end{aligned} \tag{3.19}$$

Our specification

To be more consistent with the logic of Solvency II, we propose to adopt another formulation: the SCR at time i should cover with 99.5% probability the unexpected losses on a one-year time horizon (the interval $[i, i+1]$), and we consider mortality evolution's up to time i, and from $i+1$ to T according to their best estimates:

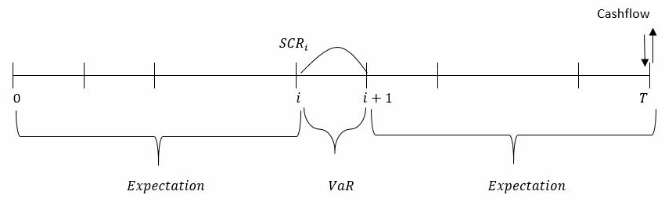

Figure 3.4: Computation of the successive SCR_i at time 0 in our specification

Then the expression of the SCR_i becomes:

$$SCR_i = \mathrm{VaR}_{99,5\%}[BE^{\mathbb{P}}_{i+1}P(i,i+1) - BE^{\mathbb{P}}_i], \tag{3.20}$$

where:

$$BE^{\mathbb{P}}_{i+1} = (I(x,i+1)E_{\mathbb{P}}(\,I(x+i+1,T-i-1) - {}_T\hat{p}_x))P(i+1,T) \tag{3.21}$$

$$BE^{\mathbb{P}}_i = (I(x,i)E_{\mathbb{P}}(\,I(x+i,T-i) - {}_T\hat{p}_x))P(i,T). \tag{3.22}$$

The SCR_i is then given by:

$$\begin{aligned} SCR_i &= P(i,T)\,\mathrm{VaR}_{99,5\%}[I(x,i+1)E_{\mathbb{P}}(\,I(x+i+1,T-i-1)) - I(x,i)E_{\mathbb{P}}(\,I(x+i,T-i))] \\ &= P(i,T)I(x,i)[\mathrm{VaR}_{99,5\%}(I(x+i,1))E_{\mathbb{P}}(\,I(x+i+1,T-i-1)) - E_{\mathbb{P}}(\,I(x+i,T-i))]. \end{aligned} \tag{3.23}$$

Therefore, the estimation of $SCR_i\mid_0$ can be given by:

$$SCR_i\mid_0 = P(i,T)E_{\mathbb{P}}(I(x,i))[\mathrm{VaR}_{99,5\%}(I(x+i,1)) - E_{\mathbb{P}}(\,I(x+i,1))]E_{\mathbb{P}}(\,I(x+i+1,T-i-1)). \tag{3.24}$$

The risk margin at time 0 is then equal to:

$$\begin{aligned} RM_0 &= 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I(x,i))[\mathrm{VaR}_{99,5\%}(I(x+i,1)) - E_{\mathbb{P}}(\,I(x+i,1))] \\ &\times \; E_{\mathbb{P}}(\,I(x+i+1,T-i-1))P(0,i+1)P(i,T). \end{aligned} \tag{3.25}$$

Finally, the price of the S-forward with the COC approach is:

$$\begin{aligned} V_{_{COC}}(0,T) &= P(0,T)\,(E_{\mathbb{P}}[\,I(x,T) - {}_T\hat{p}_x]) \\ &+ \; 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I(x,i))[\mathrm{VaR}_{99,5\%}(I(x+i,1)) - E_{\mathbb{P}}(\,I(x+i,1))] \\ & \quad E_{\mathbb{P}}(\,I(x+i+1,T-i-1))P(0,i+1)P(i,T). \end{aligned} \tag{3.26}$$

Let us now apply these methods using two particular mortality models.

HW model

To determine the explicit expression of the price, we need to compute: $\mathrm{VaR}_{99,5\%}(I(x+i,1)) = \mathrm{VaR}_{99,5\%}(e^{-\int_i^{i+1}\mu_x(s)ds})$.

In general, we have:

$$\mathrm{VaR}_{99,5\%}(I(x+t,T-t)) = \mathrm{VaR}_{99,5\%}(e^{-\int_t^T \mu_x(s)ds}) = \mathrm{VaR}_{99,5\%}(e^{X(t,T)}). \tag{3.27}$$

We know that:

$$I(x+t,T-t) = e^{-\int_t^T \mu_x^{\mathbb{P}}(u,\omega)du} = e^{X(t,T)}. \tag{3.28}$$

One can show that under the real-world risk measure $\mathbb{P}$, we have :

$$X(t,T) \sim N(m(t,T),\, n(t,T)^2). \tag{3.29}$$

So the survival index I is log-normally distributed with:

$$\begin{cases} m(t,T) = & \mu_x(t)\frac{(e^{-b(T-t)}-1)}{b} - \frac{Ae^{Bt}}{B(b+B)}(e^{B(T-t)}-1) - \frac{Ae^{Bt}}{b(b+B)}(e^{-b(T-t)}-1) \\ n^2(t,T) = & \frac{\sigma^2}{b^2}[T-t-\frac{1-e^{-b(T-t)}}{b} - \frac{(1-e^{-b(T-t)})^2}{2b}]. \end{cases} \tag{3.30}$$

Then we have:

$$\mathrm{VaR}_{99,5\%}(I(x+t,T-t)) = e^{m(t,T)+z_\eta n(t,T)}, \tag{3.31}$$

where $z_\eta = \eta$ is the quantile of a normal distribution N(0,1).
We consider $\eta = 0,995, z_\eta = 2,58$ (Solvency II standard). Finally, the price of the S-forward at time 0 becomes:

$$\begin{aligned} V_{_{COC}}^{^{HW}}(0,T) = P(0,T)(e^{\alpha_{\mathbb{P}}^{^{HW}}(0,T)-\beta_{\mathbb{P}}^{^{HW}}(0,T)\mu_x(0)} - {}_T\hat{p}_{_x}) \\ + 6\% \sum_{i=0}^{T-1}\Big[e^{\alpha_{\mathbb{P}}^{^{HW}}(0,i)-\beta_{\mathbb{P}}^{HW}(0,i)\mu_x(0)}[e^{m(i,i+1)+z_\eta n(i,i+1)} \\ - e^{\alpha_{\mathbb{P}}^{^{HW}}(i,i+1)-\beta_{\mathbb{P}}^{^{HW}}(i,i+1)\overline{\mu_x}(i)}]e^{\alpha_{\mathbb{P}}^{^{HW}}(i+1,T)-\beta_{\mathbb{P}}^{^{HW}}(i+1,T)\overline{\mu_x}(i+1)}\Big]P(0,i+1)P(i,T), \end{aligned} \tag{3.32}$$

with $\overline{\mu_x}(i) = E_{\mathbb{P}}(\mu_x(i))$ and, $\alpha_{\mathbb{P}}^{HW}(.,T), \beta_{\mathbb{P}}^{HW}(.,T)$ are given by (3.7)

CIR model

Let us here consider that the intensity of mortality follows the CIR process. Same as in the HW model, we will need to determine $\mathrm{VaR}_{99,5\%}(e^{-\int_i^{i+1}\mu_x(s)ds})$, but as it is not easy to find an analytical solution, we will need to simulate it numerically. The price of the S-forward is then given by:

$$\begin{aligned}
V_{COC}^{CIR}(0,T) &= P(0,T)(e^{\alpha_{\mathbb{P}}^{CIR}(0,T)-\beta_{\mathbb{P}}^{CIR}(0,T)\mu_x(0)} - {}_T\hat{p}_x) \\
&\quad + 6\% \sum_{i=0}^{T-1} \Big[e^{\alpha_{\mathbb{P}}^{CIR}(0,i)-\beta_{\mathbb{P}}^{CIR}(0,i)\mu_x(0)} [\mathrm{VaR}_{99,5\%}(e^{-\int_i^{i+1}\mu_x(s)ds}) \\
&\quad - e^{\alpha_{\mathbb{P}}^{CIR}(i,i+1)-\beta_{\mathbb{P}}^{CIR}(i,i+1)\overline{\mu_x}(i)}] e^{\alpha_{\mathbb{P}}^{CIR}(i+1,T)-\beta_{\mathbb{P}}^{CIR}(i+1,T)\overline{\mu_x}(i+1)} \Big] P(0,i+1)P(i,T), \quad (3.33)
\end{aligned}$$

where $\alpha_{\mathbb{P}}^{CIR}(.,T)$ and $\beta_{\mathbb{P}}^{CIR}(.,T)$ are given by (3.8).

Pricing at any time t:

Following our same reasoning, we give the expression of SCR_i at any time t $(0 < t < i)$ under the COC approach:

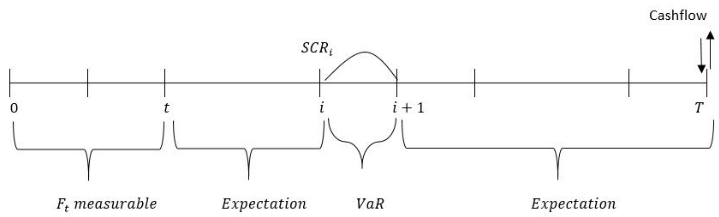

Figure 3.5: Computation of the successive SCR_i at any time t in our specification

$$\begin{aligned}
V_{COC}(t,T) &= P(t,T)I(x,t)E_{\mathbb{P}}(I(x+t,T-t)) \\
&\quad + 6\% \sum_{i=t}^{T-1} I(x,t)E_{\mathbb{P}}(I(x+t,i-t))[\mathrm{VaR}_{99,5\%}(I(x+i,1)) - E_{\mathbb{P}}(I(x+i,1))] \\
&\quad \times E_{\mathbb{P}}(I(x+i+1,T-i-1))P(i,T)P(t,i+1). \quad (3.34)
\end{aligned}$$

HW model

We determine under our philosophy the price of an S-forward at time $t(0 < t < T)$ using HW model:

$$\begin{aligned} V^{HW}_{COC}(t,T) &= P(t,T)[{}_tp^{obs}_x \cdot e^{\alpha^{HW}_{\mathbb{P}}(t,T)-\beta^{HW}_{\mathbb{P}}(t,T)\mu_x(t)} - {}_T\hat{p}_x] \\ &+ 6\% \sum_{i=t}^{T-1} \Big[{}_tp^{obs}_x \cdot e^{\alpha^{HW}_{\mathbb{P}}(t,i)-\beta^{HW}_{\mathbb{P}}(t,i)\mu_x(t)} [e^{m(i,i+1)+z_\eta n(i,i+1)} \\ &- e^{\alpha^{HW}_{\mathbb{P}}(i,i+1)-\beta^{HW}_{\mathbb{P}}(i,i+1)\overline{\mu_x}(i)}] e^{\alpha^{HW}_{\mathbb{P}}(i+1,T)-\beta^{HW}_{\mathbb{P}}(i+1,T)\overline{\mu_x}(i+1)} \Big] P(i,T)P(t,i+1). \end{aligned} \tag{3.35}$$

CIR model

Let us here consider that the intensity of mortality follows a CIR process. The price of the S-forward at time t $(0 < t < T)$ becomes:

$$\begin{aligned} V^{CIR}_{COC}(t,T) &= P(t,T)[{}_tp^{obs}_x \cdot e^{\alpha^{CIR}_{\mathbb{P}}(t,T)-\beta^{CIR}_{\mathbb{P}}(t,T)\mu_x(t)} - {}_T\hat{p}_x] \\ &+ 6\% \sum_{i=t}^{T-1} \Big[{}_tp^{obs}_x \cdot e^{\alpha^{CIR}_{\mathbb{P}}(t,i)-\beta^{CIR}_{\mathbb{P}}(t,i)\mu_x(t)} [\mathrm{VaR}_{99,5\%}(e^{-\int_i^{i+1}\mu_x(s)ds}) \\ &- e^{\alpha^{CIR}_{\mathbb{P}}(i,i+1)-\beta^{CIR}_{\mathbb{P}}(i,i+1)\mu_x(i)}] e^{\alpha^{CIR}_{\mathbb{P}}(i+1,T)-\beta^{CIR}_{\mathbb{P}}(i+1,T)\overline{\mu_x}(i+1)} \Big] P(i,T)P(t,i+1). \end{aligned} \tag{3.36}$$

3.2.4 Comparison with classical methods

We develop in this section the pricing of the S-forward contract, following other alternative theoretical formulas to compare with formula (3.34):

- The risk-neutral approach (Cairns et al. [7]);
- The Wang transform (Wang [23]);
- Sharpe ratio (Milevsky et al. [22]).

Numerical comparisons will be given in section 3.4.

Risk-neutral approach

The risk-neutral pricing is a technique widely used in quantitative finance to compute the values of derivatives, and could be adapted to price longevity-linked securities.
As discussed in some papers, (see for example Cairns et al. [8]), this approach is not ideal to price longevity derivatives as the longevity market is immature, and this method requires the availability of a considered amount of data. But it is interesting to explore the risk-neutral pricing formula for comparison purposes.
Under the risk-neutral probability measure $\mathbb{Q}_\lambda$, the price of an S-forward at time t is given by:

$$V_{\mathbb{Q}_\lambda}(t,T) = P(t,T)[{}_tp^{obs}_x E_{\mathbb{Q}_\lambda}(I(x+t,T-t)) - {}_T\hat{p}_x \mid \mathcal{F}_t], \tag{3.37}$$

where λ is the market price of longevity risk.

We first consider the HW model to get an explicit expression of the S-forward price. We assume that $\mu_x(t)$ follows the HW process given by (3.2).
Let us introduce the market prices $\lambda(t, \mu_x(t))$ associated with longevity risk, and let $w^*(t, \omega) = w(t, \omega) - \int_0^t \lambda(t, \mu_x(s))ds$.
We consider a model with a constant market price of risk $\lambda(t, \mu_x(t)) = \lambda$.
The SDE of the force of mortality under the risk-neutral measure becomes:

$$d\mu_x^{\mathbb{Q}_\lambda}(t) = (\xi(t) - b\mu_x(t) + \sigma\lambda)dt + \sigma dw^*(t). \tag{3.38}$$

From equation (3.38), we see that a positive value of λ would increase the drift of the force of mortality. To follow the same insurer's logic, by overestimating the survival process expected value and underestimating the mortality, we have to choose a negative value for λ.
Under the risk-neutral measure, the survival probability takes the form:

$$E_{\mathbb{Q}_\lambda}(I(x+t, T-t)) = e^{\alpha_{\mathbb{Q}_\lambda}^{HW}(t,T) - \beta_{\mathbb{Q}_\lambda}^{HW}(t,T)\mu_x(t)}, \tag{3.39}$$

where $\alpha_{\mathbb{Q}_\lambda}^{HW}(t,T)$ and $\beta_{\mathbb{Q}_\lambda}^{HW}(t,T)$ are given by:

$$\begin{cases} \alpha_{\mathbb{Q}_\lambda}^{HW}(t,T) & = \frac{A}{b}[e^{-bT}\frac{e^{(B+b)T}-e^{(B+b)t}}{B+b} - \frac{e^{BT}-e^{Bt}}{B}] - \frac{\sigma^2}{2b^2}[\frac{1}{b}(1-e^{-b(T-t)}) - T + t] \\ & \quad -\frac{\sigma^2}{4b^3}(1-e^{-b(T-t)})^2 - \frac{\sigma\lambda}{b}(1-\exp(-b(T-t)) \\ & \\ \beta_{\mathbb{Q}_\lambda}^{HW}(t,T) & = \frac{1}{b}(1-e^{(-b(T-t)}). \end{cases} \tag{3.40}$$

Hence, the price of the S-forward at time t with the risk-neutral approach is as follows:

$$V_{\mathbb{Q}_\lambda}^{HW}(t,T) = P(t,T)({}_tp_x^{obs}\, e^{\alpha_{\mathbb{Q}_\lambda}^{HW}(t,T) - \beta_{\mathbb{Q}_\lambda}^{HW}(t,T)\mu_x(t)} - {}_T\hat{p}_x). \tag{3.41}$$

CIR model

Let us now assume that $\mu_x(t)$ follows the CIR process given by (3.3).
We introduce the market prices $\lambda(t, \mu_x(t))$ associated with longevity risk, $w^*(t, \omega) = w(t, \omega) - \int_0^t \lambda(t, \mu_x(s))ds$.
Under the Real-world probability measure $\mathbb{P}$, w^* is a Brownian motion with drift, and therefore is not a martingale. According to Girsanov's theorem, $w^*(t, \omega)$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$.
The SDE of the force of mortality becomes:

$$d\mu_x^{\mathbb{Q}_\lambda}(t) = (\xi(t) - b\mu_x(t) + \lambda\sigma\sqrt{\mu_x(t)})dt + \sigma\sqrt{\mu_x(t)}dw^*(t). \tag{3.42}$$

The survival probability takes also the form: $E_{\mathbb{P}}(I(x+t, T-t)) = e^{\alpha_{\mathbb{Q}_\lambda}^{CIR}(t,T) - \beta_{\mathbb{Q}_\lambda}^{CIR}(t,T)\mu_x(t)}$
We assume that the market price of longevity risk has the following classical form: $\lambda = \kappa\sqrt{\mu_x(t)}$, where κ is a constant.

Equation (3.42) then becomes:

$$d\mu_x^{\mathbb{Q}_\lambda}(t) = (\xi(t) - (b - \sigma\kappa)\mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dw^*(t). \tag{3.43}$$

The parameters $\alpha^{CIR}_{\mathbb{Q}_\lambda}(t,T)$ and $\beta^{CIR}_{\mathbb{Q}_\lambda}(t,T)$ are given by:

$$\begin{cases} \alpha^{CIR}_{\mathbb{Q}_\lambda}(t,T) = 2A\int_t^T \frac{e^{Bs}(e^{\gamma(T-s)}-e^{-\gamma(T-s)})}{(2\gamma+b-\sigma\kappa)e^{\gamma(T-s)}+(2\gamma-b+\sigma\kappa)e^{-\gamma(T-s)}}ds \\ \beta^{CIR}_{\mathbb{Q}_\lambda}(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)+\frac{1}{2}(b-\sigma\kappa)\sinh(\gamma(T-t))}, \end{cases} \tag{3.44}$$

with $\gamma = \frac{((b-\sigma\kappa)^2+2\sigma^2)^{\frac{1}{2}}}{2}$.
Finally the price is given by:

$$V^{CIR}_{\mathbb{Q}_\lambda}(t,T) = P(t,T)[{}_tp_x^{obs}\, e^{\alpha^{CIR}_{\mathbb{Q}_\lambda}(t,T)-\beta^{CIR}_{\mathbb{Q}_\lambda}(t,T)\mu_x(t)} - {}_T\hat{p}_x]. \tag{3.45}$$

Wang transform approach

The Wang transform method is a distortion approach that is based on a distortion operator (Leung et al. [16]). In the insurance context, this operator converts the best estimate of the survival index into its risk equivalent using a specific price of risk. This procedure was presented by Dowd et al. [13] as a pricing method used in the pricing of several over-the-counter longevity swaps in practice, and used for instance by Denuit et al. [12])
The Wang distortion risk measure is given by:

$$\rho_{g_\delta}(x) = \int_0^\infty g_\delta(\bar{F}_x(s))ds, \tag{3.46}$$

where x is a continuous non-negative random variable.
$\bar{F}_x(s)$ is its decumulative function, and g_δ is the distortion function associated with the distortion parameter δ and given by:

$$g_\delta(s) = \Phi(\Phi^{-1}(s)+\delta), \tag{3.47}$$

where $\Phi(.)$ is the cumulative standard normal distribution
Then:

$$\rho_{g_\delta}(I(x+t,T-t)) = \int_0^\infty \Phi(\Phi^{-1}(\bar{F}_{I(x+t,T-t)}(s))+\delta)ds. \tag{3.48}$$

The price of the S-forward at time t is then given by:

$$V_{Wang}(t,T) = P(t,T)[{}_tp_x^{obs}\,\rho_{g_\delta}(I(x+t,T-t)) - {}_T\hat{p}_x \mid \mathcal{F}_t]. \tag{3.49}$$

HW model

To obtain an explicit expression of the S-forward price with HW model, we need to determine the formula of $\rho_{g_\delta}(I(x+t,T-t))$.

The Wang distortion of a log-normal distribution is still log-normal with a corrected drift parameter:

$$e^{N(m(t,T),\, n^2(t,T))} \longrightarrow e^{N(m(t,T)+\delta n(t,T), n^2(t,T))}, \tag{3.50}$$

so:

$$\rho_{g_\delta}(I(x+t, T-t)) = e^{m(t,T)+\delta n(t,T)+\frac{n(t,T)^2}{2}}. \tag{3.51}$$

The price of S-forward with Wang approach at time t is finally given by:

$$V^{HW}_{Wang}(t,T) = P(t,T)[{}_tp_x^{obs}\, e^{m(t,T)+\delta n(t,T)+\frac{n(t,T)^2}{2}} - {}_T\hat{p}_x)]. \tag{3.52}$$

CIR model

Because of its complicated distribution, it is not easy to find an explicit formula for $\rho_{g_\delta}(I(x+t, T-t))$. Hence, we will need an SDE discretization scheme.
For the CIR model, the Euler discretization scheme is as follows:

$$\mu_x(t+\Delta t) = \mu_x(t) + (\xi(t) - b\mu_x(t))\Delta t + \sigma\sqrt{\mu_x(t)}\Delta w(t). \tag{3.53}$$

In order to simulate the survival index $I(x+t, T-t)$, we make a classical assumption often encountered in life insurance theory: we suppose that the force of mortality is piecewise constant between each point computed by the above Euler discretization.

The Sharpe Ratio approach

The Sharpe ratio method was used for instance by Milevsky et al. [22] and by Loeys et al. [20] to price q-forwards. It is a standard deviation principle that does not use a transformation of survival probabilities.
In liquid financial markets, the Sharpe ratio provides a benchmark for the determination of risk premia. The Sharpe ratio is given by:

$$S^{market} = \frac{r_p - r_f}{\sigma_p}, \tag{3.54}$$

where r_p is the expected return of the portfolio, r_f is the risk free rate and σ_p is the standard deviation of the portfolio return.
Using the analogy to the Sharpe ratio in financial market, the Sharpe ratio in the insurance context (Barrieux and Verrart [1]) is equal to the excess payoff on the longevity-linked securities over its best estimate divided by its standard deviation.

$$S^{Insurance} = \frac{V(X) - E_{\mathbb{P}}(X)}{\sigma_{\mathbb{P}}(X)}. \tag{3.55}$$

According to this approach, the price of the S-forward is given by:

$$\begin{aligned} V_{Sharpe}(t,T) &= P(t,T)({}_tp_x^{obs}\, E_{\mathbb{P}}(\, I(x+t, T-t)) - {}_T\hat{p}_x \\ &+ S\,{}_tp_x^{obs}\sqrt{\mathrm{Var}_{\mathbb{P}}(I(x+t, T-t))}), \end{aligned} \tag{3.56}$$

where S is the chosen fixed Sharpe ratio, and $\text{Var}_{\mathbb{P}}(I(x+t,T-t))$ is the variance of the survival index.

HW model

The variance of the survival process is as follows:

$$\text{Var}_{\mathbb{P}}(I(x+t,T-t)) = V_{\mathbb{P}}(e^{X(t,T)}) = V_{\mathbb{P}}(e^{N(m(t,T),\,n(t,T)^2)}), \tag{3.57}$$

then:

$$\text{Var}_{\mathbb{P}}(I(x+t,T-t)) = (e^{n(t,T)^2}-1)e^{2m(t,T)+n(t,T)^2}. \tag{3.58}$$

The price of the S-forward at time t, is then given by:

$$\begin{aligned} V^{HW}_{Sharpe}(t,T) = P(t,T)\Big(& {}_t p^{obs}_x \, E_{\mathbb{P}}[\,I(x+t,T-t)] - {}_T\hat{p}_x \\ & - S\,{}_t p^{obs}_x \sqrt{(e^{n(t,T)^2}-1)e^{2m(t,T)+n(t,T)^2}}\,\Big). \end{aligned} \tag{3.59}$$

CIR model

We have:

$$\text{Var}_{\mathbb{P}}(I(x+t,T-t)) = E_{\mathbb{P}}[\,I(x+t,T-t)^2] - E_{\mathbb{P}}[\,I(x+t,T-t)]^2, \tag{3.60}$$

where:

$$E_{\mathbb{P}}[\,I(x+t,T-t)^2] = E_{\mathbb{P}}(e^{-\int_t^T 2\mu_x(s)ds}) = e^{\alpha^{CIR*}_{\mathbb{P}}(t,T)-\beta^{CIR*}_{\mathbb{P}}(t,T)2\mu_x(t)}, \tag{3.61}$$

and:

$$d2\mu_x(t) = ((2\xi(t)-2b\mu_x(t))dt + \sqrt{2}\sigma\sqrt{2\mu_x(t)}dw(t). \tag{3.62}$$

$\alpha^{CIR*}_{\mathbb{P}}(t,T)$ and $\beta^{CIR*}_{\mathbb{P}}(t,T)$ are then given by:

$$\begin{cases} \alpha^{CIR*}_{\mathbb{P}}(t,T) = 4A\int_t^T \frac{e^{Bs}(e^{\gamma^*(T-s)}-e^{-\gamma^*(T-s)})}{(2\gamma^*+b)e^{\gamma^*(T-s)}+(2\gamma^*-b)e^{-\gamma^*(T-s)}}ds \\ \beta^{CIR*}_{\mathbb{P}}(t,T) = \frac{\sinh(\gamma^*(T-t))}{\gamma^*\cosh(\gamma^*(T-t)+\frac{1}{2}b\sinh(\gamma^*(T-t))}, \end{cases} \tag{3.63}$$

with $\gamma^* = \frac{(b^2+4\sigma^2)^{\frac{1}{2}}}{2}$.

$\alpha^{CIR}_{\mathbb{P}}(t,T)$ and $\alpha^{CIR*}_{\mathbb{P}}(t,T)$ can be solved numerically. The variance of the survival index under the real world probability is:

$$\text{Var}_{\mathbb{P}}(I(x+t,T-t)) = e^{\alpha^{CIR*}_{\mathbb{P}}(t,T)-\beta^{CIR*}_{\mathbb{P}}(t,T)2\mu_x(t)} - e^{2[\alpha^{CIR}_{\mathbb{P}}(t,T)-\beta^{CIR}_{\mathbb{P}}(t,T)\mu_x(t)]}. \tag{3.64}$$

Finally, the price of the S-forward is as follows:

$$\begin{aligned} V^{CIR}_{Sharpe}(t,T) &= P(t,T)\Big({}_t p^{obs}_x \, e^{\alpha^{CIR}_{\mathbb{P}}(t,T)-\beta^{CIR}_{\mathbb{P}}(t,T)\mu_x(t)} - {}_T\hat{p}_x \\ &+ {}_t p^{obs}_x \, S\sqrt{e^{\alpha^{CIR*}_{\mathbb{P}}(t,T)-\beta^{CIR*}_{\mathbb{P}}(t,T)2\mu_x(t)} - e^{2[\alpha^{CIR}_{\mathbb{P}}(t,T)-\beta^{CIR}_{\mathbb{P}}(t,T)\mu_x(t)]}}\,\Big). \end{aligned} \tag{3.65}$$

3.3 Survival-Swaps

As we mentioned previously, the S-swap is considered as the sum of S-forward contracts.

3.3.1 Definition and structure

A Survival swap (or S-swap) is a collection of S-forwards exchanged at a predetermined calendar times $[t_1, ..., t_n]$. Schematically, an S-swap can be represented as follows:

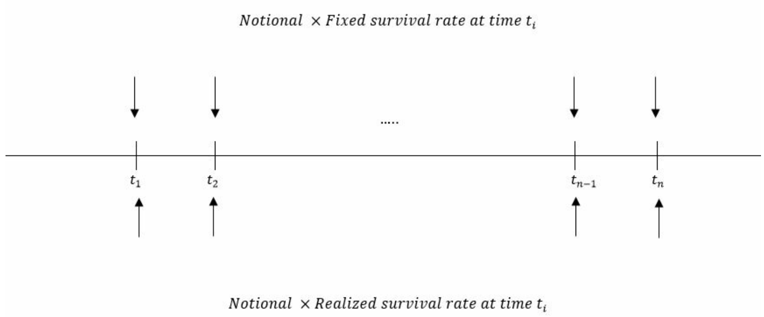

Figure 3.6: S-swap structure

3.3.2 S-swap Pricing under the cost of capital approach

Let $P(s, t_1, t_2, ..., t_n)$ denote the price of an S-swap at time s (with $s < t_1 < t < ... < t_n$). Under the assumption of non-existence of arbitrage opportunities, the price at time s of an S-Swap with $t_1, t_2, ..., t_n$ as exchange dates is given by:

$$P(s, t_1, t_2, ..., t_n) = \sum_{i=1}^{n} V(s, t_i), \tag{3.66}$$

where $V(s, t_i)$ is the S-forward price in an approach and a model that have to be specified.
Let us consider an S-swap with n cash flows. The price at time s of an S-swap is given by:

$$P_{COC}(s, t_1, t_2, ..., t_n) = \sum_{j=1}^{n} BE_{s,t_j} + \sum_{j=1}^{n} RM_{s,t_j}, \tag{3.67}$$

where BE_{s,t_j} and RM_{s,t_j} are as follows:

$$BE_{s,t_j} = P(s, t_j)[I(x, s)E_{\mathbb{P}}(\, I(x+s, t_j - s)) - {}_{t_j}\hat{p}_x], \tag{3.68}$$

and

$$RM_{s,t_j} = 6\% \sum_{i=s}^{t_j - 1} SCR_{i,t_j} \mid_s \; P(s, i+1). \tag{3.69}$$

We finally find the price of the S-swap at time s:

$$\begin{aligned} P_{COC}(s,t_1,t_2,...,t_n) &= \sum_{j=1}^{n}\Big[P(s,t_j)(I(x,s)\,E_{\mathbb{P}}(\,I(x+s,t_j-s)) - {}_{t_j}\hat{p}_x) \\ &+ 6\%\sum_{i=s}^{t_j-1}P(i,t_j)P(s,i+1)I(x,s)\,E_{\mathbb{P}}(\,I(x+s,i-s))[\mathrm{VaR}_{99,5\%}(I(x+i,1)) \\ &- E_{\mathbb{P}}(\,I(x+i,1))]E_{\mathbb{P}}(\,I(x+i+1,t_j-i-1))\Big]. \end{aligned} \quad (3.70)$$

HW model

If we consider that the force of mortality follows the HW process, the price under the Cost of Capital approach is given by:

$$\begin{aligned} P^{HW}_{COC}(s,t_1,t_2,...,t_n) &= \sum_{i=1}^{n}\Big[P(s,t_j)({}_sp_x^{obs}\,e^{\alpha_{\mathbb{P}}^{HW}(s,t_i)-\beta_{\mathbb{P}}^{HW}(s,t_i)\mu_x(s)} - {}_{t_j}\hat{p}_x) \\ &+ 6\%\sum_{i=s}^{t_j-1}P(i,t_j)P(s,i+1)_s p_x^{obs}[e^{\alpha_{\mathbb{P}}^{HW}(s,i)-\beta_{\mathbb{P}}^{HW}(s,i)\mu_x(s)}(e^{m(i,i+1)+z_\alpha n(i,i+1)}) \\ &- e^{\alpha_{\mathbb{P}}^{HW}(i,i+1)-\beta_{\mathbb{P}}^{HW}(i,i+1)\mu_x(i)}e^{\alpha_{\mathbb{P}}^{HW}(i+1,t_i)-\beta_{\mathbb{P}}^{HW}(i+1,t_i)\overline{\mu_x}(i+1)}\Big]. \end{aligned} \quad (3.71)$$

CIR model

If we consider that the force of mortality follows the CIR process, the price under the Cost of Capital approach is given by:

$$\begin{aligned} P^{CIR}_{COC}(s,t_1,t_2,...,t_n) &= \sum_{i=1}^{n}\Big[P(s,t_j)({}_sp_x^{obs}\,e^{\alpha_{\mathbb{P}}^{CIR}(s,t_i)-\beta_{\mathbb{P}}^{CIR}(s,t_i)\mu_x(s)} - {}_{t_j}\hat{p}_x) \\ &+ 6\%\sum_{i=s}^{t_j-1}P(i,t_j)P(s,i+1)_s p_x^{obs}[e^{\alpha_{\mathbb{P}}^{CIR}(s,i)-\beta_{\mathbb{P}}^{CIR}(s,i)\mu_x(s)}\,\mathrm{VaR}_{99,5\%}(e^{-\int_i^{i+1}\mu_x(s)ds}) \\ &- e^{\alpha_{\mathbb{P}}^{CIR}(i,i+1)-\beta_{\mathbb{P}}^{CIR}(i,i+1)\mu_x(i)}e^{\alpha_{\mathbb{P}}^{CIR}(i+1,t_i)-\beta_{\mathbb{P}}^{CIR}(i+1,t_i)\overline{\mu_x}(i+1)}\Big]. \end{aligned} \quad (3.72)$$

3.4 Numerical illustration

In this section we give the numerical results of the prices of different S-forwards, using the approaches studied previously for the Belgian population.
We use the following assumptions:

- An insurer with a portfolio of pure endowment contracts paying a lump sum of 1€ at maturity T, and who is interested in hedging the longevity risk;
- $N_0 = 10000$ initial policyholders for each cohort;
- Individuals aged 65, 70, or 75 years old in 2015;
- Payment of 1€ to each policyholder alive at time T;

- Calibration and fixed leg based on projected data from the IA|BE unisex projected generational mortality table[2];
- The riskless interest rate is constant (equals to 1%).

The fixed rates used for different ages and maturities are reported in Table 3.1:

T/Age	**65**	**70**	**75**
$\mu_x(0)$	0.0105677	0.0160885	0.0263359
$T=5$	0.9419321	0.9101241	0.8507335
$T=10$	0.8658090	0.7865578	0.6473974

Table 3.1: The fixed survival rates ${}_T\hat{p}_x$ for different ages and maturities.

3.4.1 Calibration

We calibrate our models to the data, using the Least Square Estimation method.

HW model

In order to determine the parameters of the survival function, we calibrate the HW model on projected mortality data for each cohort as explained in the beginning of this section.
The optimal parameters are given by Table 3.2:

Age	A	B	b	σ
65	0.00231775	0.11562220	0.25062948	0.01770006
70	0.00517446	0.11645594	0.32024870	0.02352572
75	0.01008112	0.11656453	0.32687591	0.06456541

Table 3.2: Optimal parameters values for the survival function in the HW model

CIR model

The optimal parameters for the three generations are summarized in Table 3.3:

Age	A	B	b	σ
65	0.00239811	0.11537936	0.26181448	0.00186426
70	0.00507981	0.11650159	0.31192722	0.00621368
75	0.01573756	0.11389749	0.55079961	0.02816582

Table 3.3: Optimal parameters values for the survival function in the CIR model

[2]IA|BE Mortality projection for the Belgian population, 2015. Available from: https://www.iabe.be/nl/iabe-mortality-tables

3.4.2 S-forwards COC prices

Table 3.4 gives the the prices found under the COC approach, using HW and CIR models for mortality:

	HW model					CIR model				
	BE	RM	Price	BE%	RM%	BE	RM	Price	BE%	RM%
$x_0 = 65, T = 5$	45.87512	6.75613	52.63125	87.16%	12.83%	44.28595	10.68733	54.97327	80.55%	19.44%
$x_0 = 65, T = 10$	76.53061	11.86875	88.39936	86.57%	13.42%	74.69407	22.70057	97.39464	76.69%	23.30%
$x_0 = 70, T = 5$	28.58322	21.38396	49.96718	57.20%	42.79%	28.47494	16.53768	45.01261	63.25%	36.74%
$x_0 = 70, T = 10$	26.35731	35.16336	61.52067	42.84%	57.15%	24.18322	33.45732	57.64054	41.95%	58.04%
$x_0 = 75, T = 5$	7.24736	75.30577	82.55310	91.22%	8.78%	-4.96041	55.56708	50.60666	-9.80%	109.80%
$x_0 = 75, T = 10$	-11.20288	112.83990	101.63710	-11.02%	111.02%	-14.28582	90.50064	76.21482	-18.74%	118.74%

Table 3.4: Comparison between the prices of the different S-forwards with HW and CIR models

In order to have a better idea of the magnitude of these S-forward prices, we can link them with the original price of the pure endowment contract received by the insurer. The premium P of this pure endowment contract is given by: ${}_TE_x = \frac{{}_T\hat{p}_x}{(1+i)^T}$, with i the technical interest rate.

Table 3.5 reports a comparison between the premium P and the price of the S-forward S_price, considering a technical rate of 0.5% and using COC approach for both models of longevity.

	HW model			CIR model		
	P	S_price	HC	P	S_price	HC
$x_0 = 65, T = 5$	9187.329	52.63125	0.57%	9187.329	54.97327	0.60%
$x_0 = 65, T = 10$	8444.847	88.39936	1.04%	8444.847	97.39464	1.15%
$x_0 = 70, T = 5$	8297.805	49.96718	0.60%	8297.805	45.01261	0.54%
$x_0 = 70, T = 10$	7671.854	61.52067	0.80%	7671.854	57.64054	0.75%
$x_0 = 75, T = 5$	8877.084	82.55310	0.93%	8877.084	50.60666	0.57%
$x_0 = 75, T = 10$	6314.525	101.63710	1.61%	6314.525	76.21482	1.20%

Table 3.5: Comparison between the prices of the pure endowment and S-forward contracts

We can see that the hedging cost denoted by $HC = \frac{S\_price}{P}$ is around 1%.

3.4.3 Consistency with the classical pricing methods

In this section, we compare numerically the COC approach with other methods developed in subsection 3.2.4. In order to check the consistency between COC and each of these methods, we compute the value of the parameters of the three methods; these parameters are computed such that the COC price is equal to the price given by the considered methods. We can say that a method is consistent with the COC approach if its parameters are quiet stable for different ages and maturities.

Risk-neutral approach

In this method we determine the market price of longevity risk, generating the same price as the one found with the COC method.

HW model

Under the risk-neutral method, and under the assumptions considered previously, the price at time 0 is:

$$V_{\mathbb{Q}_\lambda}^{HW}(0,T) = e^{-rT}\big(\, e^{\alpha_{\mathbb{Q}_\lambda}^{HW}(0,T)-\beta_{\mathbb{Q}_\lambda}^{HW}(0,T)\mu_x(0)} - {}_T\hat{p}_x\big). \tag{3.73}$$

The market price of risk λ is determined such that this risk-neutral price is equal to the cost of capital price. If this coefficient is stable, it means that the two methods are consistent. Table 3.6 gives the values of λ when we use the HW model for longevity:

S-forward	COC price	Market price of longevity risk
$x_0 = 65, T = 5$	52.63125	λ= -14.40%
$x_0 = 65, T = 10$	88.39936	λ= -22.61%
$x_0 = 70, T = 5$	49.96718	λ= -15.53%
$x_0 = 70, T = 10$	61.52067	λ= -25.47%
$x_0 = 75, T = 5$	82.55310	λ= -19.33%
$x_0 = 75, T = 10$	101.63710	λ= -37.53%

Table 3.6: Equivalent market price of longevity risk in the risk-neutral approach, HW model

CIR model

Under the risk-neutral approach, and considering that the intensity of mortality follows the CIR model, the price of the S-forward is:

$$V_{\mathbb{Q}_\lambda}^{HW}(0,T) = e^{-rT}\big(\, e^{\alpha_{\mathbb{Q}_\lambda}^{CIR}(0,T)-\beta_{\mathbb{Q}_\lambda}^{CIR}(0,T)\mu_x(0)} - {}_T\hat{p}_x\big). \tag{3.74}$$

Table 3.7 represents the different values of the market price of longevity risk.

S-forward	COC price	κ	Market price of longevity risk $\lambda = \kappa\sqrt{\mu_x(0)}$
$x_0 = 65, T = 5$	54.97327	-0.743	λ= -7.64%
$x_0 = 65, T = 10$	97.39464	-0.533	λ= -5.48%
$x_0 = 70, T = 5$	45.01261	-0.609	λ= -7.72%
$x_0 = 70, T = 10$	57.64054	-0.436	λ= -5.53%
$x_0 = 75, T = 5$	50.60666	-0.468	λ= -7.06%
$x_0 = 75, T = 10$	76.21482	-0.297	λ= -4.83%

Table 3.7: Equivalent market price of longevity risk in the risk-neutral approach, CIR model

Clearly, the two methods are not completely consistent, even if the parameters seem to be more stable in the CIR model.

Consistency with Sharpe Ratio

We will now compute the value of the Sharpe ratio corresponding to the S-forward price found with the COC method.

HW model

The price with this approach using HW model is given by:

$$V^{HW}_{Sharpe}(0,T) = e^{-rT}\left(e^{\alpha^{HW}_{\mathbb{P}}(0,T)-\beta^{HW}_{\mathbb{P}}(0,T)\mu_x(0)} - {}_T\hat{p}_x + S\sqrt{(e^{n(0,T)^2}-1)e^{2m(0,T)+n(0,T)^2}}\right). \tag{3.75}$$

Table 3.8 gives the equivalent Sharpe ratio generating the prices found with COC approach.

S-forward	COC price	Sharpe parameter
$x_0 = 65, T = 5$	52.63125	S=9.62%
$x_0 = 65, T = 10$	88.39936	S=9.55%
$x_0 = 70, T = 5$	49.96718	S=10.13%
$x_0 = 70, T = 10$	61.52067	S=10.43%
$x_0 = 75, T = 5$	82.55310	S=11.47%
$x_0 = 75, T = 10$	101.63710	S=13.91%

Table 3.8: Equivalent Sharpe parameters, HW model

CIR model

With the CIR model, the price of the S-forward at time 0 is given by:

$$\begin{aligned} V^{CIR}_{Sharpe}(0,T) &= e^{-rT}\Big(e^{\alpha^{CIR}_{\mathbb{P}}(0,T)-\beta^{CIR}_{\mathbb{P}}(t,T)\mu_x(t)} - {}_T\hat{p}_x \\ &+ S\sqrt{e^{\alpha^{CIR*}_{\mathbb{P}}(0,T)-\beta^{CIR*}_{\mathbb{P}}(0,T)2\mu_x(0)} - e^{2[\alpha^{CIR}_{\mathbb{P}}(0,T)-\beta^{CIR}_{\mathbb{P}}(0,T)\mu_x(0)]}}\Big) \end{aligned} \tag{3.76}$$

Table 3.9 presents the different values of the Sharpe parameter under the CIR model:

S-forward	COC price	Sharpe parameter
$x_0 = 65, T = 5$	54.97327	S=15.35%
$x_0 = 65, T = 10$	97.39464	S=17.44%
$x_0 = 70, T = 5$	45.01261	S=16.17%
$x_0 = 70, T = 10$	57.64054	S=17.55%
$x_0 = 75, T = 5$	50.60666	S=16.45%
$x_0 = 75, T = 10$	76.21482	S=17.61%

Table 3.9: Equivalent Sharpe parameters, CIR model

We can see that for both mortality models, Sharpe parameters have a better stability.

Consistency with Wang transform

We will now compute the value of the Wang transform parameter corresponding to the COC S-forward price.

HW model

Table 3.10 gives the different values of the Wang parameter under the CIR model:

S-forward	COC price	Wang parameter
$x_0 = 65, T = 5$	52.63125	$\delta = 9.62\%$
$x_0 = 65, T = 10$	88.39936	$\delta = 9.55\%$
$x_0 = 70, T = 5$	49.96718	$\delta = 10.12\%$
$x_0 = 70, T = 10$	61.52067	$\delta = 10.41\%$
$x_0 = 75, T = 5$	82.55310	$\delta = 11.44\%$
$x_0 = 75, T = 10$	101.63710	$\delta = 13.48\%$

Table 3.10: Equivalent Wang parameters, HW model

CIR model

For the CIR model we use numerical simulation to compute the Wang Transform parameters corresponding to the COC prices. We get the following results:
Table 3.11 gives the different values of the Wang parameter under the CIR model:

S-forward	COC price	Wang parameter
$x_0 = 65, T = 5$	54.97327	$\delta = 16.29\%$
$x_0 = 65, T = 10$	97.39464	$\delta = 16.49\%$
$x_0 = 70, T = 5$	45.01261	$\delta = 16.89\%$
$x_0 = 70, T = 10$	57.64054	$\delta = 17.91\%$
$x_0 = 75, T = 5$	50.60666	$\delta = 16.55\%$
$x_0 = 75, T = 10$	76.21482	$\delta = 17.75\%$

Table 3.11: Equivalent Wang parameters, CIR model

Synthesis

The Cost of Capital approach is parameterized by one variable: the Cost of Capital rate, which is fixed by regulation (6% in Solvency II nowadays). This legal and unique parameter value can be seen as a benchmark, and allows us to find equivalent values of the parameters in the three other methods. The simulations presented above show that:

- None of the classical pricing methods is completely consistent with the COC approach, which is not surprising (the COC price (formula (3.26)) is based on a multiperiod formula);

- Sharpe ratio and Wang transform methods seem closer to COC than risk-neutral approach;

- The values found for Sharpe and Wang methods are quite similar. In Appendix A we present for the HW model, complete results for ages between 55 and 85, and maturities from $T = 1$ to $T = 15$.

Remark: smoothing

Taking into account the irregular form of all these prices due to the use of gross mortality parameters for each age (formulas (3.2) and (3.3)), we can smooth these parameters. Appendix B gives the methods used for this smoothing. Appendix C gives then the smoothed prices with COC method in the HW model.

3.5 Conclusion

In this paper, we have developed in continuous-time a new way to price longevity linked securities based on the Cost of Capital approach, consistent with the directive of Solvency II. We considered two examples of stochastic time-continuous processes for longevity, Hull & White and CIR extended. This COC method has been compared with three classical methods used in finance: the risk-neutral, Wang and Sharpe methods. More precisely, we have found the parameters of these three methods giving equivalent prices to the COC approach with 6% COC charge, as mentioned in the Solvency II regulation. Even if none of these methods is consistent with the COC approach, it appears that Wang transform and Sharpe ratio methods are more similar to COC than risk-neutral method. In this paper we did not take into account the basis risk. We could use this proposed COC method to price other longevity linked securities that take into account this risk, for instance the exchange forwards and exchanged options.

Appendices

Appendix A

COC prices of different S-forwards, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	2,1768706	7,645950536	16,33313671	27,66480999	40,64993112	55,44692673	70,66711363	86,36881849	101,2928691	114,9826224	127,4397041	139,2642165	147,1956821	153,7583705	158,1290551
56	1,804452614	7,141070861	15,44932364	25,64332178	37,97904649	50,97117035	64,7213424	77,95695133	90,13311681	101,3221848	112,1431302	119,2556828	125,2720068	129,352999	129,8719672
57	1,811826103	7,648725775	16,05012504	27,16993677	39,2369969	52,26699877	64,86198942	76,29292926	86,64487884	96,5100457	102,4248649	107,0729366	109,5819853	108,2974216	104,4799611
58	1,921457249	7,920167398	18,05871237	30,2542374	44,42009981	59,00887794	73,04967574	86,6096054	100,1962391	110,158004	119,1750683	126,2555655	129,6155118	130,4268428	129,2984779
59	2,642948583	9,943257288	19,7747718	32,17534381	45,62022971	59,01751402	72,52673467	86,66302449	97,63954671	108,2011827	117,2772806	122,9959048	126,4789669	128,3496366	126,0893723
60	2,215083941	8,711412336	19,14726772	31,62991976	44,62915064	58,19946652	72,70600368	84,11846129	95,18929064	104,7365807	110,7931071	114,4571608	116,4118338	114,0732334	108,1408484
61	4,735216291	14,2038521	27,11310808	42,0329614	59,2169846	78,98996111	97,05021834	116,068152	134,6011976	150,41367	164,388671	177,0825281	185,1151056	188,6430333	187,5581031
62	6,587732845	21,23944835	39,32925734	59,44172877	80,89562805	98,73259696	115,5543037	129,8827373	139,5706718	145,7215307	149,2412294	147,4938866	141,2423595	130,6609384	117,3882424
63	5,482982277	13,92642322	25,71882976	41,11183479	55,43115179	71,53940415	87,62181391	100,9920245	112,4788675	122,7808907	128,8743127	131,1887049	129,5609005	125,440654	118,4013969
64	1,508059137	7,612287022	18,09791488	27,79522377	39,48672657	51,37172298	60,92901863	68,98958396	76,37911249	80,02344759	80,35525739	77,20176768	72,15024695	64,74788032	55,37888244
65	2,696896474	12,93272397	24,06172239	38,11826228	52,61632773	64,56731471	74,55632498	83,3903719	87,85452468	88,34422041	84,67728092	78,63810515	69,82053955	58,74166476	44,13148303
66	4,614823032	14,70019181	30,17790943	47,12023085	61,68631284	74,04400261	84,92702183	90,96105054	92,49951057	89,34859731	83,49987579	74,57252128	63,19150347	48,08872992	34,05617679
67	4,645759801	18,00554077	34,873336	50,55239268	64,70876231	77,89160123	86,41877207	90,45424822	89,6479275	86,09414769	79,31815963	69,95160513	56,6227264	44,20118202	29,39681558
68	2,842969562	11,86213427	22,31493623	33,65537116	46,25839593	55,97295051	62,53705859	65,18465349	65,89000355	63,8451608	59,47580749	51,1013859	43,49644597	33,00643291	20,33470013
69	2,423335339	9,828537047	20,1979862	33,14436205	43,82134841	51,53842873	55,23865342	56,99509509	55,8901631	52,37936921	44,69019465	37,7324747	27,66516155	15,31667731	5,200264764
70	3,66017856	12,72158932	26,38434097	39,15575211	49,83012195	56,89394113	62,38635228	65,09191902	65,33527725	61,08193416	57,24732846	49,6641262	39,18943547	30,4493177	19,14663223
71	6,437697912	19,36139302	34,51532037	50,71416162	65,91566247	81,8433937	96,38994075	107,9478129	113,8963906	118,9405067	118,4270687	113,184392	107,9791853	98,28643241	93,46497565
72	9,626547277	25,60957253	44,22020037	61,77712352	79,62024634	95,0679164	106,9139386	112,5812831	117,1533352	115,9158426	109,9975096	104,5087689	94,71621509	90,2923888	85,77119758
73	0,708690787	7,745575089	14,88434237	22,8014924	28,79186491	32,74639203	31,8996754	31,51384471	26,57459596	18,33593112	12,12373136	2,818518646	0,335971796	-1,107867918	2,703995403
74	1,324452393	6,790093136	15,94248958	24,83985865	32,66650309	35,92712688	39,71020253	38,41220462	33,27380515	29,79448071	22,42079287	21,40947446	20,73149664	24,48855191	27,4313317
75	8,96517587	26,27183403	46,65248463	67,41543229	82,55313459	96,81027227	103,9153274	105,1664864	106,3800282	101,6370614	101,7319246	100,5097345	102,0912462	101,5189943	105,4746949
76	11,37193116	31,32023918	53,85898223	71,1704332	87,66077103	96,48468211	99,14833263	101,8914638	98,35389107	99,85919291	99,95691176	102,592183	102,9073232	107,5287843	101,8696323
77	9,658499145	28,94395364	49,73262865	74,55628636	92,14289136	102,8797711	112,9217768	115,101614	121,2465898	124,5760643	128,9285846	129,7131393	133,6854071	126,1820471	116,7579353
78	-0,447350425	4,836355716	15,07039139	19,82820347	20,03731989	22,47408563	19,39108984	23,30984615	26,94946238	33,74686107	39,01944187	49,16110779	48,80895259	47,43346191	43,88340405
79	5,622704323	27,57268564	52,85582276	76,69639617	102,9355754	121,3566417	144,5677921	164,4761678	183,5556911	194,3075811	204,0164902	197,9108021	186,88201	170,7733475	150,5928846
80	-13,91536234	-22,21973085	-27,24202708	-23,81954003	-23,54800789	-13,59881353	-3,355733165	9,357538472	19,83428968	34,13432718	35,90610776	35,53945197	31,97247623	25,53022891	16,8675239
81	6,156511574	21,67351482	52,33034683	80,3326296	116,3830377	148,6407925	174,412026	190,1447675	203,0038736	197,2817564	185,2499442	167,0297243	144,14982	118,3409479	91,46531186
82	-23,40305252	-30,8710047	-38,56783304	-32,12126964	-23,02208561	-10,24862174	1,873272233	19,04155593	23,24779093	25,8751564	25,36567381	21,80115739	15,6396906	7,693179186	-0,966380885
83	-21,99351947	-37,28919886	-32,59446789	-22,3467536	-7,36699892	7,618202923	27,66777175	33,50429934	37,19812275	36,98416167	32,99747555	25,8239165	16,45087398	6,146465409	-3,73060085
84	-24,35691878	-21,96495376	-11,31965378	4,5534146	20,72940225	41,78248461	47,12300994	49,90577873	48,24442713	42,41805974	33,19167911	21,74362393	9,519340755	-1,981304238	-11,45051649
85	-31,8519033	-41,192253	-36,04810097	-23,36502	-0,732724167	8,299605391	17,03516395	22,32444309	23,62348514	21,04500877	15,29301645	7,52509245	-0,861874474	-8,49566101	-14,28804438

Figure 3.A.1: S-forward prices under the Cost of Capital approach, HW model

Consistency COC / Risk neutral, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	-8,99%	-9,51%	-10,04%	-10,59%	-11,15%	-11,73%	-12,33%	-12,94%	-13,57%	-14,21%	-14,87%	-15,54%	-16,23%	-16,93%	-17,64%
56	-8,97%	-9,44%	-9,92%	-10,42%	-10,92%	-11,44%	-11,97%	-12,51%	-13,07%	-13,64%	-14,22%	-14,81%	-15,42%	-16,03%	-16,66%
57	-9,01%	-9,64%	-10,30%	-10,98%	-11,68%	-12,41%	-13,16%	-13,94%	-14,74%	-15,56%	-16,40%	-17,26%	-18,15%	-19,05%	-19,97%
58	-9,01%	-9,65%	-10,32%	-11,01%	-11,73%	-12,47%	-13,24%	-14,04%	-14,85%	-15,69%	-16,55%	-17,44%	-18,34%	-19,27%	-20,21%
59	-8,99%	-9,50%	-10,02%	-10,55%	-11,10%	-11,67%	-12,24%	-12,84%	-13,44%	-14,06%	-14,69%	-15,34%	-16%	-16,67%	-17,36%
60	-9,03%	-9,71%	-10,43%	-11,18%	-11,95%	-12,76%	-13,59%	-14,45%	-15,34%	-16,25%	-17,19%	-18,15%	-19,13%	-20,14%	-21,16%
61	-4,15%	-4,45%	-4,78%	-5,14%	-5,54%	-5,98%	-6,49%	-7,06%	-7,71%	-8,44%	-9,28%	-10,23%	-11,21%	-12,17%	-13,12%
62	-9,28%	-10,96%	-12,81%	-14,80%	-16,94%	-19,20%	-21,57%	-24,03%	-26,56%	-29,16%	-31,81%	-34,50%	-37,22%	-39,96%	-42,73%
63	-7,07%	-7,64%	-8,28%	-8,99%	-9,80%	-10,71%	-11,67%	-12,60%	-13,53%	-14,45%	-15,38%	-16,32%	-17,27%	-18,22%	-19,19%
64	-9,04%	-9,77%	-10,53%	-11,33%	-12,16%	-13,03%	-13,92%	-14,85%	-15,81%	-16,80%	-17,81%	-18,85%	-19,91%	-21,00%	-22,11%
65	-9,15%	-10,33%	-11,60%	-12,96%	-14,40%	-15,91%	-17,49%	-19,14%	-20,85%	-22,61%	-24,41%	-26,26%	-28,15%	-30,06%	-32,01%
66	-9,29%	-11,06%	-13,02%	-15,14%	-17,41%	-19,81%	-22,33%	-24,94%	-27,63%	-30,38%	-33,18%	-36,03%	-38,90%	-41,80%	-44,71%
67	-9,30%	-11,11%	-13,10%	-15,27%	-17,58%	-20,03%	-22,60%	-25,26%	-28,00%	-30,80%	-33,65%	-36,54%	-39,46%	-42,40%	-45,36%
68	-9,16%	-10,37%	-11,66%	-13,04%	-14,50%	-16,04%	-17,65%	-19,32%	-21,05%	-22,84%	-24,67%	-26,54%	-28,45%	-30,39%	-32,36%
69	-9,23%	-10,73%	-12,36%	-14,11%	-15,98%	-17,96%	-20,04%	-22,20%	-24,43%	-26,72%	-29,06%	-31,45%	-33,87%	-36,32%	-38,79%
70	-9,22%	-10,63%	-12,16%	-13,79%	-15,53%	-17,36%	-19,28%	-21,28%	-23,35%	-25,47%	-27,64%	-29,86%	-32,11%	-34,38%	-36,69%
71	-5,42%	-6,34%	-7,39%	-8,61%	-10,01%	-11,63%	-13,46%	-15,26%	-17,07%	-18,89%	-20,72%	-22,57%	-24,44%	-26,33%	-28,22%
72	-6,34%	-7,93%	-9,84%	-12,11%	-14,79%	-17,78%	-20,82%	-23,91%	-27,05%	-30,23%	-33,43%	-36,66%	-39,89%	-43,14%	-46,39%
73	-9,43%	-11,77%	-14,40%	-17,28%	-20,36%	-23,62%	-27,02%	-30,51%	-34,08%	-37,69%	-41,35%	-45,03%	-48,73%	-52,44%	-56,15%
74	-9,40%	-11,51%	-13,86%	-16,43%	-19,18%	-22,08%	-25,12%	-28,25%	-31,46%	-34,72%	-38,03%	-41,36%	-44,72%	-48,09%	-51,47%
75	-7,51%	-9,85%	-12,70%	-15,96%	-19,33%	-22,82%	-26,41%	-30,07%	-33,78%	-37,53%	-41,31%	-45,09%	-48,88%	-52,67%	-56,46%
76	-8,54%	-11,96%	-16,03%	-20,36%	-24,92%	-29,64%	-34,45%	-39,32%	-44,22%	-49,13%	-54,04%	-58,95%	-63,85%	-68,74%	-73,63%
77	-7,08%	-9,56%	-12,59%	-16,25%	-20,08%	-24,02%	-28,05%	-32,13%	-36,26%	-40,41%	-44,56%	-48,72%	-52,87%	-57,02%	-61,16%
78	-9,78%	-13,45%	-17,66%	-22,28%	-27,17%	-32,23%	-37,39%	-42,61%	-47,85%	-53,11%	-58,38%	-63,64%	-68,90%	-74,16%	-79,41%
79	-4,08%	-5,83%	-7,98%	-10,57%	-13,60%	-17,11%	-21,10%	-25,61%	-30,61%	-35,57%	-40,51%	-45,43%	-50,31%	-55,16%	-59,99%
80	-9,29%	-11,01%	-12,91%	-14,96%	-17,15%	-19,48%	-21,91%	-24,44%	-27,04%	-29,71%	-32,42%	-35,18%	-37,97%	-40,78%	-43,60%
81	-5,58%	-7,70%	-10,30%	-13,41%	-17,07%	-21,28%	-25,51%	-29,76%	-34,02%	-38,28%	-42,52%	-46,75%	-50,96%	-55,15%	-59,31%
82	-9,35%	-11,41%	-13,70%	-16,20%	-18,88%	-21,72%	-24,68%	-27,74%	-30,89%	-34,09%	-37,34%	-40,62%	-43,92%	-47,24%	-50,57%
83	-9,34%	-11,24%	-13,32%	-15,59%	-18,01%	-20,58%	-23,26%	-26,04%	-28,90%	-31,82%	-34,78%	-37,78%	-40,80%	-43,85%	-46,90%
84	-9,44%	-11,67%	-14,17%	-16,90%	-19,82%	-22,91%	-26,13%	-29,44%	-32,83%	-36,27%	-39,74%	-43,25%	-46,77%	-50,30%	-53,83%
85	-9,34%	-11,33%	-13,53%	-15,93%	-18,51%	-21,23%	-24,08%	-27,02%	-30,05%	-33,14%	-36,27%	-39,44%	-42,63%	-45,84%	-49,06%

Figure 3.A.2: Consistency between risk-neutral and Cost of Capital method, HW model

Consistency COC / Sharpe , HW model

Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	15,34%	11,31%	9,62%	8,67%	8,06%	7,64%	7,33%	7,11%	6,94%	6,81%	6,72%	6,65%	6,59%	6,55%	6,53%
56	15,34%	11,27%	9,55%	8,57%	7,94%	7,50%	7,18%	6,94%	6,76%	6,62%	6,50%	6,42%	6,35%	6,30%	6,26%
57	15,34%	11,41%	9,78%	8,88%	8,32%	7,95%	7,68%	7,50%	7,37%	7,28%	7,22%	7,18%	7,16%	7,16%	7,16%
58	15,34%	11,42%	9,80%	8,90%	8,35%	7,97%	7,72%	7,54%	7,41%	7,33%	7,27%	7,23%	7,22%	7,22%	7,23%
59	15,35%	11,31%	9,61%	8,65%	8,03%	7,60%	7,29%	7,06%	6,89%	6,76%	6,65%	6,57%	6,52%	6,47%	6,44%
60	15,34%	11,46%	9,86%	8,99%	8,46%	8,10%	7,86%	7,70%	7,58%	7,51%	7,47%	7,45%	7,44%	7,45%	7,48%
61	7,05%	5,25%	4,52%	4,14%	3,92%	3,81%	3,76%	3,77%	3,82%	3,91%	4,03%	4,20%	4,36%	4,50%	4,62%
62	15,36%	12,30%	11,27%	10,88%	10,76%	10,80%	10,92%	11,10%	11,32%	11,55%	11,81%	12,07%	12,34%	12,61%	12,88%
63	12,01%	9,00%	7,82%	7,23%	6,92%	6,79%	6,74%	6,70%	6,67%	6,66%	6,66%	6,67%	6,69%	6,72%	6,75%
64	15,34%	11,50%	9,93%	9,08%	8,56%	8,22%	8,00%	7,85%	7,75%	7,69%	7,66%	7,66%	7,67%	7,69%	7,73%
65	15,34%	11,88%	10,58%	9,94%	9,62%	9,46%	9,40%	9,41%	9,47%	9,55%	9,66%	9,79%	9,93%	10,08%	10,23%
66	15,35%	12,36%	11,39%	11,04%	10,97%	11,04%	11,20%	11,41%	11,65%	11,92%	12,20%	12,49%	12,78%	13,07%	13,37%
67	15,36%	12,40%	11,44%	11,11%	11,04%	11,13%	11,30%	11,52%	11,77%	12,05%	12,33%	12,63%	12,93%	13,23%	13,53%
68	15,36%	11,91%	10,61%	9,99%	9,67%	9,51%	9,46%	9,47%	9,53%	9,62%	9,73%	9,86%	10,01%	10,16%	10,31%
69	15,35%	12,14%	11,02%	10,53%	10,34%	10,30%	10,36%	10,48%	10,63%	10,82%	11,02%	11,23%	11,45%	11,68%	11,91%
70	15,38%	12,09%	10,90%	10,37%	10,13%	10,06%	10,07%	10,15%	10,28%	10,43%	10,60%	10,78%	10,98%	11,18%	11,38%
71	9,07%	7,26%	6,69%	6,55%	6,62%	6,83%	7,14%	7,39%	7,63%	7,85%	8,05%	8,26%	8,45%	8,65%	8,83%
72	10,39%	8,72%	8,43%	8,61%	9,06%	9,61%	10,13%	10,61%	11,07%	11,52%	11,95%	12,38%	12,79%	13,18%	13,57%
73	15,36%	12,81%	12,16%	12,07%	12,23%	12,52%	12,87%	13,26%	13,68%	14,10%	14,52%	14,95%	15,37%	15,78%	16,19%
74	15,39%	12,66%	11,87%	11,67%	11,73%	11,93%	12,20%	12,52%	12,86%	13,22%	13,58%	13,95%	14,32%	14,68%	15,05%
75	12,18%	10,64%	10,62%	11,02%	11,47%	11,95%	12,44%	12,93%	13,42%	13,91%	14,39%	14,86%	15,32%	15,77%	16,21%
76	13,55%	12,43%	12,75%	13,31%	13,98%	14,68%	15,40%	16,11%	16,81%	17,49%	18,16%	18,80%	19,43%	20,04%	20,63%
77	11,40%	10,17%	10,33%	10,99%	11,65%	12,29%	12,92%	13,53%	14,13%	14,71%	15,27%	15,82%	16,34%	16,86%	17,35%
78	15,43%	13,83%	13,87%	14,36%	15,02%	15,75%	16,50%	17,25%	17,99%	18,71%	19,42%	20,10%	20,77%	21,42%	22,06%
79	6,44%	5,99%	6,26%	6,79%	7,48%	8,30%	9,23%	10,26%	11,38%	12,38%	13,30%	14,15%	14,94%	15,68%	16,37%
80	15,36%	12,33%	11,33%	10,95%	10,86%	10,91%	11,05%	11,24%	11,47%	11,72%	11,98%	12,25%	12,53%	12,81%	13,10%
81	8,92%	8,11%	8,32%	8,90%	9,70%	10,65%	11,49%	12,25%	12,95%	13,60%	14,21%	14,79%	15,34%	15,86%	16,35%
82	15,34%	12,58%	11,77%	11,56%	11,60%	11,79%	12,04%	12,35%	12,68%	13,03%	13,38%	13,74%	14,10%	14,45%	14,81%
83	15,39%	12,48%	11,57%	11,27%	11,23%	11,34%	11,54%	11,78%	12,05%	12,35%	12,65%	12,97%	13,28%	13,60%	13,91%
84	15,41%	12,76%	12,04%	11,90%	12,01%	12,25%	12,56%	12,91%	13,29%	13,68%	14,07%	14,47%	14,86%	15,25%	15,64%
85	15,35%	12,53%	11,68%	11,43%	11,44%	11,60%	11,83%	12,11%	12,42%	12,74%	13,08%	13,42%	13,76%	14,10%	14,44%

Figure 3.A.3: Consistency between Sharpe and Cost of Capital method, HW model

Consistency COC / Wang, HW model

Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	15,34%	11,31%	9,62%	8,66%	8,05%	7,63%	7,33%	7,11%	6,94%	6,81%	6,71%	6,64%	6,59%	6,55%	6,52%
56	15,34%	11,27%	9,55%	8,57%	7,94%	7,50%	7,18%	6,94%	6,75%	6,61%	6,50%	6,41%	6,35%	6,29%	6,26%
57	15,34%	11,41%	9,78%	8,88%	8,32%	7,94%	7,68%	7,50%	7,37%	7,27%	7,21%	7,18%	7,16%	7,15%	7,16%
58	15,34%	11,41%	9,79%	8,90%	8,34%	7,97%	7,72%	7,54%	7,41%	7,32%	7,26%	7,23%	7,21%	7,21%	7,22%
59	15,35%	11,31%	9,60%	8,64%	8,03%	7,60%	7,29%	7,06%	6,88%	6,75%	6,65%	6,57%	6,51%	6,47%	6,43%
60	15,34%	11,46%	9,86%	8,99%	8,45%	8,10%	7,86%	7,69%	7,58%	7,51%	7,46%	7,44%	7,44%	7,45%	7,47%
61	7,05%	5,25%	4,52%	4,14%	3,92%	3,81%	3,76%	3,77%	3,82%	3,91%	4,04%	4,21%	4,37%	4,52%	4,65%
62	15,36%	12,29%	11,27%	10,87%	10,76%	10,79%	10,91%	11,09%	11,30%	11,54%	11,79%	12,05%	12,31%	12,58%	12,85%
63	12,00%	9%	7,82%	7,22%	6,92%	6,78%	6,73%	6,69%	6,67%	6,66%	6,66%	6,67%	6,69%	6,72%	6,75%
64	15,34%	11,49%	9,93%	9,08%	8,56%	8,22%	7,99%	7,84%	7,75%	7,69%	7,66%	7,65%	7,66%	7,68%	7,72%
65	15,34%	11,88%	10,57%	9,94%	9,62%	9,46%	9,40%	9,41%	9,46%	9,55%	9,66%	9,78%	9,92%	10,07%	10,22%
66	15,34%	12,36%	11,39%	11,04%	10,96%	11,03%	11,19%	11,40%	11,65%	11,91%	12,19%	12,47%	12,76%	13,06%	13,35%
67	15,35%	12,39%	11,44%	11,10%	11,04%	11,12%	11,29%	11,51%	11,76%	12,03%	12,32%	12,61%	12,91%	13,20%	13,50%
68	15,36%	11,91%	10,61%	9,98%	9,66%	9,50%	9,45%	9,46%	9,52%	9,61%	9,72%	9,85%	9,99%	10,14%	10,30%
69	15,35%	12,14%	11,01%	10,53%	10,33%	10,30%	10,35%	10,47%	10,62%	10,80%	11,00%	11,22%	11,44%	11,66%	11,89%
70	15,38%	12,08%	10,90%	10,36%	10,12%	10,04%	10,06%	10,14%	10,26%	10,41%	10,58%	10,76%	10,95%	11,15%	11,35%
71	9,06%	7,25%	6,69%	6,54%	6,61%	6,83%	7,13%	7,39%	7,63%	7,85%	8,07%	8,28%	8,48%	8,68%	8,88%
72	10,38%	8,71%	8,42%	8,60%	9,04%	9,59%	10,10%	10,58%	11,04%	11,48%	11,91%	12,33%	12,74%	13,13%	13,52%
73	15,35%	12,81%	12,15%	12,06%	12,22%	12,51%	12,86%	13,25%	13,66%	14,08%	14,50%	14,92%	15,34%	15,75%	16,16%
74	15,38%	12,65%	11,86%	11,66%	11,72%	11,91%	12,18%	12,49%	12,83%	13,19%	13,55%	13,91%	14,27%	14,64%	14,99%
75	12,17%	10,62%	10,59%	11,00%	11,44%	11,91%	12,39%	12,88%	13,36%	13,84%	14,32%	14,78%	15,23%	15,68%	16,11%
76	13,53%	12,41%	12,72%	13,27%	13,93%	14,62%	15,33%	16,02%	16,71%	17,38%	18,02%	18,65%	19,26%	19,86%	20,43%
77	11,39%	10,16%	10,31%	10,95%	11,61%	12,25%	12,87%	13,47%	14,06%	14,63%	15,19%	15,72%	16,24%	16,75%	17,24%
78	15,42%	13,82%	13,85%	14,34%	14,99%	15,71%	16,45%	17,19%	17,92%	18,64%	19,33%	20,01%	20,67%	21,30%	21,93%
79	6,44%	5,99%	6,26%	6,80%	7,51%	8,34%	9,28%	10,33%	11,46%	12,48%	13,42%	14,28%	15,09%	15,85%	16,56%
80	15,36%	12,33%	11,33%	10,95%	10,85%	10,90%	11,04%	11,23%	11,45%	11,70%	11,96%	12,23%	12,51%	12,79%	13,07%
81	8,91%	8,10%	8,31%	8,90%	9,71%	10,67%	11,53%	12,30%	13,02%	13,70%	14,33%	14,94%	15,51%	16,06%	16,58%
82	15,34%	12,58%	11,77%	11,55%	11,60%	11,78%	12,04%	12,34%	12,67%	13,02%	13,37%	13,73%	14,08%	14,44%	14,79%
83	15,39%	12,48%	11,56%	11,25%	11,22%	11,33%	11,52%	11,76%	12,03%	12,32%	12,62%	12,93%	13,24%	13,55%	13,87%
84	15,40%	12,76%	12,03%	11,88%	11,99%	12,23%	12,53%	12,88%	13,25%	13,64%	14,03%	14,42%	14,81%	15,19%	15,57%
85	15,35%	12,53%	11,68%	11,42%	11,44%	11,59%	11,82%	12,10%	12,41%	12,73%	13,06%	13,40%	13,74%	14,08%	14,42%

Figure 3.A.4: Consistency between Sharpe and Cost of Capital method, HW model

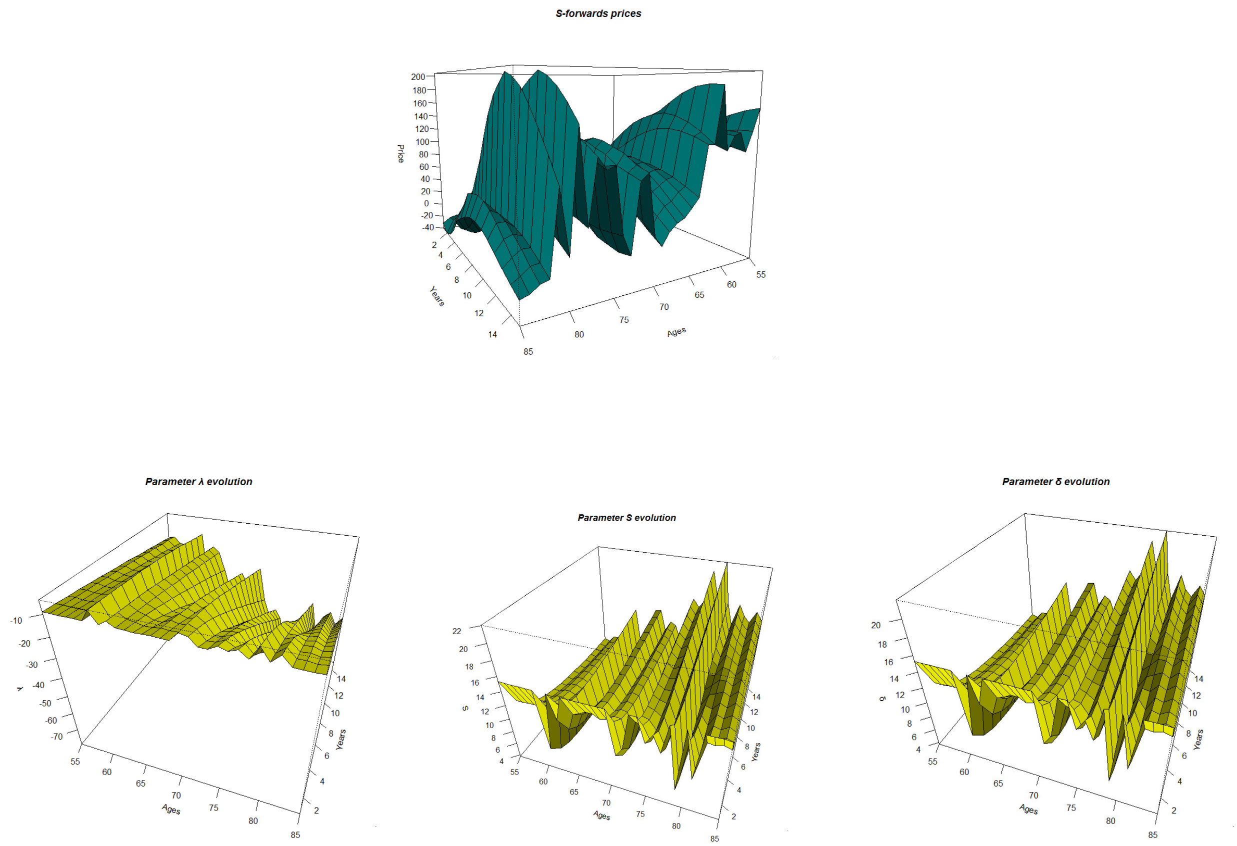

Figure 3.A.5: 3D presentation of the S-forwards prices and the corresponding parameters in the classical methods

Appendix B

Parameter	Fitted model	Error	Pseudo-R^2
A	$a + bx + cx^2$ a= 2.675e-01 b= -8.947e-03 c= 7.481e-05	0.004228	0.9300884
B	$a + bx + cx^2$ a= 1.968e-02 b= 2.872e-03 c= -2.154e-05	0.001877	0.5578862
b	ax^4 a= 1.968e-02	0.1431	0.5408143
σ	ae^{-bx} a= 0.0002570 b= -0.0560466	0.02138	0.1448673

Table 3.B.1: Non-linear regression results, HW model.

Appendix C

COC prices of different S-forwards using smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	12,8234315	15,7275763	9,46135275	-5,9316194	-31,811434	-67,77583	-115,01798	-173,36205	-243,9983	-327,33262	-423,2985	-531,18174	-654,04103	-789,02028	-936,43444
56	16,8249269	30,8657931	42,0911495	49,7938578	54,4892988	54,8564764	51,0464156	41,2684067	24,9334242	2,03595937	-26,90385	-65,33301	-110,73993	-164,0425	-226,90674
57	20,6706857	43,9019461	68,7301591	95,6801782	123,259435	151,694298	179,747187	206,811731	233,066827	259,175765	281,727279	303,440817	323,267406	339,25663	352,626129
58	23,8942982	53,6825583	89,7883655	130,419677	175,742577	224,446051	275,793804	330,095105	388,112894	446,467021	508,125776	572,400834	637,824128	705,891018	777,534527
59	25,615872	61,0889654	104,178249	154,861326	211,622125	273,452482	340,671966	414,030239	490,045902	571,835435	658,7603	749,434681	845,513061	948,188741	1055,54022
60	27,8000384	65,7858178	113,635374	169,504159	231,96268	301,231015	377,909733	458,274334	545,451106	638,695729	736,536526	840,618061	952,211603	1069,35263	1193,37817
61	27,8232919	67,394053	116,45885	173,048508	237,19974	309,2667	385,165214	467,926794	556,579108	649,438411	747,998768	853,466593	963,671435	1079,7855	1202,38476
62	28,0385395	66,751118	113,569412	168,310745	231,026523	297,200484	369,718448	447,309914	528,005642	613,081728	703,620048	797,161616	894,636687	996,360617	1104,06446
63	26,8286353	62,0545012	105,322839	156,383085	210,262895	269,692657	333,080681	398,154642	465,963457	537,464668	609,907813	683,986025	759,765352	838,864343	920,916413
64	23,6235722	55,1238222	94,0419676	134,985575	180,572631	228,905494	277,42873	326,989781	378,466198	428,854145	478,530334	526,810575	575,254254	623,327626	671,259767
65	21,1186963	48,9193328	77,7423973	110,207724	144,175282	176,858376	208,955504	241,144764	269,676037	294,874429	316,461664	336,098103	353,221207	368,153484	379,38559
66	19,3975551	38,3026818	59,6014804	81,0571117	99,7527066	115,872108	129,989896	138,719454	142,326018	140,493419	135,08204	125,58689	112,526291	94,5507928	76,4028509
67	12,8860933	26,2416496	38,0577897	45,2135255	47,8198873	46,9347294	39,1208875	24,6501729	3,22369582	-23,016652	-54,465002	-90,348773	-131,81681	-173,6766	-218,78058
68	10,2479045	16,3515347	15,4055055	7,99146678	-4,4501381	-25,292534	-54,183698	-91,35457	-134,31219	-183,31327	-237,30796	-297,24572	-357,48581	420,660199	-484,87377
69	6,20514836	2,15612113	-10,940236	-30,910099	-60,870277	-100,27494	-149,21856	-204,79087	-267,09204	-334,79064	-408,66539	-482,64994	-559,23507	-636,03223	-706,75984
70	-0,2264307	-13,556375	-36,39077	-71,218862	-117,08843	-173,85218	-238,07727	-309,69316	-387,08213	-470,89386	-554,67496	-640,86109	-726,65047	-805,26437	-878,67237
71	-6,1562934	-26,131922	-61,016829	-109,04366	-169,60117	-238,58488	-315,70242	-399,03708	-489,14758	-579,23447	-671,76735	-763,61326	-847,57572	-925,65344	-985,93039
72	-9,4078468	-38,49066	-83,810547	-143,86481	-213,59145	-292,37849	-377,96622	-470,8494	-563,90754	-659,73736	-754,9446	-841,98642	-923,02317	-985,7394	-1033,0043
73	-15,526725	-52,294961	-107,0043	-173,13369	-249,53891	-333,51768	-425,50697	-518,07298	-614,0256	-709,77272	-797,48009	-879,60531	-943,48346	-992,17608	-1018,9307
74	-21,022118	-64,999492	-122,96698	-192,84588	-271,30653	-358,69474	-447,251	-540,08123	-633,44825	-719,27575	-800,44761	-863,99121	-913,27268	-941,72827	-952,53497
75	-27,069942	-72,055917	-131,13887	-200,03433	-278,92952	-359,7121	-445,87031	-533,57153	-614,53546	-692,21775	-753,37152	-801,78222	-831,16651	-844,69088	-834,44704
76	-27,489335	-71,893635	-127,39825	-193,93997	-263,04945	-338,72324	-417,10266	-489,75299	-560,86234	-616,93826	-662,31859	-691,10647	-706,51227	-700,73631	-687,8457
77	-27,41788	-66,63359	-117,28603	-170,69225	-231,63025	-296,37548	-356,42997	-416,9238	-464,15934	-503,18758	-528,62728	-543,80744	-541,1143	-534,89247	-515,93534
78	-23,975762	-57,446044	-92,151053	-134,37257	-180,96304	-223,61098	-268,72526	-302,4751	-330,83488	-349,10459	-360,82452	-358,67806	-357,43415	-347,464	-330,2853
79	-21,784298	-38,305528	-59,332139	-83,561621	-103,60216	-127,78003	-142,33357	-154,47892	-160,41635	-164,03261	-158,44953	-159,06337	-155,85042	-150,09237	-142,15796
80	-9,892879	-12,849081	-13,02644	-6,0582118	-3,6221496	7,37876422	17,9426225	30,6121492	40,9692128	55,2586919	57,2052363	57,173457	53,9924806	47,8213268	39,1261703
81	-3,6437023	14,7385746	50,4633248	84,8545887	128,675908	169,983588	209,398916	241,63232	272,016281	283,044568	285,347579	277,687273	260,630732	235,436049	204,030103
82	8,09331233	63,4477879	130,484178	212,83905	292,881388	367,85992	430,930489	486,029555	513,970875	525,533456	519,18905	495,755357	457,175152	406,4751	347,569656
83	34,6923827	121,659527	243,335499	369,13773	489,349494	593,91095	684,409771	739,264484	769,206516	772,118337	748,864413	701,919925	635,320419	554,419173	465,434207
84	53,2660323	198,925645	372,488185	547,489146	706,242742	845,767231	940,726251	1001,40126	1024,64128	1011,07913	963,522201	886,88462	787,901934	674,594121	555,506757
85	98,3229728	299,281898	530,525075	754,167258	956,875531	1106,64508	1212,29961	1268,92233	1276,51913	1237,97809	1158,93113	1047,41265	913,239885	767,12949	619,647558

Figure 3.C.1: S-forward prices under the Cost of Capital approach with smooth parameters, HW model

Consistency COC / Risk neutral using smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	-4,04%	-5,10%	-6,29%	-7,62%	-8,68%	-9,59%	-10,43%	-11,22%	-11,98%	-12,73%	-13,48%	-14,23%	-14,98%	-15,73%	-16,48%
56	-3,83%	-4,45%	-5,14%	-5,91%	-6,78%	-7,75%	-8,83%	-9,81%	-10,75%	-11,64%	-12,52%	-13,39%	-14,25%	-15,10%	-15,96%
57	-3,74%	-4,05%	-4,38%	-4,76%	-5,18%	-5,65%	-6,18%	-6,78%	-7,46%	-8,22%	-9,09%	-10,07%	-11,12%	-12,14%	-13,15%
58	-3,77%	-3,87%	-3,97%	-4,09%	-4,21%	-4,36%	-4,52%	-4,70%	-4,91%	-5,15%	-5,42%	-5,74%	-6,10%	-6,52%	-7,00%
59	-3,89%	-3,88%	-3,87%	-3,84%	-3,81%	-3,78%	-3,74%	-3,70%	-3,65%	-3,61%	-3,56%	-3,52%	-3,48%	-3,44%	-3,40%
60	-4,10%	-4,07%	-4,02%	-3,97%	-3,90%	-3,83%	-3,74%	-3,65%	-3,55%	-3,44%	-3,32%	-3,20%	-3,06%	-2,92%	-2,77%
61	-4,38%	-4,40%	-4,41%	-4,42%	-4,42%	-4,43%	-4,44%	-4,45%	-4,47%	-4,50%	-4,53%	-4,58%	-4,65%	-4,73%	-4,83%
62	-4,74%	-4,86%	-5%	-5,15%	-5,32%	-5,52%	-5,74%	-6,00%	-6,30%	-6,65%	-7,04%	-7,50%	-8,03%	-8,63%	-9,32%
63	-5,15%	-5,44%	-5,76%	-6,13%	-6,55%	-7,03%	-7,58%	-8,21%	-8,94%	-9,76%	-10,71%	-11,79%	-13,02%	-14,41%	-15,99%
64	-5,62%	-6,11%	-6,68%	-7,32%	-8,07%	-8,92%	-9,89%	-11,01%	-12,28%	-13,74%	-15,36%	-16,98%	-18,60%	-20,23%	-21,87%
65	-6,13%	-6,87%	-7,72%	-8,70%	-9,83%	-11,12%	-12,61%	-14,24%	-15,88%	-17,53%	-19,19%	-20,86%	-22,56%	-24,28%	-26,01%
66	-6,69%	-7,70%	-8,88%	-10,23%	-11,80%	-13,44%	-15,08%	-16,74%	-18,42%	-20,13%	-21,88%	-23,65%	-25,45%	-27,27%	-29,12%
67	-7,28%	-8,60%	-10,13%	-11,79%	-13,41%	-15,06%	-16,74%	-18,47%	-20,24%	-22,06%	-23,91%	-25,80%	-27,72%	-29,66%	-31,63%
68	-7,92%	-9,55%	-11,20%	-12,81%	-14,46%	-16,17%	-17,95%	-19,78%	-21,66%	-23,60%	-25,58%	-27,59%	-29,64%	-31,71%	-33,81%
69	-8,58%	-10,26%	-11,85%	-13,50%	-15,22%	-17,03%	-18,91%	-20,87%	-22,88%	-24,95%	-27,06%	-29,21%	-31,40%	-33,61%	-35,84%
70	-9,27%	-10,76%	-12,35%	-14,07%	-15,89%	-17,81%	-19,81%	-21,90%	-24,05%	-26,26%	-28,51%	-30,81%	-33,13%	-35,49%	-37,86%
71	-9,30%	-10,87%	-12,57%	-14,40%	-16,35%	-18,40%	-20,55%	-22,78%	-25,08%	-27,44%	-29,84%	-32,28%	-34,75%	-37,25%	-39,76%
72	-9,33%	-10,99%	-12,80%	-14,75%	-16,83%	-19,03%	-21,32%	-23,70%	-26,15%	-28,66%	-31,22%	-33,81%	-36,44%	-39,08%	-41,74%
73	-9,35%	-11,11%	-13,04%	-15,12%	-17,34%	-19,68%	-22,13%	-24,67%	-27,28%	-29,94%	-32,66%	-35,40%	-38,18%	-40,97%	-43,78%
74	-9,38%	-11,24%	-13,29%	-15,50%	-17,87%	-20,36%	-22,97%	-25,67%	-28,44%	-31,27%	-34,14%	-37,05%	-39,98%	-42,92%	-45,88%
75	-9,41%	-11,38%	-13,55%	-15,90%	-18,42%	-21,08%	-23,85%	-26,71%	-29,65%	-32,65%	-35,68%	-38,75%	-41,84%	-44,94%	-48,04%
76	-9,44%	-11,52%	-13,82%	-16,32%	-19,00%	-21,82%	-24,76%	-27,80%	-30,91%	-34,07%	-37,27%	-40,50%	-43,75%	-47,00%	-50,27%
77	-9,47%	-11,67%	-14,10%	-16,76%	-19,60%	-22,59%	-25,71%	-28,93%	-32,21%	-35,54%	-38,91%	-42,30%	-45,71%	-49,12%	-52,54%
78	-9,50%	-11,82%	-14,40%	-17,21%	-20,22%	-23,40%	-26,70%	-30,09%	-33,55%	-37,05%	-40,59%	-44,15%	-47,72%	-51,30%	-54,87%
79	-9,54%	-11,98%	-14,71%	-17,69%	-20,88%	-24,23%	-27,72%	-31,29%	-34,93%	-38,61%	-42,32%	-46,04%	-49,78%	-53,52%	-57,26%
80	-9,57%	-12,14%	-15,02%	-18,18%	-21,55%	-25,10%	-28,77%	-32,53%	-36,34%	-40,20%	-44,08%	-47,98%	-51,88%	-55,78%	-59,68%
81	-9,61%	-12,31%	-15,36%	-18,69%	-22,25%	-25,99%	-29,85%	-33,80%	-37,80%	-41,83%	-45,89%	-49,95%	-54,02%	-58,09%	-62,16%
82	-9,65%	-12,49%	-15,70%	-19,22%	-22,97%	-26,91%	-30,96%	-35,10%	-39,28%	-43,50%	-47,73%	-51,97%	-56,21%	-60,44%	-64,68%
83	-9,69%	-12,68%	-16,06%	-19,77%	-23,72%	-27,85%	-32,10%	-36,43%	-40,80%	-45,20%	-49,60%	-54,02%	-58,43%	-62,83%	-67,23%
84	-9,73%	-12,87%	-16,43%	-20,33%	-24,49%	-28,82%	-33,27%	-37,79%	-42,35%	-46,92%	-51,51%	-56,10%	-60,68%	-65,26%	-69,83%
85	-9,77%	-13,06%	-16,81%	-20,92%	-25,28%	-29,82%	-34,47%	-39,18%	-43,92%	-48,68%	-53,44%	-58,21%	-62,96%	-67,71%	-72,45%

Figure 3.C.2: Consistency between risk-neutral and Cost of Capital method with smooth parameters, HW model

Consistency COC / Sharpe using smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	6,89%	6,06%	6,01%	6,21%	6,24%	6,21%	6,16%	6,12%	6,08%	6,05%	6,03%	6,02%	6,02%	6,02%	6,03%
56	6,52%	5,27%	4,89%	4,80%	4,85%	4,98%	5,18%	5,31%	5,40%	5,48%	5,54%	5,60%	5,66%	5,71%	5,76%
57	6,37%	4,78%	4,16%	3,84%	3,68%	3,61%	3,60%	3,63%	3,71%	3,83%	3,98%	4,17%	4,36%	4,53%	4,69%
58	6,40%	4,56%	3,76%	3,29%	2,98%	2,76%	2,61%	2,50%	2,42%	2,37%	2,35%	2,35%	2,36%	2,40%	2,46%
59	6,60%	4,56%	3,64%	3,07%	2,68%	2,38%	2,14%	1,95%	1,78%	1,64%	1,53%	1,42%	1,33%	1,25%	1,18%
60	6,94%	4,77%	3,77%	3,16%	2,72%	2,39%	2,13%	1,90%	1,72%	1,55%	1,41%	1,28%	1,16%	1,05%	0,95%
61	7,42%	5,15%	4,12%	3,50%	3,07%	2,75%	2,50%	2,30%	2,14%	2,01%	1,90%	1,81%	1,74%	1,68%	1,64%
62	8%	5,67%	4,65%	4,06%	3,67%	3,40%	3,21%	3,07%	2,99%	2,94%	2,92%	2,93%	2,97%	3,03%	3,12%
63	8,69%	6,32%	5,34%	4,80%	4,48%	4,29%	4,20%	4,17%	4,19%	4,27%	4,39%	4,55%	4,76%	5,00%	5,29%
64	9,46%	7,08%	6,15%	5,70%	5,48%	5,40%	5,42%	5,53%	5,71%	5,94%	6,23%	6,48%	6,72%	6,95%	7,16%
65	10,31%	7,93%	7,08%	6,73%	6,62%	6,68%	6,85%	7,09%	7,30%	7,50%	7,69%	7,88%	8,06%	8,24%	8,42%
66	11,22%	8,86%	8,10%	7,86%	7,89%	8,00%	8,12%	8,25%	8,38%	8,53%	8,68%	8,83%	9,00%	9,16%	9,32%
67	12,20%	9,86%	9,19%	8,99%	8,90%	8,89%	8,93%	9,01%	9,11%	9,24%	9,38%	9,53%	9,69%	9,85%	10,02%
68	13,23%	10,91%	10,11%	9,71%	9,52%	9,46%	9,48%	9,54%	9,65%	9,78%	9,92%	10,08%	10,25%	10,42%	10,60%
69	14,31%	11,67%	10,64%	10,16%	9,94%	9,87%	9,89%	9,97%	10,08%	10,22%	10,38%	10,56%	10,74%	10,93%	11,13%
70	15,43%	12,18%	11,02%	10,51%	10,29%	10,23%	10,26%	10,35%	10,48%	10,64%	10,82%	11,02%	11,22%	11,43%	11,64%
71	15,44%	12,26%	11,15%	10,68%	10,50%	10,47%	10,53%	10,65%	10,82%	11,00%	11,21%	11,43%	11,65%	11,88%	12,12%
72	15,45%	12,34%	11,28%	10,85%	10,71%	10,72%	10,82%	10,97%	11,16%	11,38%	11,61%	11,85%	12,10%	12,35%	12,61%
73	15,46%	12,42%	11,42%	11,04%	10,93%	10,98%	11,11%	11,30%	11,52%	11,77%	12,02%	12,29%	12,56%	12,84%	13,12%
74	15,46%	12,51%	11,56%	11,22%	11,16%	11,25%	11,42%	11,64%	11,89%	12,17%	12,45%	12,75%	13,04%	13,34%	13,64%
75	15,47%	12,60%	11,71%	11,42%	11,40%	11,53%	11,74%	11,99%	12,28%	12,58%	12,90%	13,22%	13,54%	13,86%	14,18%
76	15,48%	12,69%	11,86%	11,62%	11,65%	11,82%	12,07%	12,36%	12,68%	13,01%	13,35%	13,70%	14,05%	14,39%	14,74%
77	15,48%	12,78%	12,01%	11,83%	11,91%	12,12%	12,41%	12,73%	13,09%	13,45%	13,82%	14,20%	14,57%	14,94%	15,31%
78	15,49%	12,88%	12,18%	12,05%	12,18%	12,43%	12,76%	13,12%	13,51%	13,91%	14,31%	14,71%	15,11%	15,51%	15,90%
79	15,50%	12,98%	12,34%	12,28%	12,45%	12,75%	13,12%	13,52%	13,94%	14,37%	14,81%	15,24%	15,66%	16,08%	16,50%
80	15,51%	13,08%	12,52%	12,51%	12,74%	13,08%	13,49%	13,93%	14,39%	14,85%	15,31%	15,77%	16,23%	16,67%	17,11%
81	15,52%	13,19%	12,70%	12,75%	13,03%	13,42%	13,88%	14,36%	14,85%	15,34%	15,84%	16,32%	16,80%	17,27%	17,74%
82	15,53%	13,30%	12,88%	12,99%	13,33%	13,77%	14,27%	14,79%	15,32%	15,84%	16,37%	16,88%	17,39%	17,89%	18,37%
83	15,53%	13,41%	13,07%	13,25%	13,64%	14,13%	14,67%	15,23%	15,79%	16,36%	16,91%	17,46%	17,99%	18,51%	19,03%
84	15,54%	13,53%	13,26%	13,51%	13,96%	14,50%	15,08%	15,68%	16,28%	16,88%	17,47%	18,04%	18,60%	19,15%	19,69%
85	15,55%	13,65%	13,46%	13,77%	14,28%	14,87%	15,50%	16,14%	16,78%	17,41%	18,03%	18,63%	19,22%	19,80%	20,36%

Figure 3.C.3: Consistency between Sharpe and Cost of Capital method with smooth parameters, HW model

Consistency COC / Wang using smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	6,89%	6,06%	6,01%	6,21%	6,23%	6,20%	6,15%	6,11%	6,07%	6,05%	6,03%	6,01%	6,01%	6,01%	6,02%
56	6,52%	5,27%	4,89%	4,80%	4,84%	4,98%	5,17%	5,30%	5,40%	5,48%	5,54%	5,60%	5,66%	5,71%	5,76%
57	6,37%	4,78%	4,16%	3,84%	3,68%	3,61%	3,60%	3,63%	3,71%	3,83%	3,98%	4,17%	4,36%	4,53%	4,69%
58	6,40%	4,56%	3,75%	3,29%	2,98%	2,76%	2,61%	2,50%	2,42%	2,37%	2,35%	2,35%	2,37%	2,41%	2,46%
59	6,60%	4,56%	3,64%	3,07%	2,68%	2,38%	2,14%	1,95%	1,78%	1,65%	1,53%	1,42%	1,33%	1,25%	1,18%
60	6,94%	4,77%	3,77%	3,16%	2,72%	2,39%	2,13%	1,91%	1,72%	1,55%	1,41%	1,28%	1,16%	1,05%	0,95%
61	7,41%	5,14%	4,12%	3,50%	3,07%	2,75%	2,50%	2,30%	2,14%	2,01%	1,90%	1,81%	1,74%	1,68%	1,64%
62	8,00%	5,67%	5%	4,05%	3,66%	3,40%	3,21%	3,07%	2,99%	2,94%	2,92%	2,93%	2,97%	3,03%	3,12%
63	8,68%	6,32%	5,33%	4,80%	4,48%	4,29%	4,19%	4,17%	4,19%	4,27%	4,39%	4,55%	4,76%	5,00%	5,30%
64	9,46%	7,08%	6,15%	5,69%	5,47%	5,40%	5,42%	5,53%	5,70%	5,94%	6,22%	6,48%	6,72%	6,94%	7,15%
65	10,30%	7,93%	7,07%	6,72%	6,62%	6,67%	6,84%	7,08%	7,29%	7,49%	7,68%	7,87%	8,05%	8,23%	8,41%
66	11,22%	8,86%	8,09%	7,85%	7,88%	7,99%	8,11%	8,23%	8,37%	8,51%	8,66%	8,82%	8,98%	9,14%	9,31%
67	12,19%	9,85%	9,18%	8,98%	8,89%	8,88%	8,91%	8,99%	9,10%	9,22%	9,36%	9,51%	9,67%	9,83%	10,00%
68	13,22%	10,90%	10,09%	9,70%	9,51%	9,44%	9,46%	9,52%	9,63%	9,75%	9,90%	10,05%	10,22%	10,39%	10,57%
69	14,30%	11,66%	10,62%	10,14%	9,93%	9,85%	9,87%	9,94%	10,06%	10,20%	10,36%	10,53%	10,71%	10,90%	11,09%
70	15,43%	12,17%	11,01%	10,49%	10,27%	10,20%	10,23%	10,32%	10,46%	10,61%	10,79%	10,98%	11,18%	11,39%	11,60%
71	15,44%	12,25%	11,14%	10,66%	10,47%	10,44%	10,51%	10,63%	10,79%	10,97%	11,18%	11,39%	11,61%	11,84%	12,07%
72	15,44%	12,33%	11,27%	10,83%	10,69%	10,69%	10,79%	10,94%	11,13%	11,34%	11,57%	11,81%	12,06%	12,31%	12,56%
73	15,45%	12,41%	11,40%	11,01%	10,91%	10,95%	11,08%	11,27%	11,48%	11,73%	11,98%	12,25%	12,52%	12,79%	13,06%
74	15,45%	12,49%	11,54%	11,20%	11,14%	11,22%	11,39%	11,60%	11,85%	12,12%	12,41%	12,70%	12,99%	13,28%	13,58%
75	15,46%	12,58%	11,69%	11,40%	11,38%	11,50%	11,70%	11,95%	12,23%	12,53%	12,84%	13,16%	13,48%	13,80%	14,11%
76	15,47%	12,67%	11,84%	11,60%	11,62%	11,79%	12,03%	12,31%	12,63%	12,96%	13,30%	13,64%	13,98%	14,32%	14,66%
77	15,47%	12,76%	11,99%	11,81%	11,88%	12,09%	12,37%	12,69%	13,03%	13,39%	13,76%	14,13%	14,50%	14,86%	15,22%
78	15,48%	12,86%	12,15%	12,02%	12,14%	12,39%	12,71%	13,07%	13,45%	13,84%	14,24%	14,63%	15,03%	15,42%	15,80%
79	15,49%	12,96%	12,32%	12,25%	12,42%	12,71%	13,07%	13,47%	13,88%	14,30%	14,73%	15,15%	15,57%	15,98%	16,39%
80	15,50%	13,06%	12,49%	12,48%	12,70%	13,04%	13,44%	13,87%	14,32%	14,78%	15,23%	15,68%	16,12%	16,56%	16,99%
81	15,50%	13,17%	12,67%	12,71%	12,99%	13,38%	13,82%	14,29%	14,77%	15,26%	15,74%	16,22%	16,69%	17,15%	17,60%
82	15,51%	13,28%	12,85%	12,96%	13,29%	13,72%	14,21%	14,72%	15,24%	15,75%	16,27%	16,77%	17,27%	17,75%	18,23%
83	15,52%	13,39%	13,04%	13,21%	13,59%	14,07%	14,60%	15,15%	15,71%	16,26%	16,80%	17,33%	17,86%	18,37%	18,86%
84	15,53%	13,51%	13,23%	13,47%	13,90%	14,44%	15,01%	15,60%	16,19%	16,77%	17,35%	17,91%	18,45%	18,99%	19,51%
85	15,54%	13,62%	13,43%	13,73%	14,23%	14,81%	15,42%	16,05%	16,68%	17,29%	17,90%	18,49%	19,06%	19,62%	20,17%

Figure 3.C.4: Consistency between Wang and Cost of Capital method with smooth parameters, HW model

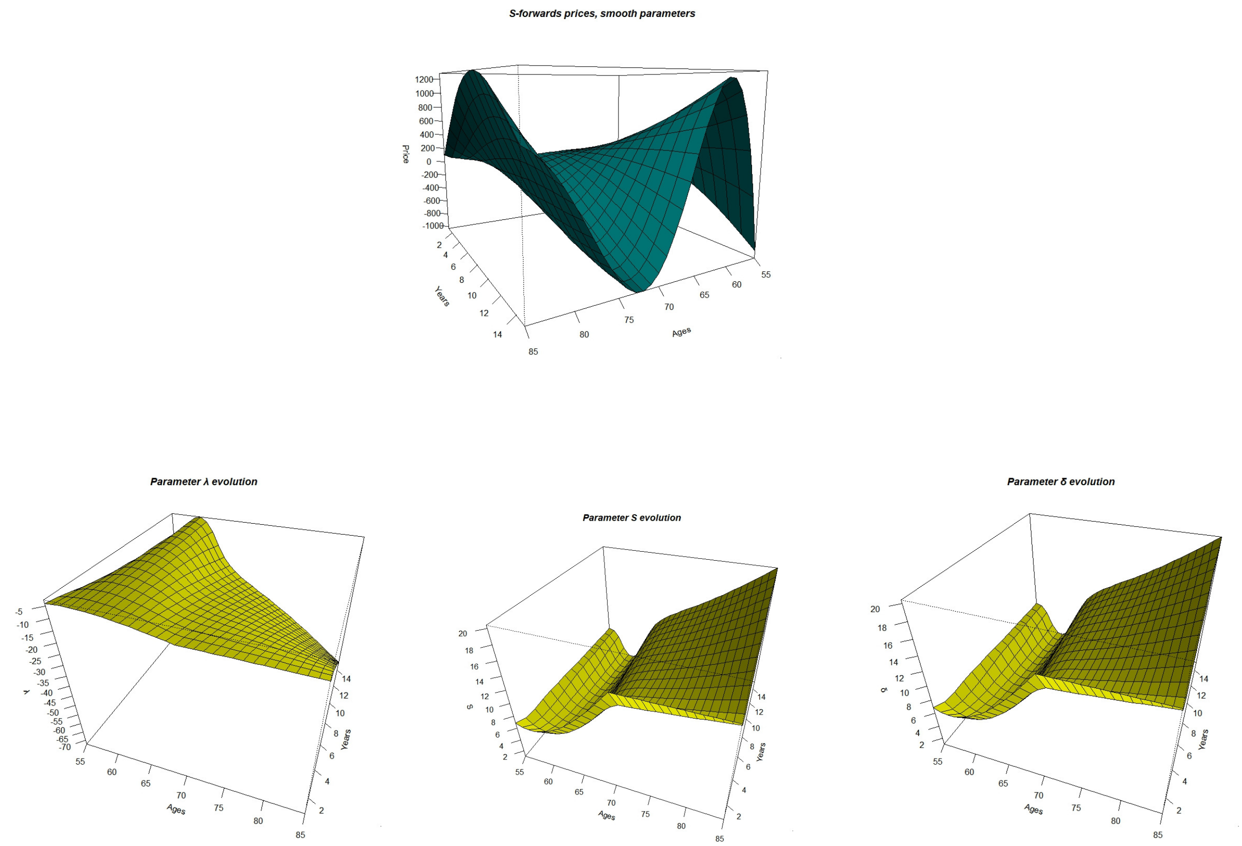

Figure 3.C.5: 3D presentation of the S-forwards prices and the corresponding parameters in the classical methods

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Chapter 4

Longevity pricing and hedging in the presence of population basis risk

Abstract

Hedging the basis risk is a challenging issue for pension funds and insurers, who are increasingly resorting to longevity-linked securities to transfer their longevity risk. These derivatives are based on population data rather than their own policy data, which may lead to a potential loss due to data mismatch. This paper proposes different possibilities to hedge the basis risk, using S-forward contracts under the Cost of Capital approach and other classical pricing methods. We assess and analyze different hedging strategies for firms facing basis risk, utilizing the Hull and White model to represent the evolution of mortality over time. We give different numerical examples to illustrate each possibility using data for the Belgian population.

4.1 Introduction

Annuity providers such as insurers and pension funds are exposed to increased longevity risk. This later can be broken down into two types of risks: the unsystematic and systematic longevity risks. The unsystematic longevity risk refers to a possible adverse development of the policyholders longevity. According to the law of large numbers, the unsystematic longevity risk can be eliminated by increasing the portfolio size. As for the systematic longevity risk, which refers to the risk associated with the overall mortality improvement across the whole population, it cannot be diversified by increasing the portfolio size, but can be mitigated by entering longevity-linked securities such as survivor bonds, q-forwards, S-forwards, or S-swaps (Blake and Burrows [5], Dowd et al. [13], Barrieu and Veraart [2], Levantesi and Menzietti [19], and Zeddouk and Devolder [31]). Some of these financial instruments have been traded over-the-counter (OTC), but because of the pricing difficulties, and the fact that these products only allow for a partial hedging of the systematic longevity risk (leaving a residual amount of risk known as basis risk), they are still not widely traded in the financial market. The payoffs of these financial instruments are determined by longevity indices based on one or more reference populations (for instance the national population). Therefore, the longevity experience of the annuity provider's population may not coincide with the reference populations, (see for example Li and Hardy [20] and Coughlan et al. [10]).

The importance of this mismatch between the two populations depends on different factors, such as demographic differences (e.g., age profile, sex, socioeconomic status), the volatility of the portfolio to be hedged in comparison with the reference population, and the difference between the payoff structures of the hedging tool and the insurer's portfolio (Li et al. [21]).
Many authors have focused on the evaluation of the basis risk. For instance, De Rosa et al. [8] provided a model for assessing the basis risk in longevity and explored its effect on the hedging strategies of pension funds and annuity providers. Haberman et al. [15] and Villegas et al. [27] proposed a framework to choose a two-population mortality projection model for assessing demographic basis risk. Plat [25] proposed a stochastic model for the spread between the national and the small populations, which allows insurers to evaluate mortality rates and assess the basis risk; see also Coughlan et al. [10], Dahl et al. [7], Li and Hardy [20], Ngai and Sherris [24], Tzeng et al. [26], and Li and Luo [22].

The main objective of our paper is to propose a simple framework to assess and evaluate the basis risk, by associating it to the payoff of a longevity-linked security: we liken this basis risk to a longevity derivative that we define and call Survival exchange forward (S-exchange), whose price corresponds to the hedging cost of this risk. To achieve this goal, we need a model able to describe the two population's mortality. Many authors have proposed stochastic multi-population models for mortality such as Bayraktar and Young [3], Barbarin [1] and Dahl et al. [7].
In our study, the mortality model should not only be able to forecast mortality, but must also enable the valuation of longevity derivatives. Affine models could be good candidates because they meet these criteria (Huang et al.[16] and Xu et al. [29]). More specifically, in our framework we use the Hull and White process to provide three multipopulation models, by modifying the trend, the volatility or both.

Once the basis risk is modelled, our other contribution consists in providing fair prices of the proposed hedging instrument in closed form when possible, using the Cost of Capital method, which is in line with the directives of Solvency II. In addition, we use some other classical pricing approaches for comparison purposes. Then, we provide different hedging options using longevity derivatives, depending on the insurer's strategy and risk aversion.

This paper in organized as follow: in Section 4.2, we propose a multi-population model to describe mortality for the reference and insurer's population. In Section 4.3, we briefly give the definition of the S-forward contract, then in Section 4.4, we define the S-exchange contract and we give closed-form formulas of its price using the Cost of Capital approach, under three different mortality spreads between the reference and the insurer's population. In Section 4.5, we compute the prices of the proposed security using the classical methods, the risk-neutral, Sharpe and Wang approaches.

In Section 4.6, we give a numerical illustration of the prices using Belgian data. We provide sensitivity tests related to the different parameters in Section 4.7, and different longevity hedging strategies are then proposed in Section 4.8. Finally, in Section 4.9, we conclude.

4.2 Multi-population modeling in continuous time

In order to assess the basis risk, we need to use a model that is able to capture the mortality trends in the reference population and in the insurer's population, whose risk is to be hedged. We will model mortality using a single explanatory variable, which is the intensity of mortality μ_x at age x defined by:

$$\mu_x = \lim_{\triangle \to 0} \frac{\mathbb{P}[x < \bar{T} \leq x + \triangle \mid \bar{T} > x]}{\triangle}, \tag{4.1}$$

where $\overline{T}$ is a random variable that represents the future lifetime of an individual.

Within the framework of our study, we are going to model the intensity of mortality $\mu_x(t)$ that represents the force of mortality for an individual aged $x + t$ at time t, as a continuous stochastic process, defined in the real space of probability $(\Omega, F, \mathbb{P})$ and adapted to its natural filtration $(F_t)_{0 \leq t \leq T}$. Thus μ_{x+t} is written as: $\mu_x(t, \omega)$, with ω an event of the space of probability $(\Omega, F, \mathbb{P})$. Therefore, the starting point of our modelling will be to describe the dynamics of the mortality through the specification of a stochastic differential equation (SDE). Afterward, we will focus on the survival process, that represents the index at time t of an individual initially aged x, alive at time t, and surviving $T - t$ years more. For one individual, the survival index is defined as follows:

$$I(x + t, T - t) = e^{-\int_t^T \mu_x(u,\omega) du}. \tag{4.2}$$

4.2.1 Assumptions

In our setting, we have chosen to use a stochastic continuous-time model for mortality, to benefit from the pricing framework used in finance. Among these approaches, we considered an affine model, whose general formula is given by:

$$d\mu_x(t) = (f_1(t) + f_2(t)\mu_x(t))dt + \sqrt{f_3(t) + f_4(t)\mu_x(t)}dw(t), \tag{4.3}$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$ are deterministic functions, and $w(t)$ a standard Brownian motion.

Our choice was motivated by the fact that with an affine model, we can get an explicit formula of the longevity index which allow the pricing of longevity derivatives. We choose in particular the Hull and White model (HW), which is a cohort mortality model[1] since Zeddouk and Devolder [30] showed that this model is suitable for describing the force of mortality of the Belgian population. In fact, they presented

[1] This model describe how the mortality of an individual aged x at time t evolves over time, rather than modelling age-period mortality like in the Lee-Carter and CBD models

a comparison between different affine processes such as Ornstein Uhlenbeck, Feller, Vasiçek and CIR extended models, and showed (using different tests such as goodness-of-fit testing and backtesting) that the HW model gives a very accurate prediction of the survival function. In addition, the HW model was also used by Zeddouk and Devolder [31] for mortality to price Survival-forwards and Survival-swaps.

- For the national reference population mortality, we consider the HW model (Hull [17]):

$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma dw(t), \tag{4.4}$$

with $\xi(t) = Ae^{Bt}$ and A, B, b, σ all positive numbers, w a standard Brownian motion under the real world probability measure $\mathbb{P}$. $\xi(t)$ is chosen such that it follows the well known Gompertz formula.
The reason behind the choice of the HW model for mortality and $\xi(t)$ as a long term target was discussed in detailed by Zeddouk and Devolder [30].

- The HW model being affine, the expectation of the survival index related to the national population is directly given by (Duffie et al. [14]):

$$E_{\mathbb{P}}(I(x+t, T-t) \mid \mathcal{F}_t) = e^{\alpha_{\mathbb{P}}(t,T) - \beta_{\mathbb{P}}(t,T)\mu_x(t)}, \tag{4.5}$$

where $\alpha(t,T)$ and $\beta(t,T)$ are given by (see for instance Zeddouk and Devolder [30]):

$$\begin{cases} \alpha(t,T) & = \frac{A}{b}[e^{-bT}\frac{e^{(B+b)T}-e^{(B+b)t}}{B+b} - \frac{e^{BT}-e^{Bt}}{B}] - \frac{\sigma^2}{2b^2}[\frac{1}{b}(1-e^{-b(T-t)}) - T + t] \\ & \quad -\frac{\sigma^2}{4b^3}(1-e^{-b(T-t)})^2 \\ & \\ \beta(t,T) & = \frac{1}{b}(1-\exp(-b(T-t)). \end{cases} \tag{4.6}$$

- The spot interest rate $r(t,\omega)$ is stochastic and is solution of an SDE of the form:

$$dr(t,\omega) = h(t,r)dt + z(t,r)d\bar{w}(t,\omega), \tag{4.7}$$

where h and z are two given functions.

- w and $\bar{w}$ are independent standard Brownian motion under the real probability measure $\mathbb{P}$.
- We denote by $V(t,T)$ the price at a given time t of an S-forward with maturity T.
- $P(t,T) = E_{\mathbb{Q}}(e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t)$ is the price at time t of a zero coupon bound with maturity T under the risk-neutral probability measure $\mathbb{Q}$.
- There is no counterparty default risk.

As mentioned in the introduction, many models have been proposed in the literature to represent the mortality evolution of two or more related populations (Villegas et al. [27]). But the majority of the models proposed were based on discrete-time stochastic mortality models such as the Lee-Carter model.

In our study, the HW model will also be used to describe the insurer's population mortality. To do that, we investigate three possibilities:

1. As the insurer's portfolio is a subpopulation of the national population, we can consider that the intensity of mortality of the two populations have the same drift and basic volatility, but the insurer's population has an additional volatility related for instance to its reduced size (extra vol).
2. We consider that the subpopulation and the national population have the same volatility, but a different long term mean-reverting target (constant shift).
3. We consider that the subpopulation has a different mean-reverting level, as well as an additional volatility (extra vol and constant shift).

4.2.2 First possibility (extra vol)

We consider that the mortality intensity of the two populations have the same drift and volatility, but the insurer's population has an additional volatility that could be related to its small size. For the insurer's population, we can then write:

$$d\mu_x'(t) = (\xi(t) - b\mu_x'(t))dt + \sigma dw(t) + \sigma^{'} dw^{'}(t), \tag{4.8}$$

where $w^{'}(t)$ is a standard Brownian motion under the real probability measure $\mathbb{P}$, related to the extra volatility of the insurer's population. Because the insurer's population is a subpopulation of the national population, it is reasonable to consider that w and $w^{'}$are dependent. Let:

$$\tilde{\sigma}\tilde{w}(t) = \sigma w(t) + \sigma^{'} w^{'}(t). \tag{4.9}$$

So the total volatility of the subpopulation is:

$$\tilde{\sigma} = \sqrt{\sigma^2 + \sigma'^2 + 2\rho\sigma\sigma'}, \tag{4.10}$$

where ρ is the correlation coefficient: $\rho(t)$ =Cov($w(t)$,$w^{'}(t)$). From (4.9) we have:

$$\tilde{w}(t) = \frac{\sigma}{\tilde{\sigma}} w(t) + \frac{\sigma^{'}}{\tilde{\sigma}} w^{'}(t). \tag{4.11}$$

Using the Cholesky decomposition for correlated Brownian motions, formula (4.11) can be written as:

$$\begin{aligned}\tilde{w}(t) &= \frac{\sigma}{\tilde{\sigma}} w(t) + \frac{\sigma^{'}}{\tilde{\sigma}}(\rho w(t) + \sqrt{1-\rho^2}\bar{w}(t)) \\ &= \frac{\sigma + \sigma'\rho}{\tilde{\sigma}} w(t) + \frac{\sigma^{'}\sqrt{1-\rho^2}}{\tilde{\sigma}}\bar{w}(t),\end{aligned} \tag{4.12}$$

where $w(t)$ and $\bar{w}(t)$ are now independent. The expectation of the survival index related to the insurer's cohort is given by:

$$E_{\mathbb{P}}(I(x+t, T-t)^{ins}) = e^{\alpha^{'}(t,T) - \beta^{'}(t,T)\mu_x'(t)}, \tag{4.13}$$

$\alpha'(t,T)$ and $\beta^{'}(t,T)$ are given by:

$$\begin{cases} \alpha^{'}(t,T) & = \frac{A}{b}\Big[e^{-bT}\frac{e^{(B+b)T}-e^{(B+b)t}}{B+b} - \frac{e^{BT}-e^{Bt}}{B}\Big] - \frac{(\sigma^2+\sigma'^2+2\rho\sigma\sigma')}{2b^2}\Big[\frac{1}{b}(1-e^{-b(T-t)}) - T + t\Big] \\ & \quad -\frac{(\sigma^2+\sigma'^2+2\rho\sigma\sigma')}{4b^3}(1-e^{-b(T-t)})^2 \\ & \\ \beta^{'}(t,T) & = \beta(t,T). \end{cases} \tag{4.14}$$

Let us look at the form of the spread between the two mortality intensities. We denote this spread by $\theta_x(t)$, defined as the difference between the mortality intensity of the insurer and the national population. We have:

$$\begin{aligned} d\theta_x(t) &= d\mu'_x(t) - d\mu_x(t) \\ &= -b(\mu'_x(t) - \mu_x(t))dt + \sigma^{'}dw^{'}(t) \\ &= -b\theta_x(t)dt + \sigma^{'}dw^{'}(t). \end{aligned} \tag{4.15}$$

The spread $\theta_x(t)$ given by (4.15) follows the Ornstein Uhlenbeck process, which is mean-reverting to 0. Its variance is bounded by its asymptotic value:

$$\lim_{t\longrightarrow+\infty} \mathrm{Var}(\theta_x(t)) = \frac{\sigma^{'2}}{2b}. \tag{4.16}$$

In this case, we have a real basis risk because the spread is stochastic.
Let us remark that the initial level of the two mortality intensities can be different: $\mu_x(0) \neq \mu'_x(0)$ (level effect). We can also observe that, since the spread has a mean reversion level equal to 0, and a bounded variance (formula (4.16)), there is no divergence between the two populations as discussed, for instance by Villegas et al. [27] and Li and Hardy [20].

4.2.3 Second possibility (constant shift)

The insurer's intensity of mortality is given by:

$$d\mu'_x(t) = (\xi'(t) - b\mu'_x(t))dt + \sigma dw(t). \tag{4.17}$$

where $\xi('t) = g \cdot \xi(t)$, and g is a positive constant.
In this case, the expression of $\alpha^{'}(.)$ to put in formula (4.5) becomes:

$$\begin{aligned} \alpha^{'}(t,T) \quad = \quad & \frac{g\cdot A}{b}\Big[e^{-bT}\frac{e^{(B+b)T}-e^{(B+b)t}}{B+b} - \frac{e^{BT}-e^{Bt}}{B}\Big] \\ & -\frac{\sigma^2}{2b^2}[\frac{1}{b}(1-e^{-b(T-t)}) - T + t] - \frac{\sigma^2}{4b^3}(1-e^{-b(T-t)})^2. \end{aligned} \tag{4.18}$$

The spread $\theta_x(t)$ between the mortality intensity of the national population, and the insurer's one in the second possibility is given by:

$$\begin{aligned} d\theta_x(t) &= (\xi^{'}(t) - \xi(t) - b(\mu'_x(t) - \mu_x(t))dt \\ &= (\xi'(t) - \xi(t) - b\theta_x(t))dt. \end{aligned} \tag{4.19}$$

In this case, we do not have a real basis risk since $\theta_x(t)$ is a deterministic mean-reverting function without any noise.

4.2.4 Third possibility (extra vol and constant shift)

The insurer's intensity of mortality is given by:

$$d\mu'_x(t,T) = (\xi'(t) - b\mu'_x(t))dt + \tilde{\sigma}d\tilde{w}(t). \tag{4.20}$$

The spread $\theta_x(t)$ between the mortality intensity of the national population and the insurer's is given by:

$$\begin{aligned} d\theta_x(t) &= (\xi'(t) - \xi(t) - b(\mu'_x(t) - \mu_x(t))dt + \sigma^{'}dw^{'}(t) \\ &= (\xi'(t) - \xi(t) - b\theta_x(t))dt + \sigma^{'}dw^{'}(t). \end{aligned} \tag{4.21}$$

$\theta_x(t)$ is a mean-reverting process to a floating target $\xi'(t) - \xi(t)$. In this case we have three effects:

- Level effect: $\mu'_x(0) \neq \mu_x(0)$
- Target effect: $\xi'(t) \neq \xi(t)$
- Volatility effect: presence of additional variance

In this case the expression of $\alpha^{'}(.)$ is given by:

$$\begin{aligned} \alpha^{'}(t,T) \quad = \quad & \frac{g\cdot A}{b}\Big[e^{-bT}\frac{e^{(B+b)T} - e^{(B+b)t}}{B+b} - \frac{e^{BT} - e^{Bt}}{B}\Big] \\ & -\frac{(\sigma^2 + \sigma'^2 + 2\rho\sigma\sigma')}{2b^2}\Big[\frac{1}{b}(1 - e^{-b(T-t)}) - T + t\Big] \\ & -\frac{(\sigma^2 + \sigma'^2 + 2\rho\sigma\sigma')}{4b^3}(1 - e^{-b(T-t)})^2. \end{aligned} \tag{4.22}$$

Remark 1 *We can alternatively choose $\xi('t)$ such that g is a time-dependent function that follows the Gompertz form, namely, $\xi'(t) = g(t)\cdot\xi(t)$, with $g(t) = g\cdot e^{\vartheta t}$, ϑ is a positive constant, in this case the expression of $\alpha^{'}(.)$ is given by:*

$$\begin{aligned} \alpha^{'}(t,T) \quad = \quad & \frac{g\cdot A}{b}\Big[e^{-bT}\frac{e^{(B+b+\vartheta)T} - e^{(B+b+\vartheta)t}}{B+b+\vartheta} - \frac{e^{(B+\vartheta)T} - e^{(B+\vartheta)t}}{B+\vartheta}\Big] \\ & -\frac{(\sigma^2 + \sigma'^2 + 2\rho\sigma\sigma')}{2b^2}\Big[\frac{1}{b}(1 - e^{-b(T-t)}) - T + t\Big] \\ & -\frac{(\sigma^2 + \sigma'^2 + 2\rho\sigma\sigma')}{4b^3}(1 - e^{-b(T-t)})^2. \end{aligned} \tag{4.23}$$

4.3 S-forward

4.3.1 Description of the product

A Survival forward (or S-forward[2]) is an agreement between two counterparties to exchange at a future date T (the maturity of the contract), an amount equal to the realized survival rate of a given population cohort (floating leg), in return for a fixed survival rate agreed at the inception of the contract (fixed rate payment).

For easiness of representation, we consider the notional amount equal to one monetary unit. The payoff of the S-forward is then given by:

$$Payoff(T) = I(x,T) - {}_T\hat{p}_x, \tag{4.24}$$

where $I(x,T)$ is the realized survival rate, and ${}_T\hat{p}_x$ is a fixed survival rate of an individual aged x at time 0 to be alive at age $x+T$ ($\mathcal{F}_0$ measurable). With this derivative, the insurer can hedge the longevity risk by being the fixed leg payer of an S-forward: if the realized survival rate is higher than the fixed rate at the maturity of the contract, he gets a positive payment which can be used to compensate the higher longevity risk arising from annuity contracts. However, the realized survival rate is usually computed based on the reference national population. Hence, by using this derivative, only a portion of the systemic risk can be covered, letting the insurer exposed to the basis risk.

4.3.2 Pricing of the product

Many authors have shed light on how to price longevity-linked securities, using different methods usually used in quantitative finance, such as risk-neutral, Sharpe ratio, and Wang approaches (Barrieu and Veraart [2]). These techniques could be adapted to price longevity-linked securities. Nevertheless, these approaches require the assessment of longevity risk parameters (market price of longevity risk (risk-neutral approach), Sharpe coefficient (Sharpe method) or Wang coefficient (Wang approach)), which is not easy due to the lack of data in the longevity derivatives market.
In the new regulatory context of Solvency II, another approach is to be more consistent with the corresponding valuation. Therefore, some authors have introduced a new methodology inspired by Solvency II: the pricing of longevity derivatives using the Cost of Capital approach. A version of this method has been proposed by Levantesi and Menzietti [19] in a discrete time model, and another version has been presented by Zeddouk and Devolder [31] in continuous time models. The overall idea behind the Cost of Capital method is based on linking the price of longevity derivatives with the capital that the insurer should hold to cover unexpected losses: according to Solvency II, insurance liabilities that cannot be hedged should be computed as the sum of the best estimate (BE) plus a risk margin (RM), which is the potential costs of transferring insurance obligations to a third party if the insurer fails. The RM at time 0 is determined by the Cost of Capital approach based on the future remuneration on the

[2]The concept of S-forward was first introduced in Life and Longevity Markets Association (LLMA), 2010

successive SCR:

$$RM_0 = C\% \sum_{i=0}^{T-1} SCR_i \mid_0 \; P(0, i+1). \tag{4.25}$$

where $SCR_i \mid_0$ is the estimation at time 0 of the solvency capital requirement to cover with 99.5% probability the unexpected losses for year i, and C is the Cost of Capital rate (6% in Solvency II). $P(0, i+1)$ is the discount factor.
The price at time 0 is then given by:

$$V_{COC}(0, T) = BE_0^{\mathbb{P}} + RM_0, \tag{4.26}$$

Namely, the insurer can by an S-forward contract to lower his exposition to the longevity risk and as a consequence, the corresponding SCR can be mitigated, or even reduced to zero if the longevity risk is completely covered, and so does the corresponding risk margin RM. Moreover, this approach has the advantage to be parameterized by the Cost of Capital rate, which is fixed by regulation (6% in Solvency II nowadays). This legal and unique value can be seen as a benchmark, which is not the case for the classical pricing methods.

4.4 S-exchange pricing with C.O.C approach

In this paper, we focus on how to cover the remaining risk called basis risk, by using another derivative called " S-exchange contract". In the literature, the price of longevity derivatives is usually computed without taking into account the basis risk. In this section, we will evaluate this risk, and determine the price of a financial tool that allows for the protection against this basic risk.

Definition

An S-exchange forward is an agreement between the insurer and an investor to exchange at a future date T (the maturity of the contract), an amount equal to the realized survival rate of a given national population, i.e the longevity index, in return for the realized survival rate of the insurer's cohort population. For easiness of representation, we consider the notional amount equal to one monetary unit. The payoff at maturity T of the S-exchange forward is then given by:

$$Payoff(T) = I(x, T)^{ins} - I(x, T), \tag{4.27}$$

where $I(x, T)^{ins} = e^{-\int_0^T \mu'_x(s)ds}$ is the realized survival rate of an individual from the insurer's cohort population in a model to be specified ($\mathcal{F}_0$ measurable), and $I(x, T) = e^{-\int_0^T \mu_x(s)ds}$ the longevity index of the national population in a model to be specified. Under the Cost of Capital approach, the price of this derivative is given by:

Proposition:

The price of the S-exchange contract at time 0 under the Cost of Capital approach is given by:

$$\begin{aligned} V_{COC}(0,T) &= P(0,T)(E_{\mathbb{P}}(I(x,T)^{ins}) - E_{\mathbb{P}}(I(x,T))) \\ &\quad + 6\% \sum_{i=0}^{T-1} \mathrm{VaR}_{99,5\%} \left[\Psi_i I(x+i,1)^{ins} - \Phi_i I(x+i,1) - \Lambda_i \right] \\ &\quad \times P(i,T)P(0,i+1), \end{aligned} \tag{4.28}$$

where:

$$\begin{aligned} \Psi_i &= E_{\mathbb{P}}(I(x,i)^{ins}) E_{\mathbb{P}}(\, I(x+i+1, T-i-1)^{ins}) \\ \Phi_i &= E_{\mathbb{P}}(I(x,i)) E_{\mathbb{P}}(\, I(x+i+1, T-i-1)) \\ \Lambda_i &= E_{\mathbb{P}}(I(x,i)^{ins}) E_{\mathbb{P}}(\, I(x+i, T-i)^{ins}) \\ &\quad - E_{\mathbb{P}}(I(x,i)) E_{\mathbb{P}}(\, I(x+i, T-i)). \end{aligned} \tag{4.29}$$

Proof:

We recall that under the COC approach, the price of any product is given by (Zeddouk and Devolder [31]):

$$V_{COC}(t,T) = BE_t^{\mathbb{P}} + RM_t, \tag{4.30}$$

where $BE_t^{\mathbb{P}}$ is the best estimate under the real-world measure, of the payoff of the S-forward at time t, and RM_t the risk margin. The price of the S-exchange forward at time 0 is given by:

$$V_{COC}(0,T) = BE_0^{\mathbb{P}} + RM_0, \tag{4.31}$$

where $BE_0^{\mathbb{P}}$ is as follows:

$$BE_0^{\mathbb{P}} = P(0,T)(E_{\mathbb{P}}(I(x,T)^{ins}) - E_{\mathbb{P}}(I(x,T))). \tag{4.32}$$

The risk margin is defined by the present value of the future remunerations on the successive SCR:

$$RM_0 = 6\% \sum_{i=0}^{T-1} SCR_i \, P(0,i+1). \tag{4.33}$$

The future SCR_i are random variables. To compute the initial RM we need to use their estimation at time 0 denoted by $SCR_i \mid_0$. The risk margin at time 0 is then given by:

$$RM_0 = 6\% \sum_{i=0}^{T-1} SCR_i \mid_0 \; P(0,i+1). \tag{4.34}$$

The SCR_i are given by:

$$SCR_i = \mathrm{VaR}_{99,5\%}[BE^{\mathbb{P}}_{i+1}P(i,i+1) - BE^{\mathbb{P}}_i], \tag{4.35}$$

where:

$$\begin{aligned} BE^{\mathbb{P}}_{i+1} &= (I(x,i+1)^{ins}E_{\mathbb{P}}(\,I(x+i+1,T-i-1)^{ins}) \\ &\quad -I(x,i+1)E_{\mathbb{P}}(\,I(x+i+1,T-i-1)))P(i+1,T), \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} BE^{\mathbb{P}}_{i} &= (I(x,i)^{ins}E_{\mathbb{P}}(\,I(x+i,T-i)^{ins}) \\ &\quad -I(x,i)E_{\mathbb{P}}(\,I(x+i,T-i))))P(i,T). \end{aligned} \tag{4.37}$$

The SCR_i are then given by:

$$\begin{aligned} SCR_i &= P(i,T)\,\mathrm{VaR}_{99,5\%}\Big[I(x,i+1)^{ins}E_{\mathbb{P}}(\,I(x+i+1,T-i-1)^{ins}) \\ &\quad -I(x,i)^{ins}E_{\mathbb{P}}(\,I(x+i,T-i)^{ins}) - I(x,i+1)E_{\mathbb{P}}(\,I(x+i+1,T-i-1)) \\ &\quad +I(x,i)E_{\mathbb{P}}(\,I(x+i,T-i)))\Big]. \end{aligned} \tag{4.38}$$

Formula (4.38) is assumed to be equivalent to:

$$\begin{aligned} SCR_i &= P(i,T)\,\mathrm{VaR}_{99,5\%}\Big[I(x,i)(I(x+i,1)^{ins}E(\,I(x+i+1,T-i-1)^{ins}) \\ &\quad -E_{\mathbb{P}}(\,I(x+i,T-i)^{ins})) - I(x,i)(I(x+i,1)(E_{\mathbb{P}}(\,I(x+i+1,T-i-1)) \\ &\quad -E_{\mathbb{P}}(\,I(x+i,T-i)))\Big]. \end{aligned} \tag{4.39}$$

Then their estimations at time 0 are given by:

$$\begin{aligned} SCR_i\mid_0 &= P(i,T)\,\mathrm{VaR}_{99,5\%}\Big[E_{\mathbb{P}}(I(x,i)^{ins})(I(x+i,1)^{ins}E_{\mathbb{P}}(\,I(x+i+1,T-i-1)^{ins}) \\ &\quad -E_{\mathbb{P}}(\,I(x+i,T-i)^{ins})) - E_{\mathbb{P}}(I(x,i))(I(x+i,1)E_{\mathbb{P}}(\,I(x+i+1,T-i-1)) \\ &\quad -E_{\mathbb{P}}(\,I(x+i,T-i)))\Big]. \end{aligned} \tag{4.40}$$

The $SCR_i\mid_0$ can be then written as:

$$SCR_i\mid_0 = P(i,T)\,\mathrm{VaR}_{99,5\%}[\Psi_i I(x+i,1)^{ins} - \Phi_i I(x+i,1) - \Lambda_i], \tag{4.41}$$

where Ψ_i,Φ_i and Λ_i are constants given by:

$$\begin{aligned} \Psi_i &= E_{\mathbb{P}}(I(x,i)^{ins})E_{\mathbb{P}}(\,I(x+i+1,T-i-1)^{ins}) \\ \Phi_i &= E_{\mathbb{P}}(I(x,i))E_{\mathbb{P}}(\,I(x+i+1,T-i-1)) \\ \Lambda_i &= E_{\mathbb{P}}(I(x,i)^{ins})E_{\mathbb{P}}(\,I(x+i,T-i)^{ins}) \\ &\quad -\,E_{\mathbb{P}}(I(x,i))E_{\mathbb{P}}(\,I(x+i,T-i)). \end{aligned} \tag{4.42}$$

Finally, the risk margin at time 0 is then equal to:

$$RM_0 = 6\% \sum_{i=0}^{T-1} \mathrm{VaR}_{99,5\%}[\Psi_i I(x+i,1)^{ins} - \Phi_i I(x+i,1) - \Lambda_i]P(i,T)P(0,i+1). \tag{4.43}$$

4.5 S-exchange pricing under other classical methods

We can also compute the S-exchange prices under three other pricing methods often used in quantitative finance:

- The risk-neutral approach (Cairns et al., [9])
- Wang transform (Wang [28])
- Sharpe ratio (Milevsky et al.[23])

The aim is to compare the results found with these methods to those found with the COC method, as developed in section 4.4.

4.5.1 Risk-neutral method

Under the risk-neutral probability measure $\mathbb{Q}_\lambda$, the price of an S-exchange contract at time t is given by:

$$\begin{aligned} V_{\mathbb{Q}_\lambda}(0,T) &= P(0,T)[E_{\mathbb{Q}_\lambda}(I^{ins}(x,T) - I(x,T))] \\ &= P(0,T)(\, e^{\alpha'_{\mathbb{Q}_\lambda}(0,T)-\beta'_{\mathbb{Q}_\lambda}(0,T)\mu_x'^{\mathbb{Q}_\lambda}(0)} - e^{\alpha_{\mathbb{Q}_\lambda}(0,T)-\beta_{\mathbb{Q}_\lambda}(0,T)\mu_x^{\mathbb{Q}_\lambda}(0)}). \end{aligned} \tag{4.44}$$

We introduce the market prices $\lambda(t, \mu_x(t))$ associated with the longevity risk, and we consider a model with a constant market price of risk $\lambda(t, \mu_x(t)) = \lambda$.
Using the HW model for longevity and considering the first possibility, the force of mortality under the risk-neutral measure for the national population $\mu_x^{\mathbb{Q}_\lambda}$ and the insurer's population $\mu_x'^{\mathbb{Q}_\lambda}$are given by:

$$d\mu_x^{\mathbb{Q}_\lambda}(t,T) = (\xi(t) - b\mu_x^{\mathbb{Q}_\lambda} + \sigma\lambda)dt + \sigma dw^*(t) \tag{4.45}$$

$$d\mu_x'^{\mathbb{Q}_\lambda}(t,T) = (\xi^{'}(t) - b\mu_x'^{\mathbb{Q}_\lambda} + \sigma\lambda)dt + \sigma dw^*(t) + \sigma'\lambda dw^{*'}(t). \tag{4.46}$$

Let $w^*(t,\omega)$= $w(t,\omega) - \int_0^t \lambda(t, \mu'_x(s))ds$, and $w^{'*}(t,\omega) = w'(t,\omega) - \int_0^t \lambda(t, \mu'_x(s))ds$.
And let:

$$\tilde{\sigma}\tilde{w^*}(t) = \sigma w^*(t) + \sigma^{'} w^{'*}. \tag{4.47}$$

Formula (4.46) becomes:

$$d\mu_x'^{\mathbb{Q}_\lambda}(t,T) = (\xi^{'}(t) - b\mu_x'^{\mathbb{Q}_\lambda} + \sigma\lambda)dt + \tilde{\sigma\lambda} d\tilde{w}^*(t). \tag{4.48}$$

So the total volatility of the subpopulation is:

$$\tilde{\sigma} = \sqrt{\sigma^2 + \sigma'^2 + 2\rho\sigma\sigma'}, \tag{4.49}$$

where ρ is the correlation coefficient: $\rho(t)$ =Cov($w^*(t)$,$w^{*'}(t)$).
From (4.47) and (4.49) we have:

$$\tilde{w^*}(t) = \frac{\sigma}{\tilde{\sigma}} w^*(t) + \frac{\sigma^{'}}{\tilde{\sigma}} w^{*'}(t). \tag{4.50}$$

Using the Cholesky decomposition for correlated Brownian motions, formula (4.50) becomes:

$$\begin{aligned}\tilde{w^*}(t) &= \frac{\sigma}{\tilde{\sigma}}w^*(t) + \frac{\sigma'}{\tilde{\sigma}}(\rho w^*(t) + \sqrt{1-\rho^2}\bar{w^*}(t)) \\ &= \frac{\sigma + \sigma'\rho}{\tilde{\sigma}}w^*(t) + \frac{\sigma'\sqrt{1-\rho^2}}{\tilde{\sigma}}\bar{w^*}(t), \end{aligned} \tag{4.51}$$

where $w^*(t)$ and $\bar{w}^*(t)$ are independent. Using the HW model, the price becomes:

$$V_{\mathbb{Q}_\lambda}(0,T) = P(0,T)(\, e^{\alpha'_{\mathbb{Q}_\lambda}(0,T)-\beta'_{\mathbb{Q}\lambda}(0,T)\mu'^{\mathbb{Q}_\lambda}_x(0)} - \, e^{\alpha_{\mathbb{Q}_\lambda}(0,T)-\beta_{\mathbb{Q}\lambda}(0,T)\mu^{\mathbb{Q}_\lambda}_x(0)}), \tag{4.52}$$

where:

$$\begin{cases}\alpha_{\mathbb{Q}_\lambda}(0,T) & = \frac{A}{b}\left[e^{-bT}\frac{e^{(B+b)T}-1}{B+b} - \frac{e^{BT}-1}{B}\right] - \frac{\sigma^2}{2b^2}\left[\frac{1}{b}(1-e^{-bT}) - T\right] \\ & \quad -\frac{\sigma^2}{4b^3}(1-e^{-bT})^2 - \frac{\lambda\sigma}{b}(1-e^{-bT}) \\ \\ \beta_{\mathbb{Q}\lambda}(0,T) & = \frac{1}{b}(1-e^{-bT}).\end{cases} \tag{4.53}$$

For $\alpha'_{\mathbb{Q}_\lambda}(0,T)$ and $\beta'_{\mathbb{Q}\lambda}(0,T)$, they take different values for each possibility. For instance, for the third possibility we obtain:

$$\begin{cases}\alpha'_{\mathbb{Q}_\lambda}(0,T) & = \frac{g\cdot A}{b}\left[e^{-bT}\frac{e^{(B+b)T}-1}{B+b} - \frac{e^{BT}-1}{B}\right] - \frac{(\sigma^2+\sigma'^2+2\rho\sigma\sigma')}{2b^2}\left[\frac{1}{b}(1-e^{-bT}) - T\right] \\ & \quad -\frac{(\sigma^2+\sigma'^2+2\rho\sigma\sigma')}{4b^3}(1-e^{-bT})^2 - \frac{\lambda\sqrt{\sigma^2+\sigma'^2+2\rho\sigma\sigma'}}{b}(1-e^{-bT})) \\ \\ \beta'_{\mathbb{Q}\lambda}(0,T) & = \frac{1}{b}(1-e^{-bT}).\end{cases} \tag{4.54}$$

4.5.2 Sharpe ratio method

The price of the S-exchange under the Sharpe approach is given by:

$$\begin{aligned}V_{Sharpe}(0,T) &= P(0,T)(\, E_{\mathbb{P}}(\, I(x,T)) - E_{\mathbb{P}}(\, I^{ins}(x,T)) \\ &\quad +S\sqrt{\mathrm{Var}_{\mathbb{P}}(I^{ins}(x,T) - I(x,T))}),\end{aligned} \tag{4.55}$$

where S is the chosen fixed Sharpe ratio, and $\mathrm{Var}_{\mathbb{P}}(I(x,T))$ is the variance of the survival index. We have:

$$\mathrm{Var}_{\mathbb{P}}(I^{ins}(x,T) - I(x,T)) = E_{\mathbb{P}}[\, I^{ins}(x,T) - I(x,T) - E_{\mathbb{P}}(\, I^{ins}(x,T) - I(x,T))]^2. \tag{4.56}$$

Under the first and the third possibility, this variance could be computed numerically, but under the second possibility we can obtain an explicit formula.

4.5.3 Wang method

We will now compute the S-exchange price under the Wang approach. The Wang transform method is a distortion approach that is based on a distortion operator. In insurance context this operator converts the best estimate of the survival index into its risk equivalent using a specific price of risk. This procedure was presented by Dowd et al. [13] as a pricing method used in the pricing of several over-the-counter longevity swaps in practice, and used for instance by Denuit et al. [12]. The Wang distortion risk measure is given by:

$$\rho_{g_\delta}(x) = \int_0^\infty g_\delta(\bar{F}_x(s))ds - \int_{-\infty}^0 (1 - g_\delta(\bar{F}_x(s)))ds, \tag{4.57}$$

where x is a continuous random variable. $\bar{F}_x(s)$ is its decumulative function, and g_δ is the distortion function associated with the distortion parameter δ and given by:

$$g_\delta(s) = \Phi(\Phi^{-1}(s) + \delta), \tag{4.58}$$

where F(.) is the cumulative standard normal distribution. then:

$$\begin{aligned} \rho_{g_\delta}(I^{ins}(x,T) - I(x,T)) \quad = \quad & \int_0^\infty \Phi(\Phi^{-1}(\bar{F}_{(I^{ins}(x,T)-I(x,T))}(s)) + \delta)ds \\ & - \int_{-\infty}^0 (1 - \Phi(\Phi^{-1}(\bar{F}_{(I^{ins}(x,T)-I(x,T))}(s)) + \delta))ds \end{aligned} \tag{4.59}$$

The price of the S-exchange is given by:

$$V_{Wang}(0,T) = P(0,T)[\rho_{g_\delta}(I^{ins}(x,T) - I(x,T))] \tag{4.60}$$

Like in the Sharpe ratio method, for the Wang approach we can obtain under the second possibility an explicit formula for the price.

4.6 Numerical illustrations

In this section, we give the numerical results of the prices of different S-exchange contracts, using the COC method and the classical approaches for the Belgian population. We use the following assumptions:

- An insurer with a portfolio of pure endowments contracts paying a lump sum of 1€ at maturity T in case of survival;
- N_0=10 000 initial policyholders for each cohort;
- Individuals aged 65 or 70 years old in 2015;
- Payment of 1€ to each policyholder alive at time T;
- Calibration is based on projected data from the IA|BE unisex projected generational mortality table.

- The risk-less interest rate is considered constant (equals to 1%);
- The parameters λ, S and δ have been chosen to be quite similar to the values usually suggested in the literature (see for instance Barrieu and Veraart [2], Cui [11]). These parameters are reported in Table 4.1:

Parameter	Values
λ	-20%
S	10%
δ	10%

Table 4.1: The classical methods' parameter values.

We calibrate the HW model on projected mortality data for each cohort using the Least Square Estimation. The optimal parameters are reported in Table 4.2:

Age	$\mu_x(0)$	A	B	b	σ
65	0.0105677	0.002398110	0.115379365	0.261814487	0.001864268
70	0.01608859	0.005079817	0.116501598	0.311927223	0.006213681

Table 4.2: Optimal parameter values for the survival function in the HW model.

Zeddouk and Devolder [31] have computed the prices of the S-forward contracts under the Cost of Capital approach, using HW model for mortality and the assumptions made in this paper. The prices as well as the fixed legs are reported in Table 4.3:

	$S - forward\,price$	$fixed\ leg$
$x_0 = 65$, $T = 5$	52.63125	0.94193
$x_0 = 65$, $T = 10$	88.39936	0.86580
$x_0 = 70$, $T = 5$	49.96718	0.91012
$x_0 = 70$, $T = 10$	61.52067	0.78655

Table 4.3: The prices and the fixed legs of the different S-forwards under the Cost of Capital approach.

4.6.1 S-exchange prices under the first possibility

The Cost of Capital approach

Using the first possibility, we determine the price of the different S-exchange derivatives under the Cost of Capital approach. We consider the following parameters:

- $\sigma' = 20\%\sigma$
- $\rho = 0.5$
- $\mu'_x(0) = \mu_x(0)$

Table 4.4 gives the comparison between the prices of the different S-forward, and S-exchange contracts under the 1^{st} possibility found under the COC approach:

	$S - forward\,price$	$S - exchange\,price$
$x_0 = 65$, $T = 5$	52.63125	2.48922
$x_0 = 65$, $T = 10$	88.39936	5.52361
$x_0 = 70$, $T = 5$	49.96718	8.89376
$x_0 = 70$, $T = 10$	61.52067	17.88005

Table 4.4: Prices of the different S-forward and S-exchange contracts under the 1^{st} possibility, COC method.

The risk-neutral approach

We use the same assumptions made in section 4.3 to compute the prices of the different S-exchange contracts. Table 4.5 presents the prices of the different contracts under the risk-neutral method, using the 1^{st} possibility to describe the force of mortality of the insurer's population:

	$S - forward\,price$	$S - exchange\,price$
$x_0 = 65$, $T = 5$	55.24155	1.12973
$x_0 = 65$, $T = 10$	86.98154	1.41983
$x_0 = 70$, $T = 5$	55.95713	3.72324
$x_0 = 70$, $T = 10$	53.61137	4.97062

Table 4.5: Prices of the different S-forward and S-exchange contracts under the 1^{st} possibility, risk-neutral method.

The Sharpe ratio method

Now we compute the price of the S-forwards and S-exchange for the Sharpe method under the 1^{st} possibility. Table 4.6 reports the S-forwards and S-exchange prices for different ages and maturities:

	$S - forward\,price$	$S - exchange\,price$
$x_0 = 65$, $T = 5$	52.88389	0.88968
$x_0 = 65$, $T = 10$	88.89517	1.50233
$x_0 = 70$, $T = 5$	49.55169	3.23218
$x_0 = 70$, $T = 10$	59.66057	5.82333

Table 4.6: Prices of the different S-forward and S-exchange contracts under the 1^{st} possibility, Sharpe method.

The Wang method

Now we compute the price of the S-forwards and S-exchange for the Wang method under the 1^{st} possibility. Table 4.7 reports the S-forwards and S-exchange prices for different ages and maturities:

	$S - forward\,price$	$S - exchange\,price$
$x_0 = 65$, $T = 5$	52.88651	0.93152
$x_0 = 65$, $T = 10$	88.90408	1.23266
$x_0 = 70$, $T = 5$	49.57395	3.00687
$x_0 = 70$, $T = 10$	59.72016	6.06159

Table 4.7: Prices of the different S-forward and S-exchange contracts under the 1^{st} possibility, Wang method.

Comparison

To summarize, we report in Table 4.8 the different S-exchange prices computed using the first possibility:

	COC	Risk-neutral	Sharpe	Wang
$x_0 = 65$, $T = 5$	2.48922	1.12973	0.88968	0.9315238
$x_0 = 65$, $T = 10$	5.52361	1.41983	1.50233	1.232661
$x_0 = 70$, $T = 5$	8.89376	3.72324	3.23218	3.006879
$x_0 = 70$, $T = 10$	17.88005	4.97062	5.82333	6.06159

Table 4.8: Comparison between S-exchange prices under the different methods, 1^{st} possibility.

Under the first possibility, the classical methods provide quite similar prices, that become higher with age and maturity. These prices are much more important under the COC method, especially for the age of 70 years old.

4.6.2 S-exchange prices under the second possibility

The Cost of Capital approach

We consider the following parameters:

- $g = 0.9$
- $\mu'_x(0) = g \cdot \mu_x(0)$

Table 4.9 gives the the prices of the contracts under the COC approach using the second possibility:

	$S - forward\,price$	$S - exchange\,price$
$x_0 = 65$, $T = 5$	52.63125	49.48473
$x_0 = 65$, $T = 10$	88.39936	105.81050
$x_0 = 70$, $T = 5$	49.96718	80.38255
$x_0 = 70$, $T = 10$	61.52067	170.68800

Table 4.9: Prices of the different S-forward and S-exchange contracts under the 2^{nd} possibility, COC method.

Risk-neutral approach

Table 4.10 reports the S-forwards and S-exchange prices under the risk neutral method, for different ages and maturities:

	$S - forward\, price$	$S - exchange\, price$
$x_0 = 65$, $T = 5$	55.24155	49.48929
$x_0 = 65$, $T = 10$	86.98154	107.25240
$x_0 = 70$, $T = 5$	55.95713	79.80213
$x_0 = 70$, $T = 10$	53.61137	172.32600

Table 4.10: Prices of the different S-forward and S-exchange contracts under the 2^{nd} possibility, risk-neutral method.

Sharpe method

Table 4.11 represents the S-forwards and S-exchange prices under Sharpe approach, for different ages and maturities:

	$S - forward\, price$	$S - exchange\, price$
$x_0 = 65$, $T = 5$	52.88389	49.47635
$x_0 = 65$, $T = 10$	88.89517	107.2783
$x_0 = 70$, $T = 5$	49.55169	79.74338
$x_0 = 70$, $T = 10$	59.66057	172.47010

Table 4.11: Prices of the different S-forward and S-exchange contracts under the 2^{nd} possibility, Sharpe method.

Wang approach

Table 4.12 reports the S-forwards and S-exchange prices under Wang approach, for different ages and maturities:

	$S - forward\, price$	$S - exchange\, price$
$x_0 = 65$, $T = 5$	52.88651	49.47637
$x_0 = 65$, $T = 10$	88.90408	107.27841
$x_0 = 70$, $T = 5$	49.57395	79.74367
$x_0 = 70$, $T = 10$	59.72016	172.47280

Table 4.12: Prices of the different S-forward and S-exchange contracts under the 2^{nd} possibility, Wang method.

Comparison

To summarize, we report in Table 4.13 the different S-exchange prices computed using the second possibility:

	COC	Risk-neutral	Sharpe	Wang
$x_0 = 65$, $T = 5$	49.48473	49.48929	49.47635	49.47637
$x_0 = 65$, $T = 10$	105.8105	107.25240	107.27830	107.2784
$x_0 = 70$, $T = 5$	80.38255	79.80213	79.743380	79.74367
$x_0 = 70$, $T = 10$	170.68800	172.32610	172.47010	172.4728

Table 4.13: Comparison between S-exchange prices under the different methods, 2^{nd} possibility.

Under this possibility, we get more or less the same S-exchange prices with the four methods.

4.6.3 S-exchange prices under the third possibility

The Cost of Capital approach

- $\sigma' = 20\%\sigma$
- $\rho = 0.5$
- $g = 0.9$
- $\mu'_x(0) = g \cdot \mu_x(0)$

Table 4.14 gives the the prices found under the COC approach, using HW for mortality:

	$S-forward\,price$	$S-exchange\,price$
$x_0 = 65$, $T = 5$	52.63125	51.95998
$x_0 = 65$, $T = 10$	88.39936	111.19450
$x_0 = 70$, $T = 5$	49.96718	88.37966
$x_0 = 70$, $T = 10$	61.52067	188.27620

Table 4.14: Prices of the different S-forward and S-exchange contracts under the 3^{rd} possibility, COC method.

The risk-neutral approach

Table 4.15 reports the S-forwards and S-exchange prices for different ages and maturities:

	$S-forward\,price$	$S-exchange\,price$
$x_0 = 65$, $T = 5$	55.24155	50.62522
$x_0 = 65$, $T = 10$	86.98154	108.69150
$x_0 = 70$, $T = 5$	55.95713	83.55947
$x_0 = 70$, $T = 10$	53.61137	177.41610

Table 4.15: Prices of the different S-forward and S-exchange contracts under the 3^{rd} possibility, risk-neutral method.

The Sharpe ratio

Table 4.16 reports the S-forwards and S-exchange prices for different ages and maturities:

	$S - forward\,price$	$S - exchange\,price$
$x_0 = 65$, $T = 5$	52.88389	53.42589
$x_0 = 65$, $T = 10$	88.89517	116.88090
$x_0 = 70$, $T = 5$	49.55169	86.51790
$x_0 = 70$, $T = 10$	59.66057	188.85620

Table 4.16: Prices of the different S-forward and S-exchange contracts under the 3^{rd} possibility, Sharpe method.

The Wang approach

Table 4.17 reports the S-forwards and S-exchange prices for different ages and maturities:

	$S - forward\,price$	$S - exchange\,price$
$x_0 = 65$, $T = 5$	52.88651	50.68662
$x_0 = 65$, $T = 10$	88.90408	108.56590
$x_0 = 70$, $T = 5$	49.57395	85.32901
$x_0 = 70$, $T = 10$	59.72016	177.48020

Table 4.17: Prices of the different S-forward and S-exchange contracts under the 3^{rd} possibility, Wang method.

Comparison

To summarize, we report in Table 4.18 the different S-exchange prices computed using the third possibility:

	COC	Risk-neutral	Sharpe	Wang
$x_0 = 65$, $T = 5$	51.95998	50.62522	53.42589	50.68662
$x_0 = 65$, $T = 10$	111.19450	108.69150	116.88090	108.5659
$x_0 = 70$, $T = 5$	88.37966	83.55947	86.51790	85.32901
$x_0 = 70$, $T = 10$	188.27620	177.41610	188.85620	177.4802

Table 4.18: Comparison between S-exchange prices under the different methods, 3^{rd} possibility.

Under this third possibility, the S-forward prices given by Wang and risk-neutral approaches are more or less similar. Overall, the Sharpe method provides the highest S-exchange prices in comparison with the four methods.

4.7 Sensitivity test

In this section, we will perform a sensitivity test for the prices, best estimates and risk margins computed under the COC approach. For each possibility, cohort and maturity, we vary one parameter and consider other parameters fixed, and we see how the price of the S-exchange contract under the COC method changes. For illustration, we provide in the Appendix section the sensitivity test for the classical methods under the third possibility.

4.7.1 First possibility

In the first possibility, to compute the prices of the S-exchange contracts we have chosen $\sigma' = 20\%\sigma$ and $\rho = 0.5$. Now we will use different values of σ' and ρ, and see how the price changes. The different prices, BE and RM are reported in Tables 4.19, 4.20 and 4.21:

	σ'				ρ		
	$10\%\sigma$	$20\%\sigma$	$30\%\sigma$	$50\%\sigma$	0	0.5	1
$x_0 = 65, T = 5$	1.2560	**2.4892**	4.2621	6.2730	2.3410	**2.4892**	2.7980
$x_0 = 65, T = 10$	2.8349	**5.5236**	8.3154	13.9834	5.3298	**5.5236**	5.7838
$x_0 = 70, T = 5$	4.2763	**8.8937**	15.3118	22.7217	7.5817	**8.8937**	9.5652
$x_0 = 70, T = 10$	7.8704	**17.8800**	24.8066	47.9071	15.0588	**17.8800**	20.4783

Table 4.19: Price sensitivity test for σ' and ρ parameters under COC approach, 1^{st} possibility.

	σ'				ρ		
	$10\%\sigma$	$20\%\sigma$	$30\%\sigma$	$50\%\sigma$	0	0.5	1
$x_0 = 65, T = 5$	0.0299	**0.0654**	0.1063	0.2045	0.0109	**0.0654**	0.1199
$x_0 = 65, T = 10$	0.10628	**0.2318**	0.3768	0.7246	0.0986	**0.2318**	0.4251
$x_0 = 70, T = 5$	0.2783	**0.6072**	0.9868	1.8978	0.1012	**0.6072**	1.1133
$x_0 = 70, T = 10$	0.8530	**1.8613**	3.0249	5.8183	0.3101	**1.8613**	3.4128

Table 4.20: Best estimate sensitivity test for σ' and ρ parameters under COC approach, 1^{st} possibility.

	σ'				ρ		
	$10\%\sigma$	$20\%\sigma$	$30\%\sigma$	$50\%\sigma$	0	0.5	1
$x_0 = 65, T = 5$	1.2260	**2.4237**	4.1558	6.0685	2.3301	**2.4237**	2.6780
$x_0 = 65, T = 10$	2.7286	**5.2917**	7.9386	13.2588	5.1925	**5.2917**	5.3587
$x_0 = 70, T = 5$	3.9979	**8.2864**	14.3250	20.8238	7.4805	**8.2864**	8.4518
$x_0 = 70, T = 10$	7.0173	**16.0186**	21.7817	42.0887	14.7486	**16.0186**	17.0654

Table 4.21: Risk margin sensitivity test for σ' and ρ parameters under COC approach, 1^{st} possibility.

4.7.2 Second possibility

In the second possibility, to compute the prices of the S-exchange contracts we have chosen $g = 0.9$. Now we will consider different values for g and see how the price changes. The results are reported in the Tables 4.22, 4.23 and 4.24:

	g			
	0.5	0.7	0.9	1.1
$x_0 = 65, T = 5$	250.2315	149.2817	**49.4847**	-49.0785
$x_0 = 65, T = 10$	543.6788	321.6998	**105.8105**	-103.7540
$x_0 = 70, T = 5$	411.7152	244.9485	**80.3825**	-78.1786
$x_0 = 70, T = 10$	905.0097	526.3405	**170.6880**	-162.5489

Table 4.22: Price sensitivity test for g parameter under COC approach, 2^{nd} possibility.

	g			
	0.5	0.7	0.9	1.1
$x_0 = 65, T = 5$	249.9182	149.1292	**49.4378**	-49.16796
$x_0 = 65, T = 10$	550.2550	325.7032	**107.1109**	-105.6799
$x_0 = 70, T = 5$	405.1099	240.8466	**79.5514**	-78.8294
$x_0 = 70, T = 10$	900.6148	527.4891	**171.6711**	-167.6422

Table 4.23: Best estimate sensitivity test for g parameter under COC approach, 2^{nd} possibility.

	g			
	0.5	0.7	0.9	1.1
$x_0 = 65, T = 5$	0.3133	0.1524	**0.0468**	0.0894
$x_0 = 65, T = 10$	-6.5761	-4.0034	**-1.3003**	1.9259
$x_0 = 70, T = 5$	6.6053	4.1018	**0.8311**	0.6508
$x_0 = 70, T = 10$	4.3949	-1.1486	**-0.9830**	5.0932

Table 4.24: Risk margin sensitivity test for g parameter under COC approach, 2^{nd} possibility.

4.7.3 Third possibility

In the third possibility, to compute the prices of the S-exchage contracts, we have chosen $g = 0.9$, $\sigma' = 20\%\sigma$ and $\rho = 0.5$. Now we will compute the price using different values of g, σ' and ρ, then we see how the price changes. The results are reported in the Tables 4.25, 4.26, 4.27:

	g				σ'				ρ		
	0.5	0.7	0.9	1.1	$10\%\sigma$	$20\%\sigma$	$30\%\sigma$	$50\%\sigma$	0	0.5	1
$x_0 = 65, T = 5$	253.0618	151.8753	**51.9599**	-46.4201	50.7853	**51.9599**	53.5352	56.3035	51.8764	**51.9599**	52.3302
$x_0 = 65, T = 10$	548.6933	326.8021	**111.1945**	-98.6251	108.3595	**111.1945**	113.5351	119.6683	110.5150	**111.1945**	111.4551
$x_0 = 70, T = 5$	424.6750	252.2611	**88.3796**	-70.4250	84.3369	**88.3796**	92.7267	103.0981	87.4360	**88.3796**	90.6162
$x_0 = 70, T = 10$	927.0974	544.8262	**188.2762**	-148.6811	179.8440	**188.2762**	197.5176	218.0175	183.5376	**188.2762**	193.0048

Table 4.25: Price sensitivity test for g, σ' and ρ parameters under COC approach, 3^{rd} possibility.

	g				σ'				ρ		
	0.5	0.7	0.9	1.1	10%σ	20%σ	30%σ	50%σ	0	0.5	1
$x_0 = 65, T = 5$	249.9851	149.1950	**49.5036**	-49.1024	49.4680	**49.5036**	49.5448	49.6435	49.4489	**49.5036**	49.5585
$x_0 = 65, T = 10$	550.5031	326.8020	**107.3459**	-98.6252	107.2186	**107.3459**	107.4928	107.8453	107.1500	**107.3459**	107.5417
$x_0 = 70, T = 5$	405.7455	241.4701	**80.1642**	-78.2276	79.8323	**80.1642**	80.5473	81.4666	80.1503	**80.1642**	80.6749
$x_0 = 70, T = 10$	902.7108	529.4879	**173.5772**	-165.8245	172.5446	**173.5772**	174.7687	177.6292	171.9880	**173.5772**	175.1659

Table 4.26: Best estimate sensitivity test for g, σ' and ρ parameters under COC approach, 3^{rd} possibility.

	g				σ'				ρ		
	0.5	0.7	0.9	1.1	10%σ	20%σ	30%σ	50%σ	0	0.5	1
$x_0 = 65, T = 5$	3.0801	2.6820	**2.4562**	2.6816	1.3173	**2.4562**	3.9903	6.6600	2.4275	**2.4562**	2.7717
$x_0 = 65, T = 10$	-1.8093	0.8587	**3.8486**	6.8260	1.1409	**3.8486**	6.0423	11.8230	3.3649	**3.8486**	3.9133
$x_0 = 70, T = 5$	18.9304	10.7905	**8.2154**	7.8026	4.5046	**8.2154**	12.1794	21.6314	7.2857	**8.2154**	9.9412
$x_0 = 70, T = 10$	24.38662	15.3382	**14.6990**	17.1434	7.2994	**14.6990**	22.7488	40.3882	11.5489	**14.6990**	17.8388

Table 4.27: Risk margin sensitivity test for g, σ' and ρ parameters under COC approach, 3^{rd} possibility.

4.8 Different longevity hedging strategies

In this section, we will define and compare different longevity hedging strategies using the Cost of Capital approach. For each strategy, we determine the Solvency Capital Required that the insurer should hold to cover unexpected losses, as well as the best estimate at time 0.

Depending on his risk aversion, the insurer can choose to hold an SCR, and/or to totally or partially hedge the longevity risk. More precisely, we will compare the following possible strategies:

1. The insurer can choose to hold an SCR, and not resort to the financial market or reinsurance to hedge the longevity risk.

2. He can choose to only hedge a portion of the systemic longevity risk by entering an S-forward, and hold the SCR corresponding to the basis risk.

3. In addition to the purchase of an S-forward, he can choose a total protection against the longevity risk by entering an S-exchange contract (assuming that such product is available in the financial market, or proposed by an investor).

4. The insurer can choose a total protection without entering an S-forward and an S-exchange, but simply by transferring the total longevity risk to a reinsurer who accepts to take it. The cost of this protection is also assumed to be computed using the Cost of Capital approach.

- **Auto-protection: holding an SCR (S1)**

In this strategy, the insurer manages himself the longevity risk by holding successive annual Solvency Capitals following the directives of Solvency II. We denote by ${}_T\hat{p}_x$ a fixed survival rate of an individual aged x at time 0 to be alive at age $x+T$ ($\mathcal{F}_0$ measurable). The premium paid by the insured is based on this fixed rate.

The best estimate in this case is given at time 0 by:

$$BE_0^{\mathbb{P}} = P(0,T)(E_{\mathbb{P}}(I(x,T)^{ins} - {}_T\hat{p}_x) \tag{4.61}$$

The expression of the successive solvency capitals under the COC approach is given by Zeddouk and Devolder [31]:

$$SCR_i = VaR_{99,5\%}[BE_{i+1}^{\mathbb{P}} P(i,i+1) - BE_i^{\mathbb{P}}] \tag{4.62}$$

where:

$$\begin{aligned} BE_{i+1}^{\mathbb{P}} &= (I^{ins}(x,i+1)E_{\mathbb{P}}(I^{ins}(x+i+1,T-i-1) - {}_T\hat{p}_x))P(i+1,T) \\ BE_i^{\mathbb{P}} &= (I^{ins}(x,i)E_{\mathbb{P}}(I^{ins}(x+i,T-i) - {}_T\hat{p}_x))P(i,T). \end{aligned} \tag{4.63}$$

Therefore, the estimation of $SCR_i \mid_0$ can be given by:

$$\begin{aligned} SCR_i \mid_0 = {} & P(i,T)E_{\mathbb{P}}(I^{ins}(x,i))[VaR_{99,5\%}(I^{ins}(x+i,1)) - E_{\mathbb{P}}(I^{ins}(x+i,1))] \\ & \times E_{\mathbb{P}}(I^{ins}(x+i+1,T-i-1)), \end{aligned} \tag{4.64}$$

and the hedging cost in this strategy is equal to 0.

- **Partial hedging: S-forward (S2)**

The insurer choose to partially hedge the longevity risk (its systemic part), by entering an S-forward. In this case, he should pay the price of the S-forward, and hold the SCR to cover the eventual losses related to the basis risk. The best estimate at time 0 is given by formula (4.32), and the estimation of the SCR$_i$ at time 0 are given by formula (4.40).

- **Total hedging: S-forward + S-exchange (S3)**

In this strategy the insurer prefers not to hold the SCR, and to be totally protected against the longevity risk. The cost of this protection is equal to the sum of the S-forward and S-exchange contracts prices.

- **Reinsurance (S4)**

Instead of buying an S-forwrad and an S-exchange, the insurer can directly resort to a reinsurer who accepts to totally cover the longevity risk. In this case, the SCR is equal to 0. For example, if the reinsurer uses the COC approach to compute the price of the total protection, the price of this protection should be equal to:

$$\begin{aligned} V_{COC}(0,T) = {} & P(0,T)\,(E_{\mathbb{P}}[I^{ins}(x,T) - {}_T\hat{p}_x]) \\ & + 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I^{ins}(x,i))[VaR_{99,5\%}(I^{ins}(x+i,1)) - E_{\mathbb{P}}(I^{ins}(x+i,1))] \\ & \times E_{\mathbb{P}}(I^{ins}(x+i+1,T-i-1))P(0,i+1)P(i,T). \end{aligned} \tag{4.65}$$

For illustration, we will now compute the different hedging strategies using the third possibility to describe the insurer's mortality, under the COC method. Our aim is to compare the different strategies that reflect the risk aversion of the insurer. We use the same data considered in the numerical illustration part of section 4.6. We restrict ourselves to two initial ages, 65 and 70 years old, for a maturity of T=5 years. Table 4.28 reports a comparison between the hedging strategies under the COC approach in the third possibility:

<table>
<tr><th></th><th colspan="4">$x_0 = 65$, $T = 5$</th><th colspan="4">$x_0 = 70$, $T = 5$</th></tr>
<tr><th></th><th>Price</th><th>BE_0</th><th colspan="2">SCR_t</th><th>Price</th><th>BE_0</th><th colspan="2">SCR_t</th></tr>
<tr><td rowspan="5">S1</td><td rowspan="5">0</td><td rowspan="5">95.31299</td><td>t=0</td><td>17.03330</td><td rowspan="5">0</td><td rowspan="5">108.13463</td><td>t=0</td><td>68.72862</td></tr>
<tr><td>t=1</td><td>17.35925</td><td>t=1</td><td>69.15855</td></tr>
<tr><td>t=2</td><td>17.43082</td><td>t=2</td><td>69.21941</td></tr>
<tr><td>t=3</td><td>17.27303</td><td>t=3</td><td>68.94691</td></tr>
<tr><td>t=4</td><td>16.90103</td><td>t=4</td><td>68.35740</td></tr>
<tr><td rowspan="5">S2</td><td rowspan="5">52.63125</td><td rowspan="5">49.50369</td><td>t=0</td><td>4.98812</td><td rowspan="5">49.96718</td><td rowspan="5">80.16426</td><td>t=0</td><td>12.48208</td></tr>
<tr><td>t=1</td><td>7.82119</td><td>t=1</td><td>23.93022</td></tr>
<tr><td>t=2</td><td>8.53356</td><td>t=2</td><td>29.12340</td></tr>
<tr><td>t=3</td><td>10.54643</td><td>t=3</td><td>39.51281</td></tr>
<tr><td>t=4</td><td>13.38823</td><td>t=4</td><td>48.05535</td></tr>
<tr><td>S3</td><td>104.59125</td><td>0</td><td colspan="2">0</td><td>138.34670</td><td>0</td><td colspan="2">0</td></tr>
<tr><td>S4</td><td>100.38690</td><td>0</td><td colspan="2">0</td><td>128.80417</td><td>0</td><td colspan="2">0</td></tr>
</table>

Table 4.28: Comparing different longevity hedging strategies using the Cost of capital approach, 3^{rd} possibility.

- In case S1, the insurer does not buy any protection, but he must hold the SCRs corresponding to the longevity risk (the systemic and the basis risks). This strategy would only cost him the cost of holding these SCRs (6% of the SCRs), but he might totally or partially lose the SCRs if things go wrong. In this strategy only, the insurer is not exposed to the credit risk.

- In case S2, the insurer enters an S-forward contract, which is an exchange traded derivative based on the national population index, covering him just partially (systemic risk). Therefore this insurer should hold smaller SCRs corresponding to the remaining basis risk, which he also risks losing totally of partially. In this strategy, the insurer is exposed to the credit risk, but this risk is limited since it is spread between a large number of investors.

- In case S3, the insurer buys two protections: he first buys an S-forward from the financial markets, then enters an S-exchange derivative which is a customized contract (OTC). Hence, he will be fully covered against the longevity risk and will not need to set aside an SCR. The credit risk is more important in this case since it is related to the financial market(S-forward) and the S-exchange's seller.

- In case S4, the insurer delegates the risk to a reinsurer, and as in case S3 there is no need to hold SCRs. The cost of this protection is also assumed to be computed using the Cost of Capital approach. This strategy represents the highest credit risk since it is related to one entity.

Besides S1, whatever the strategy the insurer chooses, the risk will not be completely mitigated as there is a credit risk that should be taken into account in the risk transfer decision. The credit risk represents a state where the counterparty (in our case the financial market or the reinsurer) is unable to fulfill its financial obligations in a timely manner, or even to pay them at all. Neither the financial market nor the reinsurer is completely free from default risk, and therefore, the insurer has to hold a capital for this extra credit risk. However, the level of this risk is more important for the reinsurer than for the financial market. The assessment of this risk is beyond the scope of this paper, we refer the interested reader to Biffis et al. [4] for more details.

The choice of the best hedging strategy depends on the insurer's risk aversion. Moreover, it is difficult to know at time 0 with certainty the best strategy in terms of the cost: if the insurer chooses S1 or S2, the cost of the hedging strategy will be known only at maturity of the contract, since it depends on whether or not the SCRs are used to cover the risk. For example, S1 would be better than S3 if the SCRs are not needed, otherwise S1 would be more expensive than S3 and S4. We can also remark that from S1 to S3, the expectation of the cost ($BE_0^{\mathbb{P}}+$ Price) increases, and this can be explained by the fact that the more the insurer protects himself against the longevity risk, the higher the cost will be.

4.9 Conclusion

In this paper, we have developed in continuous-time a framework to assess the basis risk through the pricing of a longevity derivative. We have first proposed three variants of the Hull and White process, able to capture the mortality trends in the reference population and in the insurer's population, whose risk is to be hedged. We have associated the basis risk to a longevity derivative called S-exchange, that we have priced under the Cost of Capital method consistent with Solvency II. This approach has been presented as a new benchmark, and compared with other classical pricing methods. In addition, we have proposed different hedging strategies depending on the insurer's risk aversion. For further research, we can price non-linear longevity derivatives such as longevity option.

Appendix

We provide a sensitivity test for the prices computed under the traditional pricing approaches. For each method, we vary the longevity parameter (Market price of risk, Sharpe ratio and Wang parameter) and we see how the price of the S-exchange contract under these method changes. We provide the sensitivity test for the 3^{rd} possibility, and we have fixed the values of σ', ρ and g as follows:

- $\sigma' = 20\%\sigma$
- $\rho = 0.5$
- $g = 0.9$

4.A.1 Risk-neutral

Let us now consider different values for the market risk λ and see how the prices computed under the risk-neutral method and the third possibility change:

	λ			
	-10%	-20%	-30%	-50%
$x_0 = 65, T = 5$	50.0641	**50.6252**	51.1869	52.3120
$x_0 = 65, T = 10$	108.0183	**108.6915**	109.3656	110.7164
$x_0 = 70, T = 5$	81.8591	**83.5594**	85.2652	88.6930
$x_0 = 70, T = 10$	175.4931	**177.4161**	179.3461	183.2274

Table 4.A.1: Price sensitivity test for the market price of risk λ under risk-neutral approach, 3^{rd} possibility.

4.A.2 Sharpe Ratio

Let us now consider different values for the Sharpe ratio S and see how the prices computed under the Sharpe method and the third possibility change:

	S				
	5%	10%	20%	30%	50%
$x_0 = 65, T = 5$	51.4643	**53.4258**	57.3859	61.31286	69.1344
$x_0 = 65, T = 10$	112.1124	**116.8809**	126.3752	135.9036	155.0207
$x_0 = 70, T = 5$	83.3895	**86.5179**	93.0575	99.3063	112.2471
$x_0 = 70, T = 10$	181.2285	**188.8562**	204.0879	219.3157	249.7922

Table 4.A.2: Price sensitivity test for the Sharpe ratio S, 3^{rd} possibility.

4.A.3 Wang approach

Let us now consider different values for the Wang parameter δ and see how the prices computed under the Sharpe method and the third possibility change:

	δ				
	5%	10%	20%	30%	50%
$x_0 = 65, T = 5$	49.84796	**50.6866**	51.3921	52.5178	55.3607
$x_0 = 65, T = 10$	107.7274	**108.5659**	110.9378	112.1098	115.6294
$x_0 = 70, T = 5$	82.48541	**85.3290**	86.56772	88.1180	94.6438
$x_0 = 70, T = 10$	174.9321	**177.4802**	184.0956	188.0815	200.0472

Table 4.A.3: Price sensitivity test for the Wang parameter δ under the Wang approach, 3^{rd} possibility.

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Chapter 5

Longevity modelling and pricing under a dependent multi-cohort framework[1]

Abstract

We propose a multi-cohort model that is able to capture the mortality correlation between different cohorts. The model is based on the Hull and White process to which we incorporate inter-generational risk factors, by modifying its stochastic part. We provide a pricing framework for a new survival forward contract under the Cost of Capital, risk-neutral and Sharpe approaches, allowing to cover the global multi-cohort longevity risk. We give numerical illustrations for Belgian cohorts, and we compute the price of the longevity derivative under the proposed methods, for different correlation levels.

5.1 Introduction

Insurance companies and pension funds are constantly exposed to mortality risk, and they are, therefore, becoming increasingly interested in longevity-linked securities to transfer this risk. However, only a few longevity derivatives have been launched for various reasons. One important cause is that no consensus has yet been reached regarding the best model for the mortality risk. Many continuous-time stochastic mortality models for a single generation have been proposed by a number of researchers, including [21, 7, 2, 3, 24, 19].
However, the common trend in the evolution of the longevity of different populations is relevant, and it should be taken into account by entities seeking to hedge their exposures to mortality and/or longevity risk, as discussed in detail in [6]. In this way, these entities can assess the overall longevity risk, and reduce the basis risk between their own population and population associated with the hedging instruments.
Consequently, researchers have lately become more interested in developing multi-population models for the evolution of longevity rates (see, for instance, [23, 5, 9, 13]).

[1]Zeddouk, F.; Devolder, P. Longevity Modelling and Pricing under a Dependent Multi-Cohort Framework. Risks 2020, 8, 121.

These multi-population mortality models are based on the assumption that the mortality experiences of the populations are linked together, and do not diverge in long term. According to [5], this assumption could be justified by the long-term mortality evolution and, therefore, can be applicable to longevity risk modelling.
Some multi-population models that have been proposed in the literature are based on the generalization of the well-known Carter model [15]. These models capture the mortality dependence by including an additional common factor between the multiple populations (see, for instance, the models that were presented by [18, 11, 17, 8]). In addition, [14] have proposed a model for the mortality intensity using common factors that affect all the cohorts, as well as specific factors that only affect specific cohorts.
In this paper, we consider a typical life insurer holding a portfolio of individuals with different ages, and who needs to hedge the longevity risk. We attempt to capture the eventual correlations across generations while using a multidimensional continuous-time mortality environment based on the Hull and White model. We first consider a portfolio of two different cohorts, where the correlation is based on the introduction of two risk factors modelled by independent Brownian motions; we then generalize our framework to n cohorts.
We assess the longevity risk related to these n generations through the pricing of a new longevity derivative that we call Global Survival forward contract (GS-forward). Therefore, we need to use a pricing approach in order to compute the price of this derivative. The different pricing approaches proposed in the literature are mainly based on the traditional pricing methods used in finance, such as the Sharpe ratio, the risk-neutral, and the Wang approaches, which have been adapted in the longevity context. However, these methods require the assessment of the risk premium, which is not easy due to lack of data in the longevity market. In the literature, some authors use life annuities or announced longevity bonds for calibration, or consider values usually used in finance. For instance, [4, 20] have used the longevity bond announced by BNP/EIB in 2004 for calibration. Moreover, these classical pricing methods are not necessarily consistent with the directives of Solvency II, which is an important issue for insurance companies. The Cost of Capital method (COC) allows for avoiding these issues, since this method is consistent with Solvency II, and the Cost of Capital rate is fixed by the regulator. This approach has been used by [16, 26] to price longevity derivatives, such as S-forward contracts, allowing to hedge the longevity risk for individuals belonging to one given cohort. The consistency between this Cost of Capital and the aforementioned classical methods have been discussed in detail in [26].
Our aim is to determine, in closed form, the price of the GS-forward derivative under the Cost of Capital, risk-neutral (see [4]), and Sharpe approaches [22, 2] in the case of a multi-cohort portfolio with correlated mortality experiences, as well as to compare the price of a GS-forward with the corresponding individual S-forward contracts.
This paper is organized, as follows: in Section 5.2, we present the multi-dimensional model for two cohorts, we study the correlation between these two cohorts, and we provide the general pricing framework for the GS-forward under the COC, risk-neural, and Sharpe methods. Next, we generalize this framework for n cohorts in Section 5.3. In Section 5.4, we present a numerical illustration enabling the comparison of the different GS-forward prices and also between these GS-forward prices and individual S-forwards prices. Finally, in Section 5.5 we conclude.

5.2 Cohort-Based Longevity Model: Two Cohorts

In order to describe the force of mortality, we have used an affine model that allows for the valuation of longevity derivatives ([12, 25]). In particular, we have chosen the Hull and White process (HW), which is a cohort mortality model. [27] provided a comparison between different stochastic time-continuous models, and have shown that this model can accurately predict the mortality of the Belgian population. Moreover, the HW model was also used by [26] for mortality in order to price Survival-forwards and Survival-swaps. We first focus on the case of a portfolio of two different cohorts, then we generalize our study for n cohorts.

5.2.1 Correlation between Two Forces of Mortality

We consider two different cohorts Y and Z of individuals initially aged $x_0 = y$ and $x_0 = z$ at time $t = 0$.
The evolution of the mortality intensity of the two cohorts are assumed to be given by:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y dw_y(t) \tag{5.1}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z dw_z(t), \tag{5.2}$$

where A_y, B_y, b_y, σ_y and A_z, B_z, b_z, σ_z all positive constants, w_y and w_z are two dependent Brownian motions.
Using the Cholesky decomposition for correlated Brownian motions, Equations (5.1) and (5.2) can be alternatively written as:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y(\rho_y dw_1(t) + \sqrt{1-\rho_y^2}\, dw_2(t)) \tag{5.3}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z(\rho_z dw_1(t) + \sqrt{1-\rho_z^2}\, dw_2(t)), \tag{5.4}$$

with:

$$w_y = \rho_y w_1 + \sqrt{1-\rho_y^2}\, w_2, \tag{5.5}$$

and

$$w_z = \rho_z w_1 + \sqrt{1-\rho_z^2}\, w_2, \tag{5.6}$$

where w_1 and w_2 are two independent Brownian motions (independent risk factors), and ρ_y and ρ_z are two risk parameters that link the mortality intensity of the cohorts Y and Z to the two risk factors.
The forces of mortality that are given by Equations (5.3) and (5.4) are now based on two independent Brownian motions, generating risk factors that affect the mortality of the two cohorts, and allowing to introduce the inter-generational correlation with different levels.
The solutions of the SDEs (5.3) and (5.4) are given by:

$$\mu_y(t) = \mu_y(0)e^{-b_y t} + \frac{A_y}{b_y + B_y}(e^{B_y t} - e^{-b_y t}) + \sigma_y e^{-b_y t}\int_0^t e^{b_y u} dw_y(u) \tag{5.7}$$

$$\mu_z(t) = \mu_z(0)e^{-b_z t} + \frac{A_z}{b_z + B_z}(e^{B_z t} - e^{-b_z t}) + \sigma_z e^{-b_z t}\int_0^t e^{b_z u} dw_z(u), \tag{5.8}$$

w_y and w_z being given by (5.5) and (5.6).

In order to measure the correlation between the forces of mortality of the two cohorts at any time t, we compute the correlation $corr(\mu_y(t), \mu_z(t))$ that is induced by the model.
We first determine the covariance between the two forces of mortality:

$$\begin{aligned} Cov(\mu_y(t), \mu_z(t)) &= E_{\mathbb{P}}[(\mu_y(t) - E_{\mathbb{P}}(\mu_y(t))(\mu_z(t) - E_{\mathbb{P}}(\mu_z(t)))] \\ &= \sigma_y \sigma_z e^{-(b_y+b_z)t} E_{\mathbb{P}}\left[\int_0^t e^{b_y u} dw_y(u) \cdot \int_0^t e^{b_z u} dw_z(u)\right]. \end{aligned} \tag{5.9}$$

Using the Ito multidimensional formula (see for instance [10]), formula (5.9) becomes:

$$\begin{aligned} Cov(\mu_y(t), \mu_z(t)) &= \sigma_y \sigma_z e^{-(b_y+b_z)t} \int_0^t e^{(b_y+b_z)u} \cdot \rho_{w_y,w_z} du \\ &= \frac{\sigma_y \sigma_z}{(b_y + b_z)}(1 - e^{-(b_y+b_z)t})\rho_{w_y,w_z}, \end{aligned} \tag{5.10}$$

where ρ_{w_y,w_z} is the correlation factor between w_y and w_z given by:

$$\rho_{w_y,w_z} = \rho_y \rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}. \tag{5.11}$$

Finally, the correlation $Corr(\mu_y(t), \mu_y(t))$ is:

$$\begin{aligned} Corr(\mu_y(t), \mu_z(t)) &= \frac{Cov(\mu_y(t), \mu_z(t))}{\sqrt{\mathrm{Var}_{\mathbb{P}}(\mu_y(t))}\sqrt{\mathrm{Var}_{\mathbb{P}}(\mu_z(t))}} \\ &= \rho_{w_y,w_z} \frac{2\sqrt{b_y b_z}}{b_y + b_z} \frac{1 - e^{-(b_y+b_z)t}}{\sqrt{1 - e^{-2b_y t}}\sqrt{1 - e^{-2b_z t}}} \\ &= \left(\rho_y \rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}\right)\left(\frac{2\sqrt{b_y b_z}}{b_y + b_z} \frac{1 - e^{-(b_y+b_z)t}}{\sqrt{1 - e^{-2b_y t}}\sqrt{1 - e^{-2b_z t}}}\right) \quad (5.12) \\ &= \left(\rho_y \rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}\right)\varphi_{y,z}(t). \quad (5.13) \end{aligned}$$

where $\varphi_{y,z}(t)$ is given by:

$$\varphi_{y,z}(t) = \left(\frac{2\sqrt{b_y b_z}}{b_y + b_z} \frac{1 - e^{-(b_y+b_z)t}}{\sqrt{1 - e^{-2b_y t}}\sqrt{1 - e^{-2b_z t}}}\right). \tag{5.14}$$

Remark 1 *From (5.12), we can see that, if $b_y = b_z$, which means that the reversion force of mortality is the same for the two noises, then $\varphi_{y,z}(t) = 1$, and $Corr(\mu_y(t), \mu_z(t))$ will be time-independent. In this case, we have also three possibilities:*

- *The two noises are independent, namely $\rho_y = 1$ and $\rho_z = 0$ or $\rho_y = 0$ and $\rho_z = 1$, then $Corr(\mu_y(t), \mu_z(t)) = 0$;*
- *The two noises are perfectly correlated, namely $\rho_y = \rho_z = 1$, or $\rho_y = \rho_z = 0$, then $Corr(\mu_y(t), \mu_z(t)) = 1$;*
- *The two noises are partially dependent, namely $1 < \rho_y \rho_z < 0$, then $Corr(\mu_y(t), \mu_z(t)) = \rho_y \rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}$;*

If $b_y \neq b_z$, then we have three possibilities:

- *The two noises are independent, namely* $\rho_y = 1$ *and* $\rho_z = 0$ *or* $\rho_y = 0$ *and* $\rho_z = 1$, *then* $Corr(\mu_y(t), \mu_z(t)) = 0$;
- *The two noises are perfectly correlated, namely* $\rho_y = \rho_z = 1$, *or* $\rho_y = \rho_z = 0$, *then* $Corr(\mu_y(t), \mu_z(t)) = \varphi_{y,z}(t)$ *(now not necessarily equal to 1);*
- *The two noises are partially dependent, then* $Corr(\mu_y(t), \mu_z(t))$ *becomes a function of time* t *(now not necessarily a constant).*

5.2.2 Correlation between Two Longevity Indexes

Now, let us study the correlation between two longevity indexes. The two longevity indexes related to the two cohorts Y and Z at time t are given by:

$$I_y(y+t, T-t) = e^{-\int_t^T \mu_y(u,\omega)du} \tag{5.15}$$

$$I_z(z+t, T-t) = e^{-\int_t^T \mu_z(u,\omega)du}, \tag{5.16}$$

where $I_y(y+t, T-t)$ is the longevity index of an individual of the cohort Y initially aged y, alive at time t, and surviving $T-t$ years more.
Let us compute the covariance of the two longevity indexes. This covariance is given by:

$$\begin{aligned}
Cov(I_y(y+t,T-t), I_z(z+t,T-t)) &= E_{\mathbb{P}}[(I_y(y+t,T-t) - E_{\mathbb{P}}(I_y(y+t,T-t))) \\
&\quad \times (I_z(z+t,T-t) - E_{\mathbb{P}}(I_z(z+t,T-t)))] \\
&= E_{\mathbb{P}}(I_y(y+t,T-t) I_z(z+t,T-t)) - E_{\mathbb{P}}(I_y(y+t,T-t)) \\
&\quad \times E_{\mathbb{P}}(I_z(z+t,T-t)) \\
&= E_{\mathbb{P}}(e^{-\int_t^T (\mu_y(u,\omega)+\mu_z(u,\omega))du}) - e^{\alpha_y(t,T)-\beta_y(t,T)\mu_y(t)} \\
&\quad \times e^{\alpha_z(t,T)-\beta_z(t,T)\mu_z(t)},
\end{aligned} \tag{5.17}$$

where α_y, β_y and α_z, α_z are given by:

$$\begin{cases}
\alpha_y(t,T) &= \frac{A_y}{b_y}[e^{-b_y T} \frac{e^{(B_y+b_y)T} - e^{(B_y+b_y)t}}{B_y+b_y} - \frac{e^{B_y T} - e^{B_y t}}{B_y}] - \frac{\sigma_y^2}{2b_y^2}[\frac{1}{b_y}(1-e^{-b_y(T-t)}) - T + t] \\
&\quad - \frac{\sigma_y^2}{4b_y^3}(1-e^{-b_y(T-t)})^2 \\
\beta_y(t,T) &= \frac{1}{b_y}(1-\exp(-b_y(T-t))
\end{cases} \tag{5.18}$$

and

$$\begin{cases}
\alpha_z(t,T) &= \frac{A_z}{b_z}[e^{-b_z T} \frac{e^{(B_z+b_z)T} - e^{(B_z+b_z)t}}{B_z+b_z} - \frac{e^{B_z T} - e^{B_z t}}{B_z}] - \frac{\sigma_z^2}{2b_z^2}[\frac{1}{b_z}(1-e^{-b_z(T-t)}) - T + t] \\
&\quad - \frac{\sigma_z^2}{4b_z^3}(1-e^{-b_z(T-t)})^2 \\
\beta_z(t,T) &= \frac{1}{b_z}(1-\exp(-b_z(T-t)).
\end{cases} \tag{5.19}$$

Let us now compute $E_{\mathbb{P}}(e^{-\int_t^T(\mu_y(u,\omega)+\mu_z(u,\omega))du})$.
For any given cohort of individuals aged x at time t, we have:

$$I_x(x+t,T-t)=e^{-\int_t^T\mu_x^{\mathbb{P}}(u,\omega)du}=e^{X_x(t,T)}. \tag{5.20}$$

Under the real-world risk measure $\mathbb{P}$, we have:

$$X_x(t,T)\sim N(m_x(t,T),\,n_x(t,T)^2). \tag{5.21}$$

Accordingly, the survival index I_x is log-normally distributed with X_x having a mean and a variance given by:

$$\begin{cases} m_x(t,T)= & \mu_x(t)\frac{(e^{-b(T-t)}-1)}{b}-\frac{Ae^{Bt}}{B(b+B)}(e^{B(T-t)}-1)-\frac{Ae^{Bt}}{b(b+B)}(e^{-b(T-t)}-1)\\ n_x^2(t,T)= & \frac{\sigma^2}{b^2}[T-t-\frac{1-e^{-b(T-t)}}{b}-\frac{(1-e^{-b(T-t)})^2}{2b}]. \end{cases} \tag{5.22}$$

We have:

$$X_y(t,T)\sim N(m_y(t,T),\,n_y(t,T)^2) \tag{5.23}$$
$$X_z(t,T)\sim N(m_z(t,T),\,n_z(t,T)^2),$$

where:

$$\begin{cases} m_y(t,T)= & \mu_y(t)\frac{(e^{-b_y(T-t)}-1)}{b_y}-\frac{A_ye^{B_yt}}{B_y(b_y+B_y)}(e^{B_y(T-t)}-1)-\frac{A_ye^{B_yt}}{b_y(b_y+B_y)}(e^{-b_y(T-t)}-1)\\ n_y^2(t,T)= & \frac{\sigma_y^2}{b_y^2}[T-t-\frac{1-e^{-b_y(T-t)}}{b_y}-\frac{(1-e^{-b_y(T-t)})^2}{2b_y}], \end{cases} \tag{5.24}$$

and

$$\begin{cases} m_z(t,T)= & \mu_z(t)\frac{(e^{-b_z(T-t)}-1)}{b_z}-\frac{A_ze^{B_zt}}{B_z(b_z+B_z)}(e^{B_z(T-t)}-1)-\frac{A_ze^{B_zt}}{b_z(b_z+B_z)}(e^{-b_z(T-t)}-1)\\ n_z^2(t,T)= & \frac{\sigma_z^2}{b_z^2}[T-t-\frac{1-e^{-b_z(T-t)}}{b_z}-\frac{(1-e^{-b_z(T-t)})^2}{2b_z}], \end{cases} \tag{5.25}$$

then:

$$X_y(t,T)+X_z(t,T)\sim N(m_{yz}(t,T),\,n_{yz}^2(t,T)), \tag{5.26}$$

where

$$\begin{cases} m_{yz}(t,T)= & m_y(t,T)+m_z(t,T)\\ n_{yz}^2(t,T)= & n_y^2(t,T)+n_z^2(t,T)+2Cov(X_y(t,T),X_z(t,T)). \end{cases} \tag{5.27}$$

The covariance $Cov(X_y(t,T),X_z(t,T))$ is:

$$Cov(X_y(t,T),X_z(t,T))=E_{\mathbb{P}}[X_y(t,T)-E_{\mathbb{P}}(X_z(t,T))\cdot(X_y(t,T)-E_{\mathbb{P}}(X_z(t,T)))]. \tag{5.28}$$

Using (5.7) and (5.8), we have:

$$\begin{aligned} Cov(X_y(t,T),X_z(t,T)) &= E_{\mathbb{P}}\Big[\Big(\int_t^T\sigma_ye^{-b_yt}\int_t^s e^{b_yv}dw_y(v)\Big)ds\cdot\Big(\int_t^T\sigma_ze^{-b_zt}\int_t^s e^{b_zv}dw_z(v)\Big)ds\Big]\\ &= E_{\mathbb{P}}\Big[\Big(\sigma_y\int_t^T\frac{1-e^{-b_y(T-v)}}{b_y}dw_y(v)\Big)\cdot\Big(\sigma_z\int_t^T\frac{1-e^{-b_z(T-v)}}{b_z}dw_z(v)\Big)\Big]\\ &= \sigma_y\sigma_zE_{\mathbb{P}}\Big[\Big(\int_t^T\frac{1-e^{-b_y(T-v)}}{b_y}dw_y(v)\Big)\cdot\Big(\int_t^T\frac{1-e^{-b_z(T-v)}}{b_z}dw_z(v)\Big)\Big]. \end{aligned}$$

Using the Ito multidimensional formula, we have:

$$\begin{aligned}
Cov(X_y(t,T), X_z(t,T)) &= \sigma_y\sigma_z\rho_{w_y,w_z}\left(\int_t^T \frac{1-e^{-b_y(T-v)}}{b_y}\cdot\frac{1-e^{-b_z(T-v)}}{b_z}dv\right)\\
&= \sigma_y\sigma_z\rho_{w_y,w_z}\frac{1}{b_yb_z}\left(\int_t^T (1-e^{-b_y(T-v)})\cdot(1-e^{-b_z(T-v)})dv\right)\\
&= \sigma_y\sigma_z\rho_{w_y,w_z}\frac{1}{b_yb_z}\left[(T-t)-\frac{1-e^{-b_z(T-t)}}{b_z}-\frac{1-e^{-b_y(T-t)}}{b_y}+\frac{1-e^{-(b_y+b_z)(T-t)}}{b_y+b_z}\right]\\
&= \sigma_y\sigma_z\rho_{w_y,w_z}\Psi_{y,z}(t,T). \qquad (5.29)
\end{aligned}$$

where $\Psi_{y,z}(t,T) = \frac{1}{b_yb_z}\left[(T-t)-\frac{1-e^{-b_z(T-t)}}{b_z}-\frac{1-e^{-b_y(T-t)}}{b_y}+\frac{1-e^{-(b_y+b_z)(T-t)}}{b_y+b_z}\right]$
Using (5.29), Equation (5.27) becomes:

$$\begin{cases} m_{yz}(t,T) = & m_y(t,T)+m_z(t,T)\\ n_{yz}^2(t,T) = & n_y^2(t,T)+n_z^2(t,T)+2\sigma_y\sigma_z\rho_{w_y,w_z}\Psi_{y,z}(t,T).\end{cases} \qquad (5.30)$$

Let:

$$\begin{aligned}
K(t,T) &= e^{-\int_t^T(\mu_y(u,\omega)+\mu_z(u,\omega))du}\\
&= e^{X_y(t,T)+X_z(t,T)}. \qquad (5.31)
\end{aligned}$$

$E_{\mathbb{P}}(K(t,T))$ is then given by:

$$E_{\mathbb{P}}(K(t,T)) = e^{m_{yz}(t,T)+\frac{n_{yz}^2(t,T)}{2})}. \qquad (5.32)$$

By replacing (5.32) in (5.17), we get:

$$Cov(I_y(y+t,T-t), I_z(z+t,T-t)) = e^{m_{yz}(t,T)+\frac{n_{yz}^2(t,T)}{2}} - e^{\alpha_y(t,T)-\beta_y(t,T)\mu_y(t)}\cdot e^{\alpha_z(t,T)-\beta_z(t,T)\mu_z(t)}. \quad (5.33)$$

The correlation $Corr(I_y(y+t,T-t), I_z(z+t,T-t))$ is given by:

$$Corr(I_y(y+t,T-t), I_z(z+t,T-t)) = \frac{Cov(I_y(y+t,T-t), I_z(z+t,T-t))}{\bar{n}_y(t,T)\cdot\bar{n}_z(t,T)}, \qquad (5.34)$$

where $\bar{n}_y(t,T)$ and $\bar{n}_z(t,T)$ are the standard deviations that are related to the two longevity indexes I_y and I_z.
Using (5.24) and (5.25), we can easily compute $\bar{n}_y(t,T)$ and $\bar{n}_z(t,T)$:

$$\begin{aligned}
\bar{n}_y^2(t,T) &= e^{2m_y(t,T)+n_y^2(t,T)}\cdot(e^{n_y^2(t,T)}-1)\\
\bar{n}_z^2(t,T) &= e^{2m_z(t,T)+n_z^2(t,T)}\cdot(e^{n_z^2(t,T)}-1).
\end{aligned}$$

We finally get:

$$Corr(I_y(y+t,T-t), I_z(z+t,T-t)) = \frac{e^{m_{yz}(t,T)+\frac{n_{yz}^2(t,T)}{2}} - e^{\alpha_y(t,T)-\beta_y(t,T)\mu_y(t)}\cdot e^{\alpha_z(t,T)-\beta_z(t,T)\mu_z(t)}}{\sqrt{e^{2m_y(t,T)+n_y^2(t,T)}(e^{n_y^2(t,T)}-1)}\sqrt{e^{2m_z(t,T)+n_z^2(t,T)}(e^{n_z^2(t,T)}-1)}}. \qquad (5.35)$$

Remark 2 *We remark that, if the two noises are independent, namely $\rho_y = 1$ and $\rho_z = 0$ or $\rho_y = 0$ and $\rho_z = 1$, then:*

- $Cov(X_y(t,T), X_z(t,T)) = 0$;
- *By elementary calculation, we find:*

$$m_{yz}(t,T) + \frac{n_{yz}^2(t,T)}{2} = \alpha_y(t,T) - \beta_y(t,T)\mu_y(t) + \alpha_z(t,T) - \beta_z(t,T)\mu_z(t),$$

which means that:

$$Corr(I_y(y+t, T-t), I_z(z+t, T-t)) = 0.$$

5.2.3 S-Forward Pricing

An S-forward (called individual S-forward in this paper) is a financial product exchanging at a fixed maturity T, the realized survival rate of a given population, in return for a fixed rate that was agreed at inception.
For a notional amount equal to one monetary unit, the payoff of this product at maturity becomes:

$$Payoff(T) = I(x,T) - {}_T\hat{p}_x, \tag{5.36}$$

where $I(x,T)$ is the realized survival rate at maturity, and ${}_T\hat{p}_x$ is a fixed survival rate that represents the probability to be alive at age $x+T$ for an individual initially aged x.
This derivative allows to hedge the longevity risk that is related to an individual belonging to a given cohort. The pricing of this derivative under the COC approach as well as under the classical methods (risk-neutral, Sharpe, and Wang) was discussed in detail in [26].

5.2.4 GS-Forward Pricing

We consider the case of an insurer holding a portfolio of policyholders belonging to the cohort Y or Z. We assume that this insurer should pay one monetary unit to each individual alive at maturity T.
The payoff of this global S-forward (that we call GS-forward) at time t is given by:

$$Payoff(T) = I_y(y,T) + I_z(z,T) - ({}_T\hat{p}_y + {}_T\hat{p}_z).$$

where ${}_T\hat{p}_y$ and ${}_T\hat{p}_z$ are two fixed survival rates related to cohorts Y and Z, of an individual aged $y+T$ and $z+T$, respectively, at time 0, to be alive at age y and z, respectively, (F_0 measurable).

GS-Forward Pricing under the COC Approach

The price at time 0 under the COC approach is given by:

$$V_{COC}(0,T) = BE_0^{\mathbb{P}} + RM_0, \tag{5.37}$$

where $BE_0^{\mathbb{P}}$ is as follows:

$$BE_0^{\mathbb{P}} = P(0,T)E_{\mathbb{P}}[I_y(y,T) + I_z(z,T) - ({}_T\hat{p}_y + {}_T\hat{p}_z)].$$

The risk margin is equal to the present value of the required returns on the future Solvency Capital Requirements (SCRs):

$$RM_0 = C\% \sum_{i=0}^{T-1} SCR_i\, P(0,i+1),$$

where SCR_i is the Solvency Capital Requirement that corresponds to the Value-at-Risk at a confidence level of 99.5% on a one-year period, and C is the Cost of Capital nowadays equal to 6% (Solvency II requirements).
The future SCRs being stochastic, we need to use an estimation of their future values at time 0. This estimation is denoted by: $\hat{SCR}_i{=}SCR_i \mid_0$.
The initial risk margin is then given by:

$$RM_0 = C\% \sum_{i=0}^{T-1} \hat{SCR}_i\, P(0,i+1).$$

Taking into account the Solvency II definition of the SCRs (Value-at-Risk at 99.5% on a one year), the expression of the SCR_i becomes:

$$SCR_i = \mathrm{VaR}_{99,5\%}[BE_{i+1}^{\mathbb{P}} P(i,i+1) - BE_i^{\mathbb{P}}],$$

where:

$$\begin{aligned} BE_{i+1}^{\mathbb{P}} &= (I_y(y,i+1)E_{\mathbb{P}}(\, I_y(y+i+1,T-i-1) + I_z(z,i+1)E_{\mathbb{P}}(\, I_z(z+i+1,T-i-1)) \\ &\quad - ({}_T\hat{p}_y + {}_T\hat{p}_z))P(i+1,T) \end{aligned}$$

$$BE_i^{\mathbb{P}} = (I_y(y,i)E_{\mathbb{P}}(\, I_y(y+i,T-i) + I_z(z,i)E_{\mathbb{P}}(\, I_z(z+i,T-i) - ({}_T\hat{p}_y + {}_T\hat{p}_z))P(i,T).$$

The SCR_i is then given by:

$$\begin{aligned} SCR_i &= P(i,T)\,\mathrm{VaR}_{99,5\%}[I_y(y,i+1)E_{\mathbb{P}}(\, I_y(y+i+1,T-i-1)) + I_z(z,i+1)E_{\mathbb{P}}(\, I_z(z+i+1,T-i-1)) \\ &\quad - I_y(y,i)E_{\mathbb{P}}(\, I_y(y+i,T-i)) - I_z(z,i)E_{\mathbb{P}}(\, I_z(z+i,T-i))] \\ \\ &= P(i,T)[\mathrm{VaR}_{99,5\%}[I_y(y,i)I_y(y+i,1)E_{\mathbb{P}}(\, I_y(y+i+1,T-i-1)) + I_z(z,i)I(z+i,1) \\ &\quad \times E_{\mathbb{P}}(\, I_z(z+i+1,T-i-1)) - I_y(y,i)E_{\mathbb{P}}(\, I_y(y+i,T-i)) - I_z(z,i)E_{\mathbb{P}}(\, I_z(z+i,T-i))] \\ \\ &= P(i,T)[\mathrm{VaR}_{99,5\%}[I_y(y,i)I_y(y+i,1)E_{\mathbb{P}}(\, I_y(y+i+1,T-i-1)) - I_y(y,i)E_{\mathbb{P}}(\, I_y(y+i,T-i)) \\ &\quad + I_z(z,i)I_z(z+i,1)E_{\mathbb{P}}(\, I_z(z+i+1,T-i-1)) - I_z(z,i)E_{\mathbb{P}}(\, I_z(z+i,T-i))] \\ \\ &= P(i,T)[\mathrm{VaR}_{99,5\%}[I_y(y,i)(I_y(y+i,1)E_{\mathbb{P}}(\, I_y(y+i+1,T-i-1)) - E_{\mathbb{P}}(\, I_y(y+i,T-i)) \\ &\quad + I_z(z,i)(I_z(z+i,1)E_{\mathbb{P}}(\, I_z(z+i+1,T-i-1)) - E_{\mathbb{P}}(\, I_z(z+i,T-i))]. \end{aligned}$$

Therefore, the estimation of $SCR_i \mid_0$ can be given by:

$$
\begin{aligned}
SCR_i\,|_0 &= P(i,T)[\mathrm{VaR}_{99,5\%}[E_{\mathbb{P}}(I_y(y,i))\big((I_y(y+i,1)E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_y(y+i,T-i)))\\
&+ E_{\mathbb{P}}(I_z(z,i))((I_z(z+i,1)E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_z(z+i,T-i)))]\\
\\
&= P(i,T)[\mathrm{VaR}_{99,5\%}[E_{\mathbb{P}}(I_y(y,i))\big((I_y(y+i,1)\\
&\times E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_y(y+i,1))E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)))\\
&+ E_{\mathbb{P}}(I_z(z,i))((I_z(z+i,1)E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1))\\
&- E_{\mathbb{P}}(\,I_z(z+i,1))E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)))]\\
\\
&= P(i,T)[\mathrm{VaR}_{99,5\%}(\nu\cdot(I_y(y+i,1) - E_{\mathbb{P}}(\,I_y(y+i,1)))\cdot\theta + \xi\cdot(I_z(z+i,1) - E_{\mathbb{P}}(\,I_z(z+i,1)))\cdot\eta)],
\end{aligned}
$$

where ν, θ, ξ, and η are constants given by:

$$
\begin{aligned}
\nu &= E_{\mathbb{P}}(I_y(y,i))\\
\theta &= E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1))\\
\xi &= E_{\mathbb{P}}(I_z(z,i))\\
\eta &= E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)).
\end{aligned}
$$

The risk margin at time 0 is then equal to:

$$
RM_0 = 6\%\sum_{i=0}^{T-1} P(0,i{+}1)P(i,T)[\mathrm{VaR}_{99,5\%}(\nu\cdot(I_y(y{+}i,1) - E_{\mathbb{P}}(\,I_y(y{+}i,1)))\cdot\theta + \xi\cdot(I_z(z{+}i,1) - E_{\mathbb{P}}(\,I_z(z{+}i,1)))\cdot\eta)].
$$

The price of the GS-forward under the COC approach is finally given by:

$$
\begin{aligned}
V^{\,y,z}_{\,COC}(0,T) &= P(0,T)(E_{\mathbb{P}}(I_y(y,T)) + E_{\mathbb{P}}(I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z))\\
&+ 6\%\sum_{i=0}^{T-1} P(0,i+1)P(i,T)[\mathrm{VaR}_{99,5\%}(\nu\cdot(I_y(y+i,1) - E_{\mathbb{P}}(\,I_y(y+i,1)))\cdot\theta + \xi\cdot(I_z(z+i,1)\\
&- E_{\mathbb{P}}(\,I_z(z+i,1)))\cdot\eta)].
\end{aligned} \tag{5.38}
$$

GS-Forward Pricing under the Risk-Neutral Approach

The price of a GS-forward at time t under the risk-neutral probability measure $\mathbb{Q}$, is given by:

$$
V^{y,z}_{\mathbb{Q}_{\lambda\lambda'}}(0,T) = P(0,T)[E_{\mathbb{Q}_{\lambda\lambda'}}(I_y(y,T) + I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z)\mid \mathcal{F}_0], \tag{5.39}
$$

where λ and λ' are the two market prices of longevity risk that are linked to the two independent risk factors, modelled by the two Brownian motions w_1 and w_2 (formulas (5.5) and (5.6)).
We assume that the market price of risk is constant ($\lambda(t) = \lambda$ and $\lambda'(t) = \lambda'$).
The two SDEs of the forces of mortality under the real-word measure $\mathbb{P}$ are given by formulas (5.3) and (5.4). Under the risk-neutral measure, these formulas become:

$$
d\mu_y^{\mathbb{Q}_{\lambda\lambda'}}(t) = (A_y e^{B_y t} - b_y\mu_y(t) + \tau\sigma_y)dt + \sigma_y(\rho_y dw_1^*(t) + \sqrt{1-\rho_y^2}\,dw_2^*(t)) \tag{5.40}
$$

$$d\mu_z^{\mathbb{Q}_{\lambda\lambda'}}(t) = (A_z e^{B_z t} - b_z\mu_z(t) + \tau'\sigma_z)dt + \sigma_z(\rho_z dw_1^*(t) + \sqrt{1-\rho_z^2}dw_2^*(t)), \tag{5.41}$$

where:

- $w_1^*(t)= w_1(t)-\lambda t$, and $w_2^*(t)= w_2(t)-\lambda' t$;
- $\tau = \lambda\rho_y + \lambda'\sqrt{1-\rho_y^2}$ and $\tau' = \lambda\rho_z + \lambda'\sqrt{1-\rho_z^2}$.

We have:

$$\begin{aligned} E_{\mathbb{Q}_{\lambda\lambda'}}(I(y,T)) &= e^{\alpha_y^{\mathbb{Q}_{\lambda\lambda'}}(0,T)-\beta_y^{\mathbb{Q}_{\lambda\lambda'}}(0,T)\mu_y^{\mathbb{Q}_{\lambda\lambda'}}(0)} \\ E_{\mathbb{Q}_{\lambda\lambda'}}(I(z,T)) &= e^{\alpha_z^{\mathbb{Q}_{\lambda\lambda'}}(0,T)-\beta_z^{\mathbb{Q}_{\lambda\lambda'}}(0,T)\mu_z^{\mathbb{Q}_{\lambda\lambda'}}(0)}, \end{aligned}$$

where $\alpha_y^{\mathbb{Q}_{\lambda\lambda'}}, \beta_y^{\mathbb{Q}_{\lambda\lambda'}}$ and $\alpha_z^{\mathbb{Q}_{\lambda\lambda'}}, \beta_z^{\mathbb{Q}_{\lambda\lambda'}}$ are given by:

$$\begin{cases} \alpha_y^{\mathbb{Q}_{\lambda\lambda'}}(t,T) &= \frac{A_y}{b_y}[e^{-b_y T}\frac{e^{(B_y+b_y)T}-e^{(B_y+b_y)t}}{B_y+b_y} - \frac{e^{B_y T}-e^{B_y t}}{B_y}] - \frac{\sigma_y^2}{2b_y^2}[\frac{1}{b_y}(1-e^{-b_y(T-t)}) - T + t] \\ & -\frac{\sigma_y^2}{4b_y^3}(1-e^{-b_y(T-t)})^2 - \frac{\sigma_y\tau}{b_y}(1-e^{-b_y(T-t)}) \\ \\ \beta_y^{\mathbb{Q}_{\lambda\lambda'}}(t,T) &= \frac{1}{b_y}(1-e^{-b_y(T-t)}) \end{cases}$$

and

$$\begin{cases} \alpha_z^{\mathbb{Q}_{\lambda\lambda'}}(t,T) &= \frac{A_z}{b_z}[e^{-b_z T}\frac{e^{(B_z+b_z)T}-e^{(B_z+b_z)t}}{B_z+b_z} - \frac{e^{B_z T}-e^{B_z t}}{B_z}] - \frac{\sigma_z^2}{2b_z^2}[\frac{1}{b_z}(1-e^{-b_z(T-t)}) - T + t] \\ & -\frac{\sigma_z^2}{4b_z^3}(1-e^{-b_z(T-t)})^2 - \frac{\sigma_z\tau'}{b_z}(1-e^{-b_z(T-t)}) \\ \\ \beta_z^{\mathbb{Q}_{\lambda\lambda'}}(t,T) &= \frac{1}{b_z}(1-e^{-b_z(T-t)}). \end{cases}$$

GS-Forward Pricing under the Sharpe Approach

Let us now compute the GS-forward price under the Sharpe approach. The price at time 0 is:

$$V_{Sharpe}^{y,z}(0,T) = P(0,T)\Big[E_{\mathbb{P}}(I_y(y,T)) + E_{\mathbb{P}}(I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z) + S\sqrt{\mathrm{Var}_{\mathbb{P}}(I_y(y,T)+I_z(z,T))}\Big], \tag{5.42}$$

where S represents the Sharpe ratio, and $\mathrm{Var}_{\mathbb{P}}(I_y(y,T))$ is the variance of the survival index. Because $I_y(y,T)$ and $I_z(z,T)$ are dependent, we have:

$$\mathrm{Var}_{\mathbb{P}}(I_y(y,T)+I_z(z,T)) = \mathrm{Var}_{\mathbb{P}}(I_y(y,T))+\mathrm{Var}_{\mathbb{P}}(I_z(z,T))+2Cov(I_y(y,T),I_z(z,T)). \tag{5.43}$$

The price under the Sharpe approach is finally given by:

$$\begin{aligned} V_{Sharpe}^{y,z}(0,T) = P(0,T)\Big[&E_{\mathbb{P}}(I_y(y,T)) + E_{\mathbb{P}}(I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z) \\ &+ S\sqrt{\mathrm{Var}_{\mathbb{P}}(I_y(y,T)) + \mathrm{Var}_{\mathbb{P}}(I_z(z,T)) + 2Cov(I_y(y,T),I_z(z,T))}\Big]. \end{aligned} \tag{5.44}$$

From (5.23), $\text{Var}_{\mathbb{P}}(I_y(y,T))$ and $\text{Var}_{\mathbb{P}}(I_z(z,T))$ are given by:

$$\text{Var}_{\mathbb{P}}(I_y(y,T)) = e^{(2m_y(0,T)+n_y^2(0,T))} \cdot (e^{n_y^2(0,T)} - 1)$$
$$\text{Var}_{\mathbb{P}}(I_z(z,T)) = e^{(2m_z(0,T)+n_z^2(0,T))} \cdot (e^{n_z^2(0,T)} - 1).$$

We have the expression of the covariance $Cov(I_y(y,T), I_z(z,T))$ given by formula (5.33). The expression of the variance $\text{Var}_{\mathbb{P}}(I_y(y,T) + I_z(z,T))$ is given by:

$$\begin{aligned}\text{Var}_{\mathbb{P}}(I_y(y,T) + I_z(z,T)) &= e^{(2m_y(0,T)+n_y^2(0,T))} \cdot (e^{n_y^2(0,T)} - 1) + e^{(2m_z(0,T)+n_z^2(0,T))}(e^{n_z^2(0,T)} - 1)\\ &+ 2e^{m_{yz}(0,T)+\frac{n^2(0,T)}{2})} - e^{\alpha_y(0,T)-\beta_y(0,T)\mu_x^y(t)} \cdot e^{\alpha_z(0,T)-\beta_z(0,T)\mu_x^z(0)}.\end{aligned}$$

5.2.5 Consistency between S-Forward and GS-Forward Pricing Methods

Let us now derive the price of the individual S-forward contract from the GS-forward's price formulas given in Section 5.2.4.

Under the real-world measure $\mathbb{P}$, the mortality intensity formulas (5.3) and (5.4) can be written as:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y d\hat{w}_y(t) \tag{5.45}$$
$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z d\hat{w}_z(t), \tag{5.46}$$

where $\hat{w}_y(t) = \rho_y w_1(t) + \sqrt{1-\rho_y^2}\, w_2(t)$ and $\hat{w}_z(t) = \rho_z w_1(t) + \sqrt{1-\rho_z^2}\, w_2(t)$.

For each of the three pricing methods, we can determine the price of the individual S-forward contract directly from the GS-forward price formula, by considering just one cohort.

- Cost of Capital approach:

The prices of the individual S-forwards under the Cost of Capital approach for cohorts Y and Z are given by:

$$\begin{aligned}V^y_{COC}(0,T) &= P(0,T)\,(E_{\mathbb{P}}[\,I_y(y,T) - {}_T\hat{p}_y]) + 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I_y(y,i))[\text{VaR}_{99,5\%}(I_y(y+i,1))\\ &- E_{\mathbb{P}}(\,I_y(y+i,1))]E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1))P(0,i+1)P(i,T)\end{aligned} \tag{5.47}$$

$$\begin{aligned}V^z_{COC}(0,T) &= P(0,T)\,(E_{\mathbb{P}}[\,I_z(z,T) - {}_T\hat{p}_z]) + 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I_z(z,i))[\text{VaR}_{99,5\%}(I_z(z+i,1))\\ &- E_{\mathbb{P}}(\,I_z(z+i,1))]E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1))P(0,i+1)P(i,T)\end{aligned} \tag{5.48}$$

- Sharpe approach:

The prices of the individual S-forwards under the Sharpe approach for cohorts Y and Z are given by:

$$V^y_{Sharpe}(0,T) = P(0,T)\Big[E_{\mathbb{P}}(I_y(y,T)) - {}_T\hat{p}_y + S\sqrt{\text{Var}_{\mathbb{P}}(I_y(y,T))}\Big]$$
$$V^z_{Sharpe}(0,T) = P(0,T)\Big[E_{\mathbb{P}}(I_z(z,T)) - {}_T\hat{p}_z + S\sqrt{\text{Var}_{\mathbb{P}}(I_z(z,T))}\Big]$$

- Risk-neutral

The prices of the individual S-forwards under the risk-neutral approach for cohorts Y and Z are given by:

$$V^y_{\mathbb{Q}_{\lambda\lambda'}}(0,T) = P(0,T)[E_{\mathbb{Q}_{\lambda\lambda'}}(I_y(y,T)) - {}_T\hat{p}_y) \mid \mathcal{F}_0]$$
$$V^z_{\mathbb{Q}_{\lambda\lambda'}}(0,T) = P(0,T)[E_{\mathbb{Q}_{\lambda\lambda'}}(I_z(z,T)) - {}_T\hat{p}_z) \mid \mathcal{F}_0].$$

Under the risk-neutral approach, formulas (5.45) and (5.46) can be written as:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y\mu_y(t) + \tau\sigma_y)dt + \sigma_y d\overline{w}_y(t) \tag{5.49}$$
$$d\mu_z(t) = (A_z e^{B_z t} - b_z\mu_z(t) + \tau'\sigma_z)dt + \sigma_z d\overline{w}_z(t), \tag{5.50}$$

where $\overline{w}_y(t) = \rho_y w^*_1(t) + \sqrt{1-\rho_y^2}\, w^*_2(t)$, and $\overline{w}_z(t) = \rho_z w^*_1(t) + \sqrt{1-\rho_z^2}\, w^*_2(t)$.
We can see that under the risk-neutral measure $\mathbb{Q}_{\lambda\lambda'}$, the drifts have been changed by adding $\tau\sigma_y$ and $\tau'\sigma_z$ where $\tau = \lambda\rho_y + \lambda'\sqrt{1-\rho_y^2}$ and $\tau' = \lambda\rho_z + \lambda'\sqrt{1-\rho_z^2}$.

If we consider the special case where $\rho_y = 1$, and only one risk parameter $\rho_z = \rho$, τ and τ' become:

$$\tau = \lambda$$
$$\tau' = \lambda\rho + \lambda'\sqrt{1-\rho^2}.$$

We define the market price of risk λ^* for the two cohorts, as follows:

$$\lambda = \lambda^*$$
$$\tau' = \lambda\rho + \lambda'\sqrt{1-\rho^2} = \lambda^*.$$

We find that:

$$\lambda = \lambda^*$$
$$\lambda' = \lambda^*\sqrt{\frac{1-\rho}{1+\rho}}.$$

For illustration, if we consider $\rho = 0.95$ and $\lambda = -20\%$, we find that $\lambda' = -3.2\%$. We can see that the values of λ and λ' are very different, but the market prices of risk τ and τ' on which the final price depends are equal.
We conclude that COC and Sharpe methods are coherent between global and individual models; however, in the risk-neutral approach, we can also obtain coherency between the models, but we have to impose conditions on the values of λ and λ'.

5.3 Cohort-Based Longevity Model: n Cohorts

Let us now generalize our study to n cohorts. We consider n different cohorts of individuals initially aged x_1, x_2....x_n at time $t = 0$, and we use the HW model for describing the evolution of mortality of each cohort.
The mortality intensities that are related to these cohorts are given by:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1e^{B_1t} - b_1\mu_{x_1}(t) \\ A_2e^{B_2t} - b_2\mu_{x_2}(t) \\ . \\ . \\ . \\ A_ne^{B_nt} - b_n\mu_{x_n}(t) \end{pmatrix} \mathrm{dt}+ \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} & . & . & \rho_{1,n} \\ \rho_{2,1} & 1 & \rho_{2,3} & . & . & \rho_{2,n} \\ . & \rho_{3,2} & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & . & . \\ \rho_{n,1} & \rho_{n,2} & \rho_{n,3} & . & . & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_2(t) \\ . \\ . \\ . \\ \sigma_n dw_n(t) \end{pmatrix}. \tag{5.51}$$

where w_1, $w_2...w_n$ are independent.

We denote $M = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} & . & . & \rho_{1,n} \\ \rho_{2,1} & 1 & \rho_{2,3} & . & . & \rho_{2,n} \\ . & \rho_{3,2} & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & . & . \\ \rho_{n,1} & \rho_{n,2} & \rho_{n,3} & . & . & 1 \end{pmatrix}$

M is a symmetric $n \times n$ Matrix that represents the impact of the risk factors between two cohorts $\rho_{i,j}$.
Formula (5.51) is the general case, since it includes n independent risk factors that affect each cohort. We consider three particular cases in order to limit the number of risk factors, and also to look at extreme situations. We study three possibilities that express how the cohorts are correlated:

1. One common risk factor: the cohorts are equally dependent, which means that the forces of mortality of these cohorts are given by:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1e^{B_1t} - b_1\mu_{x_1}(t) \\ A_2e^{B_2t} - b_2\mu_{x_2}(t) \\ . \\ . \\ . \\ A_ne^{B_nt} - b_n\mu_{x_n}(t) \end{pmatrix} \mathrm{dt}+ \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_1(t) \\ . \\ . \\ . \\ \sigma_n dw_1(t) \end{pmatrix}.$$

In this case, the correlation is given by:

$$Corr(\mu_{x_k}(t), \mu_{x_l}(t)) = \varphi_{x_k,x_l}(t),$$

where $\varphi_{x_k,x_l}(t)$ is given by:

$$\varphi_{x_k,x_l}(t) = \Big(\frac{2\sqrt{b_{x_k}b_{x_l}}}{b_{x_k}+b_{x_l}} \frac{1-e^{-(b_{x_k}+b_{x_l})t}}{\sqrt{1-e^{-2b_{x_k}t}}\sqrt{1-e^{-2b_{x_l}t}}}\Big).$$

2. Two common risk factors: the cohorts are dependent with different degrees:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1e^{B_1t} - b_1\mu_{x_1}(t) \\ A_2e^{B_2t} - b_2\mu_{x_2}(t) \\ . \\ . \\ . \\ A_ne^{B_nt} - b_n\mu_{x_n}(t) \end{pmatrix} dt+ \begin{pmatrix} \rho_1 & \sqrt{1-\rho_1^2} & 0 & . & . & 0 \\ \rho_2 & \sqrt{1-\rho_2^2} & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \rho_n & \sqrt{1-\rho_n^2} & 0 & . & . & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_2(t) \\ . \\ . \\ . \\ \sigma_n dw_n(t) \end{pmatrix},$$

where w_1, w_2 ...w_n are n independent Brownian motions. In this case, the correlation is given by:

$$Corr(\mu_{x_k}(t), \mu_{x_l}(t)) = \Big(\rho_k\rho_l + \sqrt{1-\rho_k^2}\sqrt{1-\rho_l^2}\Big)\varphi_{x_k,x_l}(t),$$

3. One risk factor by cohort: the cohorts are completely independent:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1 e^{B_1 t} - b_1\mu_{x_1}(t) \\ A_2 e^{B_2 t} - b_2\mu_{x_2}(t) \\ . \\ . \\ . \\ A_n e^{B_n t} - b_n\mu_{x_n}(t) \end{pmatrix} \text{dt} + \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_2(t) \\ . \\ . \\ . \\ \sigma_n dw_n(t) \end{pmatrix},$$

where w_1, w_2 ...w_n are n-independent Brownian motions. In this case, the correlation is given by:

$$Corr(\mu_{x_k}(t), \mu_{x_l}(t)) = 0.$$

5.3.1 GS-Forward Pricing

We consider the case of an insurer holding a portfolio of policyholders belonging to the cohorts X_1, X_2 , X_3....X_n. We assume that this insurer should pay one monetary unit to each individual alive at maturity T.
The payoff of this GS-forward at time T that is evaluated at time 0 is given by:

$$Payoff(T) = \sum_{k=1}^{n} \big(I_{x_k}(x_k, T) - {}_T\hat{p}_{x_k}\big),$$

where ${}_T\hat{p}_{x_k}$ is the fixed survival rate related to cohort X_k, and $I_{x_k}(x_k, T)$ the survival index of an individual aged x_k at time 0, to be alive at age $x_k + T$.

GS-Forward Pricing under the COC Approach

The price at time 0 under the COC approach is given by:

$$V^{G}_{COC}(0,T) = BE_0^{\mathbb{P}} + RM_0 \tag{5.52}$$

where $BE_0^{\mathbb{P}}$ is as follows:

$$BE_0^{\mathbb{P}} = P(0,T)E_{\mathbb{P}}\big[\sum_{j=1}^{n}(I_{x_j}(x_j,T) - {}_T\hat{p}_{x_j})\big].$$

The risk margin at time 0 is:

$$RM_0 = C\% \sum_{i=0}^{T-1} S\hat{C}R_i\, P(0, i+1).$$

Subsequently, the expression of the SCR_i is given by:

$$SCR_i = \text{VaR}_{99,5\%}\left[BE_{i+1}^{\mathbb{P}} P(i,i+1) - BE_i^{\mathbb{P}}\right],$$

where:

$$BE^{\mathbb{P}}_{i+1} = \Big(\sum_{k=1}^{n} I_{x_k}(x_k, i+1) E_{\mathbb{P}}(I_{x_k}(x_k+i+1, T-i-1) - {}_T\hat{p}_{x_k}) P(i+1, T)\Big)$$
$$BE^{\mathbb{P}}_{i} = \Big(\sum_{k=1}^{n} I_{x_k}(x_k, i) E_{\mathbb{P}}(I_{x_k}(x_k+i, T-i) - {}_T\hat{p}_{x_k}) P(i, T).$$

The estimation of $SCR_i \mid_0$ can be given by:

$$\begin{aligned} SCR_i \mid_0 &= P(i,T)\Big[\mathrm{VaR}_{99{,}5\%} \big[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k, i))((I_{x_k}(x_k+i, 1) \\ &\times E_{\mathbb{P}}(I_{x_k}(x_k+i+1, T-i-1)) - E_{\mathbb{P}}(I_{x_k}(x_k+i, T-i))\big]\Big]. \end{aligned}$$

RM_0 is then equal to:

$$\begin{aligned} RM_0 &= 6\% \sum_{i=0}^{T-1} P(0, i+1) P(i,T)\Big[\mathrm{VaR}_{99{,}5\%} \big[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k, i))((I_{x_k}(x_k+i, 1) \\ &\times E_{\mathbb{P}}(I_{x_k}(x_k+i+1, T-i-1)) - E_{\mathbb{P}}(I_{x_k}(x_k+i, T-i))\big]\Big]. \end{aligned}$$

Finally, the price of the GS-forward with the COC approach is:

$$\begin{aligned} V^{G}_{COC}(0,T) &= P(0,T) E_{\mathbb{P}}\big[\sum_{k=1}^{n} (I_{x_k}(x_k, T) - {}_T\hat{p}_{x_k})\big] + 6\% \sum_{i=0}^{T-1} P(0, i+1) P(i,T) \\ &\times \Big[\mathrm{VaR}_{99{,}5\%} \big[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k, i))((I_{x_k}(x_k+i, 1) \\ &\times E_{\mathbb{P}}(I_{x_k}(x_k+i+1, T-i-1)) - E_{\mathbb{P}}(I_{x_k}(x_k+i, T-i))\big]\Big]. \end{aligned} \tag{5.53}$$

GS-Forward Pricing under the Risk-Neutral Approach

The price of the GS-forward at time 0 is given by:

$$V^{G}_{\mathbb{Q}_{\lambda_k}}(0,T) = P(0,T)\Big[\sum_{k=1}^{n} E_{\mathbb{Q}_{\lambda_k}}(I_{x_k}(x_k, T) - {}_T\hat{p}_{x_k} \mid \mathcal{F}_0\Big], \tag{5.54}$$

where λ_k is the market price of risk associated to the cohort x_k.

GS-Forward Pricing under the Sharpe Approach

The price of the GS-forward at time 0 under the Sharpe approach is given by:

$$V^{G}_{Sharpe}(0,T) = P(0,T)\left[\sum_{k=1}^{n} (E_{\mathbb{P}}(I_{x_k}(x_k, T)) - {}_T\hat{p}_{x_k} + S\sqrt{\mathrm{Var}_{\mathbb{P}}(\sum_{k=1}^{n} I_{x_k}(x_k, T))}\right], \tag{5.55}$$

where S is the Sharpe ratio, and ${}_T\hat{p}_{x_k}$ is the fixed rate that is related to the cohort X_k.

Because $I_{x_k}(x_k,T)$ are dependent, we have:

$$\mathrm{Var}_{\mathbb{P}}\Big(\sum_{k=1}^{n} I_{x_k}(x_k,T)\Big) = \sum_{k=1}^{n} \mathrm{Var}_{\mathbb{P}}\Big(I_{x_k}(x_k,T)\Big) + 2\sum_{k=1}^{n}\sum_{\substack{l\neq k\\ l=1}}^{n} Cov\Big(I_{x_k}(x_k,T), I_{x_l}(x_l,T)\Big). \quad (5.56)$$

The price at time 0 under the Sharpe approach is finally given by:

$$V^{G}_{Sharpe}(0,T) = P(0,T)\left[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k,T)) - {}_T\hat{p}_{x_k} + S\sqrt{\sum_{k=1}^{n}\mathrm{Var}_{\mathbb{P}}(I_{x_k}(x_k,T)) + 2\sum_{k=1}^{n}\sum_{\substack{l\neq k\\ l=1}}^{n} Cov\big(I_{x_k}(x_k,T), I_{x_l}(x_l,T)\big)}\,\right], \quad (5.57)$$

where $Cov(I_{x_k}(x_k,T), I_{x_l}(x_l,T))$ is given by (5.33).

5.3.2 Application to a Portfolio of Annuities

Life and pension annuities are typical products that allow retirees to obtain lifelong incomes. For fixed lifetime annuities, the longevity risk is completely supported by the insurer or the pension fund. Therefore, it is crucial for the annuity provider to hedge this risk, which is not easy given the limited solutions available on the market.

Our pricing framework has the advantage to be flexible, as it can also be generalized to hedge a portfolio of annuities: we consider a pension fund holding a portfolio of policyholders that belong to the cohorts $X_1, X_2, ..X_n$, to whom he provides fixed annuities of one monetary unit at each date $t_1, t_2, ..., t_m$ if the policyholder is alive.

For each cohort X_k, the annuity provider can mitigate the corresponding longevity risk by entering a survival swap (S-swap).

This S-swap is considered to be an aggregation of S-forward contracts (see for instance [26]). We denote, by $P^{x_k}(0, t_1, t_2, ..., t_m)$, the price at time 0 of an S-swap for the individuals belonging to the cohort X_k, with $t_1, t_2, ..., t_m$ as exchange dates ($0 < t_1 < t_2 < ... < t_m$). Without arbitrage opportunities, the price of this S-swap is equal to the sum of the prices of the corresponding individual S-forward contracts (see, for instance, [26]):

$$P^{x_k}(0, t_1, t_2, ..., t_m) = \sum_{i=1}^{m} V^{x_k}(0, t_i), \quad (5.58)$$

where $V^{x_k}(0, t_i)$ is the price at time 0 of the individual S-forward contract related to a given cohort X_k for a maturity t_i.

Therefore, if we do not take into account the eventual correlation between cohorts, the price of the S-swap for individuals that belong to the different cohorts $X_1, X_2, ..X_n$ is given by:

$$\bar{P}^{x_k}(0, t_1, t_2, ..., t_m) = \sum_{k=1}^{n}\sum_{i=1}^{m} V^{x_k}(0, t_i) \quad (5.59)$$

Alternatively, if such product exists, the annuity provider can cover the longevity risk for all individuals that belong to the different cohorts $X_1, X_2, ..X_n$, by entering one derivative instead of a derivative by cohort, while taking the eventual correlation between cohorts into account. The price at time 0 of this Global S-swap (GS-swap) is given by:

$$P^G(0, t_1, t_2, ..., t_m) = \sum_{i=1}^{m} V^G(0, t_i) \tag{5.60}$$

where $V^G(0, t_i)$ is the price of the GS-forward contract.

5.4 Numerical Simulation

In this section, we compute the price of different GS-forward contracts based on the Belgian population data, and using the different pricing methods.

5.4.1 Assumptions

To price the GS-forwards, we consider these assumptions:

- An insurer with a portfolio of pure endowment contracts paying 1€ at maturity T;
- Two maturities: $T = 5$ and $T = 10$[2].;
- N_0=10 000 initial policyholders for each cohort;
- We consider two examples: individuals initially aged $y = 55$ and $z = 60$, or individuals initially aged $y = 60$ and $z = 65$, old in 2015;
- According to the literature, the values of ρ are around 0.98 [14]. However, to have a better idea on the price evolution, we consider the following values for ρ: 0.95, 0.98, 1 (in view of illustration, we also consider the extreme value $\rho = 0$);
- Data from the Belgian IA|BE unisex mortality table[3];
- The risk-less interest rate denoted r, is considered constant, $r = 1\%$;
- The optimal parameters for the HW model are given by Table 5.1:

Age in 2015	$\boldsymbol{\mu_x(0)}$	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{b}$	$\boldsymbol{\sigma}$
55	0.00466531	0.00042258	0.11428187	0.11669113	0.00200113
60	0.00722197	0.00089226	0.11571836	0.15355787	0.00166015
65	0.01056770	0.00231775	0.11562220	0.25062948	0.01770006

Table 5.1: Optimal parameters value for the different cohorts, HW model.

[2] We limit ourselves to a short maturity of 5 or 10 years, since the market is reluctant to longer maturities given the uncertainty related to longevity.

[3] IA|BE mortality projection for the Belgian population (2015) available at www.iabe.be.

- The fixed rates are reported in Table 5.2:

T/Age	55	60	65
$T = 5$	0.9737899	0.9605744	0.9419321
$T = 10$	0.9395278	0.9107331	0.8658090

Table 5.2: The fixed survival rates ${}_T\hat{p}_x$ for different ages and maturities.

- For simplification purpose, we chose $\rho_y = 1$, and we consider only one risk parameter $\rho_z = \rho$

The two mortality intensities for cohorts Y and Z given by (5.3) and (5.4) now become:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y dw_1(t) \tag{5.61}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z(\rho dw_1(t) + \sqrt{1-\rho^2} dw_2(t)) \tag{5.62}$$

ρ_{w_y,w_z} and the correlation given by formula (5.13) become:

$$\rho_{w_y,w_z} = \rho \tag{5.63}$$

$$Corr(\mu_y(t), \mu_z(t)) = \rho \varphi_{y,z}(t)$$

5.4.2 Correlation between Cohorts

Before computing the prices of the GS-forward contracts, let us determine the correlation between the cohorts. To do so, we first compute $\varphi_{y,z}(t)$ for the two portfolios considered:

- x_0= 55 and x_0= 60 in 2015 (denoted $\varphi_{55,60}(t)$); and,
- x_0= 60 and x_0= 65 in 2015 (denoted $\varphi_{60,65}(t)$).

We use Belgian data from IA|BE and compute $\varphi_{55,60}(t)$ and $\varphi_{60,65}(t)$ for $t = 1$ to $t = 10$. Table 5.3 presents the results:

	$\varphi_{55,60}(t)$	$\varphi_{60,65}(t)$
$t = 1$	0.9999436	0.9995160
$t = 2$	0.9997768	0.9981148
$t = 3$	0.9995068	0.9959374
$t = 4$	0.9991450	0.9931869
$t = 5$	0.9987058	0.9900917
$t = 6$	0.9982057	0.9868707
$t = 7$	0.9976623	0.9837081
$t = 8$	0.9970927	0.9807412
$t = 9$	0.9965130	0.9780582
$t = 10$	0.9959375	0.9757037

Table 5.3: Some values of $\varphi_{y,z}(t)$.

Now that we have the values of $\varphi_{y,z}(t)$, we can compute the correlation between the forces of mortality, and between the longevity indexes.
For ease of notation, we write: $Corr(\mu_y(t), \mu_z(t)) = Corr(\mu_{y,z}(t))$
The correlation between the forces of mortality are reported in Table 5.4:

	$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	$Corr(\mu_{55,60}(T))$	$Corr(\mu_{60,65}(T))$	$Corr(\mu_{55,60}(T))$	$Corr(\mu_{60,65}(T))$	$Corr(\mu_{55,60}(T))$	$Corr(\mu_{60,65}(T))$
$T = 5$	0.9487705	0.9405871	0.9787317	0.9702899	0.9987058	0.9900917
$T = 10$	0.9461406	0.9269185	0.9760188	0.9561896	0.9959375	0.9757037

Table 5.4: Values of $Corr(\mu_{y,z}(T))$.

The correlation between longevity indexes is given by (5.35). For ease of notation, we write: $Corr(I_y(y,T), I_z(z,T)) = Corr(I_{y,z}(T))$. Table 5.5 reports the results:

	$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	$Corr(I_{55,60}(T))$	$Corr(I_{60,65}(T))$	$Corr(I_{55,60}(T))$	$Corr(I_{60,65}(T))$	$Corr(I_{55,60}(T))$	$Corr(I_{60,65}(T))$
$T = 5$	0.9498743	0.9490736	0.9798716	0.9790453	0.9998698	0.9990265
$T = 10$	0.9495996	0.9475659	0.9795936	0.9774933	0.9995898	0.9974450

Table 5.5: Values of $Corr(I_{y,z}(T))$.

5.4.3 Pricing GS-Forwards

Let us now compute the prices of the different GS-forward contracts under the different methods.

Pricing GS-Forwards under the Risk-Neutral Approach

Because the risk-neutral method is based on the expectation of the survival index, the price of the GS-forward under this approach does not depend on ρ. This price is equal to the sum of the prices of the individual S-forwards that are related to the corresponding cohorts:

$$
\begin{aligned}
V^y(0,T) + V^z(0,T) &= P(0,T)[E_{\mathbb{Q}_\lambda}(I_y(y,T)) - {}_T\hat{p}_y) \mid \mathcal{F}_0] + P(0,T)[E_{\mathbb{Q}_{\lambda'}}(I_z(z,T)) - {}_T\hat{p}_z) \mid \mathcal{F}_0] \\
&= P(0,T)[E_{\mathbb{Q}_\lambda}(I_y(y,T)) + E_{\mathbb{Q}_{\lambda'}}(I_z(z,T)) - {}_T\hat{p}_y - {}_T\hat{p}_z) \mid \mathcal{F}_0] \\
&= P(0,T)[E_{\mathbb{Q}_\lambda}(I_y(y,T) + I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z) \mid \mathcal{F}_0] \\
&= V^{y,z}(0,T) \qquad (5.64)
\end{aligned}
$$

Thus, we will only focus on the COC and Sharpe approaches.

Pricing GS-Forwards under the COC Approach

Let us now compute the price of the GS-forward while using the HW model under the COC approach.
Table 5.6 reports the prices of the different GS-forwards $V^{y,z}_{COC}(0,T)$ for different ages, maturities, and correlation levels:

		$\rho = 0$		$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	BE	***RM***	***Price***	***RM***	***Price***	***RM***	***Price***	***RM***	***Price***
$V_{COC}^{55,60}(0,5)$	71.0608	25.5553	96.6161	31.3349	102.3958	31.5374	102.5983	31.6411	102.7020
$V_{COC}^{55,60}(0,10)$	193.7744	63.4303	257.2047	76.1547	269.9291	76.5697	270.3441	76.7914	270.5658
$V_{COC}^{60,65}(0,5)$	84.1478	22.0132	106.1611	27.4256	111.5735	27.5616	111.7095	27.6549	111.8028
$V_{COC}^{60,65}(0,10)$	169.7714	18.2327	188.0041	27.1015	196.8729	27.5122	197.2836	27.6512	197.4226

Table 5.6: The prices of the different GS-forwards for different ages, maturities, and correlations, Cost of Capital (COC) approach.

Pricing GS-Forwards under the Sharpe Approach

Let us now compute the price of the GS-forward under the Sharpe approach, while using $S = 10\%$, which is consistent with the values found in the literature (see for instance [1], and [26]). Table 5.7 reports the prices of the different GS-forwards $V_{Sharpe}^{y,z}(0,T)$, as well as the best estimates and the premiums P for different ages, maturities, and correlation levels:

		$\rho = 0$		$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	BE	***P***	***Price***	***P***	***Price***	***P***	***Price***	***P***	***Price***
$V_{Sharpe}^{55,60}(0,5)$	71.0608	12.3187	83.3796	17.0625	88.1234	17.1910	88.2519	17.2761	88.3370
$V_{Sharpe}^{55,60}(0,10)$	193.7744	26.1532	219.9276	36.0616	229.8361	36.3305	230.1050	36.5088	230.2832
$V_{Sharpe}^{60,65}(0,5)$	84.1479	10.2784	94.4263	14.3410	98.4889	14.4507	98.5986	14.5234	98.6713
$V_{Sharpe}^{60,65}(0,10)$	169.7714	19.6761	189.4476	27.3093	197.0808	27.5159	197.2874	27.6528	197.4243

Table 5.7: The prices of the different GS-forwards for different ages, maturities, and correlations, Sharpe method.

5.4.4 Comparison between Individual and GS-Forward Prices

In our framework, we consider an insurer with a portfolio of individuals that belong to two different cohorts. Hence, to hedge the longevity risk, this insurer needs to buy a GS-forward based on these two cohorts. Such a product can be available as an OTC derivative, but if it is not the case, the insurer should enter two different S-forwards that correspond to the two different cohorts. It could be interesting to check if these two situations are equivalent. To do so, we compare the prices of the individual S-forward contracts to the prices of the GS-forward. We denote, by $V^y(0,T)$ and $V^z(0,T)$, the prices of these S-forward contracts at time 0 for a maturity T, simultaneously for cohorts Y and Z.
In order to measure the difference between buying a GS-forward for the whole portfolio, and buying an individual S-forward by cohort, let us define $\Delta_\rho^{y,z}$ by the following:

$$\Delta_\rho^{y,z}(0,T) = \frac{V^y(0,T) + V^z(0,T) - V^{y,z}(0,T)}{V^{y,z}(0,T)}.$$

We compute $\Delta\rho^{y,z}(0,T)$ for the Cost of Capital and Sharpe approaches.

Cost of Capital Approach

Under the COC approach, the prices of the individual S-forward contracts for each cohort Y and Z are reported in Table 5.8:

	Prices
$V^{55}_{COC}(0,5)$	50.3640
$V^{55}_{COC}(0,10)$	143.2363
$V^{60}_{COC}(0,5)$	52.3390
$V^{60}_{COC}(0,10)$	127.3379
$V^{65}_{COC}(0,5)$	59.4699
$V^{65}_{COC}(0,10)$	108.2631

Table 5.8: The prices of the different individual S-forward contracts under the COC approach.

We report the values of $\Delta_\rho^{y,z}$ in Table 5.9:

	$\rho = 0$	$\rho = 0.95$	$\rho = 0.98$	$\rho = 1$
$\Delta_\rho^{55,60}(0,5)$	6.300%	0.300%	0.102%	0.00098%
$\Delta_\rho^{55,60}(0,10)$	5.198%	0.231%	0.085%	0.00310%
$\Delta_\rho^{60,65}(0,5)$	5.320%	0.211%	0.089%	0.00550%
$\Delta_\rho^{60,65}(0,10)$	5.020%	0.289%	0.080%	0.00978%

Table 5.9: $\Delta_\rho^{y,z}(0,T)$ values for different ages and maturities, COC approach.

Sharpe Approach

Under the Sharpe approach, the prices of these S-forward contracts for each cohort Y and Z without the correlation effect are reported in Table 5.10:

	Prices
$V^{55}_{Sharpe}(0,5)$	42.5466
$V^{55}_{Sharpe}(0,10)$	121.7403
$V^{60}_{Sharpe}(0,5)$	45.7909
$V^{60}_{Sharpe}(0,10)$	108.5467
$V^{65}_{Sharpe}(0,5)$	52.8839
$V^{65}_{Sharpe}(0,10)$	88.8951

Table 5.10: The prices of the different individual S-forward contracts under Sharpe approach.

We report the values of $\Delta_\rho^{y,z}$ in Table 5.11.

	$\rho = 0$	$\rho = 0.95$	$\rho = 0.98$	$\rho = 1$
$\Delta_\rho^{55,60}(0,5)$	5.940%	0.240%	0.097%	0.00062%
$\Delta_\rho^{55,60}(0,10)$	4.710%	0.196%	0.079%	0.00160%
$\Delta_\rho^{60,65}(0,5)$	4.490%	0.188%	0.077%	0.00350%
$\Delta_\rho^{60,65}(0,10)$	4.220%	0.183%	0.078%	0.00880%

Table 5.11: $\Delta_\rho^{y,z}(0,T)$ values for different ages and maturities, Sharpe approach.

For the two approaches, we can see that:

- if $\rho = 0$, we observe a strong non-additive effect for the two random variables;
- if $0 < \rho < 1$, the prices reflect the non-additive effect as well as the dependence effect between cohorts;
- if $\rho = 1$, the values of $\Delta_\rho^{y,z}(0,T)$ are almost equal to 0, which means that COC and Sharpe approaches are additive when the two cohorts are completely correlated; and,
- when ρ decreases, the GS-forward contract becomes less expensive than individual S-forwards.

5.5 Conclusions

In this paper, we have explored the correlation between the mortality of different cohorts, and its impact in the pricing of the GS-forward contract under three pricing approaches. To describe mortality, we have used the Hull and White model, into which we incorporate risk factors, which allow for the introduction of inter-generational correlations with different levels. We have provided the GS-forward price in closed form under the Cost of Capital, risk-neutral and Sharpe methods for different dependence levels. In addition, to measure the impact of including this dependence, we have compared both GS-forward and individual S-forward prices. For the risk-neutral approach, the correlation between generations does not have an effect on the GS-forward price, whereas, for the Cost of Capital and Sharpe approaches, we have observed a significant correlation effect on the prices. For these two pricing methods, if the cohorts are not perfectly correlated, the GS-forward becomes less expensive than the sum of the respective individual S-forward contracts.

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Chapter 6

General conclusion and perspectives

Longevity risk is a fundamental concern for life insurers, whose primary role is to guarantee, in the case of the policyholder's survival, the payment of a lump sum or an annuity in exchange for a premium. The increase in life expectancy has pushed reinsurers and the financial markets to propose longevity derivatives that allow insurers to transfer their longevity risk. In this thesis, we have addressed different aspects of this longevity risk, such as the modelling and pricing issues, from various angles.

6.1 Summary of the main results

The first important step in our risk assessment framework was to determine an appropriate model allowing to predict mortality accurately. In **Chapter 2**, we provided a comparison between mean and non-mean reversion continuous time models using statistical methodologies, such as calibration and backtesting. We have considered five different models, including non-mean reversion models (Ornstein-Uhlenbeck and Feller), a mean reversion model with a fixed target (Vasiçek), and mean reversion processes with a time-dependent long-term level. For the reversion level, we considered an exponential form in line with the Gompertz law. The Vasiçek model provided poor calibration results due to its fixed target, so it was eliminated from the comparison. The calibration test and the backtesting indicated that the CIR extended and Hull and White models perform better, and it can therefore be concluded that incorporating a time-dependent target into the Ornstein-Uhlenbeck and Feller models significantly improves their performance.

Subsequently, in **Chapter 3**, we used these two selected models to price two longevity derivatives: survival forwards and survival swaps. We discussed the difficulties generated by the use of the classical pricing methods used in finance (risk-neutral, Sharpe ratio and Wang approaches) for pricing longevity securities, which are due to parameter calibration issues related to the lack of data and instruments on the market. To solve this problem, we proposed a pricing approach based on the Cost of Capital methodology consistent with the directive of Solvency II. This technique enabled us to easily compute the prices of the longevity derivatives without the need for market data, since the Cost of Capital rate is fixed by the regulator. In fact, the Cost of Capital approach is parameterized by one variable, the Cost of Capital rate, which is fixed by regulation (6% in Solvency II nowadays). This legal and unique parameter value can be seen as a benchmark, and allows to find equivalent values of the parameters in the three other methods. The results obtained show that the classical methods considered are completely different from the Cost of Capital method. This could be explained by the fact that these methods are based on different methodologies and conceptions. It is important for insurers to respect the directives of the regulator, and if they enter a longevity derivative to cover the longevity risk instead of holding an SCR, the pricing methodology used to compute the price and to determine the SCR should be coherent.

Hence, the Cost of Capital method approach is more appropriate for pricing longevity derivatives. These longevity derivatives were based on the national population data rather than the insurer's own data, which may lead to a potential loss due to data mismatch, called the basis risk. To measure this basis risk, in **Chapter 4** we defined an S-exchange contract, whose price corresponds to the hedging cost of this risk. We then developed different multi-population models based on the Hull and White model by modifying the trend and/or the volatility of the mortality intensity process, and we compared the results with the classical pricing methods. Based on these results, we proposed different longevity hedging strategies depending on the insurer's risk aversion.

Finally, in **Chapter 5** we took into account the eventual correlation between generations and its impact in pricing the S-forwards contracts. The correlation between generations was captured using a multi-cohort model based on the Hull and White process, into which we incorporated inter-generational risk factors. We showed that when the cohorts are considered correlated, the price of the global S-forward contract under the cost of capital and Sharpe approaches becomes less expensive than the sum of the prices of the corresponding individual S-forwards contracts.

6.2 General discussion: potential application

Longevity risk is a major issue for life insurers and reinsurers, especially given its systematic aspect. Moreover, in the context of the public social security schemes and the aging of the population, the demand for fully funded pension products should increase in the next decades. Therefore, it is crucial for life insurers to find appropriate solutions to hedge this longevity risk from a marketing and a technical point of view. One possible solution is to transfer this risk (partially or totally) to the financial markets. The advantage of this approach for insurers and reinsurers is of course quite clear. At the same time, from the investor side, a big interest could also emerge because of the independence (or weak dependence) between the financial risks and the longevity risk. Another important point is the huge capacity of transactions from the financial markets, in comparison with the available equities of the reinsurers, limited by definition. Nowadays, we have to admit that securitization of longevity risk is not yet widely developed. Clearly, a first step would be to develop traded products based on national population indices permitting the financial markets to trust the official outcomes of the product. Pricing these products remains a challenge; the use of pricing solutions inspired more by finance than insurance (such as the cost of capital proposed in this thesis) has two main advantages:

- For the buyer, its consistency with the solvency requirement of Solvency II

- For the seller, its financial logic, closer to the philosophy of financial markets than other pricing methods (financialization of the economy).

It will then result only a partial protection for the life insurer. Measuring and eventually hedging the remaining part of the risk (called the basis risk) is also a subsequent question to address. The logic proposed in this thesis could be then to stimulate the purchase of an additional OTC product which is a customized solution that can be added to a more basic product based on national population. In this thesis we have developed some new innovative OTC derivatives in that direction (S-exchange contract).

6.3 Perspective and further research

The present work can be extended, among others, in five possible directions:

- Compared to the discrete-time models like Lee-Carter, the stochastic time-continuous models facilitate the integration of jumps. Indeed, the stochastic differential equation of the force of mortality can contain a jump part such as a Poisson process. Therefore, a first promising area is to extend the considered mortality processes to a more general setting, able to capture sudden events by including a jump effect, using a Poisson process or a more general Levy process. By introducing jumps into a mortality model, the intensity of mortality can suddenly change (either improving or worsening) due to unexpected external events such as the current Covid-19 pandemic. These jumps can be positive if the mortality worsens (in the case of wars or epidemics, for instance), while they can be negative if mortality improves (in the case of medical progress, for example). The jump component can be modelled in a way such that it has permanent effects on mortality rates, or a transitory effect, where mortality rates pop up when the catastrophic events happen, and fall back to the normal level when the events end. This latter is designed to capture short-term catastrophic events such as the 1918 Spanish flu and the 2004 earthquake. It would be interesting to compare the models with permanent jump effects to the models with transitory jump effects, and assess their effectiveness in forecasting mortality, as well as their appropriateness in longevity securitization pricing.

- In Chapter 2, we have chosen the Gompertz model as a target for the Hull and White and CIR extended model, which is an intuitive choice given the increasing trend of mortality over time. The calibration has also showed a stable behaviour of the exponential coefficient, which confirms the appropriateness of this choice. Nevertheless, It could be interesting to consider other time-dependent or even stochastic targets for these models, and compare the results obtained. For instance, we can use as a target the Svensson-Nelson-Siegel model that is widely used in finance especially for interest rate structure.

- In our framework, we have only focused on pricing longevity forward and swap contracts, which are linear derivatives. Our choice was motivated by the fact that these kinds of assets are the most frequently traded and can form the basis for other longevity securities. However, our pricing approach can also be utilized to price other longevity derivatives, such as survival bonds, or non-linear derivatives, such as survival options.

- The Cost of Capital method can be considered for derivatives subject to European regulations (Solvency II). It would be interesting to see how to adapt this method for derivatives subject to other regulations. For instance, it may be challenging to use this Cost of Capital approach to price a longevity product based on the spread between two longevity indices, one of which concerns a non-European population. This is the case, for example, with the Kortis/Swiss Re bond issued in 2010. The pricing of such a hybrid product could still be based on returns on shareholders' needed capital. However, the way to compute the required own funds and the remuneration rate should then be adapted.
- Finally, in our pricing setup, we adopted the classical independence assumption usually made in the actuarial literature between mortality risk and interest rate risk. This assumption may seem acceptable in the short term; however, it also seems natural that catastrophic risks, for instance, an earthquake or a severe pandemic such as the COVID-19 crisis, will affect the economy and the financial markets. Furthermore, in the long term, it is intuitive that demographic changes can affect the global economy.