CORE DISCUSSION PAPER
9920

COALITIONAL NEGOTIATION

Ana Mauleon[¤] and Vincent Vannetelbosch[y]

March 1999

Abstract

We develop a two-stage negotiation model to study the impact of costly inspections on both the coalition formation outcome and the per-member payo®s. In the ¯rst stage, the players are forming coalitions and inside each coalition formed the members share the coalition bene¯ts. We adopt the largest consistent set (LCS) to predict which coalition structures are possibly stable. We also introduce a re¯nement, the largest cautious consistent set (LCCS). In the second stage, the inspection game takes place inside each coalition. For games with positive spillovers, many coalition structures may belong to the LCS under costless inspection. The grand coalition, which is the e±cient coalition structure, always belongs to the LCS and is the unique one to belong to the LCCS. Under costly inspection, the grand coalition does not always belong to the LCS. Nevertheless, there exists inspection cost parameters such that the LCS singles out the grand coalition.

Keywords: coalition formation, inspections, positive spillovers, largest consistent set.
JEL Classi¯cation: C70, C71, C72, C78.

[¤]Department of Economic Analysis, Basque Country University.
[y]Institute of Public Economics, Basque Country University. E-mail: vv@bl.ehu.es

We wish to thank Francis Bloch for helpful comments. The research of Vincent Vannetelbosch started at CORE and has been made possible by a fellowship of the Basque Country government.
This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's O±ce, Science Policy Programming. The scienti¯c responsability is assumed by the authors.

1 Introduction

Coalitional bargaining models ask how multiple players agree on a payo® distribution and which coalition structure emerges at equilibrium. See Binmore [4], Chatterjee et al. [8], Okada [17] among others.

Much of real-world negotiations involve not only bargaining but also the compliance with the agreement. Signatories may be unwilling to comply with the agreement reached, unless additional measures are taken. To control the compliance with the agreement inspection systems can be implemented. See Avenhaus et al. [1] for a survey of the recent literature on inspection games. Inspections systems are mathematical models of a situation where an inspector veri¯es that another party adheres to certain legal rules. Various inspection systems may be considered as shown in Kilgour [14].

Up to now in the analysis of economic situations such as environmental agreements or cartel formations, the compliance problem has been neglected. See Barrett [2], Bloch [5],[6] or Yi [24]. It has been implicitly assumed that inspections were perfect and costless. Costless perfect inspection means that someone who violates the agreement is detected with probability one and without cost.

Moreover, it has been shown empirically that both inspections and the threat of an inspection have a strong impact on the compliance with the agreement. For the case of the pulp and paper industry in Quebec, Laplante and Rilstone [16] have found that the threat of an inspection as well as actual inspections has a signi¯cant negative impact on pollution emissions. Similar results are obtained by Helland [12] for the pulp and paper industry in USA.

What is the impact of costly inspections on both the coalition formation outcome and the per-member payo®s? To answer this question we develop a two-stage negotiation model. In the ¯rst stage, players are forming coalitions and inside each coalition formed the members share the coalition bene¯ts. In the second stage, given the coalition structure formed, the inspection game takes place inside each coalition.

Regarding the coalition formation, we adopt the largest consistent set (LCS) de¯ned by Chwe [9] to predict which coalition structures are possibly stable. The LCS presents some advantages compared to noncooperative simultaneous or sequential games of coalition formation[1]. Firstly, the LCS

[1] For a survey of the noncooperative models of coalition formation in games with spillovers, the reader is referred to Bloch (1997). The most disturbing feature of simultaneous coalition formation games (see e.g. Hart and Kurz [11], Yi [24]) is that the players cannot be farsighted. Sequential games (see Bloch [6]) remedy to this drawback,

does not rely on a very detailed description of the coalition formation process. No commitment assumption is imposed. Secondly, the LCS incorporates the farsightedness of the coalitions. That is, a coalition considers the possibility that, once it acts, another coalition might react, a third coalition might in turn react, and so on without limit. The LCS aims to be a weak concept which rules out with con¯dence, but is not so good at picking out. Hence, we introduce a re¯nement, called the largest cautious consistent set (LCCS).

Without too much loss of generality, we assume equal sharing of the coalition bene¯ts. Indeed, this sharing rule could be derived endogenously. Also, four conditions are imposed on the bene¯ts derived from the cooperation. A ¯rst condition restricts the analysis to games with positive spillovers, where the formation of a coalition by other players increases the bene¯t of an agent. A second condition imposes that, in any coalition structure, small coalitions have higher per-member bene¯ts than big coalitions. A third condition assumes free-riding incentives: a player becomes better o® leaving any coalition to be alone. A fourth condition says that the grand coalition is e±cient with respect to the sum of the players' bene¯ts. An economic situation satisfying these four conditions is the cartel formation game under Cournot competition.

The second stage of the negotiation model is the inspection stage. We model it as a noncooperative game of complete information. Inside each coalition, the coalition members have to choose simultaneously between Violation and Compliance with the agreement as well as between Inspection and No inspection of the coalition partners. The inspection system is assumed to be perfect and costly. Since each coalition member has to choose between inspecting all his partners and inspecting nobody, the inspection cost is increasing with the size of coalition. In order to obtain a closed form solution, we assume that a violation within a coalition does not a®ect the other coalitions. The two-stage negotiation model is solved backwards. Within each coalition, the inspection game has a unique symmetric Nash equilibrium where only the strategies (Compliance, No inspection), (Compliance, Inspection), (Violation, No Inspection) are played with positive probabilities. Each per-member expected payo® is equal, at equilibrium, to the per-member bene¯t reduced by an amount larger than the inspection cost.

---

but are quite sensitive to the coalition formation procedure and rely on the commitment assumption. Once some players have agreed to form a coalition they are committed to remain in that coalition.

As a benchmark we first consider the results on costless inspection. Many coalition structures may belong to the LCS in situations satisfying the four conditions imposed. Nevertheless, the stand-alone coalition structure (where all players are singletons) is never stable, while the grand coalition always belongs to the LCS. On the contrary, the LCCS singles out the grand coalition.

The main results about how costly inspection affects the stability of coalition structures are the following ones. Firstly, there always exists cost parameters such that there is a unique coalition structure belonging to the LCS and it is the grand coalition. That is, costly inspection might be used as a selection device. In a cartel formation game, beside selecting the grand coalition, costly inspection might also improve the welfare. Secondly, costly inspection favors the occurrence of stable symmetric coalition structures (where all coalitions have the same size). Thirdly, whether or not the stand-alone coalition structure and the grand coalition are in the LCS depends on the magnitude of the inspection cost. Finally, the LCS might select a non-efficient coalition structure and even delete the efficient one with respect to the sum of the players' payoffs.

To illustrate the main results, we consider two economic examples: a cartel formation game with Cournot competition and a public goods model with congestion. For these two examples, we compare the outcomes obtained under the LCS (and LCCS) with those obtained under Bloch's [6] sequential game of coalition formation. In the cartel formation game with costless inspection, the coalition structures supported by any symmetric stationary perfect equilibria of Bloch's sequential game always belong to the LCS.

2 Farsighted coalition formation

2.1 Symmetric players, equal sharing and positive spillovers

We develop a two-stage negotiation model. In the first stage, the coalition formation takes place. In the second stage, given the coalition structure, the inspection game takes place inside each coalition.

Definition 1 A coalition structure $P = \{S_1, S_2, ..., S_m\}$ is a partition of the player set $N = \{1, 2, ..., n\}$, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{m} S_i = N$.

Let $\mathcal{P}$ be the finite set of coalition structures. We denote by $B(S_i; P)$ the benefits obtained from cooperation by coalition S_i in the coalition structure P. We assume that all players are identical or symmetric. Therefore, without too much loss of generality, we assume equal sharing of the coalition benefits among coalition members.[2] Let $|S_i|$ be the cardinality of coalition S_i.

Assumption In any coalition S_i belonging to P, the benefits are divided equally among its members: $b_j(S_i;P) = b_l(S_i;P) = |S_i|^{-1} \cdot B(S_i;P)$ for all $j,l \in S_i$, $i = 1,\ldots,m$.

So, we denote by $b(S_i;P)$ the benefit of any player belonging to S_i in the coalition structure P. Throughout the paper, we focus on games satisfying the following conditions on the per-member benefits. Condition (P.1) restricts our analysis to games with positive spillovers, where the formation of a coalition by other players increases the benefit of an agent. Formally,

(P.1) $b(S_i; P\setminus\{S_1,S_2\} \cup \{S_1 \cup S_2\}) > b(S_i;P)$ for all $j \in S_i$, $S_i \neq S_1, S_2$.

Condition (P.2) imposes that, in any coalition structure, small coalitions have higher per-member benefits than big coalitions.

(P.2) $b(S_i;P) < b(S_j;P)$ if and only if $|S_i| > |S_j|$.

Condition (P.3) is related to the existence of individual free-riding incentives. That is, if a player leaves any coalition to be alone, then he is better off.

(P.3) $b(\{j\}; P\setminus\{S_i\} \cup \{S_i \setminus \{j\}, \{j\}\}) > b(S_i;P)$ for all $j \in S_i$, $S_i \in P$.

Finally, condition (P.4) assumes that the grand coalition is efficient with respect to the benefits. We denote by $b(N)$ the benefits of any player belonging to the grand coalition $\{N\}$.

(P.4) $\nexists\, P = \{S_1, S_2, \ldots, S_m\} \in \mathcal{P}$ such that $P \neq \{N\}$ and $\sum_{i=1}^{m} b(S_i;P) \cdot |S_i| \geq b(N) \cdot |N|$.

---

[2] By introducing a bargaining stage between the coalition formation and inspection stages, the equal sharing rule emerges as the symmetric Nash bargaining solution or as the limit solution of a noncooperative game that sets out a particular bargaining procedure (see e.g. Krishna and Serrano [15], Vannetelbosch [22]). Recently, Ray and Vohra [20] have provided another justification for this assumption of equal sharing rule. Indeed, in an infinite-horizon model of coalition formation among symmetric players with endogenous bargaining, they have shown that in any equilibrium without delay there is equal sharing.

Under perfect and costless inspection, the benefits of a player belonging to S in P are equal to his expected payoffs. But once we allow for costly inspection, we distinguish between benefits and expected payoffs in order to analyze how costly inspection affects the coalition formation. Moreover, since the rule of division of the benefits among coalition members is fixed and the inspection game has a unique symmetric Nash equilibrium (as shown later on), each player is able to evaluate directly the expected payoff he obtains in different coalition structures. So, the gains from forming coalitions can be represented by a valuation V. By $V_j(S_i; P)$ we denote the expected payoff of player j belonging to S_i in the coalition structure P.

Definition 2 A valuation V is a mapping which associates to each coalition structure P a vector of individual expected payoffs in $\mathbb{R}^n$.

2.2 Farsighted coalitional stability

We adopt the largest consistent set (LCS) defined by Chwe [9] to predict which coalition structures are possibly stable.

2.2.1 The effectiveness relation

How the coalition formation proceeds? What coalitions can do if and when they form is specified by $\{\to_S\}_{S \subseteq N, S \neq \emptyset}$, where $\{\to_S\}$, $S \subseteq N$, is an effectiveness relation on $\mathcal{P}$. For any $P, P' \in \mathcal{P}$, $P \to_S P'$ means that if the coalition structure P is the status-quo, coalition S can make the coalition structure P' the new status-quo. After S deviates to P' from P, coalition S' might move to P'' where $P' \to_{S'} P''$, etc. All actions are public. If a status-quo P is reached and no coalition decides to move from P, then P is a stable coalition structure. Restrictions are imposed on the coalition formation process. Indeed, we define the effectiveness relation $\{\to_S\}$ as follows: $P \to_S P'$ if and only if (i) $\{S_i \setminus (S_i \cap S) : S_i \in P\} = \{S_i' \in P' : S_i' \subseteq N \setminus S\}$ and (ii) $\exists \{S_1', \dots, S_l'\} \subseteq P'$ such that $\bigcup_{j=1}^{l} S_j' = S$. Condition (i) simply means that no simultaneous deviations are possible. If the players in S deviate leaving their coalition(s) in P, the non-deviating players do not move. Nevertheless, once S has moved, the players not in S can react to the deviation of S. Condition (ii) simply allows the deviating players in S to form one or several coalitions in the new status-quo P'. Non-deviating players do not belong to those new coalitions.

2.2.2 The largest consistent set (LCS)

Based on the indirect dominance relation, Chwe [9] defined the largest consistent set (LCS) assuming that players have strict preferences over outcomes or coalition structures. But the coalition formation games we study do not exclude that some players are indifferent among some coalition structures. So, we define the indirect weak dominance relation. A coalition structure P' indirectly weakly dominates P if P' can replace P in a sequence of moves, such that at each move all deviators weakly prefer the ending coalition structure P' to the status-quo they face, and at least one deviator strictly prefers P'. Formally, indirect weak dominance is defined as follows.

Definition 3 The coalition structure P is indirectly weakly dominated by P', or $P \ll P'$, if there exists a sequence $P^0, P^1, \ldots, P^m$ (where $P^0 = P$ and $P^m = P'$) and a sequence $S_0, S_1, \ldots, S_{m-1}$ such that $P^j \rightarrow_{S_j} P^{j+1}$, $V_i(\cdot, P') \geq V_i(\cdot, P^j)$ for all $i \in S_j$, and $V_i(\cdot, P') > V_i(\cdot, P^j)$ for some $i \in S_j$, for $j = 0, 1, \ldots, m-1$.

Direct weak dominance, by setting $m = 1$ in Definition 3, entails that coalitions look only at the next step. But farsighted coalitions do not just look at the next step. The coalition structure P is directly weakly dominated by P', or $P < P'$, if there exists a coalition S such that $P \rightarrow_S P'$, $V_i(S, P') \geq V_i(S, P)$ for all $i \in S$ and $V_i(S, P') > V_i(S, P)$ for some $i \in S$. Obviously, if $P < P'$ then $P \ll P'$. Notice that, if P is indirectly dominated by P', then P is indirectly weakly dominated by P'. Of course the reverse is not true.

To check if a coalition structure P is stable, consider a deviation by coalition S to P'. There might be further deviations which end up at P'', where $P' \ll P''$. There might not be any further deviations, in which case the ending coalition structure $P'' = P'$. In any case, the ending coalition structure P'' should itself be stable. If some member of coalition S does not weakly prefer P'' to the original coalition structure P, then the deviation is deterred. A coalition structure P is stable if every deviation is deterred. As in Chwe [9] the LCS can be defined in an iterative way.[3]

Definition 4 Let $Y^0 \equiv \mathcal{P}$. Then, Y^k $(k = 1, 2, \ldots)$ is inductively defined as follows: $P \in \mathcal{P}$ belongs to Y^k if and only if $\forall\, P', S$ such that $P \rightarrow_S P'$, $\exists\, P'' \in Y^{k-1}$, where $P' = P''$ or $P' \ll P''$; such that we do not have $V_i(\cdot, P) \leq V_i(\cdot, P'')$ for all $i \in S$ and $V_i(\cdot, P) < V_i(\cdot, P'')$ for some $i \in S$.

[3] Chwe [9] has shown that there uniquely exists a LCS defined from the indirect dominance relation. Xue [23] has extended Chwe's nonemptiness result of the LCS by relaxing the assumption of countability on the set $\mathcal{P}$.

Since $\mathcal{P}$ is finite, there exists $n \in \mathbb{N}$ such that $Y^k = Y^{k+1}$ for all $k \geq n$, and Y^n is the LCS, Γ.

If a coalition structure is not in the LCS, it cannot be stable. The LCS is the set of all coalition structures which can be possibly stable.

2.2.3 The largest cautious consistent set (LCCS)

In most situations, like the rationalizability concepts, the LCS does not determine what will happen but what can possibly happen. To give sharper predictions it is necessary to refine the LCS. Applying the spirit of some refinements of the rationalizability concept (see Bernheim [3], Herings and Vannetelbosch [13], Pearce [18]) to the LCS leads to the definition of the largest cautious consistent set (LCCS).

Definition 5 Let $Z^0 \equiv \mathcal{P}$. Then, Z^k $(k = 1, 2, \ldots)$ is inductively defined as follows: $P \in \mathcal{P}$ belongs to Z^k if and only if $\forall\, P', S$ such that $P \rightarrow_S P'$, $\exists\, \alpha = (\alpha(P_1), \ldots, \alpha(P_m))$ satisfying $\sum_{j=1}^{m} \alpha(P_j) = 1$, $\alpha(P_j) \in (0,1)$, that gives positive weight to each $P_j \in Z^{k-1}$, where $P' = P_j$ or $P' \ll P_j$, such that we do not have

$$V_i(\cdot, P) \leq \sum_{\substack{P_j \in Z^{k-1} \\ P' = P_j \text{ or } P' \ll P_j}} \alpha(P_j) \cdot V_i(\cdot, P_j) \text{ for all } i \in S, \text{ and}$$

$$V_i(\cdot, P) < \sum_{\substack{P_j \in Z^{k-1} \\ P' = P_j \text{ or } P' \ll P_j}} \alpha(P_j) \cdot V_i(\cdot, P_j) \text{ for some } i \in S.$$

Since $\mathcal{P}$ is finite, there exists $n \in \mathbb{N}$ such that $Z^k = Z^{k+1}$ for all $k \geq n$, and Z^n is the LCCS, Φ.

The idea behind the LCCS is that once a coalition S deviates from P to P', this coalition S should contemplate the possibility to end with positive probability at any coalition structure P'' not ruled out and such that $P' = P''$ or $P' \ll P''$. Hence, a coalition structure P is never stable if a coalition S can engage a deviation from P to P' and doing so there is no risk that some coalition members will end worse off. Obviously, the LCCS is a refinement of LCS, $\Phi(\mathcal{P}) \subseteq \Gamma(\mathcal{P})$.

3 The inspection game

3.1 Strategies, outcomes and payoffs

The second stage of the negotiation model is the inspection stage. Coalition members may be unwilling to comply with the agreement reached, unless additional measures are taken. To enforce the compliance inspection systems can be implemented. As discussed in Kilgour [14], two major features characterize the inspection systems. A first one is the quality of information: inspection results may be ambiguous or imperfect. A second one is the inspection cost.

We model the inspection stage as a noncooperative game of complete information played among the coalition members. Inside each coalition $S \in P$, the coalition members have to choose simultaneously between Violation and Compliance with the agreement as well as between Inspection and No inspection of the coalition partners.

Single level of violation. There is only one level of violation: each coalition member who is violating tries to appropriate all the coalition benefits.

So, a player who violates has no further choices, such as when, where, or how to violate. Thus, a player is deprived of any actions that might help to conceal a violation, and one possible cause of ambiguity, a low level of violation, is not considered.

Single level of inspection. There is only one level of inspection which consists in inspecting all the coalition partners.

The inspection system is assumed to be perfect but costly. Perfect inspection means that someone who violates the agreement is detected with probability one. Since each coalition member has to choose between inspecting all his partners and inspecting nobody, the inspection cost is increasing with the size of coalition S. We denote by $c(S)$ the inspection cost supported by each player inspecting and belonging to S.

In the inspection game within coalition S, each coalition member has four pure strategies: "Compliance, No inspection", "Compliance, Inspection", "Violation, No Inspection" and "Violation, Inspection". We denote these four strategies by (C,N), (C,I), (V,N) and (V,I), respectively. In order to obtain a closed form solution, we simplify as much as possible the inspection game. Thus, we assume that a violation within a coalition $S \in P$ does not affect the other coalitions $S' \neq S; S' \in P$. That is, the scope for violating is limited to the benefits of the coalition. We classify the outcomes and its associated payoffs in four categories.

I. No violation. Each member $i \in S$ obtains as payoff $b(S;P)$ if he does not inspect or $b(S;P) - c(S)$ if he inspects.

Given the single level of violation, a violation is said to be successful if and only if one player violates while the others do not inspect.

II. Successful violation. The player who is violating will grab all the benefits and will get as payoff $B(S;P)$ (if he does not inspect) or $B(S;P) - c(S)$ (if he inspects), while his coalition partners obtain 0 as payoff.

Two kinds of unsuccessful violation are considered, with or without apprehension.

III. Unsuccessful violation without apprehension. That is, more than one coalition members is violating but nobody is inspecting. Given the single level of violation, all the members of coalition S get 0 as payoff.

The players who are complying lose all their benefits because of the violation, while the violating players obtain also zero as payoff because all of them want to appropriate the same benefits. A possible interpretation is that the benefits are passing from hand to hand among the violating players without stopping. So, the assumption of zero payoff for everybody does not rely on the total destruction of the coalition benefits.

IV. Unsuccessful violation with apprehension. A player who violates and is apprehended obtains as payoff 0 (if he does not inspect) or $-c(S)$ (if he inspects); a player who inspects and is not apprehended obtains $\gamma \cdot b(S;P) - c(S)$ with $\gamma \in [0;1)$; a player who complies and does not inspect obtains 0 as payoff.

The amount $\gamma \cdot b(S;P)$ can be interpreted as a reward for detecting and reporting violations. The reward is assumed to be smaller than the agreed benefits due to legal proceedings costs, for example. Passivity (relying on somebody else to inspect) is not remunerated. Notice that a violating player can be apprehended by another player who is also violating. This last player will be rewarded only if nobody else is inspecting.

3.2 The Symmetric Nash Equilibrium

The two-stage negotiation model is solved backwards. Given the assumptions of symmetric players and equal sharing of the coalition benefits, we look only for a symmetric Nash equilibrium of the inspection game played within each coalition S. Let β_1^i be the probability of player i of playing the pure strategy (C,I); β_2^i be the probability of player i of playing the pure strategy (C,N); β_3^i be the probability of player i of playing the pure strategy (V,I); and β_4^i be the probability of player i of playing the pure strategy (V,N). A symmetric Nash equilibrium implies that $\beta_1^i = \beta_1$, $\beta_2^i = \beta_2$, $\beta_3^i = \beta_3$, $\beta_4^i = \beta_4$ for all $i \in S$. Then, the expected payoffs are, respectively, for (C,I), (C,N), (V,I), (V,N):

$$\left[(1-\gamma)(\beta_1+\beta_2)^{|S|-1}+\gamma\right] b(S;P) - c(S);$$

$$(\beta_1+\beta_2)^{|S|-1} b(S;P);$$

$$(\beta_2)^{|S|-1} B(S;P) - c(S) + \left[\sum_{m=1}^{|S|-1}\binom{|S|-1}{m}(\beta_2)^{|S|-1-m}(\beta_4)^m\right]\gamma b(S;P);$$

$$(\beta_2)^{|S|-1} B(S;P);$$

with the binomial coefficient

$$\binom{|S|-1}{m} = \frac{(|S|-1)!}{m!\cdot(|S|-1-m)!}.$$

For $\gamma \cdot b(S;P) > c(S)$, the symmetric Nash equilibrium $(\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*)$ of the inspection stage is

$$\beta_1^* = \left(1-\left(\frac{b(S;P)}{B(S;P)}\right)^{\frac{1}{|S|-1}}\right)\left(\frac{\gamma\cdot b(S;P)-c(S)}{\gamma\cdot b(S;P)}\right)^{\frac{1}{|S|-1}}; \quad \beta_3^* = 0;$$

$$\beta_2^* = \left(\frac{\gamma\cdot b(S;P)-c(S)}{\gamma\cdot B(S;P)}\right)^{\frac{1}{|S|-1}}; \quad \beta_4^* = 1 - \left(\frac{\gamma\cdot b(S;P)-c(S)}{\gamma\cdot b(S;P)}\right)^{\frac{1}{|S|-1}};$$

and, for $\gamma \cdot b(S;P) \le c(S)$, the symmetric Nash equilibrium $(\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*)$ is $\beta_1^* = \beta_2^* = \beta_3^* = 0$ and $\beta_4^* = 1$. Other asymmetric Nash equilibria may exist in the inspection game. In the rest of the analysis it is assumed that only the symmetric Nash equilibrium will occur. Therefore, the expected payoff (at equilibrium) of player i belonging to S in the coalition structure P is given by

$$V_i(S;P) = \begin{cases} b(S;P) - \gamma^{-1}\cdot c(S) & \text{if positive} \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } i \in S.$$

and denoted $V(S;P)$ from now on[4], with $b(S;P) = B(S;P)\cdot|S|^{-1}$.

[4] In other words, costly perfect inspection provides the foundation for introducing coalition formation costs or organizational costs which increase with the size of the coalition.

4 Farsighted stable agreements

In order to characterize the stable coalition structures for the two-stage negotiation model, beside the conditions imposed on the benefits $b(S;P)$, an additional assumption on the inspection cost is introduced.

Assumption For all $S \subseteq N$, $c(S) = \begin{cases} c \cdot (|S| - 1)^{\phi} & \text{for } |S| > 1 \\ 0 & \text{for } |S| = 1 \end{cases}$, with $c, \phi > 0$.

For $c = 0$, the inspection is said to be costless. For $c > 0$, the inspection is said to be costly and its cost increases with the coalition size. Before stating the results, we introduce some definitions or notations. A coalition structure P is symmetric if and only if $|S_i| = |S_j|$ for all $S_i, S_j \in P$. By $P^* = \{N\}$ we denote the grand coalition; P^* is a symmetric coalition structure. We denote by $\overline{P}$ the stand-alone coalition structure: $\overline{P} = \{S_1, \ldots, S_n\}$ with $|S_i| = 1$ for all $S_i \in \overline{P}$.

4.1 Stable coalition structures with costless inspection

The following three lemmas partially characterize the LCS for the two-stage negotiation model under costless inspection. Lemma 1 states that any coalition structure, wherein some coalition members would receive less than in the stand-alone coalition structure, is never stable.

Lemma 1 Under (P.1)-(P.4) and costless inspection, if there exists $S \in P$ such that $V(S;P) < V(S';\overline{P})$, then $P \notin \Gamma(\mathcal{P})$.

Proof. Under costless inspection ($c = 0$), $V(S;P) = b(S;P) = |S|^{-1} \cdot B(S;P)$. Condition (P.2) implies that in any coalition structure P, $V(S_i;P) < V(S_j;P)$ if and only if $|S_i| > |S_j|$. To prove Lemma 1, we proceed by steps.

Step one. Firstly, we show that all coalition structures $P \in \mathcal{P}$ containing only one coalition S with $|S| > 1$ and $V(S;P) < V(S';\overline{P})$, do not belong to $\Gamma(\mathcal{P})$. Obviously, $P < \overline{P}$ and the deviation $P \to_S \overline{P}$ cannot be deterred. Indeed, any deviation from $\overline{P}$ of players that did not belong to S in P will improve the payoff of players that were in S (in P) and are singletons in $\overline{P}$. Therefore, $P \notin \Gamma(\mathcal{P})$.

Step two. Secondly, we show that all coalition structures $P \in \mathcal{P}$ containing only two coalitions S_1, S_2 with $|S_1| \geq |S_2| > 1$ and $V(S_1;P) < V(S';\overline{P})$, do not belong to $\Gamma(\mathcal{P})$. Condition (P.1) implies that the coalition S_1 has incentives to split into singletons. Indeed, $V(S_1, P') > V(S', \overline{P})$ and $P \ll P'$ where $P' = P \setminus S_1 \cup \{j\}_{j \in S_1}$. The deviation $P \to_{S_1} P'$ cannot be deterred. Indeed,

- if $V(S_2, P') < V(S', \overline{P})$, then using the argumentation of step one, the deviation $P' \to_{S_2} \overline{P}$ is not deterred and $\overline{P} \gg P$. Therefore, $P \notin \Gamma(\mathcal{P})$.

- if $V(S_2, P') > V(S', \overline{P})$, we have to show that any deviation from P' of players in $N \setminus S_1$ will never make players in S_1 worse off than in P. Two kind of deviations are possible. First, the players in S_2 form a bigger coalition with players not in S_1. Then, by condition (P.1), the players in S_1 that now are singletons obtain a payoff even greater than in P'. Second, some player(s) leave(s) S_2 to form singleton(s). Then, the players that were in S_1 are worse than in P' but, by (P.1), they are better off than in $\overline{P}$, and $V(S_1, P) < V(S', \overline{P})$. Therefore, there is no other coalition structure P'' such that $P'' \gg P'$ and $V(\{j\}, P'') < V(S_1, P)$ for some $j \in S_1$. Hence, $P \notin \Gamma(\mathcal{P})$.

Step three. Thirdly, proceeding as above, we can show that all coalition structures $P \in \mathcal{P}$ containing only three coalitions S_1, S_2, S_3 with $|S_1| \geq |S_2| \geq |S_3| > 1$ and $V(S_1, P) < V(S', \overline{P})$, do not belong to $\Gamma(\mathcal{P})$. And so on. ■

In Lemma 2, we show that the stand-alone coalition structure, i.e. the coalition structure consisting only of singletons, is never stable under costless inspection.

Lemma 2 Under (P.1)-(P.4) and costless inspection, $\overline{P} \notin \Gamma(\mathcal{P})$.

Proof. From Definition 4 and Lemma 1, we have that $Y^0 \equiv \mathcal{P}$ and $Y^1 = \{P \in \mathcal{P} : \forall P', S$ such that $P \to_S P'$, $\exists P'' \in Y^0$, where $P' = P''$ or $P' \ll P''$, we do not have $V_i(\cdot, P) \leq V_i(\cdot, P'')$ for all $i \in S$ and $V_i(\cdot, P) < V_i(\cdot, P'')$ for some $i \in S\} \subseteq \mathcal{P} \setminus \{P \in \mathcal{P} : \exists S \in P$ for which $V(S, P) < V(S', \overline{P})\}$. Next we show that $\overline{P} \notin Y^2 = \{P \in \mathcal{P} : \forall P', S$ such that $P \to_S P'$, $\exists P'' \in Y^1$, where $P' = P''$ or $P' \ll P''$, we do not have $V_i(\cdot, P) \leq V_i(\cdot, P'')$ for all $i \in S$ and $V_i(\cdot, P) < V_i(\cdot, P'')$ for some $i \in S\}$. Any coalition structure $P \in Y^1$ is such that $\forall S \in P$: $V(S, P) \geq V(S', \overline{P})$. By (P.2) and (P.4), the coalition structure $P^* = \{N\}$ is efficient and $V(N) > V(S', \overline{P})$ for all $i \in N$. Therefore, $\overline{P} \notin Y^2$ because the deviation $\overline{P} \to_N P^*$ cannot be deterred. Indeed, for all $P'' \in Y^1$, where $P^* = P''$ or $P^* \ll P''$, we have $V_i(\cdot, \overline{P}) \leq V_i(\cdot, P'')$ for all $i \in N$ and $V_i(\cdot, \overline{P}) < V_i(\cdot, P'')$ for some $i \in N$, by (P.1). ■

On the contrary to the stand-alone coalition structure, the grand coalition structure (which is the efficient one) always belongs to the LCS, and is possibly stable.

Lemma 3 Under (P.1)-(P.4) and costless inspection, $P^* \in \Gamma(\mathcal{P})$.

Proof. To prove that $P^* \in Y^k$ $(k \geq 1)$ we have to show that for all $P \neq P^*$ we have $P \ll P^*$. That is, we show that P^* could be stable since any deviation $P^* \to_S P$ can be deterred by the threat of ending in P^*. The proof is done in two steps.

Step A. By (P.2) and (P.4) the players belonging to the biggest coalition (in size) in any $P \neq P^*$ are worse than in P^*. Also, all players prefer P^* to $\overline{P}$, and $P^* > \overline{P}$.

Step B. Take the sequence of moves where at each move one player belonging to the biggest coalition (in the current coalition structure) deviates to form a singleton, until the coalition structure $\overline{P}$ is reached. From $\overline{P}$ occurs the deviation $\overline{P} \to_N P^*$.

Therefore, (A)-(B) imply that $P^* \gg P$ for all $P \neq P^*$. ■

From these three lemmas, we obtain a sufficient condition such that the LCS singles out the grand coalition under costless inspection.

Proposition 1 Under (P.1)-(P.4) and costless inspection ($c = 0$), if each non-symmetric coalition structure $P \in \mathcal{P}$ is such that there exists $S \in P$ satisfying $V(S; P) < V(S'; \overline{P})$, then $\Gamma(\mathcal{P}) = \{P^*\}$.

Proof. Lemma 1 tells us that coalition structures $P \in \mathcal{P}$, where $\exists\, S \in P$ such that $V(S; P) < V(S'; \overline{P})$, do not belong to $\Gamma(\mathcal{P})$. Lemma 2 tells us that $\overline{P} \notin \Gamma(\mathcal{P})$. The conditions (P.2) and (P.4) imply that all symmetric coalition structures $P (\neq P^*)$ are such that $V(S_i; P) = V(S_j; P)$ and $V(N) > V(S_i; P)$ for all $S_i; S_j \in P$ (it implies that $P^* > P$ for all $P (\neq P^*)$ symmetric). So, the deviation $P \to_N P^*$ (where P symmetric) cannot be deterred since $\nexists P'$ such that $P' \gg P^*$ and $P' \in Y^1$. Therefore, $\Gamma(\mathcal{P}) = \{P^*\}$. ■

But in most economic situations satisfying the conditions (P.1)-(P.4), many coalition structures belong to the LCS. Indeed, the LCS aims to be a weak concept which rules out with confidence. On the contrary, the LCCS aims to be better at picking out. Under costless inspection, the LCCS singles out the grand coalition.

Proposition 2 Under (P.1)-(P.4) and costless inspection, $\Phi(\mathcal{P}) = \{P^*\}$.

Proof. From Definition 5 we have $Z^0 = \mathcal{P}$. By (P.2) and (P.4) the players belonging to the biggest coalition (in size) in any $P \in \mathcal{P}\setminus\{P^*\}$ are worse than in P^*.

Step one. From Lemma 1 and Definition 5, it is straightforward that the set of coalition structures $\{P \in \mathcal{P} : \exists\, S \in P$ such that $V(S; P) < V(S'; \overline{P})\}$ does not belong to Z^1. From Lemma 2 and Definition 5, it is also straightforward that $\overline{P} \notin Z^2$ ($\overline{P} \notin \Phi(\mathcal{P})$).

Step two. Take the coalition structure(s) $P \in Z^2 = \mathcal{P}\setminus\{\overline{P}\} \cup \{P \in \mathcal{P} : \exists\, S \in P$ such that $V(S; P) < V(S'; \overline{P})\}$ containing the coalition S that obtains the smallest payoff. Obviously, $P \notin Z^3$ ($P \notin \Phi(\mathcal{P})$) since for all $P'; S$ such that $P \to_S P'$, the expected payoff obtained by assigning positive probabilities to all coalition structures $P'' \in Z^2$, with $P' = P''$ or $P' \ll P''$, is strictly preferred to $V(S; P)$ for some player(s) in S and weakly preferred to $V(S; P)$ for the other player(s) in S.

Step three. Take the coalition structure(s) $P \in Z^3$ containing the coalition S that obtains the smallest payoff. Obviously, $P \notin Z^4$ ($P \notin \Phi(\mathcal{P})$) since for all $P'; S$ such that $P \to_S P'$, the expected payoff obtained by assigning positive probabilities to all coalition structures $P'' \in Z^3$, with $P' = P''$ or $P' \ll P''$, is strictly preferred to $V(S; P)$ for some player(s) in S and weakly preferred to $V(S; P)$ for the other player(s) in S. And so on.

Since $\mathcal{P}$ is finite, we obtain (after finitely many steps) that $\Phi(\mathcal{P}) = \{P^*\}$. ■

This result is due to the basic idea behind the LCCS. Intuitively, at each iteration in the definition of the LCCS, we rule out the coalition structure wherein some players receive less or equal than what they could obtain in all candidates to be stable (i.e. all coalition structures not ruled out yet) since these players cannot end worse off by engaging a move.

4.2 Stable coalition structures with costly inspection

In this section we study how costly inspection affects the stability of coalition structures. The main result we obtain is the following one. There always exists cost parameters such that there is a unique coalition structure belonging to the LCS and it is the grand coalition. In other words, costly inspection might be used as a selection device to singles out the grand coalition.

Proposition 3 Under (P.1)-(P.4) and costly inspection, $\exists\, c; \varphi > 0$ such that $\Gamma(\mathcal{P}) = \{P^*\}$.

Proof. For $c = 0$, the conditions (P.2) and (P.4) imply that the players belonging to the biggest coalition (in size) in any $P \in \mathcal{P} \setminus \{P^*\}$ are worse than in P^*. For $c, \varepsilon > 0$, as $\varepsilon \to 0_+$ we have that $[|S - 1|^{\varepsilon}] \to 1_+$ for all coalition S with cardinality $|S| > 1$. That is, as $\varepsilon \to 0_+$ the inspection cost approaches a constant: $\gamma^{-1} \cdot c \cdot |S - 1|^{\varepsilon} \to \gamma^{-1} \cdot c$. Therefore, as $\varepsilon \to 0_+$ there exists an interval $(\underline{c}; \overline{c})$ such that for all $c \in (\underline{c}; \overline{c})$ we have that each $P \in \mathcal{P}$ $(P \neq P^*; \overline{P})$ contains (at least) one coalition S such that $V(S;P) < V(S'; \overline{P}) < V(N)$. Applying Lemmas 1-2-3, we have $\Gamma(\mathcal{P}) = \{P^*\}$. ■

We denote by $\underline{V}(\cdot;P)$ the smallest per-member payoff obtained in P; $\underline{V}(S;P)$ simply says that it is the members of S who obtain the smallest per-member payoff in P. Let $\mathcal{P}^* \in \mathcal{P}$ be defined as the set of coalition structures such that $\underline{V}(\cdot;P) \geq \underline{V}(\cdot;P')$ for all $P' \in \mathcal{P}$.

Proposition 4 Under (P.1)-(P.4) and costly inspection, $\Phi(\mathcal{P}) \subseteq \mathcal{P}^*$.

Proof. From Definition 5 we have $Z^0 = \mathcal{P}$.

Step one. Take the coalition structures $P \in Z^0$ such that $\underline{V}(\cdot;P) \leq \underline{V}(\cdot;P')$ for all $P' \in Z^0$. Obviously, $P \notin Z^1$ since for all $P'; S$ such that $P \to_S P'$, the expected per-member payoff obtained by assigning positive probabilities to all coalition structures $P'' \in Z^0$, with $P' = P''$ or $P' \ll P''$, is strictly preferred to $V(S;P)$ for some player(s) in S and weakly preferred to $V(S;P)$ for the other player(s) in S. Then, $Z^1 = Z^0 \setminus \{P \in Z^0 : \underline{V}(\cdot;P) \leq \underline{V}(\cdot;P')$ for all $P' \in Z^0\}$.

Step k. For $k \geq 1$, take the coalition structures $P \in Z^k$ such that $\underline{V}(\cdot;P) \leq \underline{V}(\cdot;P')$ for all $P' \in Z^k$. Obviously, $P \notin Z^{k+1}$ since for all $P'; S$ such that $P \to_S P'$, the expected per-member payoff obtained by assigning positive probabilities to all coalition structures $P'' \in Z^k$, with $P' = P''$ or $P' \ll P''$, is strictly preferred to $V(S;P)$ for some player(s) in S and weakly preferred to $V(S;P)$ for the other player(s) in S. Then, $Z^{k+1} = Z^k \setminus \{P \in Z^k : \underline{V}(\cdot;P) \leq \underline{V}(\cdot;P')$ for all $P' \in Z^k\}$.

Since $\mathcal{P}$ is finite, we obtain that, for some $k^* \in \mathbb{N}$, $Z^{k^*} = \mathcal{P}^*$. Therefore, $\Phi(\mathcal{P}) \subseteq \mathcal{P}^*$. ■

Some remarks about how costly inspection affects the LCS can be made. First, Lemma 1 is always true, while Lemma 2 and Lemma 3 are not. Second, symmetric coalition structures are more likely to be stable. Third, the largest consistent set could select a non-efficient coalition structure and even delete the efficient one, where efficiency is defined with respect to the sum of the players' payoffs.

5 Applications and comments

5.1 Cartel formation under Cournot competition

As in Bloch [7] and Yi [24] we consider a Cournot oligopoly with inverse demand $p(q) = a - q$, where q is the industry output. The industry consists of $|N|$ identical firms. In stage one the cartel formation takes place. Inside each cartel, we assume equal sharing of the benefits obtained from the cartel's production. In stage two, once stable agreements on cartel formation have been reached, we observe a Cournot competition among the cartels and the inspection game takes place. Assuming the symmetric Nash equilibrium of the inspection stage, the expected payoff for each firm in each possible coalition structure is well defined. Firm i's cost function is given by $d \cdot q_i$, where q_i is firm i's output and d $(a > d)$ is the common constant marginal cost. As a result, the per-member expected payoff in a cartel of size $|S|$ are given by

$$V(S;P) = \frac{(a-d)^2}{|S| \cdot (|P|+1)^2} - \frac{(|S|-1)^{\phi} \cdot c}{\gamma} \text{ for all } i \in S, \tag{1}$$

where $(a-d)^2 \cdot (|S| \cdot (|P|+1)^2)^{-1} = b(S;P)$ and $|P|$ is the number of cartels within P.

Lemma 4 Output cartels in a Cournot oligopoly with the inverse demand function $p(q) = a - q$ and the cost function $d(q_i) = d \cdot q_i$ satisfy (P.1)-(P.4):

Yi [24] asserted that conditions (P.1) and (P.2) are satisfied. Moreover, it is straightforward to show that (P.3) and (P.4) are also satisfied. Next, we provide some very interesting examples.

Example 1. $|N| = 4$, $c = 0$, $d = 0$, $a = 1$.

Partitions	Payoffs
$\{4\}$	$(.0625; .0625; .0625; .0625)$
$\{3;1\}$	$(.037; .037; .037; .111)$
$\{2;2\}$	$(.0556; .0556; .0556; .0556)$
$\{2;1;1\}$	$(.03125; .03125; .0625; .0625)$
$\{1;1;1;1\}$	$(.04; .04; .04; .04)$

In the first round of the iterative procedure to compute the LCS, we eliminate the coalition structures $\{2;1;1\}$ and $\{3;1\}$. Indeed, the deviations $\{2;1;1\} \to \{1;1;1;1\}$ and $\{3;1\} \to \{1;1;1;1\}$ are not deterred. In the second round, we eliminate $\{1;1;1;1\}$ and $\{2;2\}$. Indeed, the deviations

$\{1,1,1,1\} \to \{4\}$ and $\{2,2\} \to \{4\}$ are not deterred. Therefore, the largest consistent set is the grand coalition $\{\{4\}\}$.

This example highlights Proposition 1. Also, it tells us that, under costless inspection and for $|N| = 4$, the LCS singles out the grand coalition $P^* = \{N\}$. The next proposition gives us the full characterization of the LCS under costless inspection.

Proposition 5 In the cartel formation game under Cournot competition and costless inspection, $\Gamma(P) = \{P^*\}$ for $|N| \leq 4$, and $\Gamma(P) = P \setminus \{\{\overline{P}\} \cup \{P \in P : \exists S \in P \text{ such that } V(S;P) < V(S';\overline{P})\}\}$ for $|N| \geq 5$.

The proof of this proposition, as well as the other proofs not in the main text may be found in the appendix. Some remarks can be made. Firstly, P^* always belongs to the LCS (Lemma 3), while $\overline{P}$ never belongs to the LCS (Lemma 2). Secondly, for $|N| \geq 5$, all symmetric coalition structures, except $\overline{P}$, belong to the LCS. Finally, all non-symmetric coalition structures P such that $\nexists S \in P$ with $V(S;P) < V(S';\overline{P})$ belong to the LCS.

Example 2a. $|N| = 6$, $d = 0$, $a = 1$ and $c = 0$.

Partitions	Payoffs
$\{6\}$	(.0417; .0417; .0417; .0417; .0417; .0417)
$\{5;1\}$	(.0222; .0222; .0222; .0222; .0222; .111)
$\{4;2\}$	(.0278; .0278; .0278; .0278; .0556; .0556)
$\{3;3\}$	(.0370; .0370; .0370; .0370; .0370; .0370)
$\{4;1;1\}$	(.0156; .0156; .0156; .0156; .0625; .0625)
$\{3;2;1\}$	(.0208; .0208; .0208; .0312; .0312; .0625)
$\{2;2;2\}$	(.0312:.0312; .0312; .0312; .0312; .0312)
$\{3;1;1;1\}$	(.0133; .0133; .0133; .0400; .0400; .0400)
$\{2;2;1;1\}$	(.0200; .0200; .0200; .0200; .0400; .0400)
$\{2;1;1;1;1\}$	(.0139; .0139; .0278; .0278; .0278; .0278)
$\{1;1;1;1;1;1\}$	(.0204; .0204; .0204; .0204; .0204; .0204)

In the first round of the iterative procedure to compute the LCS, we eliminate the coalition structures $\{2;1;1;1;1\}$, $\{3;1;1;1\}$, $\{4;1;1\}$. The deviations $\{2;1;1;1;1\} \to \{1;1;1;1;1;1\}$, $\{3;1;1;1\} \to \{1;1;1;1;1;1\}$, $\{4;1;1\} \to \{1;1;1;1;1;1\}$ are not deterred. Also, we can eliminate $\{2;2;1;1\}$: the deviation $\{2;2;1;1\} \to \{2;1;1;1;1\}$ is not deterred. In the second round, we delete the coalition structure $\{1;1;1;1;1;1\}$: the deviation $\{1;1;1;1;1;1\} \to \{3;3\}$ is not deterred. No more coalition structures can be eliminated at the next rounds. For example, the deviation from $\{2;2;2\}$ to $\{6\}$ is deterred by the further deviation to $\{5;1\}$. Therefore, $\{\{6\}, \{5;1\}, \{4;2\}, \{3;3\},$

$\{3,2,1\}$, $\{2,2,2\}\}$ is the largest consistent set. Total expected payoffs associated to coalition structures $\{6\}$, $\{5,1\}$, $\{2,2,2\}$ are $.2502$, $.222$, $.1872$, respectively.

Example 2b. $|N| = 6$, $d = 0$, $a = 1$, $c = .00035$, $\phi = .5$ and $\gamma = .808$.

Partitions	Payoffs
$\{6\}$	$(.0320; .0320; .0320; .0320; .0320; .0320)$
$\{5,1\}$	$(.0135; .0135; .0135; .0135; .0135; .1110)$
$\{4,2\}$	$(.0203; .0203; .0203; .0203; .0513; .0513)$
$\{3,3\}$	$(.0309; .0309; .0309; .0309; .0309; .0309)$
$\{4,1,1\}$	$(.0081; .0081; .0081; .0081; .0625; .0625)$
$\{3,2,1\}$	$(.0147; .0147; .0147; .0269; .0269; .0625)$
$\{2,2,2\}$	$(.0269: .0269; .0269; .0269; .0269; .0269)$
$\{3,1,1,1\}$	$(.0072; .0072; .0072; .0400; .0400; .0400)$
$\{2,2,1,1\}$	$(.0157; .0157; .0157; .0157; .0400; .0400)$
$\{2,1,1,1,1\}$	$(.0096; .0096; .0278; .0278; .0278; .0278)$
$\{1,1,1,1,1,1\}$	$(.0204; .0204; .0204; .0204; .0204; .0204)$

Applying the iterative procedure to compute the LCS, we get that the largest consistent set is $\{\{6\}\}$. Total expected payoff associated to coalition structure $\{6\}$ is $.192$.

This example shows that, introducing costly inspection singles out a unique stable coalition structure (and the efficient one). Moreover, the welfare (i.e. the sum of the players' payoffs) is bigger than the one associated to some stable coalition structures when inspection is costless. So, costly inspection might be used as a selection device as well as a welfare improving mechanism. A natural question is whether this result holds in general for the cartel formation game under Cournot competition. A positive answer is given in Proposition 6.

Let P^* be the coalition structure in P such that all coalitions $S \in P^*$ have cardinality $|S| = 2$ if $|N|$ is even, while $(|N| - 1)/2$ coalitions $S \in P^*$ have cardinality $|S| = 2$ if $|N|$ is odd. Also, we denote by

$$W(P) = \sum_{S \in P} |S| \cdot b(S; P)$$

the sum of the benefits in P. From Proposition 5 and simple computations, we obtain the following corollary.

Corollary 1 In the cartel formation game under Cournot competition and costless inspection, if $|N| \geq 5$ then $P^* \in \Gamma(P)$.

Corollary 1 tells us that, whenever the industry consists of more than four firms, the coalition structure composed of (almost) only coalitions of cardinality two always belongs to the LCS. This corollary is useful to prove our next proposition (see the appendix).

Proposition 6 *In the cartel formation game under Cournot competition and costly inspection, if $|N| \geq 5$ then $\exists\, c, \phi > 0$ such that $V(N) \cdot |N| > W(P^*)$ and $\Gamma(P) = \{P^*\}$.*

Proposition 6 tells us the following interesting message for the cartel formation game. Whenever the industry consists of more than four firms, there always exists inspection cost parameters $(c, \phi > 0)$ such that, the grand coalition structure is selected (and is the efficient one) and the welfare might be improved compared to the costless case.

5.2 Pure public goods with congestion

A second application of games with positive spillovers are economies with pure public goods. The model we study is inspired from Bloch [7] wherein we introduce congestion as in Gallastegui et al. [10]. The economy consists of $|N|$ agents. At cost $d_i(q_i)$, agent i can provide q_i units of the public good. Let $q = \sum_i q_i$ be the total amount of public good. The utility each agent obtains from the public good depends positively on the total amount of public good provided, but negatively on the number of coalition partners: $U_i(q) = (|S|)^{-\alpha} \cdot q$ for all $i \in S$, where parameter $\alpha > 0$ measures the degree of congestion. Each agent owns a technology to produce the public good, and the cost of producing the amount q_i of the public good is given by $d_i(q_i) = \frac{1}{2}(q_i)^2$. Since individual cost functions are convex and exhibit decreasing returns to scale, it is cheaper to produce an amount q of public goods using all technologies than using a single technology. In stage one the coalition formation takes place. Inside each coalition, we assume equal sharing of the production. Once stable agreements on coalition formation have been reached, each coalition of agents acts noncooperatively. On the contrary, inside every coalition, agents act cooperatively and the level of public good is chosen to maximize the sum of utilities of the coalition members. That is, for any coalition structure $P = \{S_1, S_2, \ldots, S_m\}$, the level of public good q_i chosen by the coalition S_i solves

$$\max_{q_i} |S_i| \cdot \left[(|S_i|)^{-\alpha} \left(q_i + \sum_{j \neq i} q_j \right) - \frac{1}{2} \left(\frac{q_i}{|S_i|} \right)^2 \right]$$

yielding a total level of public good provision for the coalition S_i equal to $q_i = (|S_i|)^{2-\alpha}$, $i = 1;...;m$. In stage two the inspection game takes place. Assuming the symmetric Nash equilibrium of the inspection stage, the expected payoff for each agent in each possible coalition structure is well defined. As a result, the per-member expected payoff in a coalition of size $|S_i|$ is given by

$$V(S_i;P) = (|S_i|)^{-\alpha} \cdot \sum_{j=1}^{m} (|S_j|)^2 - \frac{1}{2}(|S_i|)^{2-2\alpha} - \frac{(|S_i - 1|)^{\phi} \cdot c}{\gamma}, \qquad (2)$$

where $b(S_i;P) = (|S_i|)^{-\alpha} \cdot \sum_{j=1}^{m} (|S_j|)^2 - \frac{1}{2}(|S_i|)^{2-2\alpha}$ for all agents belonging to S_i, $i = 1;...;m$.

Lemma 5 Public goods coalitions with utility function $U_i(q) = (|S|)^{-\alpha} \cdot q$ for all $i \in S$ and cost function $d_i(q_i) = \frac{1}{2}(q_i)^2$ satisfy (P.1)-(P.4) if $|N| \in [4;10]$ and $\alpha \in (.335;.36)$.

Notice that, for $|N| < 4$ the condition (P.3) is violated, while for $|N| > 10$ it is (P.4) which is violated. It can be shown that $\Gamma(P) = \mathcal{P} \setminus \{\{\overline{P}\} \cup \{P \in \mathcal{P} : \exists S \in P \text{ such that } V(S;P) < V(S';\overline{P})\}\}$ for $|N| \in [4;10]$ and $\alpha \in (.335;.36)$ under costless inspection. The next example captures the fact that, whenever the inspection is costly, stable coalition structures consist of smaller coalitions.

Example 3. $|N| = 4$ and $\alpha = .35$.

Partitions	3a. Payoffs ($c = 0$)	3b. Payoffs ($\phi = 1; c = .8; \gamma = .8$)
$\{4\}$	(6.82; 6.82; 6.82; 6.82)	(3.82; 3.82; 3.82; 3.82)
$\{3;1\}$	(4.72; 4.72; 4.72; 9.5)	(2.72; 2.72; 2.72; 9.5)
$\{2;2\}$	(5.05; 5.05; 5.05; 5.05)	(4.05; 4.05; 4.05; 4.05)
$\{2;1;1\}$	(3.48; 3.48; 5.5; 5.5)	(2.48; 2.48; 5.5; 5.5;)
$\{1;1;1;1\}$	(3.5; 3.5; 3.5; 3.5)	(3.5; 3.5; 3.5; 3.5)

Under costless inspection, the LCS is $\Gamma(P) = \{\{4\};\{3;1\};\{2;2\}\}$ while the LCCS is $\Pi(P) = \{\{4\}\}$. Under costly inspection ($\phi = 1$, $c\gamma^{-1} = 1$), $\Gamma(P) = \Pi(P) = \{\{2;2\}\}$.

5.3 Sequential formation of coalitions

For the cartel formation game and the public good model, we compare the outcomes obtained under the LCS (and LCCS) with those obtained under a sequential game of coalition formation with fixed payoff division proposed by

Bloch [6]. Bloch's [6] sequential game extends Rubinstein's [21] alternating-offer bargaining model to the formation of coalitions. A fixed protocol is assumed and the sequential game proceeds as follows. Player 1 proposes the formation of a coalition S_1 to which he belongs. Each prospective player answers the proposal in the order fixed by the protocol. If one prospective player rejects the proposal, then he makes a counter-proposal to which he belongs. If all prospective players accept, then the coalition S_1 is formed. All players in S_1 withdraw from the game, and the game proceeds among the players belonging to $N \setminus S_1$. Like in Rubinstein's bargaining model this sequential game has an infinite horizon, but the players do not discount the future. The players who do not reach an agreement in finite time receive a payoff of zero. Contrary to the LCS, this sequential game relies on the commitment assumption. Once some players have agreed to form a coalition they are committed to remain in that coalition.

Consider the following finite procedure to form coalitions. First, player 1 starts the game and chooses an integer s_1 in the interval $[1; |N|]$. Second, player $s_1 + 1$ chooses an integer s_2 in $[1; |N| - s_1]$. Third, player $s_1 + s_2 + 1$ chooses an integer s_3 in $[1; |N| - s_1 - s_2]$. The game goes on until the sequence $(s_1; s_2; s_3; \ldots)$ satisfies $\sum_j s_j = |N|$. For symmetric valuations, if the finite procedure yields as subgame perfect equilibrium a coalition structure with the property that payoffs are decreasing in the order in which coalitions are formed, then this coalition structure is supported by the (generically) unique symmetric stationary perfect equilibrium (SSPE) of the sequential game (see Bloch [6]). This result makes easy the characterization of the SSPE outcome of the cartel formation game under costless inspection.

Lemma 6 (Bloch, 1996) In Bloch's sequential coalition formation game under Cournot competition and costless inspection, any symmetric stationary perfect equilibria (SSPE) is characterized by $P = \{S^*; \{i\}_{i \in S_1}\}$ where $|S^*|$ is the first integer following $(2n + 3 - y)\frac{1}{2}$, where $y = \sqrt{4n + 5}$. If y is an integer, $|S^*|$ can take on the two values $(2n + 3 - y)\frac{1}{2}$ and $(2n + 5 - y)\frac{1}{2}$.

Intuitively, in the sequential game, firms commit to stay out of the cartel until the number of remaining firms equals the minimal profitable cartel size (this is the smallest coalition size for which a coalition member obtains a higher payoff than if all coalitions are singletons, and is equal to $|S^*|$). Then, from Proposition 5 and Lemma 6, the relationship between LCS and SSPE follows straightforwardly.

Proposition 7 In the cartel formation game under Cournot competition and costless inspection, the coalition structures supported by any symmetric stationary perfect equilibria (SSPE) of Bloch's sequential game always belong to the largest consistent set (LCS).

Also, it is easy to verify that this relationship also holds in the public good model with congestion and costless inspection for $|N| \in [4;10]$ and $\alpha \in (.335;.36)$. But, under costly inspection, there is no relationship between LCS and SSPE (see Example 3b).

	SSPE	LCS	LCCS
Example 1	$\{\{4\}\}$	$\{\{4\}\}$	$\{\{4\}\}$
Example 2a	$\{\{5;1\}\}$	$\left\{ \{5;1\};\{4;2\};\{3;3\}; \{6\};\{3;2;1\};\{2;2;2\} \right\}$	$\{\{6\}\}$
Example 2b	$\{\{6\}\}$	$\{\{6\}\}$	$\{\{6\}\}$
Example 3a	$\{\{3;1\}\}$	$\left\{ \{4\};\{3;1\}; \{2;2\} \right\}$	$\{\{4\}\}$
Example 3b	$\{\{4\}\}$	$\{\{2;2\}\}$	$\{\{2;2\}\}$

Table 1: The LCS, LCCS and SSPE in our examples.

In Table 1, we report the coalition structures supported by SSPE, LCS or LCCS in our previous examples. Except the earlier mentioned relationships between these solution concepts, these examples show that there are no other relationships.[5] Some remarks can be made. Firstly, whenever the inspection cost is not too large, the LCS tends to select coalition structures that were possibly stable ones under costless inspection, while SSPE may support coalition structures that were even not stable. Secondly, the LCCS and SSPE are better at picking out compared to the LCS. Thirdly, costly inspection favors the occurrence of stable symmetric coalition structures whatever the solution concept used.

6 Conclusion

Much of real-world negotiations involve not only bargaining but also the compliance with the agreement. Up to now the compliance issue has been neglected in coalitional bargaining and its applications to economic situations such as environmental agreements, cartel formation or public goods

[5] There are no relationships between Ray and Vohra's [19] equilibrium binding agreements (EBA) concept and the LCS or the LCCS. Also, EBA do not support the efficient coalition structure while the LCS does in most examples under costless inspection.

coalitions. To study the impact of the compliance issue on both the coalition formation outcome and the per-member payoffs, we have developed a two-stage negotiation model.

The main contributions of this paper are the following ones. For games with positive spillovers, many coalition structures may belong to the largest consistent set (LCS) under costless inspection. The coalition structure consisting only of singletons is never stable, while the grand coalition, which is the efficient coalition structure, always belongs to the LCS. The LCS is the solution concept we used to predict which coalition structures are possibly stable. A refinement, the largest cautious consistent set (LCCS), singles out the grand coalition. But, under costly inspection, the grand coalition does not always belong to the LCS. Nevertheless, there always exists inspection cost parameters such that there is a unique coalition structure belonging to the LCS and it is the grand coalition. Two economic applications with positive spillovers have been considered: a cartel formation game and a public goods economy with congestion. In the cartel formation game, beside selecting the grand coalition, costly inspection might also improve the welfare. Finally, in the cartel formation game with costless inspection, the coalition structures supported by any symmetric stationary perfect equilibria of Bloch's [6] sequential game of coalition formation always belong to the LCS.

Some extensions may be worthwhile. A first one is to allow the violation within a coalition to affect the others coalition. Such extension would link and complexify substantially the inspection games. Hence, only through simulations results could be obtained. A second one is to consider heterogenous players. In this paper, we have assumed symmetric players, an assumption common to the recent literature on coalition formation (see Bloch [7]).

A Appendix

Proof of Proposition 5.

Part 1: $|N| \le 4$. Simple computations show that each non-symmetric coalition structure $P \in \mathcal{P}$ is such that there exists $S \in P$ with $V(S;P) < V(S';\overline{P})$ (Example 2 gives us the case for $|N| = 4$). Then, from Proposition 1, we have $\Gamma(\mathcal{P}) = \{P^*\}$ for $|N| \le 4$.

Part 2: $|N| \ge 5$. From Lemmas 1 - 2 - 3, we have $\{P \in \mathcal{P} : \exists S \in P$ such that $V(S;P) < V(S';\overline{P})\} \notin \Gamma(\mathcal{P})$, $\overline{P} \notin \Gamma(\mathcal{P})$ and $P^* \in \Gamma(\mathcal{P})$, respectively.

To prove that $\{P \in \mathcal{P} : P \neq \overline{P}; P^*$ and $V(S;P) \ge V(S';\overline{P})$ for all

$S \in P\} \subset \Gamma(P)$, we have to show that all possible deviations from P can be deterred. Two kind of possible deviations that benefit the deviating players have to be considered.

Firstly, we consider the spliting deviations $P \to_S P'$ such that $|P'| > |P|$. The condition (P.1) implies that the players in $N \backslash S$ are worse off in P'. Then, conditions (P.1) and (P.3) imply that further spliting deviations of players in $N \backslash S$ can occur and lead to some P'' where $P'' \to_N P$ and $V(\cdot; P'') < V(\cdot; P)$ for all $i \in N$. Therefore, $P' \succ P$ and the deviation $P \to_S P'$ is deterred.

Secondly, we consider the enlarging deviations $P \to_S P'$ such that $|P'| < |P|$. Only numerically it can be shown that there always exist a coalition structure P'' such that: (i) $P'' > P'$, (ii) $V(\cdot; P'') < V(\cdot; P)$ for some $i \in S$, i.e. for some player(s) belonging to the coalition deviating from P to P', and (iii) $V(S; P'') \geq V(S'; \overline{P})$ for all $S \in P''$. Therefore, the enlarging deviation $P \to_S P'$ is deterred. ■

Proof of Proposition 6. Assume $|N| \geq 5$ and remember that $\tilde{P} \in \Gamma(P)$ under costless inspection (see Corollary 1). We denote by $\underline{b}(\cdot; P)$ the smallest per-member benefit obtained in P; $\underline{b}(S; P)$ simply says that it is the members of S who obtain the smallest per-member benefit in P. Proposition 6 is proved in two parts: (1) $|N|$ even, (2) $|N|$ odd.

Part 1: $|N|$ even. Let $\hat{P} \in P$ be the coalition structure such that $\left|\hat{P}\right| = 2$ and $|S_1| = \frac{1}{2}|N| - 1$, $|S_2| = \frac{1}{2}|N| + 1$, $S_1; S_2 \in \hat{P}$. It is straightforward that $b(S_2; \hat{P}) > \underline{b}(S; P)$ for all non-symmetric coalition structures P. From Expression (1) we obtain

$$
\begin{aligned}
V\left(S_2; \hat{P}\right) &= \frac{(a-d)^2}{\left(\frac{1}{2}|N| + 1\right)^2} - \gamma^{-1} \cdot c \cdot \left(\frac{1}{2}|N|\right)^{\varepsilon} = \underline{V}\left(\cdot; \hat{P}\right); \\
V\left(S'; \overline{P}\right) &= \frac{(a-d)^2}{(|N| + 1)^2}; \\
V(P^*) &= \frac{(a-d)^2}{4|N|} - \gamma^{-1} \cdot c \cdot (|N| - 1)^{\varepsilon}; \\
b\left(S; \tilde{P}\right) &= \frac{(a-d)^2}{2\left(\frac{1}{2}|N| + 1\right)^2}.
\end{aligned}
$$

Intuitively, we look for bounds ($\underline{c}$ and $\overline{c}$) on c such that, as $\varepsilon \to 0_+$, the LCS singles out P^* and the sum of the per-member payoffs for P^* is greater than the sum of the per-member benefits for $\tilde{P}$ under costless inspection. Notice that, as $\varepsilon \to 0_+$, introducing costly inspections tends to decrease by a constant the per-member payoffs of all coalitions of size bigger than one. Let $\underline{c}$ and $\overline{c}$ be defined as the values of c (with $\varepsilon = 0$) such that $V\left(S_2;\hat{P}\right) = V\left(S^0;\overline{P}\right)$ and $V(P^*) = b\left(S;\tilde{P}\right)$, respectively. Hence,

$$\underline{c} = \gamma(a-d)^2\left[\frac{1}{\left(\frac{1}{2}|N|+1\right)9} - \frac{1}{(|N|+1)^2}\right];$$

$$\overline{c} = \gamma(a-d)^2\,\frac{1}{|N|}\left[\frac{1}{4} - \frac{2|N|}{(|N|+2)^2}\right].$$

That is, $\underline{c}$ is the minimum value for c such that all non-symmetric coalition structures do not belong to the LCS as $\varepsilon \to 0_+$; and, $\overline{c}$ is the maximum value for c such that the per-member payoff in P^* is greater than the per-member benefit in $\tilde{P}$ as $\varepsilon \to 0_+$. In other words, $\overline{c}$ is the maximum value for c such that, as $\varepsilon \to 0_+$, a welfare improvement could occur by introducing costly inspections. Notice that, as $\varepsilon \to 0_+$, $V(P^*) > V(\cdot;P)$ for all symmetric coalition structures $P \neq P^*;\overline{P}$ and $V(P^*) > V(\cdot;\overline{P})$ for all $c < \overline{c}$. Numerically, it is true that $\overline{c} > \underline{c}$ for all $|N|$ even. Therefore, as $\varepsilon \to 0_+$, there always exists $c \in (\underline{c};\overline{c})$ such that P^* is the unique coalition structure belonging to the LCS and $V(P^*)\cdot|N| > W(\tilde{P}) = b(\tilde{P})\cdot\left|\tilde{P}\right|$.

Part 2: $|N|$ odd. Let $\hat{P} \in \mathcal{P}$ be the coalition structure such that $\left|\hat{P}\right| = 2$ and $|S_1| = \frac{1}{2}(|N|-1)$, $|S_2| = \frac{1}{2}(|N|+1)$, $S_1;S_2 \in \hat{P}$. It is straightforward that $b(S_2;\hat{P}) > \underline{b}(S;P)$ for all non-symmetric coalition structures P. From Expression (1) we obtain

$$V\left(S_2;\hat{P}\right) = \frac{(a-d)^2}{(|N|+1)\frac{9}{2}} - \gamma^{-1}\cdot c\cdot\left(\frac{1}{2}(|N|-1)\right)^{\varepsilon} = \underline{V}\left(\cdot;\hat{P}\right);$$

$$V\left(S^0;\overline{P}\right) = \frac{(a-d)^2}{(|N|+1)^2};$$

$$V(P^*) = \frac{(a-d)^2}{4|N|} - \gamma^{-1}\cdot c\cdot(|N|-1)^{\varepsilon};$$

$$b\left(S;\tilde{P}\right) = \frac{(a-d)^2}{2\left(\frac{1}{2}(|N|+3)\right)^2} \text{ if } |S| = 2;$$

$$b\left(S;\tilde{P}\right) = \frac{(a-d)^2}{\left(\frac{1}{2}(|N|+3)\right)^2} \text{ if } |S| = 1;$$

$$W\left(\hat{P}\right) = \left(\frac{\left|\hat{P}\right| + 1}{2}\right)\frac{(a - d)^2}{\left(\frac{1}{2}(|N| + 3)\right)^2}.$$

Intuitively, we look for bounds ($\underline{c}$ and $\bar{c}$) on c such that, as $\varepsilon \to 0_+$, the LCS singles out P^* and the sum of the per-member payoffs for P^* is greater than the sum of the per-member benefits for $\hat{P}$ under costless inspection. Notice that, as $\varepsilon \to 0_+$, introducing costly inspections tends to decrease by a constant the per-member payoffs of all coalitions of size bigger than one. Let $\underline{c}$ and $\bar{c}$ be defined as the values of c (with $\varepsilon = 0$) such that $V(S_2; \hat{P}) = V(S'; \bar{P})$ and $V(P^*) \cdot |N| = W(\hat{P})$, respectively. Hence,

$$\begin{aligned}
\underline{c} &= \gamma (a - d)^2 \left[\frac{1}{(|N| + 1)\frac{9}{2}} - \frac{1}{(|N| + 1)^2}\right]; \\
\bar{c} &= \gamma (a - d)^2 \frac{1}{|N|}\left[\frac{1}{4} - \frac{2(|N| + 1)}{(|N| + 3)^2}\right].
\end{aligned}$$

That is, $\underline{c}$ is the minimum value for c such that all non-symmetric coalition structures do not belong to the LCS as $\varepsilon \to 0_+$; and, $\bar{c}$ is the maximum value for c such that the sum of the per-member payoffs in P^* is greater than the sum of the per-member benefits in $\hat{P}$ as $\varepsilon \to 0_+$. In other words, $\bar{c}$ is the maximum value for c such that, as $\varepsilon \to 0_+$, a welfare improvement could occur by introducing costly inspections. Notice that, as $\varepsilon \to 0_+$, $V(P^*) > V(\cdot; P)$ for all symmetric coalition structures $P \neq P^*; \bar{P}$ and $V(P^*) > V(\cdot; \bar{P})$ for all $c < \bar{c}$. Numerically, it is true that $\bar{c} > \underline{c}$ for all $|N| \geq 5$ odd. Therefore, as $\varepsilon \to 0_+$, there always exists $c \in (\underline{c}; \bar{c})$ such that P^* is the unique coalition structure belonging to the LCS and $V(P^*) \cdot |N| > W(\hat{P})$. $\blacksquare$

References

[1] Avenhaus, R., B. von Stengel and S. Zamir, 1998, "Inspection Games," to appear in Volume III of the Handbook of Game Theory, edited by R.J. Aumann and S. Hart.

[2] Barrett, S., 1997, "Towards a Theory of International Environmental Cooperation," in New Directions in the Economic Theory of the Environment (C. Carraro and D. Siniscalco, Eds.), pp.239-280, Cambridge: Cambridge University Press.

[3] Bernheim, D., 1984, "Rationalizable Strategic Behavior," Econometrica 52, 1007-1028.

[4] Binmore, K.G., 1985, "Bargaining and Coalitions," in Game Theoretic Models of Bargaining (A.E. Roth, Ed.), pp.269-304, Cambridge: Cambridge University Press.

[5] Bloch, F., 1995, "Endogenous Structures of Association in Oligopolies," RAND Journal of Economics 26 (3), 537-556.

[6] Bloch, F., 1996, "Sequential Formation of Coalitions in Games with Externalities and Fixed Payo® Division," Games and Economic Behavior 14, 90-123.

[7] Bloch, F., 1997, "Non-Cooperative Models of Coalition Formation in Games with Spillovers," in New Directions in the Economic Theory of the Environment (C. Carraro and D. Siniscalco, Eds.), pp.311-352, Cambridge: Cambridge University Press.

[8] Chatterjee, K., B. Dutta, D. Ray, and K. Sengupta, 1993, "A Noncooperative Theory of Coalitional Bargaining," Review of Economic Studies 60, 463-477.

[9] Chwe, M.S., 1994, "Farsighted Coalitional Stability," Journal of Economic Theory 63, 299-325.

[10] Gallastegui, M.C., F. Grafe and E. Inarra, 1997, "Congestion E®ects in a Public-Good Economy," Journal of Economics 66, 189-204.

[11] Hart, S. and M. Kurz, 1983, "Endogenous Formation of Coalitions," Econometrica 51, 1047-1064.

[12] Helland, E., 1998, "The Enforcement of Pollution Control Laws: Inspections, Violations, and Self-Reporting," The Review of Economics and Statistics 80, 141-153.

[13] Herings, P.J.J. and V.J. Vannetelbosch, 1999, "Re¯nements of Rationalizability for Normal-Form Games," International Journal of Game Theory 28, 53-68.

[14] Kilgour, D.M., 1994, "The Use of Costless Inspection in Enforcement," Theory and Decision 36, 207-232.

[15] Krishna, V. and R. Serrano, 1996, "Multilateral Bargaining," Review of Economic Studies 63, 61-80.

[16] Laplante B. and P. Rilstone, 1996, "Environmental Inspections and Emissions of the Pulp and Paper Industry in Quebec," Journal of Environmental Economics and Management 31, 19-36.

[17] Okada, A., 1996, "A Noncooperative Coalitional Bargaining Game with Random Proposers," Games and Economic Behavior 16, 97-108.

[18] Pearce, D.G., 1984, "Rationalizable Strategic Behavior and the Problem of Perfection," Econometrica 52, 1029-1050.

[19] Ray, D. and R. Vohra, 1997, "Equilibrium Binding Agreements," Journal of Economic Theory 73, 30-78.

[20] Ray, D. and R. Vohra, 1999, "A Theory of Endogenous Coalition Structures," Games and Economic Behavior 26, 286-336.

[21] Rubinstein, A., 1982, "Perfect Equilibrium in a Bargaining Model," Econometrica 50, 97-109.

[22] Vannetelbosch, V.J., 1998, "Rationalizability and Equilibrium in N-Person Sequential Bargaining," forthcoming in Economic Theory.

[23] Xue, L., 1997, "Nonemptiness of the Largest Consistent Set," Journal of Economic Theory 73, 453-459.

[24] Yi, S.S., 1997, "Stable Coalition Structures with Externalities," Games and Economic Behavior 20, 201-237.