On the impact of the atmosphere on the time-varying age of a passive tracer in the ocean

Eric Deleersnijder, 1 January 2017

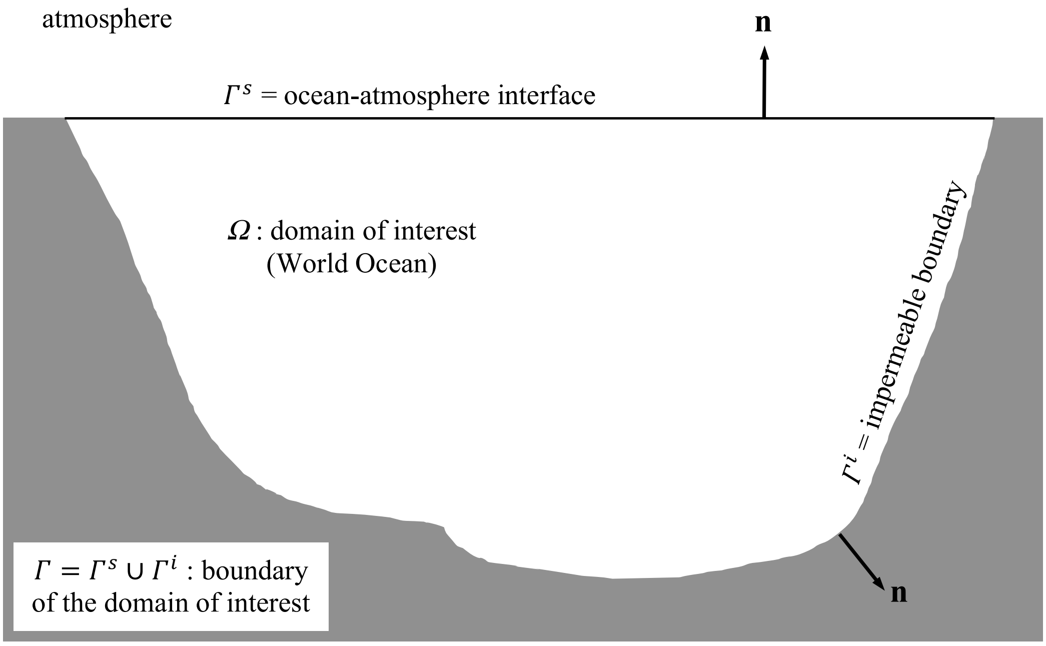
A gas originating from the atmosphere progressively enters the ocean where it is dissolved in the water. The dissolved gas is subject to advection and diffusion as well as a first-order decay process. The flux at the ocean-atmosphere interface is modelled by means of a Robin boundary condition that involves a suitable piston velocity.

The abovementioned phenomena are diagnosed by evaluating the age of the tracer penetrating the ocean. In the atmosphere, the age of the gas is assumed to be known. It will be seen that modifying the atmospheric age by a constant value leads to a shift of the oceanic tracer age of the same value.

This result holds valid if all the variables and coefficients of the model are functions of time and location, with the noticeable exception of the prescribed shift of the age of the gas in the atmosphere, which must be a constant.

Tracer concentration in the ocean

The domain of interest, the World Ocean, is denoted Ω. Its boundary is surface Γ . The latter consists of two parts, Γ^s and Γ^i , with $\Gamma = \Gamma^s \cup \Gamma^i$ and $\Gamma^s \cap \Gamma^i = \varnothing$. The former part of the boundary, the atmosphere-ocean interface, allows gas exchanges to take place, whilst the remainder of the boundary, Γ^i , is impermeable. The outward unit normal vector to the boundary is denoted **n**, with $|\mathbf{n}| = 1$. See figure below.



Let t and $\mathbf{x}$ denote the time and the position vector. The velocity vector, $\mathbf{v}(t,\mathbf{x})$, is divergence-free ($\nabla \bullet \mathbf{v} = 0$). The velocity satisfies the following boundary condition

$$[\mathbf{v} \bullet \mathbf{n}]_{\mathbf{x}\in\Gamma} = 0 \quad . \tag{1}$$

Diffusive processes are taken into account with the help of diffusivity tensor $\mathbf{K}(t,\mathbf{x})$, which is symmetric and positive definite.

In the ocean the concentration of the tracer under consideration, $C(t,\mathbf{x})$, is governed by the reactive transport equation

$$\frac{\partial C}{\partial t} = -\gamma C - \nabla \bullet (C\mathbf{v} - \mathbf{K} \bullet \nabla C) \quad , \tag{2}$$

where γ (≥0) is the rate of decay of the tracer, which may be a function of time, position and other variables, excluding the age of the tracer. Clearly, if the tracer is passive, the decay rate must be prescribed to be zero. At the ocean surface, the following Robin boundary condition is to be enforced

$$\left[(\mathbf{K} \bullet \nabla C) \bullet \mathbf{n} + \chi (C - C^a)\right]_{\mathbf{x}\in\Gamma^s} = 0 \quad , \tag{3}$$

where $\chi(t,\mathbf{x}^s)$ (≥0) is the relevant piston velocity, whilst $C^a(t,\mathbf{x}^s)$ is the surface tracer concentration at saturation. On Γ^i, a no-flux boundary condition is prescribed:

$$\left[(\mathbf{K} \bullet \nabla C) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \quad . \tag{4}$$

At the initial instant, no tracer is assumed to be present in the ocean:

$$C(0,\mathbf{x}) = 0 \quad . \tag{5}$$

The concentration $C(t,\mathbf{x})$ of the tracer is obtained at any time and position by solving partial differential problem (2)-(5).

Age of the tracer

In the atmosphere, the age $\tau(t,\mathbf{x})$ of the gas is assumed to be a known function of time and position. Then, in accordance with the Constituent-oriented Age and Residence time Theory (CART, www.climate.be/cart_flyer), the age concentration in the ocean, $\alpha(t,\mathbf{x})$, is the solution of the following partial differential problem:

$$\frac{\partial \alpha}{\partial t} = C - \gamma\alpha - \nabla \bullet (\alpha\mathbf{v} - \mathbf{K} \bullet \nabla \alpha) \quad , \tag{6}$$

$$\left[(\mathbf{K} \bullet \nabla \alpha) \bullet \mathbf{n} + \chi (\alpha - C^a \tau)\right]_{\mathbf{x}\in\Gamma^s} = 0 \quad , \tag{7}$$

$$\left[(\mathbf{K} \bullet \nabla \alpha) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \quad , \tag{8}$$

$$\alpha(0,\mathbf{x}) = 0 \quad . \tag{9}$$

It must be pointed out that surface boundary condition (7) is based on the assumptions that the tracer particles leaving the ocean through the ocean-atmosphere interface take their age along with them, whilst those entering the ocean have the age $\tau(t,\mathbf{x}^s)$ at the instant and position they cross the ocean surface.

In the ocean, CART provides that the age of the tracer, $a(t,\mathbf{x})$, is the ratio of the age concentration to the concentration:

$$a(t,\mathbf{x}) = \frac{\alpha(t,\mathbf{x})}{C(t,\mathbf{x})} \quad . \tag{10}$$

Modifying the prescribed atmospheric age

Dr Anne Mouchet (Personal communication, December 2016) suggested that it might be worth studying the consequences of a shift of the prescribed atmospheric age. Accordingly, the age of the gas in the atmosphere is modified to $\tau(t,\mathbf{x})+\delta\tau$, where $\delta\tau$ is a constant. No other alteration is made in the model developed above.

The modified age concentration is

$$\tilde{\alpha}(t,\mathbf{x}) = \alpha(t,\mathbf{x}) + C(t,\mathbf{x})\delta\tau \quad , \tag{11}$$

where the tilde "~" is introduced in order to identify all of the variables associated with the situation in which the atmospheric age has been modified. To demonstrate that this expression holds valid, it must be seen that (11) is the solution of the partial differential problem ensuing from the relevant transformation of (6)-(9), i.e.

$$\frac{\partial\tilde{\alpha}}{\partial t} = C - \gamma\tilde{\alpha} - \nabla\bullet(\tilde{\alpha}\mathbf{v} - \mathbf{K}\bullet\nabla\tilde{\alpha}) \quad , \tag{12}$$

$$\left[(\mathbf{K}\bullet\nabla\tilde{\alpha})\bullet\mathbf{n} + \chi[\tilde{\alpha} - C^a(\tau+\delta\tau)]\right]_{\mathbf{x}\in\Gamma^s} = 0 \quad , \tag{13}$$

$$\left[(\mathbf{K}\bullet\nabla\tilde{\alpha})\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \quad , \tag{14}$$

$$\tilde{\alpha}(0,\mathbf{x}) = 0 \quad . \tag{15}$$

Substituting (11) into (12)-(15) yields

$$\underbrace{\frac{\partial\alpha}{\partial t} - C + \gamma\alpha + \nabla\bullet(\alpha\mathbf{v} - \mathbf{K}\bullet\nabla\alpha)}_{=0,\ \text{see (6)}} = -\delta\tau\underbrace{\left[\frac{\partial C}{\partial t} + \gamma C + \nabla\bullet(C\mathbf{v} - \mathbf{K}\bullet\nabla C)\right]}_{=0,\ \text{see (2)}} \quad , \tag{16}$$

$$\underbrace{\left[(\mathbf{K}\bullet\nabla\alpha)\bullet\mathbf{n} + \chi(\alpha - C^a\tau)\right]_{\mathbf{x}\in\Gamma^s}}_{=0,\ \text{see (7)}} = -\delta\tau\underbrace{\left[(\mathbf{K}\bullet\nabla C)\bullet\mathbf{n} + \chi(C - C^a)\right]_{\mathbf{x}\in\Gamma^s}}_{=0,\ \text{see (3)}} \quad , \tag{17}$$

$$\underbrace{\left[(\mathbf{K}\bullet\nabla\alpha)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^i}}_{=0,\ \text{see (8)}} = -\delta\tau\underbrace{\left[(\mathbf{K}\bullet\nabla C)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^i}}_{=0,\ \text{see (4)}} \quad , \tag{18}$$

$$\underbrace{\alpha(0,\mathbf{x})}_{\substack{=0\\ \text{see (9)}}} = -\delta\tau\underbrace{C(0,\mathbf{x})}_{\substack{=0\\ \text{see (5)}}} \quad . \tag{19}$$

QED.

The modified age is readily seen to be

$$\tilde{a}(t,\mathbf{x}) = \frac{\tilde{\alpha}(t,\mathbf{x})}{C(t,\mathbf{x})} = \frac{\alpha(t,\mathbf{x}) + C(t,\mathbf{x})\delta\tau}{C(t,\mathbf{x})} = a(t,\mathbf{x}) + \delta\tau \tag{20}$$

Shifting the atmospheric age by $\delta\tau$ simply causes the age of the tracer in the ocean to be shifted by the same amount of time, i.e. $\delta\tau$. This is because the age of the tracer particles entering the ocean is shifted by $\delta\tau$ and the age of the tracer particles leaving the ocean take their age along with them. This is readily understood.

Concluding remark

Result (20) about the shift of the age is far from unexpected. It is simple, if not trivial, and its physical interpretation seems to be straightforward. It must be underscored, however, that it holds true for a model in which all variables and parameters may depend on time and position, except the shift of the age of the gas in the atmosphere, which must be constant. Taking into account space or time variations of the atmospheric age shift would demand much more sophisticated mathematical developments, probably involving the suitable Green's function, and no result so simple as (20) would be arrived at.
