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**FUNCTIONAL CONVERGENCE
OF QUANTILE-TYPE FRONTIERS
WITH APPLICATION
TO PARAMETRIC APPROXIMATIONS**

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Functional Convergence of Quantile-type Frontiers with Application to Parametric Approximations

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Abstract

Nonparametric estimators of the upper boundary of the support of a multivariate distribution are very appealing because they rely on very few assumptions. But in productivity and efficiency analysis, this upper boundary is a production (or a cost) frontier and a parametric form for it allows for a richer economic interpretation of the production process under analysis. On the other hand, most of the parametric approaches rely on often too restrictive assumptions on the stochastic part of the model and are based on standard regression techniques fitting the shape of the center of the cloud of points rather than its boundary. To overcome these limitations, Florens and Simar (2005) propose a two-stage approach which tries to capture the shape of the cloud of points near its frontier by providing parametric approximations of a nonparametric frontier. In this paper we propose an alternative method using the nonparametric quantile-type frontiers introduced in Aragon, Daouia and Thomas-Agnan (2005) for the nonparametric part of our model. These quantile-type frontiers have the superiority of being more robust to extremes. Our main results concern the functional convergence of the quantile-type frontier process and its uniform complete convergence. Then we provide convergence and asymptotic normality of the resulting estimators of the parametric approximation. We also improve some convergence results obtained by Florens and Simar. The approach is illustrated through simulated and real data sets.

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1 Basic Concepts and Notations

Let Ψ be the support of the joint distribution of a random vector $(X, Y) \in \mathbb{R}_+^p \times \mathbb{R}_+$ defined on the probability space $(\Omega, \mathcal{A}, P)$. Consider the problem of estimating the upper boundary of Ψ from a sample $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$ of independent random vectors with the same distribution as (X, Y) . If $\mathcal{X} = \{x \in \mathbb{R}_+^p \mid Y(x) \neq \emptyset\}$ where $Y(x) = \{y \in \mathbb{R}_+ \mid (x, y) \in \Psi\}$, then define $\varphi(x) = \max Y(x)$ for any $x \in \mathcal{X}$. The upper boundary of Ψ can be then defined as the set $\{(x, \varphi(x)) \mid x \in \mathcal{X}\}$. This boundary is assumed to be nondecreasing in the sense that the frontier function $\varphi(x)$ is nondecreasing¹ in x .

This kind of problem appears naturally to be useful when analyzing production performance of firms, where X represents the vector of inputs (resources of production) and Y is the output (a quantity of produced goods). In this context, $\varphi(x)$ is the production frontier, i.e., the maximal achievable level of output for a firm working at the level of inputs x . The production efficiency of a firm operating at the level (x, y) can be then measured by the relative comparison of its output y with the reference frontier $\varphi(x)$. These methods have been widely applied to examine efficiency in a variety of industries; see Lovell (1993) and Seiford (1996) for comprehensive bibliographies of these applications. Aside from the production setting, the problem of estimating monotone boundaries also naturally occurs in portfolio management. In capital asset pricing models (CAPM), the objective is to analyze the performance of investment portfolios. Risk and average return on a portfolio are analogous to inputs and outputs in models of production; in CAPM, the boundary of the attainable set of portfolios gives a benchmark relative to which the efficiency of a portfolio can be measured. These models were developed by Markovitz (1959) and others; Sengupta (1991) and Sengupta and Park (1993) provide links between CAPM and nonparametric estimation of frontiers.

We will follow the probabilistic formulation of the frontier problem proposed by Cazals, Florens and Simar (2002, CFS hereafter). Let us denote by $F(y|x) = F(x, y)/F_X(x)$ the nonstandard² conditional distribution function of Y given $X \leq x$, where F is the joint distribution function of (X, Y) and $F_X(x) = F(x, \infty)$. From now on we assume that $x \in \mathbb{R}_+^p$

¹For two vectors x and x' in $\mathbb{R}^p$ the inequality $x \leq x'$ has to be understood componentwise. A real valued function r on $\mathbb{R}^p$ is then said to be nondecreasing with respect to this partial order if $x \leq x'$ implies $r(x) \leq r(x')$.

²We use the word “nonstandard” to focus on the unusual condition $X \leq x$ in place of the more common $X = x$.

is such that $F_X(x) > 0$. The monotone upper surface of Ψ can be then characterized through the graph of the frontier function

$$\varphi(x) = \inf\{y \in \mathbb{R}_+ | F(y|x) = 1\}.$$

This value represents the upper boundary of the support of the nonstandard conditional probability measure of Y given $X \leq x$.

In our approach, we focus on a deterministic frontier model where the support Ψ contains all the observations (X_i, Y_i) with probability 1. Parametric approaches have been first proposed in the econometric litterature (this includes, among others, Aigner and Chu, 1968 and Greene, 1980). Here we assume the frontier function $\varphi(x)$ can be written as a specified analytical function $\varphi_\theta(x)$ depending on a finite number of parameters $\theta \in \mathbb{R}^k$. We denote this parametric model by

$$Y_i = \varphi_\theta(X_i) - U_i \quad \text{with} \quad U_i \geq 0 \quad \text{a.s.}, \quad (1.1)$$

If we specify an appropriate stochastic models for the error terms U_i , Greene (1980) proposes estimators of θ based on OLS (shifted or corrected to take into account for the positiveness of the error term) or by MLE. On one hand parametric models are very appealing because the parameters are easy to estimate and to interpret but on the other hand their properties rely on restrictive assumptions on the stochastic part of the model (parametric specification of the law of U , homoscedasticity, ...) and they are based on standard regression tools which capture the shape of the cloud of points $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$ near its center rather than near its optimal boundary (see Florens and Simar, 2005 for a careful discussion).

To overcome these drawbacks, Simar (1992) proposed (without analyzing the statistical properties of the obtained estimators) to identify in a first step, where is located the upper frontier by using a nonparametric method, then in a second step, to adjust the parametric model to the obtained nonparametric frontier. From an economic point of view, the first step can be viewed as a kind of “filtering” for eliminating from the sample clearly inefficient units which certainly do not provide substantial information to analyse how to transform efficiently inputs into output.

We can use in this first step filtering the flexible nonparametric Free Disposal Hull (FDH) estimator (Deprins, Simar and Tulkens, 1984),

$$\widehat{\varphi}_n(x) = \inf\{y \in \mathbb{R}_+ | \widehat{F}_n(y|x) = 1\} = \max_{i|X_i \leq x} Y_i,$$

where $\widehat{F}_n(y|x) = \widehat{F}_n(x, y) / \widehat{F}_{X,n}(x)$, with $\widehat{F}_n(x, y) = (1/n) \sum_{i=1}^n 1(X_i \leq x, Y_i \leq y)$ and $\widehat{F}_{X,n}(x) = \widehat{F}_n(x, \infty)$. The asymptotic of $\widehat{\varphi}_n(x)$ was first derived by Korostolev, Simar and

Tsybakov (1995) for the consistency and by Park, Simar and Weiner (2000) for the asymptotic sampling distribution. The drawback of the FDH estimator lies in the fact that it is, by construction, very sensitive to extreme values. CFS have proposed alternatives which are shown to be more robust than the FDH estimator. They are based on the concept of partial order- m expected frontiers defined as $\varphi_m(x) = E(\max(Y^1, \dots, Y^m) \mid X \leq x)$, where for an integer $m \geq 1$, $(Y^1, \dots, Y^m)$ are m independent identically distributed random variables generated by the distribution of Y given $X \leq x$. It turns out that

$$\varphi_m(x) = \int_0^{\varphi(x)} (1 - [F(y|x)]^m) dy,$$

which can be nonparametrically estimated by

$$\hat{\varphi}_{m,n}(x) = \int_0^{\hat{\varphi}_n(x)} (1 - [\hat{F}_n(y|x)]^m) dy.$$

The order m can be viewed as a trimming parameter which tunes the robustness of the estimator since, as pointed in CFS, when $m \rightarrow \infty$, $\varphi_m(x)$ converges to the full frontier $\varphi(x)$ and $\hat{\varphi}_{m,n}(x)$ converges to the FDH estimator $\hat{\varphi}_n(x)$.

Florens and Simar (2005) provide the asymptotic properties of the resulting estimators of θ when these two nonparametric estimators of the frontier are used in the first “filtering” stage. In this paper we propose an alternative method using the nonparametric quantile-type frontiers introduced in Aragon, Daouia and Thomas-Agnan (2002). The interest is that these quantile-type estimators are more robust to extremes than the CFS order- m estimators as established in Daouia and Ruiz-Gazin (2003).

Similarly to the order- m partial frontier, the order- α partial frontier function increases w.r.t. the continuous order $\alpha \in [0, 1]$ and converges to the full frontier $\varphi(x)$ as $\alpha \nearrow 1$. It is defined, for a given level x , by the conditional α -quantile of the distribution of Y given $X \leq x$,

$$q_\alpha(x) := F^{-1}(\alpha|x) = \inf\{y \in \mathbb{R}_+ \mid F(y|x) \geq \alpha\}.$$

A simple estimator of $q_\alpha(x)$, which increases and converges to the FDH $\hat{\varphi}_n(x)$ as $\alpha \nearrow 1$, is easily derived by inverting the empirical version $\hat{F}_n(y|x)$ of $F(y|x)$,

$$\hat{q}_{\alpha,n}(x) := \hat{F}_n^{-1}(\alpha|x) = \inf\{y \in \mathbb{R}_+ \mid \hat{F}_n(y|x) \geq \alpha\}.$$

This estimator is very fast to compute, very easy to interpret and satisfies very similar statistical properties to those of the CFS estimator. In summary, it converges at the rate $\sqrt{n}$, is asymptotically unbiased and normally distributed. Moreover, when the order α is considered as a function of n such that $n^{(p+2)/(p+1)}(1 - \alpha(n)) \rightarrow 0$ as $n \rightarrow \infty$, $\hat{q}_{\alpha(n),n}(x)$ estimates

the true frontier function $\varphi(x)$ and shares the same asymptotic distribution (Weibull) of the FDH estimator $\widehat{\varphi}_n(x)$, as described in Park *et al.* (2000).

The paper is organized as follows. Section 2 is our basic contribution: it extends previous results of convergence of $\widehat{q}_{\alpha,n}(x)$, by establishing functional convergence as $\sqrt{n}(\widehat{q}_{\alpha,n} - q_\alpha)$ being a process indexed by x as well as its uniform complete convergence. Section 3 proposes then, in the spirit of Florens and Simar (2005), the parametric approximations of the quantile frontier and develops the asymptotic properties of the resulting estimators, for linear parametric model as well as for general parametric models. It also provides new results for estimating the parametric model for the full frontier itself, by using the more robust order- m or order- α partial frontiers in the first step. Indeed it establishes the complete convergence of the estimators of the parametric models for the full frontier, by choosing m and α as appropriate functions of n . Section 4 illustrates and compare the different approaches with simulated and real data sets. Section 5 concludes.

2 Convergence Theorems of Quantile-type Frontiers

Throughout this section the order α is arbitrarily fixed in $]0, 1[$ and $K \subset \mathbb{R}_+^p$ is an arbitrarily fixed set such that $\inf_{x \in K} F_X(x) > 0$. The support of Y is assumed to have a finite upper boundary $\nu > 0$. The domain $\mathbb{D} \subset \mathbb{R}^{p+1}$ is used to denote any fixed compact subset which contains both $K \times [-\nu, \nu]$ and the support Ψ of (X, Y) .

2.1 Functional Convergence

Let x be a fixed value such that $F_X(x) > 0$. Assume that the conditional distribution function $F(\cdot|x)$ is differentiable at $q_\alpha(x)$ with derivative $f(q_\alpha(x)|x) > 0$. Aragon *et al.* (2005) have proved

$$\widehat{q}_{\alpha,n}(x) - q_\alpha(x) = \frac{\alpha \widehat{F}_{X,n}(x) - \widehat{F}_n(x, q_\alpha(x))}{f(q_\alpha(x)|x) F_X(x)} + R_n^\alpha(x) \quad (2.1)$$

where the remainder term $R_n^\alpha(x)$ becomes negligible as $n \rightarrow \infty$. More precisely, they have shown $\sqrt{n} R_n^\alpha(x) = o_p(1)$ as $n \rightarrow \infty$. This result has been improved in Daouia (2005) who extends (2.1) to a Bahadur-type asymptotic representation, namely $R_n^\alpha(x) = O(n^{-3/4}(\log n)^{1/2}(\log \log n)^{1/4})$ as $n \rightarrow \infty$, with probability 1. He also proves the functional convergence of $\sqrt{n}(\widehat{q}_{\alpha,n}(x) - q_\alpha(x))$ as a process indexed by the order α in the space $L^\infty([0, 1])$ of bounded functions on $]0, 1[$. However, Daouia (2005) raises the question of finding the asymptotic limit of $\sqrt{n}(\widehat{q}_{\alpha,n}(x) - q_\alpha(x))$ as a process indexed by $x \in \mathbb{R}^p$. In this section we address this issue in our basic Theorem 2.1 below.

To explain the rationale behind this theorem, define the domain $\mathbb{G}$ of distribution functions $G(\cdot, \cdot)$ on $\mathbb{R}_+^p \times \mathbb{R}_+$ whose supports are also contained in $\mathbb{D}$ and such that

$$G(x, \infty) > 0 \quad \forall x \in K, \quad \inf\{y \geq 0 \mid G(\infty, y) = 1\} \leq \nu.$$

For any $G \in \mathbb{G}$ and any $x \in K$, let $G^{-1}(\alpha|x) := \inf\{y \geq 0 \mid G(y|x) \geq \alpha\}$ denotes the α th quantile of the conditional distribution function $G(\cdot|x) = G(x, \cdot)/G(x, \infty) \equiv G(x, \cdot)/G(x, \nu)$. Then, for any $G \in \mathbb{G}$ and any $x \in K$, we have

$$0 \leq G^{-1}(\alpha|x) \leq G^{-1}(1|x) \leq \inf\{y \geq 0 \mid G(\infty, y) = 1\} \leq \nu.$$

Hence the conditional quantile transformation $G^{-1}(\alpha|\cdot) : x \mapsto G^{-1}(\alpha|x)$ as a map $K \rightarrow [0, \nu]$ is bounded on K and so, we write $G^{-1}(\alpha|\cdot) \in L^\infty(K)$. Now consider $\phi : G \mapsto G^{-1}(\alpha|\cdot)$ as a map $\mathbb{G} \subset L^\infty(\mathbb{D}) \rightarrow L^\infty(K)$.

Lemma 2.1. *Assume that $F(\cdot|x)$ is differentiable for any $x \in K$, with derivative $f(\cdot|x)$ such that $\sup_{(x,y) \in K \times \mathbb{R}} f(y|x) < \infty$, $\inf_{x \in K} f(q_\alpha(x)|x) > 0$ and the differentiability of $F(\cdot|x)$ at $q_\alpha(x)$ is uniform in $x \in K$, i.e.,*

$$\sup_{x \in K} \left| \frac{F(q_\alpha(x) + \xi|x) - F(q_\alpha(x)|x)}{\xi} - f(q_\alpha(x)|x) \right| \rightarrow 0 \quad \text{as } \xi \rightarrow 0.$$

Then ϕ is Hadamard-differentiable at $F \in \mathbb{G}$ tangentially to the set $C(\mathbb{D})$ of continuous functions on $\mathbb{D}$. The derivative is the map $\phi'_F : h \mapsto \phi'_F(h)$ as a map $C(\mathbb{D}) \rightarrow L^\infty(K)$, where

$$\phi'_F(h) : x \mapsto \frac{\alpha h(x, \nu) - h(x, q_\alpha(x))}{f(q_\alpha(x)|x)F_X(x)}. \quad (2.2)$$

Proof. Let $h_t \rightarrow h$ uniformly in $L^\infty(\mathbb{D})$, where $h \in C(\mathbb{D})$ and $F + th_t \in \mathbb{G}$ for all small $t > 0$. Write $q_{\alpha t}$ for $\phi(F + th_t)$. Following the definition of the Hadamard differentiability (see van der Vaart (1998), p.296), we shall show that $(q_{\alpha t}(x) - q_\alpha(x))/t$ converges to $(\alpha h(x, \nu) - h(x, q_\alpha(x)))/f(q_\alpha(x)|x)F_X(x)$ as $t \searrow 0$, uniformly in $x \in K$. By the definition of ϕ , we have for every $x \in K$ and every $\varepsilon > 0$,

$$(F + th_t)(q_{\alpha t}(x) - \varepsilon|x) < \alpha \leq (F + th_t)(q_{\alpha t}(x)|x).$$

We choose $\varepsilon = \varepsilon_t = o(t)$ as $t \searrow 0$. Because $G(x, \infty) = G(x, \nu) > 0$ for any $G \in \mathbb{G}$ and any $x \in K$, we have $(F + th_t)(x, q_{\alpha t}(x) - \varepsilon_t) < \alpha(F + th_t)(x, \nu) \leq (F + th_t)(x, q_{\alpha t}(x))$. Then

$$\begin{aligned} F(q_{\alpha t}(x) - \varepsilon_t|x) + \frac{th_t(x, q_{\alpha t}(x) - \varepsilon_t) - \alpha th_t(x, \nu)}{F_X(x)} &< \alpha \\ &\leq F(q_{\alpha t}(x)|x) + \frac{th_t(x, q_{\alpha t}(x)) - \alpha th_t(x, \nu)}{F_X(x)}. \end{aligned} \quad (2.3)$$

Since $\varepsilon_t \searrow 0$ as $t \searrow 0$, $\varepsilon_t < \nu$ for all t small enough, and so $q_{\alpha t}(x) - \varepsilon_t \in [-\nu, \nu]$ for every $x \in K$ and all t sufficiently small. Therefore $\sup_{x \in K} |h_t(x, q_{\alpha t}(x) - \varepsilon_t) - h(x, q_{\alpha t}(x) - \varepsilon_t)| \leq \sup_{(x,y) \in K \times [-\nu, \nu]} |h_t(x, y) - h(x, y)|$, for all small t . Since h_t converges uniformly to h on $K \times [-\nu, \nu] \subset \mathbb{D}$, we obtain $th_t(x, q_{\alpha t}(x) - \varepsilon_t) = th(x, q_{\alpha t}(x) - \varepsilon_t) + o(t)$ uniformly in $x \in K$. Likewise, $th_t(x, q_{\alpha t}(x)) = th(x, q_{\alpha t}(x)) + o(t)$ and $-\alpha th_t(x, \nu) = -\alpha th(x, \nu) - \alpha o(t)$, where the $o(t)$ terms are uniform in $x \in K$. By using the fact that $\sup_{(x,y) \in K \times \mathbb{R}} f(y|x) < \infty$, we also have $F(q_{\alpha t}(x) - \varepsilon_t|x) = F(q_{\alpha t}(x)|x) + O(\varepsilon_t)$, uniformly in $x \in K$. Hence

$$\begin{aligned} F(q_{\alpha t}(x)|x) + O(\varepsilon_t) + \frac{th(x, q_{\alpha t}(x) - \varepsilon_t) + o(t) - \alpha th(x, \nu) - \alpha o(t)}{F_X(x)} &< \alpha \\ &\leq F(q_{\alpha t}(x)|x) + \frac{th(x, q_{\alpha t}(x)) + o(t) - \alpha th(x, \nu) - \alpha o(t)}{F_X(x)}. \end{aligned}$$

By replacing α in the middle of inequalities by $F(q_{\alpha}(x)|x)$, we get

$$\begin{aligned} -\frac{th(x, q_{\alpha t}(x)) + o(t) - \alpha th(x, \nu) - \alpha o(t)}{F_X(x)} &\leq F(q_{\alpha t}(x)|x) - F(q_{\alpha}(x)|x) \\ &< -O(\varepsilon_t) - \frac{th(x, q_{\alpha t}(x) - \varepsilon_t) + o(t) - \alpha th(x, \nu) - \alpha o(t)}{F_X(x)}. \end{aligned} \quad (2.4)$$

Let us show that $q_{\alpha t}(x) \rightarrow q_{\alpha}(x)$ uniformly in $x \in K$, as $t \searrow 0$. In view of the uniform differentiability, in $x \in K$, of $F(\cdot|x)$ at $q_{\alpha}(x)$, there exists a $\xi_0 > 0$ such that for all $|\xi| < \xi_0$ we have $\sup_{x \in K} \left| \frac{F(q_{\alpha}(x) + \xi|x) - \alpha}{\xi} - f(q_{\alpha}(x)|x) \right| < \inf_{x \in K} f(q_{\alpha}(x)|x)/2$. Let $0 < \delta < \xi_0$. Then $\inf_{x \in K} f(q_{\alpha}(x)|x)/2 < \frac{F(q_{\alpha}(x) + \delta|x) - \alpha}{\delta}$ and $\inf_{x \in K} f(q_{\alpha}(x)|x)/2 < \frac{\alpha - F(q_{\alpha}(x) - \delta|x)}{\delta}$, for all $x \in K$. Whence $\sup_{x \in K} F(q_{\alpha}(x) - \delta|x) < \alpha < \inf_{x \in K} F(q_{\alpha}(x) + \delta|x)$. Let $\Delta_{\delta} = [\inf_{x \in K} F(q_{\alpha}(x) + \delta|x) - \alpha] \wedge [\alpha - \sup_{x \in K} F(q_{\alpha}(x) - \delta|x)]$. We have

$$\begin{aligned} \sup_{x \in K} \frac{|h_t(x, q_{\alpha t}(x)) - \alpha h_t(x, \nu)|}{F_X(x)} &\leq \left[\sup_{x \in K} |h_t(x, q_{\alpha t}(x)) - h(x, q_{\alpha t}(x))| + \sup_{x \in K} |h(x, q_{\alpha t}(x))| \right. \\ &\quad \left. + \sup_{x \in K} |h_t(x, \nu) - h(x, \nu)| + \sup_{x \in K} |h(x, \nu)| \right] / \inf_{x \in K} F_X(x). \end{aligned}$$

The fact that $q_{\alpha t}(x) \in [0, \nu]$, for every $x \in K$, implies that $\sup_{x \in K} |h(x, q_{\alpha t}(x))|$ is majored by $\sup_{(x,y) \in K \times [0, \nu]} |h(x, y)|$ which is finite since h is bounded on $K \times [-\nu, \nu] \subset \mathbb{D}$. Likewise $\sup_{x \in K} |h(x, \nu)|$ is finite. Hence, the uniform convergence of h_t to h yields

$$\sup_{x \in K} \frac{|th_t(x, q_{\alpha t}(x)) - \alpha th_t(x, \nu)|}{F_X(x)} \longrightarrow 0 \quad \text{as } t \searrow 0. \quad (2.5)$$

Since $q_{\alpha t}(x) - \varepsilon_t \in [-\nu, \nu]$ for every $x \in K$ and all small t , $\sup_{x \in K} |h(x, q_{\alpha t}(x) - \varepsilon_t)|$ is majored by $\sup_{(x,y) \in K \times [-\nu, \nu]} |h(x, y)|$ which is finite since $h \in L^{\infty}(\mathbb{D})$. Thus, we obtain in the same way as for (2.5) that

$$\sup_{x \in K} \frac{|th_t(x, q_{\alpha t}(x) - \varepsilon_t) - \alpha th_t(x, \nu)|}{F_X(x)} \longrightarrow 0 \quad \text{as } t \searrow 0. \quad (2.6)$$

Regarding (2.5) and (2.6), there exists $t_\delta > 0$ such that for all $t < t_\delta$ and for every $x \in K$,

$$\begin{aligned} F(q_\alpha(x) - \delta|x) &\leq \alpha - \Delta_\delta < \\ \alpha - \frac{th_t(x, q_{\alpha t}(x)) - \alpha th_t(x, \nu)}{F_X(x)} &\quad \text{and} \quad \alpha - \frac{th_t(x, q_{\alpha t}(x) - \varepsilon_t) - \alpha th_t(x, \nu)}{F_X(x)} \\ &< \alpha + \Delta_\delta \leq F(q_\alpha(x) + \delta|x). \end{aligned}$$

By using the inequalities (2.3), we get $F(q_\alpha(x) - \delta|x) < F(q_{\alpha t}(x)|x)$ and $F(q_{\alpha t}(x) - \varepsilon_t|x) < F(q_\alpha(x) + \delta|x)$. It follows by the monotonicity of $F(\cdot|x)$ that $-\delta < q_{\alpha t}(x) - q_\alpha(x) < \delta + \varepsilon_t$ uniformly in $x \in K$, and thus we conclude that

$$\sup_{x \in K} |q_{\alpha t}(x) - q_\alpha(x)| \longrightarrow 0 \quad \text{as } t \searrow 0. \quad (2.7)$$

It can be then easily seen from the uniform differentiability of $F(\cdot|x)$ at $q_\alpha(x)$ that

$$\sup_{x \in K} \left| \frac{F(q_{\alpha t}(x)|x) - F(q_\alpha(x)|x)}{q_{\alpha t}(x) - q_\alpha(x)} - f(q_\alpha(x)|x) \right| \longrightarrow 0 \quad \text{as } t \searrow 0. \quad (2.8)$$

This yields $F(q_{\alpha t}(x)|x) - F(q_\alpha(x)|x) = (q_{\alpha t}(x) - q_\alpha(x))f(q_\alpha(x)|x) + (q_{\alpha t}(x) - q_\alpha(x))o(1)$, uniformly in $x \in K$. Therefore

$$\begin{aligned} & - \frac{(q_{\alpha t}(x) - q_\alpha(x))}{tf(q_\alpha(x)|x)}o(1) - \frac{th(x, q_{\alpha t}(x)) + o(t) - \alpha th(x, \nu) - \alpha o(t)}{tf(q_\alpha(x)|x)F_X(x)} \leq \frac{(q_{\alpha t}(x) - q_\alpha(x))}{t} \\ & < - \frac{(q_{\alpha t}(x) - q_\alpha(x))}{tf(q_\alpha(x)|x)}o(1) - \frac{O(\varepsilon_t)}{tf(q_\alpha(x)|x)} - \frac{th(x, q_{\alpha t}(x) - \varepsilon_t) + o(t) - \alpha th(x, \nu) - \alpha o(t)}{tf(q_\alpha(x)|x)F_X(x)}. \end{aligned}$$

Thus we arrive at

$$\begin{aligned} \sup_{x \in K} \left| \frac{q_{\alpha t}(x) - q_\alpha(x)}{t} - \frac{\alpha h(x, \nu) - h(x, q_\alpha(x))}{f(q_\alpha(x)|x)F_X(x)} \right| &\leq \frac{1}{\inf_{x \in K} f(q_\alpha(x)|x)F_X(x)} \\ &\times \left(\sup_{x \in K} |h(x, q_{\alpha t}(x) - \varepsilon_t) - h(x, q_\alpha(x))| + \sup_{x \in K} |h(x, q_{\alpha t}(x)) - h(x, q_\alpha(x))| + \frac{o(t)}{t} \right) \\ &+ \frac{1}{\inf_{x \in K} f(q_\alpha(x)|x)} \left(\sup_{x \in K} \frac{|q_{\alpha t}(x) - q_\alpha(x)|}{t} o(1) + \frac{O(\varepsilon_t)}{t} \right). \end{aligned} \quad (2.9)$$

Because h is uniformly continuous on the compact $\mathbb{D}$ and $\varepsilon_t \rightarrow 0$, it follows from (2.7) that

$$\sup_{x \in K} |h(x, q_{\alpha t}(x)) - h(x, q_\alpha(x))| \rightarrow 0, \quad \sup_{x \in K} |h(x, q_{\alpha t}(x) - \varepsilon_t) - h(x, q_\alpha(x))| \rightarrow 0 \quad (2.10)$$

as $t \searrow 0$. Now, let us show that $\sup_{x \in K} |q_{\alpha t}(x) - q_\alpha(x)|/t$ is bounded as $t \searrow 0$. It can be easily seen from (2.4) that

$$\begin{aligned} \inf_{x \in K} \left| \frac{F(q_{\alpha t}(x)|x) - F(q_\alpha(x)|x)}{q_{\alpha t}(x) - q_\alpha(x)} \right| &\times \sup_{x \in K} \frac{|q_{\alpha t}(x) - q_\alpha(x)|}{t} \\ &\leq \frac{O(\varepsilon_t)}{t} + \frac{1}{\inf_{x \in K} F_X(x)} \left[\sup_{x \in K} |h(x, q_{\alpha t}(x)) - h(x, q_\alpha(x))| + \frac{o(t)}{t} + \sup_{x \in K} |h(x, \nu)| \right. \\ &\quad \left. + \sup_{x \in K} |h(x, q_\alpha(x))| + \sup_{x \in K} |h(x, q_{\alpha t}(x) - \varepsilon_t) - h(x, q_\alpha(x))| \right]. \end{aligned} \quad (2.11)$$

Since $q_\alpha(x) \in [0, \nu]$ for every $x \in K$, $\sup_{x \in K} |h(x, q_\alpha(x))|$ is majored by $\sup_{(x,y) \in \mathbb{D}} |h(x, y)|$ which is finite. Likewise $\sup_{x \in K} |h(x, \nu)|$ is finite. Furthermore, we have by (2.8),

$$\inf_{x \in K} \left| \frac{F(q_{\alpha t}(x)|x) - F(q_\alpha(x)|x)}{q_{\alpha t}(x) - q_\alpha(x)} \right| \longrightarrow \inf_{x \in K} f(q_\alpha(x)|x) > 0 \quad \text{as } t \searrow 0.$$

Hence, by using these results in conjunction with (2.10) and (2.11), we get

$$q_{\alpha t}(x) - q_\alpha(x) = O(t), \text{ uniformly in } x \in K. \quad (2.12)$$

Finally, by combining (2.9), (2.10) and (2.12), we obtain the desired uniform convergence in $x \in K$ of $(q_{\alpha t}(x) - q_\alpha(x)) / t$ to $(\alpha h(x, \nu) - h(x, q_\alpha(x))) / f(q_\alpha(x)|x) F_X(x)$, as $t \searrow 0$. $\square$

The empirical process $\sqrt{n}(\widehat{F}_n - F)$ converges in distribution in $L^\infty(\overline{\mathbb{R}}^{p+1})$ to $\mathbb{F}$, a $p+1$ dimensional F -Brownian bridge (see , e.g., van der Vaart and Wellner, 1996, p. 82). $\mathbb{F}$ is a Gaussian process with zero mean and covariance function $E(\mathbb{F}(t_1)\mathbb{F}(t_2)) = F(t_1 \wedge t_2) - F(t_1)F(t_2)$, for all $t_1, t_2 \in \overline{\mathbb{R}}^{p+1}$. By making use of this functional convergence and applying the functional delta method in conjunction with Lemma 2.1, we can easily prove the main result of this section.

Theorem 2.1. *Under the conditions of Lemma 2.1, if furthermore F is continuous, then the conditional empirical quantile process $\sqrt{n}(\widehat{q}_{\alpha,n} - q_\alpha)$, indexed by $x \in K$, converges in distribution in $L^\infty(K)$ to $\phi'_F(\mathbb{F})$, a centered Gaussian process with covariance function defined, for any $x^1, x^2 \in K$, by*

$$\begin{aligned} \Gamma_\alpha(x^1, x^2) = & \left\{ [f(q_\alpha(x^1)|x^1)F_X(x^1)] [f(q_\alpha(x^2)|x^2)F_X(x^2)] \right\}^{-1} \times \\ & \left\{ \alpha^2 F_X(x^1 \wedge x^2) - \alpha F(x^1 \wedge x^2, q_\alpha(x^2)) - \alpha F(x^1 \wedge x^2, q_\alpha(x^1)) + F(x^1 \wedge x^2, q_\alpha(x^1) \wedge q_\alpha(x^2)) \right\}. \end{aligned} \quad (2.13)$$

Proof. Since $\sqrt{n}(\widehat{F}_n - F)$ converges in distribution to the F -Brownian bridge $\mathbb{F}$ in $L^\infty(\overline{\mathbb{R}}^{p+1})$ and since $L^\infty(\overline{\mathbb{R}}^{p+1}) \subset L^\infty(\mathbb{D})$, we obtain in view of Lemma 18.13 of van der Vaart (1998, p.261) that $\sqrt{n}(\widehat{F}_n - F)$ also converges in distribution to $\mathbb{F}$ in $L^\infty(\mathbb{D})$. Because $F \in C(\mathbb{D})$, the sample paths of the F -Brownian bridge are continuous on $\mathbb{D}$.

On the other hand, by Lemma 2.1, the map $\phi : \mathbb{G} \subset L^\infty(\mathbb{D}) \longrightarrow L^\infty(K)$ is Hadamard-differentiable at F tangentially to $C(\mathbb{D})$. Thus, the functional delta method (see Theorem 20.8 of van der Vaart, 1998, p.297) can be applied. It implies that $\sqrt{n}(\phi(\widehat{F}_n) - \phi(F)) = \sqrt{n}(\widehat{q}_{\alpha,n} - q_\alpha)$ converges in distribution to $\phi'_F(\mathbb{F})$ in $L^\infty(K)$. Since the process $\mathbb{F}$ is Gaussian and the operator ϕ'_F is linear, the limiting process $\phi'_F(\mathbb{F})$ is Gaussian with zero mean and covariance function given by

$$\Gamma_\alpha(x^1, x^2) = E \left[\frac{\alpha \mathbb{F}(x^1, \nu) - \mathbb{F}(x^1, q_\alpha(x^1))}{f(q_\alpha(x^1)|x^1)F_X(x^1)} \times \frac{\alpha \mathbb{F}(x^2, \nu) - \mathbb{F}(x^2, q_\alpha(x^2))}{f(q_\alpha(x^2)|x^2)F_X(x^2)} \right]$$

for every $x^1, x^2 \in K$. $\square$

Under the conditions of Lemma 2.1, the remainder term $R_n^\alpha(x)$ of (2.1), considered as a process indexed by $x \in K$, can be expressed in the following way

$$\sqrt{n}R_n^\alpha = \sqrt{n}(\hat{q}_{\alpha,n} - q_\alpha) - \phi'_F \left(\sqrt{n}(\hat{F}_n - F) \right) \quad (2.14)$$

where $\phi'_F(h)$ is described in (2.2) for any $h \in L^\infty(\overline{\mathbb{R}}^{p+1})$. Because the linear operator ϕ'_F is continuous from $L^\infty(\overline{\mathbb{R}}^{p+1})$ into $L^\infty(K)$, it follows from the continuous mapping theorem that the process $\phi'_F \left(\sqrt{n}(\hat{F}_n - F) \right)$, indexed by $x \in K$, converges in distribution to the tight Gaussian process $\phi'_F(\mathbb{F})$ in $L^\infty(K)$.

Therefore, according to Theorem 1.5.4 of van der Vaart and Wellner (1996, p.35), both processes $\phi'_F \left(\sqrt{n}(\hat{F}_n - F) \right)$ and $\sqrt{n}(\hat{q}_{\alpha,n} - q_\alpha)$ are asymptotically tight under the conditions of Theorem 2.1. The difference process $\sqrt{n}R_n^\alpha$, given by (2.14), is then asymptotically tight under the same conditions (see, *e.g.*, Theorem 1.5.6 of van der Vaart and Wellner, p.36). Moreover, its marginals $(\sqrt{n}R_n^\alpha(x^1), \dots, \sqrt{n}R_n^\alpha(x^r))$ converge in probability to $0 \in \mathbb{R}^r$, for every $x^1, \dots, x^r \in K$. Thus the nuisance process $\sqrt{n}R_n^\alpha$, indexed by $x \in K$, converges in distribution to the process zero in $L^\infty(K)$ in view of the second part of Theorem 1.5.4 of van der Vaart and Wellner (1996). Since the map $g \mapsto \sup_{x \in K} |g(x)|$ from $L^\infty(K)$ into $\mathbb{R}$ is continuous with respect to the supremum norm, the continuous-mapping theorem immediately implies the following result.

Corollary 2.1. *Under the conditions of Theorem 2.1, we have*

$$\sup_{x \in K} \sqrt{n}|R_n^\alpha(x)| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

This corollary will also serve our purposes in Section 3 to prove the asymptotic normality of the resulting “semi-parametric” estimators.

2.2 Uniform Complete Convergence

First, by using the same technique of proof of Lemma 1 of Csörgő and Révész (1978, p. 888), it is easy to prove the following preliminary result.

Lemma 2.2. *Let $\kappa(x) := \sup\{y \geq 0 | F(y|x) = 0\}$ and assume that $F(\cdot|x)$ is twice differentiable on $]\kappa(x), \varphi(x)[$ for any $x \in K$, with first derivative $f(\cdot|x) > 0$ and second derivative $f'(\cdot|x)$. Assume furthermore that for some $\gamma > 0$*

$$\sup_{x \in K} \sup_{\kappa(x) < y < \varphi(x)} F(y|x)(1 - F(y|x)) \frac{|f'(y|x)|}{f^2(y|x)} \leq \gamma. \quad (2.15)$$

Then, for any $x \in K$ and any $\alpha_1, \alpha_2 \in]0, 1[$,

$$\frac{f(q_{\alpha_1}(x)|x)}{f(q_{\alpha_2}(x)|x)} \leq \left\{ \frac{\alpha_1 \vee \alpha_2}{\alpha_1 \wedge \alpha_2} \frac{1 - (\alpha_1 \wedge \alpha_2)}{1 - (\alpha_1 \vee \alpha_2)} \right\}^\gamma.$$

The analogue of this lemma in the non-conditional quantile literature was used, *e.g.*, by Csörgő and Révész (1978) to establish a Brownian bridge approximation of the sample quantile process indexed by its order, and recently by Cheng and Parzen (1997) to investigate the asymptotic properties of their unified smoothed estimator of the quantile function. Further examples on the use of the non-conditional version of Condition (2.15) and its close link with the so-called score function (see, *e.g.*, Hájek and Sidak, 1967, p. 19) can be found in Csörgő (1983, Chapter 1). See also Parzen (1979, 1980) for a discussion of tail monotonicity assumptions of extreme value theory as related to this condition.

In our setup, Lemma 2.2 allows to establish the uniform complete convergence of the normed conditional quantile process $\{f(q_\alpha(x)|x)(\widehat{q}_{\alpha,n}(x) - q_\alpha(x)); x \in K\}$. We recall that a sequence of random variables $\{Z_n\}$ converges completely to a random variable Z ($Z_n \xrightarrow{co} Z$ as $n \rightarrow \infty$) if $\sum_{n=1}^\infty \text{Prob}(|Z_n - Z| > \varepsilon) < \infty$ for every $\varepsilon > 0$. In particular, the complete convergence implies convergence with probability 1 by Borell-Cantelli Lemma. See Hsu and Robbins (1947) for further discussions.

Theorem 2.2. *If the conditions of Lemma 2.2 hold, then*

$$\sup_{x \in K} f(q_\alpha(x)|x)|\widehat{q}_{\alpha,n}(x) - q_\alpha(x)| \xrightarrow{co} 0 \quad \text{as } n \rightarrow \infty.$$

Proof. For all $x \in K$, we have by the smoothness of $F(\cdot|x)$,

$$\widehat{q}_{\alpha,n}(x) - q_\alpha(x) = F^{-1}(F(\widehat{q}_{\alpha,n}(x)|x)|x) - F^{-1}(\alpha|x) = \frac{F(\widehat{q}_{\alpha,n}(x)|x) - \alpha}{f(q_{\alpha_{x,n}}(x)|x)}$$

where $F(\widehat{q}_{\alpha,n}(x)|x) \wedge \alpha < \alpha_{x,n} < F(\widehat{q}_{\alpha,n}(x)|x) \vee \alpha$. Hence

$$\sup_{x \in K} f(q_\alpha(x)|x)|\widehat{q}_{\alpha,n}(x) - q_\alpha(x)| \leq \sup_{x \in K} |F(\widehat{q}_{\alpha,n}(x)|x) - \alpha| \sup_{x \in K} \frac{f(q_\alpha(x)|x)}{f(q_{\alpha_{x,n}}(x)|x)}. \quad (2.16)$$

Let $N_x = n\widehat{F}_{X,n}(x)$. Since $\widehat{F}_n(\widehat{q}_{\alpha,n}(x)|x) = \frac{k}{N_x}$ for $\frac{k-1}{N_x} < \alpha \leq \frac{k}{N_x}$ and $k = 1, \dots, N_x$ (see Aragon et al., 2005, Equation (5)), we have $\widehat{F}_n(\widehat{q}_{\alpha,n}(x)|x) - \alpha = \sum_{k=1}^{N_x} (\frac{k}{N_x} - \alpha) \mathbb{I}_{\frac{k-1}{N_x}, \frac{k}{N_x}}(\alpha)$. This yields $0 \leq \widehat{F}_n(\widehat{q}_{\alpha,n}(x)|x) - \alpha \leq 1/N_x \leq \{n \inf_{x \in K} \widehat{F}_{X,n}(x)\}^{-1}$. Since $\inf_{x \in K} F_X(x) > 0$, it can be easily seen from Daouia and Simar (2003, see the last part of the proof of Lemma 3.2 from Equation (3)) that $n^{-1}/\inf_{x \in K} \widehat{F}_{X,n}(x) \xrightarrow{co} 0$ as $n \rightarrow \infty$, and so

$$\sup_{x \in K} |\widehat{F}_n(\widehat{q}_{\alpha,n}(x)|x) - \alpha| \xrightarrow{co} 0 \quad \text{as } n \rightarrow \infty. \quad (2.17)$$

We also can easily show that

$$|\widehat{F}_n(\widehat{q}_{\alpha,n}(x)|x) - F(\widehat{q}_{\alpha,n}(x)|x)| \leq \frac{\|\widehat{F}_n - F\|_\infty + \|\widehat{F}_{X,n} - F_X\|_\infty}{\inf_{x \in K} \widehat{F}_{X,n}(x)}$$

where $\|\cdot\|_\infty$ denotes the sup-norm. As established in Daouia and Simar (2003, see the part of the proof of Lemma 3.2 between Equations (2)-(3)), the term on the right-hand side of the last inequality converges completely to 0 as $n \rightarrow \infty$, and therefore

$$\sup_{x \in K} |\widehat{F}_n(\widehat{q}_{\alpha,n}(x)|x) - F(\widehat{q}_{\alpha,n}(x)|x)| \xrightarrow{co.} 0 \quad \text{as } n \rightarrow \infty. \quad (2.18)$$

By combining (2.17) and (2.18), we get

$$\sup_{x \in K} |F(\widehat{q}_{\alpha,n}(x)|x) - \alpha| \xrightarrow{co.} 0 \quad \text{as } n \rightarrow \infty. \quad (2.19)$$

On the other hand, we have by Lemma 2.2 and the definition of $\alpha_{x,n}$, for any $x \in K$

$$\begin{aligned} \frac{f(q_{\alpha_{x,n}}(x)|x)}{f(q_\alpha(x)|x)} \quad \text{and} \quad & \frac{f(q_\alpha(x)|x)}{f(q_{\alpha_{x,n}}(x)|x)} \leq \sup_{x \in K} \left\{ \frac{\alpha \vee \alpha_{x,n}}{\alpha \wedge \alpha_{x,n}} \frac{1 - (\alpha \wedge \alpha_{x,n})}{1 - (\alpha \vee \alpha_{x,n})} \right\}^\gamma \\ & \leq \sup_{x \in K} \left\{ \frac{F(\widehat{q}_{\alpha,n}(x)|x) \vee \alpha}{F(\widehat{q}_{\alpha,n}(x)|x) \wedge \alpha} \frac{1 - (F(\widehat{q}_{\alpha,n}(x)|x) \wedge \alpha)}{1 - (F(\widehat{q}_{\alpha,n}(x)|x) \vee \alpha)} \right\}^\gamma \\ & \leq \sup_{x \in K} \left\{ \frac{F(\widehat{q}_{\alpha,n}(x)|x)}{\alpha} \frac{1 - \alpha}{1 - F(\widehat{q}_{\alpha,n}(x)|x)} \right\}^\gamma \\ & \vee \sup_{x \in K} \left\{ \frac{\alpha}{F(\widehat{q}_{\alpha,n}(x)|x)} \frac{1 - F(\widehat{q}_{\alpha,n}(x)|x)}{1 - \alpha} \right\}^\gamma \\ & \leq \left\{ \frac{\sup_{x \in K} F(\widehat{q}_{\alpha,n}(x)|x)}{\alpha} \frac{1 - \alpha}{\inf_{x \in K} (1 - F(\widehat{q}_{\alpha,n}(x)|x))} \right\}^\gamma \\ & \vee \left\{ \frac{\alpha}{\inf_{x \in K} F(\widehat{q}_{\alpha,n}(x)|x)} \frac{\sup_{x \in K} (1 - F(\widehat{q}_{\alpha,n}(x)|x))}{1 - \alpha} \right\}^\gamma. \end{aligned} \quad (2.20)$$

Since $|\sup_{x \in K} F(\widehat{q}_{\alpha,n}(x)|x) - \alpha|$, $|\inf_{x \in K} F(\widehat{q}_{\alpha,n}(x)|x) - \alpha|$, $|\sup_{x \in K} (1 - F(\widehat{q}_{\alpha,n}(x)|x)) - (1 - \alpha)|$ and $|\inf_{x \in K} (1 - F(\widehat{q}_{\alpha,n}(x)|x)) - (1 - \alpha)|$ are smaller than or equal to the quantity $\sup_{x \in K} |F(\widehat{q}_{\alpha,n}(x)|x) - \alpha|$ which converges completely to 0 in view of (2.19), the term on the right-hand side of inequality (2.20) converges completely to 1 (it can be easily seen that if $\{V_n\}$ is a sequence of random variables such that $|V_n| \xrightarrow{co.} \alpha$, then $|V_n|/\alpha \xrightarrow{co.} 1$, $\alpha/|V_n| \xrightarrow{co.} 1$ and $(\alpha/|V_n|)^\gamma \xrightarrow{co.} 1$; if $V_n \xrightarrow{co.} 1$ and $W_n \xrightarrow{co.} 1$ then $V_n \times W_n \xrightarrow{co.} 1$ and $V_n \vee W_n \xrightarrow{co.} 1$). Therefore $\sup_{x \in K} f(q_\alpha(x)|x)/f(q_{\alpha_{x,n}}(x)|x) \xrightarrow{co.} 1$, and thus $\sup_{x \in K} f(q_\alpha(x)|x)|\widehat{q}_{\alpha,n}(x) - q_\alpha(x)| \xrightarrow{co.} 0$ in view of (2.16) and (2.19). This completes the proof of Theorem 2.2. $\square$

3 Parametric Approximations of Quantile Frontiers

Throughout this section μ is a finite positive measure on the support of X , the order α is fixed in $]0, 1[$ and the set $K \subset \mathbb{R}^p$ is such that $\inf_{x \in K} F_X(x) > 0$.

Consider a parametric family of functions $\varphi_\theta(x)$ defined on $\mathbb{R}_+^p$ and depending on a finite number of parameters $\theta \in \Theta \subset \mathbb{R}^k$. We will consider two cases: we first want to estimate the best approximation of the partial order- α frontier function q_α in the parametric family $\{\varphi_\theta | \theta \in \Theta\}$, and then to estimate in a robust way the following pseudo-true value of θ :

$$\theta(K, \mu) = \arg \min_{\theta \in \mathbb{R}^k} \int_K (\varphi(x) - \varphi_\theta(x))^2 d\mu(x), \quad (3.1)$$

which defines the best parametric approximation of the full frontier function φ in the parametric family $\{\varphi_\theta | \theta \in \mathbb{R}^k\}$, in the $L^2(K, \mu)$ norm. If it is the unique solution of (3.1), then it coincides with the true value θ of the model (1.1). A weakly consistent estimator of $\theta(K, \mu)$ has been suggested in Florens and Simar (2005) and defined as follows:

$$\hat{\theta}_n(K, \mu) = \arg \min_{\theta \in \mathbb{R}^k} \int_K (\hat{\varphi}_n(x) - \varphi_\theta(x))^2 d\mu(x). \quad (3.2)$$

But this estimator suffers from the dramatic lack of robustness of the FDH frontier $\hat{\varphi}_n$.

In place of estimating $\theta(K, \mu)$, Florens and Simar (2005) rather propose to estimate the pseudo-true value for the order- m frontier:

$$\theta_m(K, \mu) = \arg \min_{\theta \in \mathbb{R}^k} \int_K (\varphi_m(x) - \varphi_\theta(x))^2 d\mu(x), \quad (3.3)$$

where $m \geq 1$ is an integer by its nonparametric estimator:

$$\hat{\theta}_{m,n}(K, \mu) = \arg \min_{\theta \in \mathbb{R}^k} \int_K (\hat{\varphi}_{m,n}(x) - \varphi_\theta(x))^2 d\mu(x). \quad (3.4)$$

As an alternative, we rather propose in this paper to estimate a pseudo-true value for the order- α frontier, defined as

$$\theta_\alpha(K, \mu) = \arg \min_{\theta \in \mathbb{R}^k} \int_K (q_\alpha(x) - \varphi_\theta(x))^2 d\mu(x). \quad (3.5)$$

Our quantile-based procedure has the merit to be more robust to extremes since the order- α quantile frontiers have the superiority of being more resistant to extremes than the FDH $\hat{\varphi}_n$ and the order- m frontiers $\hat{\varphi}_{m,n}$ when estimating the true full frontier φ as showed in Daouia and Ruiz-Gazen (2003). By a plug-in argument of the robust quantile frontier $\hat{q}_{\alpha,n}$, we obtain the estimator

$$\hat{\theta}_{\alpha,n}(K, \mu) = \arg \min_{\theta \in \mathbb{R}^k} \int_K (\hat{q}_{\alpha,n}(x) - \varphi_\theta(x))^2 d\mu(x). \quad (3.6)$$

As pointed in Florens and Simar (2005), the existence and unicity of the pseudo-true values (3.1), (3.3) and (3.5) are based on technical conditions (integrability, identification, structure of the functional space $\{\varphi_\theta \mid \theta \in \Theta\}$). We will not explicit these technical hypothesis here but the implicit set of functions we consider is the set of square integrable functions with respect to the density $d\mu(x)$. This set is an Hilbert space provided by its norm topology. If, for instance, the subset $\{\varphi_\theta \mid \theta \in \Theta\}$ is closed and convex, then the pseudo-true values exist and are unique. So, in what follows, we assume that these pseudo-true values exist and are unique.

Moreover, for the sample values $\hat{\theta}_n(K, \mu)$, $\hat{\theta}_{m,n}(K, \mu)$, $\hat{\theta}_{\alpha,n}(K, \mu)$ and other estimators defined bellow derived from $\hat{\theta}_{m,n}(K, \mu)$ and $\hat{\theta}_{\alpha,n}(K, \mu)$ to converge completely, the pseudo-true values $\theta(K, \mu)$, $\theta_m(K, \mu)$ and $\theta_\alpha(K, \mu)$ should be well-separated points of minimum of the integrals in equations (3.1), (3.3) and (3.5). In other words, if we denote by $M(\theta)$ the integral in each one of these equations and by θ_0 the pseudo-true value which minimizes the map $M: \Theta \rightarrow \overline{\mathbb{R}}$, then it will be required that

$$M(\theta_0) < \inf \{M(\theta) \mid \theta \in \Theta : d(\theta, \theta_0) \geq \varepsilon\}, \quad (3.7)$$

for every $\varepsilon > 0$, where $d(\cdot, \cdot)$ is the Euclidean metric on $\mathbb{R}^k$. The following lemma shows how this condition will be used to prove complete convergence of our estimators. It also extends Theorem 5.7 in van der Vaart (1998, p. 45) from the weak convergence to the complete convergence.

Lemma 3.1. *Let M_n be random functions and let M be a fixed function of $\theta \in \Theta$; a metric space endowed with the metric d . Consider the values θ_0 and θ_n that minimize $\theta \mapsto M(\theta)$ and $\theta \mapsto M_n(\theta)$ respectively. If the condition (3.7) holds and $\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \xrightarrow{co} 0$, then $\theta_n \xrightarrow{co} \theta_0$.*

Proof. First let us show that $M(\theta_n) \xrightarrow{co} M(\theta_0)$. Putting $Z_n = M_n(\theta_0) - M(\theta_0)$, we have $Z_n \xrightarrow{co} 0$ from the uniform complete convergence of M_n to M . Since $M_n(\theta_n) \leq M_n(\theta_0)$ by definition of θ_n , we then obtain $M_n(\theta_n) \leq M(\theta_0) + Z_n$. Therefore

$$0 \leq M(\theta_n) - M(\theta_0) \leq M(\theta_n) - M_n(\theta_n) + Z_n \leq \sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| + Z_n \xrightarrow{co} 0.$$

Thus $M(\theta_n) \xrightarrow{co} M(\theta_0)$. Let us now show that $\sum_{n=1}^{\infty} P\{d(\theta_n, \theta_0) \geq \varepsilon\} < \infty$, for every $\varepsilon > 0$. Because $\inf_{d(\theta, \theta_0) \geq \varepsilon} M(\theta) > M(\theta_0)$, there exists a number $\eta_\varepsilon > 0$ such that $\inf_{d(\theta, \theta_0) \geq \varepsilon} M(\theta) > M(\theta_0) + \eta_\varepsilon$. It follows that the event $\{d(\theta_n, \theta_0) \geq \varepsilon\}$ is contained in $\{M(\theta_n) > M(\theta_0) + \eta_\varepsilon\}$. Hence $\sum_{n=1}^{\infty} P\{d(\theta_n, \theta_0) \geq \varepsilon\} \leq \sum_{n=1}^{\infty} P\{M(\theta_n) - M(\theta_0) > \eta_\varepsilon\} < \infty$ by the complete convergence of $M(\theta_n)$ to $M(\theta_0)$. $\square$

A natural choice for μ is the law $\mathbb{P}_X$ of X , but since $\mathbb{P}_X$ is usually unknown, we can define sample versions of the pseudo-true values by using the empirical probability measure $\widehat{\mathbb{P}}_{X,n} = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$, putting a mass $1/n$ at each X_i ,

$$\theta_\alpha(K, \widehat{\mathbb{P}}_{X,n}) = \arg \min_{\theta \in \mathbb{R}^k} \frac{1}{n} \sum_{i=1}^n (q_\alpha(X_i) - \varphi_\theta(X_i))^2 \mathbb{I}_K(X_i),$$

$$\widehat{\theta}_{\alpha,n}(K, \widehat{\mathbb{P}}_{X,n}) = \arg \min_{\theta \in \mathbb{R}^k} \frac{1}{n} \sum_{i=1}^n (\widehat{q}_{\alpha,n}(X_i) - \varphi_\theta(X_i))^2 \mathbb{I}_K(X_i).$$

In order to simplify the presentation, we first restrict our parametric family to the class of linear models

$$\varphi_\theta(x) = g'(x)\theta,$$

where $g'(x) = (g_1(x), \dots, g_k(x))$, with $g_j(\cdot)$ being known scalar functions.

Theorem 3.1. *Assume that the scalar functions $g_1(\cdot), \dots, g_k(\cdot)$ are bounded on K .*

1. *If the conditions of Theorem 2.2 hold with $\inf_{x \in K} f(q_\alpha(x)|x) > 0$, then*

$$\widehat{\theta}_{\alpha,n}(K, \widehat{\mathbb{P}}_{X,n}) - \theta_\alpha(K, \widehat{\mathbb{P}}_{X,n}) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

2. *If the conditions of Theorem 2.1 hold, then*

$$\sqrt{n} \left(\widehat{\theta}_{\alpha,n}(K, \widehat{\mathbb{P}}_{X,n}) - \theta_\alpha(K, \widehat{\mathbb{P}}_{X,n}) \right) \xrightarrow{\mathcal{L}} N(0, M \Sigma^\alpha M) \quad \text{as } n \rightarrow \infty,$$

where

$$M = \{E[\mathbb{I}_K(X_1)g(X_1)g'(X_1)]\}^{-1}, \quad \Sigma^\alpha = \text{Var}_F \{E_F[H((X_2, Y_2), (X_1, Y_1)) | (X_1, Y_1)]\}$$

with $H(\cdot, \cdot)$ being described in (3.11).

Proof. The values $\theta_\alpha(K, \widehat{\mathbb{P}}_{X,n})$ and $\widehat{\theta}_{\alpha,n}(K, \widehat{\mathbb{P}}_{X,n})$ have the following explicit expressions

$$\begin{aligned} \widehat{\theta}_{\alpha,n}(K, \widehat{\mathbb{P}}_{X,n}) &= \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)g'(X_i) \right\}^{-1} \left[\frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)\widehat{q}_{\alpha,n}(X_i)g(X_i) \right], \\ \theta_\alpha(K, \widehat{\mathbb{P}}_{X,n}) &= \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)g'(X_i) \right\}^{-1} \left[\frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)q_\alpha(X_i)g(X_i) \right]. \end{aligned}$$

The estimation error is then given by

$$\begin{aligned} \widehat{\theta}_{\alpha,n}(K, \widehat{\mathbb{P}}_{X,n}) - \theta_\alpha(K, \widehat{\mathbb{P}}_{X,n}) &= \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)g'(X_i) \right\}^{-1} \\ &\quad \times \left[\frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)(\widehat{q}_{\alpha,n}(X_i) - q_\alpha(X_i)) \right]. \end{aligned} \tag{3.8}$$

We have from the asymptotic representation (2.1), for any $x \in K$,

$$\begin{aligned}\widehat{q}_{\alpha,n}(x) - q_\alpha(x) &= \frac{1}{n} \sum_{j=1}^n (1/f(q_\alpha(x)|x)F_X(x)) \\ &\quad \times \{\alpha \mathbb{I}(X_j \leq x) - \mathbb{I}(X_j \leq x, Y_j \leq q_\alpha(x))\} + R_n^\alpha(x).\end{aligned}\tag{3.9}$$

Then

$$\begin{aligned}V_n &:= \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)(\widehat{q}_{\alpha,n}(X_i) - q_\alpha(X_i)) \\ &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n H((X_i, Y_i), (X_j, Y_j)) + \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)R_n^\alpha(X_i)\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}H((X_i, Y_i), (X_j, Y_j)) &= \mathbb{I}_K(X_i)g(X_i)(1/f(q_\alpha(X_i)|X_i)F_X(X_i)) \\ &\quad \times \{\alpha \mathbb{I}(X_j \leq X_i) - \mathbb{I}(X_j \leq X_i, Y_j \leq q_\alpha(X_i))\}.\end{aligned}\tag{3.11}$$

Since $\sum_{i=1}^n \sum_{j=1}^n H((X_i, Y_i), (X_j, Y_j)) = \sum_{i=1}^n \sum_{j=1}^n H((X_j, Y_j), (X_i, Y_i))$, we have

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n H((X_i, Y_i), (X_j, Y_j)) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \mathcal{H}((X_i, Y_i), (X_j, Y_j))$$

where $\mathcal{H}((X_i, Y_i), (X_j, Y_j)) = (H((X_i, Y_i), (X_j, Y_j)) + H((X_j, Y_j), (X_i, Y_i)))/2$ is a symmetric kernel. Hence, V_n is by definition (see, *e.g.*, Serfling, 1980, p. 174) the Von Mises statistic associated to the parametric function

$$\begin{aligned}\theta(F) &= E_F[\mathcal{H}((X_1, Y_1), (X_2, Y_2))] \\ &= [E_F H((X_1, Y_1), (X_2, Y_2)) + E_F H((X_2, Y_2), (X_1, Y_1))]/2.\end{aligned}$$

Let

$$H_1(X_1, Y_1) = E_F[H((X_1, Y_1), (X_2, Y_2)) | (X_1, Y_1)].$$

For any (x_1, y_1) , we have $H_1(x_1, y_1) = E_F[H((x_1, y_1), (X_2, Y_2))] = 0$. Then $H_1(X_1, Y_1) = 0$, whence $E_F[H_1(X_1, Y_1)] = 0$, i.e., $E_F[H((X_1, Y_1), (X_2, Y_2))] = 0$. We obtain in a similar way $E_F[H((X_2, Y_2), (X_1, Y_1))] = 0$, which gives $\theta(F) = 0$. Consequently, we have by Theorem A and Lemma 5.7.3 of Serfling (1980, p. 190 and 206),

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n H((X_i, Y_i), (X_j, Y_j)) \xrightarrow{P} 0 \quad \text{as } n \longrightarrow \infty.$$

On the other hand, we have from (3.9),

$$\sup_{x \in K} |R_n^\alpha(x)| \leq \sup_{x \in K} |\hat{q}_{\alpha,n}(x) - q_\alpha(x)| + \frac{\|\hat{F}_{X,n} - F_X\|_\infty + \|\hat{F}_n - F\|_\infty}{\inf_{x \in K} f(q_\alpha(x)|x)F_X(x)}$$

where $\|\cdot\|_\infty$ denotes the sup-norm. Since $\|\hat{F}_{X,n} - F_X\|_\infty$ and $\|\hat{F}_n - F\|_\infty$ converge in probability to 0 by Kiefer's inequality (see, *e.g.*, Serfling, 1980, Theorem B, p. 61), it follows from Theorem 2.2 that $\sup_{x \in K} |R_n^\alpha(x)| \xrightarrow{P} 0$ as $n \rightarrow \infty$. Hence, since the scalar functions $g_j(\cdot)$ are bounded on K , the vector $\frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)R_n^\alpha(X_i)$ converges in probability to 0 as $n \rightarrow \infty$, and thus

$$V_n \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

Finally, since $\{\frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)g'(X_i)\}^{-1}$ converges to $\{E[\mathbb{I}_K(X_1)g(X_1)g'(X_1)]\}^{-1}$ in probability via the law of large numbers, we complete the proof of the first result.

Let us now prove the asymptotic normality of $\sqrt{n}(\hat{\theta}_{\alpha,n}(K, \hat{\mathbb{P}}_{X,n}) - \theta_\alpha(K, \hat{\mathbb{P}}_{X,n}))$. We have from (3.8)-(3.11),

$$\begin{aligned} \sqrt{n}(\hat{\theta}_{\alpha,n}(K, \hat{\mathbb{P}}_{X,n}) - \theta_\alpha(K, \hat{\mathbb{P}}_{X,n})) &= \sqrt{n}V_n \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)g'(X_i) \right\}^{-1} \\ &= \left[\sqrt{n}W_n + \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i) \left\{ \sqrt{n}R_n^\alpha(X_i) \right\} \right] \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)g'(X_i) \right\}^{-1} \end{aligned}$$

where $W_n = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n H((X_i, Y_i), (X_j, Y_j))$. We obtain via Theorem A and Lemma 5.7.3 of Serfling (1980, p. 192 and 206) that $\sqrt{n}W_n \xrightarrow{\mathcal{L}} N(0, 4\zeta_1)$ as $n \rightarrow \infty$, where

$$\begin{aligned} \zeta_1 &= \text{Var}_F \{E_F[\mathcal{H}((X_1, Y_1), (X_2, Y_2)) | (X_1, Y_1)]\} \\ &= \text{Var}_F \{E_F[H((X_2, Y_2), (X_1, Y_1)) | (X_1, Y_1)]\} / 4. \end{aligned}$$

We get this last result by using the fact that $H_1(X_1, Y_1) = 0$. On the other hand, if the conditions of Theorem 2.1 hold, then $\sup_{x \in K} \sqrt{n}|R_n^\alpha(x)| \xrightarrow{P} 0$ as $n \rightarrow \infty$ by Corollary 2.1. If furthermore the scalar functions $g_j(\cdot)$ are bounded on K , then the vector $\frac{1}{n} \sum_{i=1}^n \mathbb{I}_K(X_i)g(X_i)\{\sqrt{n}R_n^\alpha(X_i)\}$ converges in probability to 0 as $n \rightarrow \infty$. This ends the proof. $\square$

Let us now turn to the more general case where we consider a general parametric model and a general measure μ . From now on, we will assume that the condition (3.7) holds for each integral $M(\theta)$ and its corresponding pseudo-true value θ_0 in Equations (3.1), (3.5) and (3.3). Let us start by the complete convergence of $\hat{\theta}_{\alpha,n}(K, \mu)$.

Theorem 3.2. Assume that φ and $\{\varphi_\theta, \theta \in \Theta\}$ are bounded on K and that the conditions of Theorem 2.2 hold with $\inf_{x \in K} f(q_\alpha(x)|x) > 0$. Then

$$\widehat{\theta}_{\alpha,n}(K, \mu) \xrightarrow{co.} \theta_\alpha(K, \mu) \quad \text{as } n \rightarrow \infty.$$

Proof. Following Lemma 3.1, we shall prove that

$$\left| \int_K (\widehat{q}_{\alpha,n}(x) - \varphi_\theta(x))^2 d\mu(x) - \int_K (q_\alpha(x) - \varphi_\theta(x))^2 d\mu(x) \right|$$

converges completely and uniformly in θ to 0 as $n \rightarrow \infty$. Since φ and $\varphi_\theta, \theta \in \mathbb{R}^k$, are bounded on K and $\mu(K)$ is finite, it suffices to show that $\sup_{\theta \in \mathbb{R}^k} \left| \|\widehat{q}_{\alpha,n} - \varphi_\theta\|_{L^2(K, \mu)} - \|q_\alpha - \varphi_\theta\|_{L^2(K, \mu)} \right|$ converges completely to 0 as $n \rightarrow \infty$. We have

$$\sup_{\theta \in \mathbb{R}^k} \left| \|\widehat{q}_{\alpha,n} - \varphi_\theta\|_{L^2(K, \mu)} - \|q_\alpha - \varphi_\theta\|_{L^2(K, \mu)} \right| \leq \|\widehat{q}_{\alpha,n} - q_\alpha\|_{L^2(K, \mu)}$$

with

$$\|\widehat{q}_{\alpha,n} - q_\alpha\|_{L^2(K, \mu)}^2 = \int_K (\widehat{q}_{\alpha,n}(x) - q_\alpha(x))^2 d\mu(x) \leq \left(\sup_{x \in K} |\widehat{q}_{\alpha,n}(x) - q_\alpha(x)| \right)^2 \mu(K).$$

Since $\inf_{x \in K} f(q_\alpha(x)|x) > 0$, we get $\sup_{x \in K} |\widehat{q}_{\alpha,n}(x) - q_\alpha(x)| \xrightarrow{co.} 0$ in view of Theorem 2.2, and thus we obtain the complete convergence of $\widehat{\theta}_{\alpha,n}(K, \mu)$ to $\theta_\alpha(K, \mu)$. $\square$

Note that in the preceding theorem, if we replace the conditions of Theorem 2.2 by those of Theorem 2.1, then by following the same arguments we could obtain $\widehat{\theta}_{\alpha,n}(K, \mu) \xrightarrow{P} \theta_\alpha(K, \mu)$ as $n \rightarrow \infty$.

By making use of Theorem 2.1 in conjunction with a technique of proof of Florens and Simar (2005), the next theorem shows the asymptotic normality of $\widehat{\theta}_{\alpha,n}(K, \mu)$.

Theorem 3.3. Assume that φ and $\{\varphi_\theta, \theta \in \Theta\}$ are bounded on K and that the conditions of Theorem 2.1 hold. Assume furthermore that $\theta \mapsto \varphi_\theta(x)$ is differentiable on Θ for any $x \in K$, that the differentiability at $\theta_\alpha(K, \mu)$ is uniform in $x \in K$, i.e.,

$$\sup_{x \in K} \left| \frac{\varphi_\theta(x) - \varphi_{\theta_\alpha(K, \mu)}(x)}{\theta - \theta_\alpha(K, \mu)} - \frac{\partial \varphi_\theta}{\partial \theta}(x)|_{\theta=\theta_\alpha(K, \mu)} \right| \longrightarrow 0 \quad \text{as } \theta \rightarrow \theta_\alpha(K, \mu),$$

and that A, A^{-1}, B and $\int_K \frac{\partial \varphi_\theta}{\partial \theta}(x)|_{\theta=\theta_\alpha(K, \mu)} d\mu(x)$ exist, where

$$A = \int_K \frac{\partial \varphi_\theta}{\partial \theta}(x)|_{\theta=\theta_\alpha(K, \mu)} \frac{\partial \varphi_\theta}{\partial \theta'}(x)|_{\theta=\theta_\alpha(K, \mu)} d\mu(x)$$

$$B = \int_K \int_K \frac{\partial \varphi_\theta}{\partial \theta}(x)|_{\theta=\theta_\alpha(K, \mu)} \frac{\partial \varphi_\theta}{\partial \theta'}(z)|_{\theta=\theta_\alpha(K, \mu)} \Gamma_\alpha(x, z) d\mu(x) d\mu(z),$$

with $\Gamma_\alpha(\cdot, \cdot)$ being described in (2.13). Then,

$$\sqrt{n} \left(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_\alpha(K, \mu) \right) \xrightarrow{\mathcal{L}} N(0, A^{-1} B A^{-1}).$$

Proof. We have in view of the first order conditions which define $\widehat{\theta}_{\alpha,n}(K, \mu)$,

$$\int_K \left(\widehat{q}_{\alpha,n}(x) - \varphi_{\widehat{\theta}_{\alpha,n}(K, \mu)}(x) \right) \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\widehat{\theta}_{\alpha,n}(K, \mu)} d\mu(x) = 0.$$

It follows

$$\begin{aligned} & \sqrt{n} \int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\widehat{\theta}_{\alpha,n}(K, \mu)} (\widehat{q}_{\alpha,n}(x) - q_{\alpha}(x)) d\mu(x) \\ & + \sqrt{n} \int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\widehat{\theta}_{\alpha,n}(K, \mu)} (q_{\alpha}(x) - \varphi_{\theta_{\alpha}(K, \mu)}(x)) d\mu(x) \\ & = \sqrt{n} \int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\widehat{\theta}_{\alpha,n}(K, \mu)} \left(\varphi_{\widehat{\theta}_{\alpha,n}(K, \mu)}(x) - \varphi_{\theta_{\alpha}(K, \mu)}(x) \right) d\mu(x). \end{aligned}$$

In this equality, $\frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\widehat{\theta}_{\alpha,n}(K, \mu)}$ can be replaced by $\frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)}$ and we note that the second term on the left hand side is then equal to zero, by the definition of $\theta_{\alpha}(K, \mu)$. By Young's form of Taylor's expansion of the first order, we have for any $x \in K$,

$$\begin{aligned} \varphi_{\widehat{\theta}_{\alpha,n}(K, \mu)}(x) - \varphi_{\theta_{\alpha}(K, \mu)}(x) &= \frac{\partial \varphi_{\theta}}{\partial \theta'}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} \left(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu) \right) \\ &+ \varepsilon(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu)) \left(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu) \right) \end{aligned}$$

where $\varepsilon(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu)) = o_p(1)$ as $n \rightarrow \infty$, uniformly in $x \in K$, since $\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu) \xrightarrow{P} 0$ and the differentiability of $\theta \mapsto \varphi_{\theta}(x)$ at $\theta_{\alpha}(K, \mu)$ is uniform in $x \in K$.

Therefore

$$\begin{aligned} & \sqrt{n} \int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} \left(\varphi_{\widehat{\theta}_{\alpha,n}(K, \mu)}(x) - \varphi_{\theta_{\alpha}(K, \mu)}(x) \right) d\mu(x) \\ &= \left(\int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} \frac{\partial \varphi_{\theta}}{\partial \theta'}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} d\mu(x) \right) \left\{ \sqrt{n} \left(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu) \right) \right\} \\ &+ \left(\int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} d\mu(x) \right) \varepsilon(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu)) \left\{ \sqrt{n} \left(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu) \right) \right\}. \end{aligned}$$

It follows

$$\begin{aligned} \sqrt{n} \left(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu) \right) &= \left\{ \int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} d\mu(x) \right. \\ &\quad \left. + \int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} d\mu(x) \varepsilon(\widehat{\theta}_{\alpha,n}(K, \mu) - \theta_{\alpha}(K, \mu)) \right\}^{-1} \\ &\quad \times \int_K \frac{\partial \varphi_{\theta}}{\partial \theta}(x) \Big|_{\theta=\theta_{\alpha}(K, \mu)} \left\{ \sqrt{n} (\widehat{q}_{\alpha,n}(x) - q_{\alpha}(x)) \right\} d\mu(x). \end{aligned}$$

We finally obtain the desired convergence by using Theorem 2.1. $\square$

Note that for practical purposes (construction of confidence intervals, ...), we could use bootstrap estimates of the variance matrix $A^{-1}BA^{-1}$, as suggested in details in Florens and Simar (2005, Appendix)

Convergence to the parametric approximation of the full frontier itself

Florens and Simar (2005) show that the estimator $\hat{\theta}_n(K, \mu)$ defined in (3.2), converges in probability to $\theta(K, \mu)$. We will show below how this weak consistency can be easily extended to the complete convergence sense. On the other hand, Cazals *et al.* (2002) derived an estimator $\hat{\varphi}_{m(n),n}$ of the full frontier function φ by an appropriate choice of the order m as a function of the sample size n . This estimator shares the same asymptotic distribution as the FDH estimator $\hat{\varphi}_n$, but is more robust to extremes. Likewise, Aragon *et al.* (2005) suggested a quantile-type estimator $\hat{q}_{\alpha(n),n}$ of φ , which keeps the same asymptotic distribution as $\hat{\varphi}_n$ and $\hat{\varphi}_{m(n),n}$. In our next theorem, we will also show the complete convergence of the corresponding pseudo-true values $\hat{\theta}_{m(n),n}(K, \mu)$ and $\hat{\theta}_{\alpha(n),n}(K, \mu)$ when estimating $\theta(K, \mu)$ itself. For this we need the following lemma.

Lemma 3.2. *Assume that F_X , φ and φ_m are continuous on a compact subset K interior to the support of X , for every $m \geq 1$, and that the upper boundary of the support of Y is finite. Then, $\sup_{x \in K} |\hat{\varphi}_n(x) - \varphi(x)|$, $\sup_{x \in K} |\hat{\varphi}_{m(n),n}(x) - \varphi(x)|$ and $\sup_{x \in K} |\hat{q}_{\alpha(n),n}(x) - \varphi(x)|$ converge completely to 0 as $n \rightarrow \infty$, where $m(n)$ and $\alpha(n)$ are such that*

$$\lim_{n \rightarrow \infty} m(n) = \infty, \quad \lim_{n \rightarrow \infty} m(n) (\log n/n)^{1/2} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} n(1 - \alpha(n)) = 0.$$

This lemma is proved in Daouia and Simar (2005, see Theorems 2.1, 2.3 and Lemma 3.3).

Theorem 3.4. *Assume that the parametric functions $\{\varphi_\theta, \theta \in \mathbb{R}^k\}$ are bounded on K and that the conditions of Lemma 3.2 hold. Then, $\hat{\theta}_n(K, \mu)$, $\hat{\theta}_{m(n),n}(K, \mu)$ and $\hat{\theta}_{\alpha(n),n}(K, \mu)$ converge completely to $\theta(K, \mu)$ as $n \rightarrow \infty$. In particular, if the parametric model (1.1) is correctly specified for the frontier, then the three estimators converge completely to the value θ such that $\varphi = \varphi_\theta$.*

Proof. It can be easily seen that

$$\begin{aligned} \sup_{\theta \in \mathbb{R}^k} \left| \|\hat{\varphi}_n - \varphi_\theta\|_{L^2(K, \mu)} - \|\varphi - \varphi_\theta\|_{L^2(K, \mu)} \right| &\leq \|\hat{\varphi}_n - \varphi\|_{L^2(K, \mu)} \leq \sup_{x \in K} |\hat{\varphi}_n(x) - \varphi(x)| (\mu(K))^{1/2} \\ \sup_{\theta \in \mathbb{R}^k} \left| \|\hat{\varphi}_{m(n),n} - \varphi_\theta\|_{L^2(K, \mu)} - \|\varphi - \varphi_\theta\|_{L^2(K, \mu)} \right| &\leq \sup_{x \in K} |\hat{\varphi}_{m(n),n}(x) - \varphi(x)| (\mu(K))^{1/2} \\ \sup_{\theta \in \mathbb{R}^k} \left| \|\hat{q}_{\alpha(n),n} - \varphi_\theta\|_{L^2(K, \mu)} - \|\varphi - \varphi_\theta\|_{L^2(K, \mu)} \right| &\leq \sup_{x \in K} |\hat{q}_{\alpha(n),n}(x) - \varphi(x)| (\mu(K))^{1/2}. \end{aligned}$$

The proof follows by using the same technique of proof of Theorem 3.2 in conjunction with these inequalities and Lemma 3.2. $\square$

Note that the full frontier $\varphi(\cdot)$ is monotone and it is natural to try to incorporate this information into the estimation procedure. Daouia and Simar (2005) proposed nondecreasing estimators $\hat{\varphi}_{m(n),n}^\#$ and $\hat{q}_{\alpha(n),n}^\#$ of φ by isotonizing the original estimators $\hat{\varphi}_{m(n),n}$ and $\hat{q}_{\alpha(n),n}$. These monotone estimators appear to be more stable and more robust to extreme values than the initial versions. Furthermore, they converge uniformly and completely to φ under the same conditions of Lemma 3.2 (see Daouia and Simar, 2005, Theorems 2.1-2.3) and so, we can introduce the new estimators $\hat{\theta}_{m(n),n}^\#(K, \mu)$ and $\hat{\theta}_{\alpha(n),n}^\#(K, \mu)$ of $\theta(K, \mu)$ and easily establish their uniform complete convergence in the same way as the original ones $\hat{\theta}_{m(n),n}(K, \mu)$ and $\hat{\theta}_{\alpha(n),n}(K, \mu)$.

Note also that the analysis of the asymptotic distributions of $\hat{\theta}_n(K, \mu)$, $\hat{\theta}_{m(n),n}(K, \mu)$, $\hat{\theta}_{\alpha(n),n}(K, \mu)$ and their isotonized versions is more complexe since there is no functional convergence theorem for $\hat{\varphi}_n$, $\hat{\varphi}_{m(n),n}$ and $\hat{q}_{\alpha(n),n}$ to φ . Only punctual convergence results to a Weibull distribution for $\hat{\varphi}_n(x)$, $\hat{\varphi}_{m(n),n}(x)$ and $\hat{q}_{\alpha(n),n}(x)$ are available, respectively, in Park *et al.* (2000), Cazals *et al.* (2002) and Aragon *et al.* (2005). We shall return to this topic in another paper.

All the results proved in our approach remain valid if we use $w(x)dx$ instead of $d\mu(x)$, where $w(x)$ is a given weight function which can be viewed as a density on x weighting the error term.

4 Numerical Illustrations

In this section, we present three simulated examples as in Florens and Simar (2005) to illustrate our procedure, we also present a real data example on the activity of Air Controlers in Europe. We confirm the advantages of the robust semi-parametric estimators in the presence of outliers over their full frontier alternative (either using Shifted-OLS or the FDH filter as a first stage) and we compare in these examples the parametric quantile-type frontier approximations with the order- m frontier approximations, indicating some advantages of the former over the latter.

4.1 Example 1

We first consider a case where the frontier function φ is linear. We choose (X, Y) uniformly distributed over the region $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$. Here $\varphi(x) = x$, $\varphi_\theta(x) = \theta x$ with $0 \leq \theta \leq 1$, and $F(y|x) = 2x^{-1}y - x^{-2}y^2$, for $0 < x \leq 1$ and $0 \leq y \leq x$. The conditional

α -quantile is $q_\alpha(x) = x(1 - \sqrt{1 - \alpha})$, for $0 < x \leq 1$ and $0 \leq \alpha \leq 1$, and the order- m frontier can be computed as $\varphi_m(x) = x(1 - A_m)$, where $A_m = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} 2^j / (2m - j + 1)$. Therefore $\theta_\alpha = 1 - \sqrt{1 - \alpha}$ and $\theta_m = 1 - A_m$.

The parametric approximations $\{\hat{\theta}_{m,n}\}_n$ and $\{\hat{\theta}_{\alpha,n}\}_n$ do not estimate in general the same quantity, except for the limiting case where m tends to infinity and α to one. But in this particular example, if $\alpha = 1 - A_m^2$, then both partial pseudo-true values θ_α and θ_m coincide and so, the robust proposals of the two sequences of parametric approximations can be compared.

We illustrate in Figure 1 one sample of 100 observations generated according the above scenario and we compare the results when adding 3 outliers. As expected and already pointed in Florens and Simar (2005), in this scenario, the shifted-OLS behaves very badly. The interest here is focused on the comparison of the order- m frontier with the α -quantile, with their common FDH limit. When there are no outliers, the two fitted parametric frontier are, as expected by our choice of m and α , very similar but when the 3 outliers are added, the α -quantile seems to resist more easily to the presence of the 3 serious outliers. We see also that the fit obtained by the FDH estimator is too sensitive to these outliers.

4.2 Example 2

We choose here a concave frontier given by the Cobb-Douglas model $Y = X^{1/2} \exp(-U)$, where X is uniform on $[0, 1]$ and U , independent of X , is Exponential with parameter $\lambda = 3$. Here $\varphi(x) = x^{1/2}$ and $\varphi_\theta(x) = \theta x^{1/2}$ with $0 \leq \theta \leq 1$. The α -quantile frontier is given by $q_\alpha(x) = x^{1/2} \theta_\alpha$, where $\theta_\alpha = \cos\left(\frac{\arccos(1-2\alpha)+4\pi}{3}\right) + \frac{1}{2}$, and the order- m frontier can be computed as $\varphi_m(x) = x^{1/2} \theta_m$, where $\theta_m = 1 - B_m$ with $B_m = \sum_{j=0}^m \binom{m}{j} (-2)^{m-j} 3^j / (3m - j + 1)$. In this particular case, if $\alpha = \frac{1}{2}(1 - \cos[3 \arccos(\frac{1}{2} - B_m) - 4\pi])$, then $\theta_\alpha \equiv \theta_m$ can be estimated by $\{\hat{\theta}_{m,n}\}_n$ as well as $\{\hat{\theta}_{\alpha,n}\}_n$.

We illustrate again, in Figure 2, one sample of 100 observations and we compare the results when adding 3 outliers to the same sample. Here, the shifted-OLS behaves better since the inefficiency terms U are independent of X . The comparison of the order- m frontier with the α -quantile leads to similar conclusion than in the preceding example: as expected, similar results when there are no outliers and again, the α -quantile seems to resist more to the presence of the 3 outliers.

4.3 Example 3

We choose here, as in Florens and Simar (2005) the same scenario as in the preceding example but we introduce heterogeneity in the inefficiency term (U is not independent of X): here

we choose $E(U|X) = \frac{2}{3}(1 - X)$ so that $E(U) = 1/3$ as above in Example 2. We do not have explicit analytical expressions for the $\varphi_m(x)$ nor for $q_\alpha(x)$. But for ease of comparison, we chose the same values for m and α as above.

Figure 3 displays the results (again, one sample of 100 observations without and with 3 added outliers). Here, the shifted-OLS is a catastrophe (as pointed in Florens and Simar, 2005). The comparison of the order- m frontier with the α -quantile leads again to similar conclusion than in the two preceding examples: similar results when there are no outliers and better robustness to the three outliers for the α -quantile fitted frontier.

4.4 A real data set

We could use our approach to any model with multiple inputs but in order to provide pictures in two dimensions, we illustrate our estimators with a real data coming from the efficiency analysis of Air Controllers in Europe (Mouchart and Simar, 2002). We have data on activity of 37 European air controllers in the year 2000. The activity of each controller can be described by one input (an aggregate factor of different kind of labor) and one output (an aggregate factor of the activity produced, based on the number of air movements controlled, the number flight hours controlled, ...).

Here we have $n = 37$ observations and we choose $m = 20$ and $\alpha = 0.95$, only for illustrative purpose. The computation time is so fast (less than 0.15 seconds for producing the estimates), that a grid of values for the tuning parameters (m and α) can be chosen and then the practitioner could proceed to a sensitivity analysis. We also choose a Cobb-Douglas specification for the frontier model. Figure 4 displays the obtained results.

Clearly, the shifted-OLS seems to be outperformed by the other estimators, since it is based on a restrictive hypothesis of homogeneous distribution of the inefficiencies. The fit obtained through the first step FDH filter provide a sensible estimate but is very sensitive to extreme values (one around $x = 2.8$ and the other on the right around $x = 7.5$). Again, the fit based on the order- m filter is more robust to these extreme points but as expected, this is still more true with the fit obtained with the α -quantile frontier.

Just for illustrative purpose, we reproduce in Table 1 the point estimates of the fitted Cobb-Douglas production frontier. As explained in Florens and Simar (2005), standard deviations of the parameters obtained through the order- m and the α -quantile frontier can be obtained by simple bootstrap. The other two approaches (full-frontier estimates) cannot provide, up to now, estimates of the standard deviations of the estimates.

method	Intercept	Elasticity
Shifted-OLS	0.8238	0.8833
FDH	0.5886	0.8838
α -quantile	0.5554	0.7798
order- m	0.5583	0.8191

Table 1: *Point estimates of the Cobb-Douglas parameters following the different approaches.*

5 Conclusion

In this paper we have shown how the parametric approximation of α -quantile frontiers offers an attractive alternative to the approximation of the order- m frontier, as proposed by Florens and Simar (2005). The former benefits from better robustness properties to outliers or extreme values than the latter, both approaches being more robust than the parametric approximation of the full frontier using the FDH as first step.

It confirms also that these approaches are much more appealing than the classical parametric estimators of the frontier, that are mostly based on regression ideas and fit the shape of the center of the cloud of points rather than its boundary.

The paper provides all the statistical theory (asymptotic). We first prove functional convergence results of the quantile-type frontier and establish its uniform complete convergence. Then we prove the root- n consistency and asymptotic normality of the resulting estimators of the parametric approximation of the partial order- α frontier.

We also improve some convergence results obtained by Florens and Simar (2005) when estimating the parameters of the full frontier. We provide new results for estimating these parameters, by using the more robust order- m or order- α partial frontiers in the first step: we show the complete convergence of these estimators, by choosing m and α as appropriate functions of n . Finally the good behavior of our method in finite samples is illustrated through simulated and real data sets.

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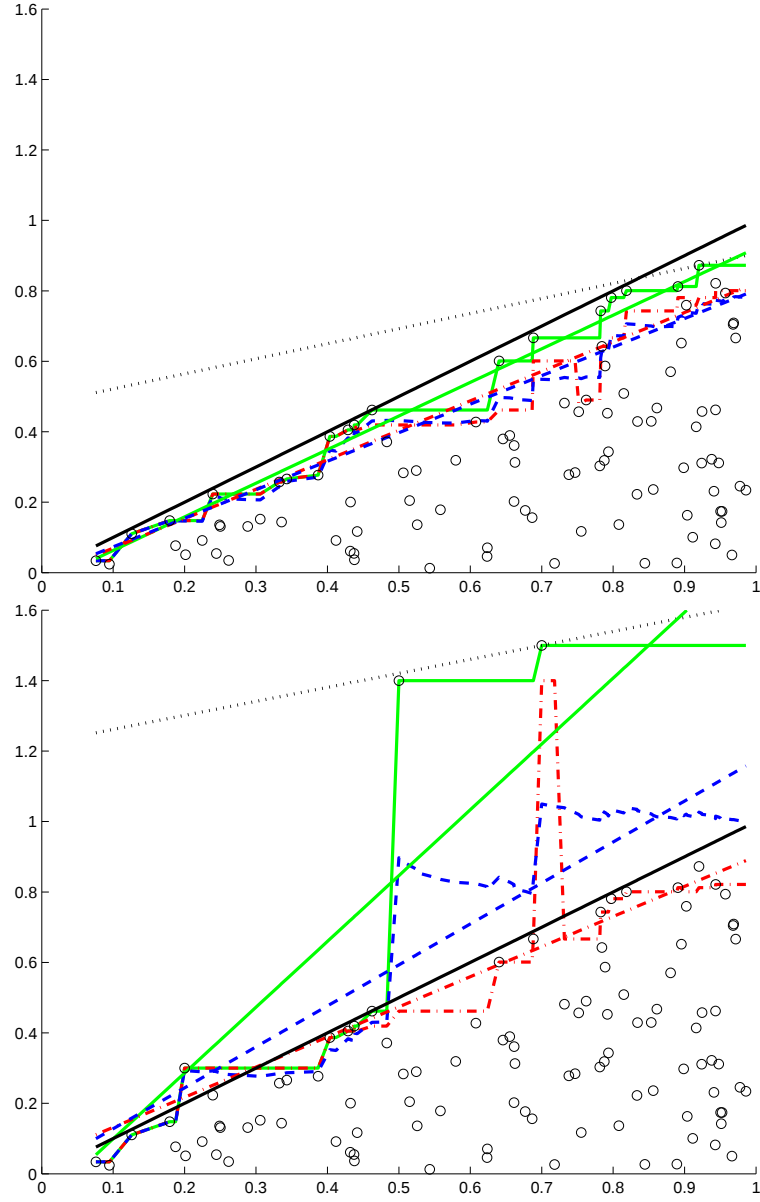


Figure 1: *Results for Example 1. In solid black line, the true frontier $y = x$. In cyan solid, the FDH frontier estimate and its corresponding parametric fit, in blue dashed the order- m frontier $\hat{\varphi}_{m,n}$, and its corresponding parametric fit and in dash-dot red the order- α frontier $\hat{q}_{\alpha,n}$ and its corresponding parametric fit. Here, $m = 20$ and $\alpha = .9622$. From top to bottom 100 simulated observations without outliers, 100 simulated observations with 3 outliers included. In black dotted, the shifted OLS estimate.*

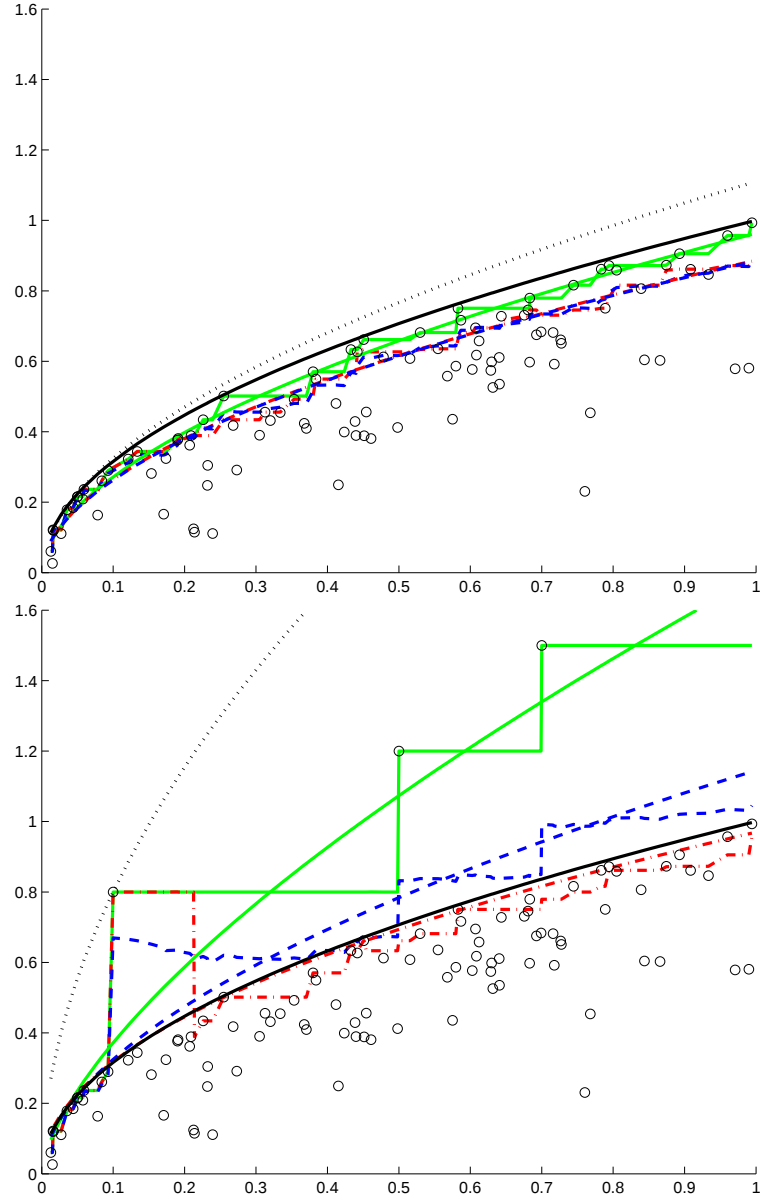


Figure 2: *Results for Example 2. In solid black line, the true frontier $y = x^{0.5}$. In cyan solid, the FDH frontier estimate and its corresponding parametric fit, in blue dashed the order- m frontier $\hat{\varphi}_{m,n}$, and its corresponding parametric fit and in dash-dot red the order- α frontier $\hat{q}_{\alpha,n}$ and its corresponding parametric fit. Here, $m = 20$ and $\alpha = .9622$. From top to bottom 100 simulated observations without outliers, 100 simulated observations with 3 outliers included. In black dotted, the shifted OLS estimate.*

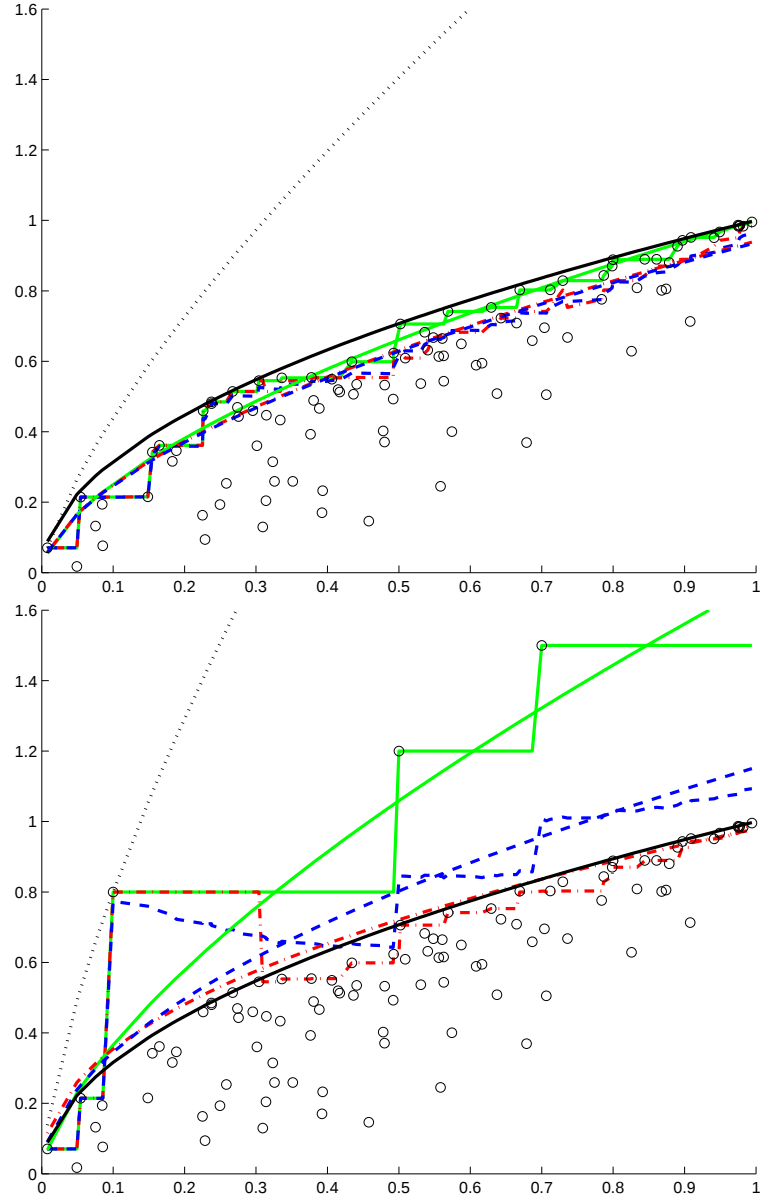


Figure 3: *Results for Example 3. In solid black line, the true frontier $y = x^{0.5}$. In cyan solid, the FDH frontier estimate and its corresponding parametric fit, in blue dashed the order- m frontier $\hat{\varphi}_{m,n}$, and its corresponding parametric fit and in dash-dot red the order- α frontier $\hat{q}_{\alpha,n}$ and its corresponding parametric fit. Here, $m = 20$ and $\alpha = .9622$. From top to bottom 100 simulated observations without outliers, 100 simulated observations with 3 outliers included. In black dotted, the shifted OLS estimate.*

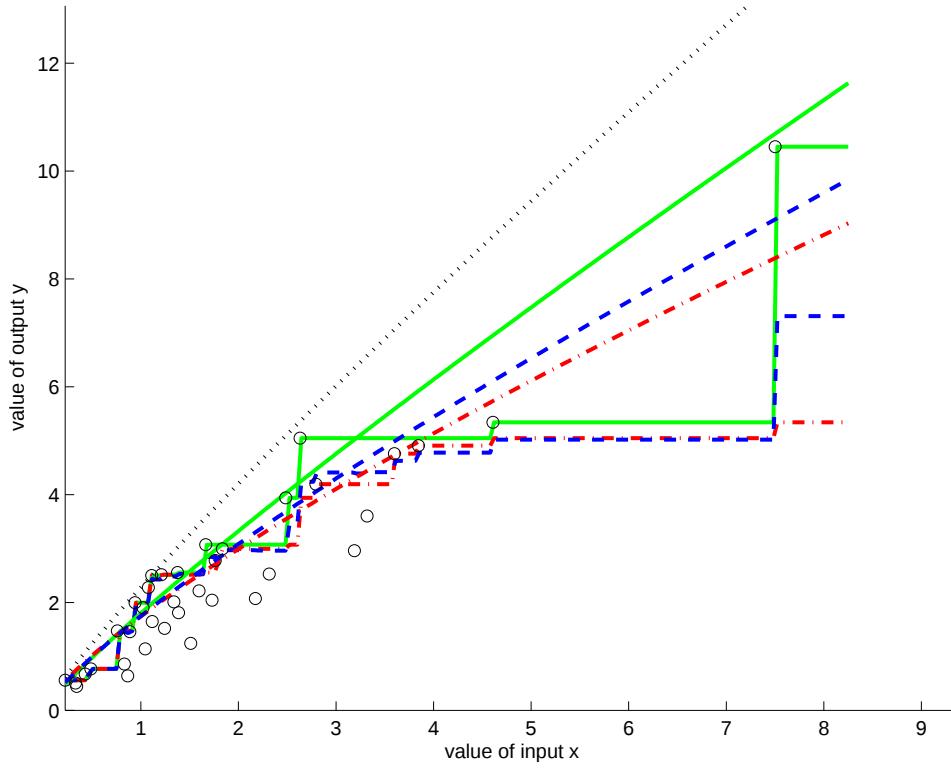


Figure 4: *Air controllers example.* In cyan solid, the *FDH* frontier estimate and its corresponding parametric fit, in blue dashed the order- m frontier $\hat{\varphi}_{m,n}$, and its corresponding parametric fit and in dash-dot red the order- α frontier $\hat{q}_{\alpha,n}$ and its corresponding parametric fit. Here, $m = 20$ and $\alpha = .95$. In black dotted, the shifted *OLS* estimate.