

Two-step nilpotent Leibniz algebras

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Part 1: Classification of two-step nilpotent Leibniz algebras

- Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F}) \neq 2$. A *left Leibniz algebra* over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ endowed of a bilinear map (called *commutator* or *bracket*) $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies the *left Leibniz identity*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad \forall x, y, z \in \mathfrak{g}.$$

- We can define a *right Leibniz algebra*, using the right Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \quad \forall x, y, z \in \mathfrak{g}.$$

- A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*.
- We assume that $\dim_{\mathbb{F}} \mathfrak{g} < \infty$.

- The *Leibniz kernel*

$$\text{Leib}(\mathfrak{g}) = \text{Span}_{\mathbb{F}} \{[x, x] \mid x \in \mathfrak{g}\}$$

is the smallest bilateral ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\text{Leib}(\mathfrak{g})$ is a Lie algebra.

- The *left* and the *right center* of $\mathfrak{g}$ are

$$Z_l(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}$$

$$Z_r(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [\mathfrak{g}, x] = 0\}$$

- The *center* of $\mathfrak{g}$ is $Z(\mathfrak{g}) = Z_l(\mathfrak{g}) \cap Z_r(\mathfrak{g})$.
- $\text{Leib}(\mathfrak{g}) \subseteq Z_l(\mathfrak{g})$.

Definition

Let $\mathfrak{g}$ be a left Leibniz algebra and let

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}], \quad \forall k \geq 0,$$

be the *lower central series* of $\mathfrak{g}$. $\mathfrak{g}$ is *n -step nilpotent* if $\mathfrak{g}^{(n-1)} \neq 0$ and $\mathfrak{g}^{(n)} = 0$.

Proposition

If $\mathfrak{g}$ is a left two-step nilpotent Leibniz algebra, then $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$ and $\mathfrak{g}$ is symmetric.

Proposition

If $\mathfrak{g}$ is a left nilpotent Leibniz algebra with $\dim_{\mathbb{F}} \mathfrak{g}^{(1)} = 1$, then $\mathfrak{g}^{(1)} \subseteq Z(\mathfrak{g})$ and $\mathfrak{g}$ is symmetric.

- Let $\mathfrak{g}$ be a two-step nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = \text{Span}_{\mathbb{F}}\{z_1, \dots, z_t\}$, then

$$[x, y] = \phi_1(x, y)z_1 + \dots + \phi_t(x, y)z_t, \quad \forall x, y \in \mathfrak{g}$$

where $\phi_i : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ is a bilinear form, $\forall i = 1, \dots, t$.

- Our purpose is to reduce such a given bilinear form to a suitable canonical representation.

- Let $\mathfrak{g}$ be a nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}z$ and

$$[x, y] = \phi(x, y)z, \quad \forall x, y \in \mathfrak{g}$$

- If $\mathfrak{g}$ is not a Lie algebra, then $0 \neq \text{Leib}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$.
- $\phi = \phi^- + \phi^+$, where

$$\phi^- = \frac{\phi - \phi^t}{2}, \quad \phi^+ = \frac{\phi + \phi^t}{2}.$$

- We use the simultaneous reduction to canonical form of a pair of bilinear forms.

Definition

A Kronecker module is a quadruple (V_1, V_2, f_1, f_2) , where V_1, V_2 are vector spaces over $\mathbb{F}$ and $f_1, f_2 : V_1 \rightarrow V_2$ are linear maps.

- Given a pair of bilinear forms $\phi^-, \phi^+ : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ as above, we define the Kronecker module $(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+})$, where

$$\overline{\phi^-}(x) : z \mapsto \phi^-(x, z), \quad \overline{\phi^+}(y) : z \mapsto \phi^+(y, z), \quad \forall z \in \mathfrak{g}.$$

- $\mathfrak{g}$ is *decomposable* if $\mathfrak{g} = U \oplus U^\perp$.
- We can assume that $\mathfrak{g}$ is indecomposable.

The indecomposable modules $(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+})$ turn out to be one of the following three pairs

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & C_{f(x)^m} \\ C_{f(x)^m}^t & 0 \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} 0 & J_0 \\ -J_0^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J_2 \\ J_2^t & 0 \end{pmatrix} \quad (3)$$

where $f(x) \in \mathbb{F}[x]$ is a monic irreducible polynomial and J_1, J_2 are the matrices associated with

$$F_1, F_2 : \mathbb{F}^n \rightarrow \mathbb{F}^{n+1}$$

defined by

$$F_1(x_1, \dots, x_n) = (x_1, \dots, x_n, 0),$$

$$F_2(x_1, \dots, x_n) = (0, x_1, \dots, x_n).$$

Definition

Let $f(x) \in \mathbb{F}[x]$ be a monic irreducible polynomial. Let $n \in \mathbb{N}$ and let $A = C_{f(x)^n}$. The *Heisenberg* Leibniz algebra $\mathfrak{l}_{2n+1}^A$ is the $(2n+1)$ -dimensional indecomposable Leibniz algebra with associated Kronecker module of type (1).

- If $A = (a_{ij}) \in M_n(\mathbb{F})$ and $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^A$, then the non-trivial commutators are

$$[e_i, f_j] = (\delta_{ij} + a_{ij}) h, \quad [f_j, e_i] = (-\delta_{ij} + a_{ij}) h, \quad \forall i, j = 1, \dots, n$$

- If $(a_{ij}) = 0$, then we obtain the classical Heisenberg algebra $\mathfrak{h}_{2n+1}$.

Definition

Let $n \in \mathbb{N}$ and let $A = C_{\chi^n}$. The *Kronecker* Leibniz algebra $\mathfrak{k}_n$ is the $(2n + 1)$ -dimensional indecomposable Leibniz algebra with associated Kronecker module of type (2).

Definition

The *Dieudonné* Leibniz algebra $\mathfrak{d}_n$ is the $(2n + 2)$ -dimensional Leibniz algebra with associated Kronecker module of type (3).

The case $\mathbb{F} = \mathbb{C}$

Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$. Then

$$C((x-a)^n) \sim J_a = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & a \end{pmatrix}$$

Thus $\mathfrak{l}_{2n+1}^A \cong \mathfrak{l}_{2n+1}^{J_a}$ and the non-trivial Leibniz brackets are

$$[e_i, f_i] = (1+a)h, \quad [f_i, e_i] = (-1+a)h \quad \forall i = 1, \dots, n$$

$$[e_i, f_{i-1}] = h, \quad [f_{i-1}, e_i] = h, \quad \forall i = 2, \dots, n$$

where $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^{J_a}$.

Proposition (G. La Rosa, M. M., 2022)

Let $a \in \mathbb{C}$. The Heinseberg Leibniz algebras $\mathfrak{l}_{2n+1}^J$ and $\mathfrak{l}_{2n+1}^{J-a}$ are isomorphic.

Proposition (G. La Rosa, M. M., 2022)

Let $a, a' \in \mathbb{C}$. Then $\mathfrak{l}_3^a \cong \mathfrak{l}_3^{a'} \iff a' = \pm a$.

Part 2: *Coquecigrue Problem* for nilpotent real Leibniz algebras

- It is the problem, formulated by J.-L. Loday, of finding a generalization to Leibniz algebras of Lie's third theorem, which associates a Lie local group with any real or complex Lie algebra.
- Given any Leibniz algebra $\mathfrak{g}$, one wants to find a manifold endowed with a smooth map which plays the role of Ad for Lie groups, and such that the tangent space at a distinguished point, endowed with the differential ad of Ad , is a Leibniz algebra isomorphic to $\mathfrak{g}$.
- In the special case of Lie algebras $\mathfrak{g}$, one wants to obtain the usual simply connected Lie group associated with $\mathfrak{g}$.
- We assume now that $\mathbb{F} = \mathbb{R}$.

Definition

A (left) *rack* is a set X with a binary operation $\triangleright : X \times X \rightarrow X$ which is (left) autodistributive

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z), \quad \forall x, y, z \in X$$

and such that $x \triangleright - : X \rightarrow X$ is a bijection $\forall x \in X$.

A rack is *pointed* if $\exists 1 \in X$ such that $1 \triangleright x = x$ and $x \triangleright 1 = 1$, $\forall x \in X$.

A rack $(X, \triangleright)$ is a *quandle* if $x \triangleright x = x$, $\forall x \in X$.

Example

Every group endowed with the conjugation is a pointed quandle.

Definition

A *Lie rack* is a pointed rack $(X, \triangleright, 1)$ such that X is a smooth manifold, $\triangleright$ is a smooth map and $x \triangleright -$ is a diffeomorphism $\forall x \in X$.

The tangent space of a Lie rack

Proposition (M.-K. Kinyon, 2007)

If X is a Lie rack, then the bracket

$$[x, y] = \text{ad}_x(y),$$

where

- $\text{ad} : T_1X \rightarrow \mathfrak{gl}(T_1X)$ is the differential of $\Phi : X \mapsto \text{GL}(T_1X)$;
- $\Phi(x) = T_1\phi(x)$, $\forall x \in X$;
- $\phi(x) = x \triangleright -$, $\forall x \in X$

defines on T_1X a Leibniz algebra structure.

- M.-K. Kinyon solves the *coquecigrue problem* for the class of *split Leibniz algebras*, which are Leibniz algebras $\mathfrak{g}$ with a bilateral ideal $\text{Leib}(\mathfrak{g}) \subseteq \mathfrak{a} \subseteq Z_l(\mathfrak{g})$ and with a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus_s \mathfrak{a}$$

and

$$[(x, a), (y, b)] = ([x, y], \rho_x(b)), \quad \forall (x, a), (y, b) \in \mathfrak{h} \oplus \mathfrak{a}$$

where $\rho : \mathfrak{h} \times \mathfrak{a} \rightarrow \mathfrak{a}$ is the action on the $\mathfrak{h}$ -module $\mathfrak{a}$.

Proposition (M.-K. Kinyon, 2007)

Let $\mathfrak{g}$ be a split Leibniz algebra. Then there exists a Lie rack X such that T_1X is isomorphic to $\mathfrak{g}$.

The local integration of a Leibniz algebra

- S. Covez gives a local solution to the coquegrue problem: he shows how to integrate every Leibniz algebra into a Lie local rack.
- The central point of his result is to see every Leibniz algebra $\mathfrak{g}$ as an abelian extension with kernel $Z_1(\mathfrak{g})$ and to integrate explicitly the corresponding Leibniz algebra 2-cocycle into a Lie local rack 2-cocycle.

- We associate with $\mathfrak{g}$ the short exact sequence

$$0 \rightarrow Z_I(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \rightarrow 0$$

where $\mathfrak{g}_0 = \mathfrak{g} / Z_I(\mathfrak{g})$.

- $Z_I(\mathfrak{g})$ is a $\mathfrak{g}_0$ -module $\Rightarrow \exists \omega \in \text{ZL}^2(\mathfrak{g}_0, Z_I(\mathfrak{g}))$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus_{\omega} Z_I(\mathfrak{g})$$

with

$$[(x, a), (y, b)] = ([x, y]_{\mathfrak{g}_0}, \rho_x(b) + \omega(x, y)),$$

where $\rho : \mathfrak{g}_0 \times Z_I(\mathfrak{g}) \rightarrow Z_I(\mathfrak{g})$ is the action induced by the $\mathfrak{g}_0$ -module structure of $Z_I(\mathfrak{g})$.

Theorem (S. Covez, 2013)

Every Leibniz algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus_{\omega} \mathfrak{a}$ can be integrated into a Lie local rack of the form

$$G_0 \times_f \mathfrak{a}$$

with operation defined by

$$(g, a) \triangleright (h, b) = \left(ghg^{-1}, \phi_g(b) + f(g, h) \right), \quad \forall (g, a), (h, b) \in G_0 \times \mathfrak{a}$$

and unit $(1, 0)$, where G_0 is a Lie group such that $\text{Lie}(G_0) = \mathfrak{g}_0$, ϕ is the exponentiation of the action ρ , $f : G_0 \times G_0 \rightarrow \mathfrak{a}$ is the Lie local racks 2-cocycle defined by

$$f(g, h) = \int_{\gamma_h} \left(\int_{\gamma_g} \tau^2(\omega)^{\text{eq}} \right)^{\text{eq}}, \quad \forall g, h \in G_0$$

and $\tau^2(\omega) \in \text{ZL}^1(\mathfrak{g}_0, \text{Hom}(\mathfrak{g}_0, \mathfrak{a}))$ is defined by $\tau^2(\omega)(x)(y) = \omega(x, y)$, $\forall x, y \in \mathfrak{g}_0$.

The case of quandles

Theorem (G. La Rosa, M. M., 2022)

Let R be a Lie rack integrating a Leibniz algebra $\mathfrak{g}$. R is a quandle if and only if $\mathfrak{g}$ is a Lie algebra. In particular $R = \text{Conj}(G)$, where $\text{Lie}(G) = \mathfrak{g}$.

$G = G_0 \times_F Z_l(\mathfrak{g})$ is the Lie group with operation

$$(g, a)(h, b) = (gh, a + b + F(g, h)), \quad \forall (g, a), (h, b) \in G_0 \times Z_l(\mathfrak{g})$$

where $F : G_0 \times G_0 \rightarrow Z_l(\mathfrak{g})$ is a Lie group 2-cocycle such that

$$f(g, h) = F(g, h) - F(g, g^{-1}) + F(gh, g^{-1}), \quad \forall g, h \in G_0.$$

Theorem (G. La Rosa, M. M., 2022)

Every nilpotent real Leibniz algebra $\mathfrak{g}$ has a global integration into a Lie rack.

Sketch of the proof

- $\mathfrak{g} = \mathfrak{g}_0 \oplus_{\omega} Z_I(\mathfrak{g})$
- $\mathfrak{g}$ is nilpotent $\Rightarrow \mathfrak{g}_0$ is a nilpotent Lie algebra and

$$\rho_x(a) = \text{ad}_x(a), \quad \forall x \in \mathfrak{g}_0, a \in Z_I(\mathfrak{g}),$$

can be represented as strictly lower triangular matrix.

- If G_0 is the simply connected Lie group integrating $\mathfrak{g}_0$, then $\phi_{\exp(x)} = \exp(\rho_x) \in \text{Aut}(Z_I(\mathfrak{g}))$ is a unitriangular matrix whose entries are polynomial expressions.
- Fixed $\gamma_g(s) = g^s, \gamma_h(t) = h^t, \forall g, h \in G_0, \forall s, t \in [0, 1]$, we have that

$$f(g, h) = \int_{\gamma_h} \left(\int_{\gamma_g} \tau^2(\omega)^{eq} \right)^{eq} \quad \forall g, h \in G_0$$

is everywhere defined because it involves the integration of matrices with polynomial entries in $\mathbb{R}[s]$ and $\mathbb{R}[t]$.

Integration of two-step nilpotent Leibniz algebras

Theorem (G. La Rosa, M. M., 2022)

Let $(\mathfrak{g}, [-, -])$ be a two-step nilpotent Leibniz algebra and let $\omega : \mathfrak{g}_0 \times \mathfrak{g}_0 \rightarrow [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{g}_0 = \mathfrak{g} / [\mathfrak{g}, \mathfrak{g}]$, be the Leibniz algebras 2-cocycle associated with the short exact sequence

$$0 \rightarrow [\mathfrak{g}, \mathfrak{g}] \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \rightarrow 0.$$

Then the multiplication

$$(x, a) \triangleright (y, b) = (y, b + \omega(x, y)), \quad \forall (x, a), (y, b) \in \mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$$

defines a Lie global rack structure on $\mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$, such that $T_{(0,0)}(\mathfrak{g}_0 \times_{\omega} [\mathfrak{g}, \mathfrak{g}], \triangleright)$ is a Leibniz algebra isomorphic to $\mathfrak{g}$.

Example

Let $A = (a_{ij})_{i,j} \in M_n(\mathbb{R})$ and let $\mathfrak{g} = \mathfrak{l}_{2n+1}^A$ be the $(2n+1)$ -dimensional Heisenberg Leibniz algebra with basis $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ and bracket

$$\begin{aligned} &[(x_1, \dots, x_n, y_1, \dots, y_n, z), (x'_1, \dots, x'_n, y'_1, \dots, y'_n, z')] = \\ &= \left(0, \dots, 0, \sum_{i,j=1}^n (\delta_{ij} + a_{ij}) x_i y'_j + \sum_{i,j=1}^n (-\delta_{ij} + a_{ij}) x'_i y_j \right) \end{aligned}$$

We have that $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}h \subseteq Z(\mathfrak{g})$, thus the Lie global rack integrating $\mathfrak{g}$ is $R_{2n+1}^A = (\mathfrak{g}_0 \times_f \mathbb{R}h, \triangleright)$ with multiplication

$$\begin{aligned} &(x_1, \dots, x_n, y_1, \dots, y_n, z) \triangleright (x'_1, \dots, x'_n, y'_1, \dots, y'_n, z') = \\ &= \left(x'_1, \dots, x'_n, y'_1, \dots, y'_n, z' + \sum_{i,j=1}^n [(\delta_{ij} + a_{ij}) x_i y'_j + (-\delta_{ij} + a_{ij}) x'_i y_j] \right) \end{aligned}$$

Definition

We define $(R_{2n+1}^A, \triangleright)$ as the *Heisenberg rack*.

- The multiplication $\triangleright$ in R_{2n+1}^A is a *perturbation* of the conjugation of the Heisenberg Lie group H_{2n+1} .
- We use the canonical matrix representation

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \leq \mathrm{GL}_{n+2}(\mathbb{R})$$

The conjugation formula for two matrices in H_{2n+1} is given by

$$\begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 1 & x' & z' + \sum_{i=1}^n (x_i y'_i - y_i x'_i) \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix}$$

With the same representation, the multiplication $\triangleright$ of R_{2n+1}^A turns into

$$\begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \triangleright \begin{pmatrix} 1 & x' & z' \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & x' & z' + \sum_{i,j=1}^n [(\delta_{ij} + a_{ij}) x_i y'_j + (\delta_{ij} - a_{ij}) x'_i y_j] \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix},$$

hence for $A = 0_{n \times n}$, it holds $R_{2n+1}^{0_{n \times n}} = \text{Conj}(H_{2n+1})$.

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Thank you for your attention!