

Stochastic rationality and Möbius inverse*

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Abstract

Discrete choice theory is very much dominated by the paradigm of the maximization of a random utility, thus implying that the probability of choosing an alternative in a given set is equal to the sum of the probabilities of all the rankings for which this alternative comes first. This property is called stochastic rationality. In turn, the choice probability system is said to be stochastically rationalizable if and only if the Block-Marschak polynomials are all nonnegative. In this paper, we show that the Block-Marschak polynomials can be defined as the probabilities that the decision maker has to delete each alternative from the choice set when the choice probability system is stochastically rationalizable.

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1 Introduction

Discrete choice theory is very much dominated by the paradigm of the maximization of a random utility.¹ As shown by Block and Marschak (1960), this implies that the choice probability system is stochastically rationalizable: the probability of choosing an alternative in a given set is equal to the sum of the probabilities of all the rankings for which this alternative comes first. Falmagne (1978) has shown later that the choice probability system is stochastically rationalizable if and only if the corresponding Block-Marschak polynomials are all nonnegative.² This amounts to imposing some regularity on the changes in the choice probabilities as more alternatives are deleted from the choice set. In particular, removing one alternative never decreases the probability of choosing a remaining alternative. Furthermore, the marginal effect of removing an alternative increases as the choice set shrinks. However, higher-order conditions are difficult to interpret, thus showing a need for a better understanding of what these polynomials are.

In this note, we build on Billot and Thisse (1999) with the aim of proposing a behavioral interpretation of the Block-Marschak polynomials when the choice probability system is stochastically rationalizable. Specifically, we will see that they can be considered as *the probabilities that the decision maker has to delete each alternative from the choice set*. In other words, a choice probability system is consistent with the maximization of a random utility if and only if the Block-Marschak polynomials are equal to the probabilities that the decision maker ascribes to the fact of choosing from a set that offers her less diversity. In a way that will be made clear below, this interpretation is consistent with a contextual approach of individual choice. Moreover, we also identify the expressions that describe the one-to-one relationship between these probabilities and the Block-Marschak polynomials, thus allowing to compute the former from the later (and vice versa).

2 The Block-Marschak-Falmagne Approach

2.1 Choice probability system and stochastic rationality

Let A be a finite set of n alternatives and $2_{-\emptyset}^A$ the set of all nonempty subsets of A . For any subset $S \in 2_{-\emptyset}^A$ and any alternative $a \in A$, we denote $A \setminus S$ by $\overline{S}$, the set $S \setminus \{a\}$ by S_{-a} and the set $S \cup \{a\}$ by S_{+a} . Let also 2_{-a} be the set of all subsets of A_{-a} .

For any subset $S \in 2_{-\emptyset}^A$, let $(a, S) \rightarrow p_S(a)$ be a mapping from the subset of $A \times 2_{-\emptyset}^A$ containing all the (a, S) such that $a \in S$, into the interval

¹See Anderson *et al.* (1992) for a survey and various economic applications, as well as McFadden (2001) for a very insightful discussion of discrete choices in economic modeling.

²Falmagne's theorem has been developed independently by Barbera and Pattanaik (1986) in a slightly more general setting.

$[0, 1]$. Assume that every p_S satisfies the axioms of a probability over S . We interpret $p_S(a)$ as the probability that an individual *wants to choose* the alternative a when facing the choice set S .

Definition 1 : A choice probability system (CPS) is a pair $(A, \mathbf{P})$ where $\mathbf{P}$ is defined as $(p_S : S \in 2_{-\emptyset}^A)$.

Let $R(A)$ be the set of all possible strict rankings (there is no tie) of the alternatives belonging to A . If the cardinality of A is n , then there are $n!$ possible strict rankings. Define a probability distribution over $R(A)$. Then, a CPS $(A, \mathbf{P})$ is *stochastically rationalizable* if there exists a probability distribution over $R(A)$ for which $p_S(a)$ is equal to the sum of the probabilities associated with the rankings of $R(A)$ in which the alternative a is ranked above all the other alternatives of S , whenever $a \in A$ and $S \in 2_{-\emptyset}^A$.

2.2 A necessary and sufficient condition for stochastic rationality

For each subset $S \subset A_{-a}$, a *Block-Marschak polynomial* $\phi(a, S)$ of the CPS $(A, \mathbf{P})$ is defined as follows:

$$\phi(a, S) = \sum_{k=0}^s (-1)^k \sum_{T \in \mathcal{F}(S, s-k)} p_{\overline{T}}(a) \quad (1)$$

where $\mathcal{F}(S, s-k)$ is the family of subsets of S whose cardinal is equal to $s-k$.

Theorem 1 (Falmagne) : A CPS $(A, \mathbf{P})$ is stochastically rationalizable if and only if all Block-Marschak polynomials $\phi(a, S)$ defined for any alternative $a \in A$ and any $S \subset A_{-a}$ are nonnegative.

3 The Möbius Approach

3.1 A reinterpretation of the Block-Marschak polynomials

In the theorem below, we show that any CPS $(A, \mathbf{P})$ can always be *dually* associated with a unique set of coefficients, which we call the *Möbius coefficients* because they are obtained by means of the dual of the Möbius inverse (Billot and Thisse, 1999).

Theorem 2 : Let a CPS $(A, \mathbf{P})$. Then, for each alternative $a \in A$ and each $S \subset A_{-a}$, $S \neq \emptyset$, there exists a unique set π^a of coefficients $(\pi_T^a : T \supset S)$ such that :

$$p_{S+a}(a) = \sum_{T \supset S} \pi_T^a \quad (2)$$

if and only if each coefficient of the set π^a is given by

$$\pi_T^a = \sum_{R \supset S} (-1)^{r-s} p_{R+a}(a) \quad (3)$$

where r and s stand for the cardinalities of R and S , respectively.

Proof. It involves three steps.

Step 1. We show that

$$\sum_{T \supset S} (-1)^t = \begin{cases} (-1)^n & \text{if } S = A, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for $s < n$, we have:

$$\begin{aligned} \sum_{T \supset S} (-1)^t &= (-1)^s + (-1)^{s+1} + \dots + (-1)^n \\ &= (-1)^s [(-1)^0 + \dots + (-1)^{n-s}] \\ &= (-1)^s (1 - 1)^{n-s} = 0. \end{aligned}$$

For $s = n$, we have $\sum_{T \supset S} (-1)^t = (-1)^n$ because $t = n$.

Step 2. We now show that

$$\sum_{T \supset R \supset S} (-1)^r = \begin{cases} (-1)^t & \text{if } S = T, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, since $T \cup \bar{T} = A \supset R \cup \bar{T} \supset S \cup \bar{T}$, then $A \supset M \supset S \cup \bar{T}$ where $M \equiv R \cup \bar{T}$. Hence, $r = m - \bar{t}$ and we have:

$$\begin{aligned} \sum_{T \supset R \supset S} (-1)^r &= \sum_{M \supset S \cup \bar{T}} (-1)^{m-\bar{t}} \\ &= (-1)^{-\bar{t}} \sum_{M \supset S \cup \bar{T}} (-1)^m \\ &= (-1)^{t-n} \sum_{M \supset S \cup \bar{T}} (-1)^m \\ &= (-1)^{t-n} (-1)^n \quad (\text{by Step 1 where } S = T) \\ &= (-1)^t. \end{aligned}$$

Otherwise, the argument used in Step 1 still applies.

Step 3. We first show that (3) implies (2):

$$\begin{aligned}
\sum_{T \supset S} (-1)^{t-s} p_{T+a}(a) &= (-1)^{-s} \sum_{T \supset S} (-1)^t p_{T+a}(a) \\
&= (-1)^{-s} \sum_{T \supset S} (-1)^t \sum_{R \supset T} \pi_R^a \\
&= (-1)^{-s} \sum_{R \supset S} \pi_R^a \sum_{R \supset T \supset S} (-1)^t \\
&= (-1)^{-s} \pi_S^a (-1)^s \quad (\text{by Step 2}) \\
&= \pi_S^a.
\end{aligned}$$

It remains to prove that (2) implies (3). We have:

$$\begin{aligned}
\sum_{T \supset S} \pi_T^a &= \sum_{T \supset S} \sum_{R \supset T \supset S} (-1)^{t-s} p_{R+a}(a) \\
&= \sum_{R \supset S} (-1)^{-s} p_{R+a}(a) \sum_{R \supset T \supset S} (-1)^t \\
&= (-1)^{-s} p_{S+a}(a) (-1)^s \quad (\text{by Step 2}) \\
&= p_{S+a}(a).
\end{aligned}$$

■

Definition 2 : A Möbius coefficient system (MCS) is a pair (A, Π) where Π is defined as $(\pi^a : a \in A)$.

Note that a Möbius coefficient π_T^a need not be nonnegative. Then, Theorem 1 has the following direct implication:

Corollary 1 : Any CPS $(A, \mathbf{P})$ is equivalent to a MCS (A, Π) .

We now show that any Block-Marschak polynomial can be rewritten as a Möbius coefficient.

Lemma 1 : Let a CPS $(A, \mathbf{P})$. Then, for any alternative $a \in A$ and any subset $S \subset A_{-a}$, we have:

$$\pi_{S_{-a}}^a = \phi(a, S).$$

Proof. Consider any polynomial $\phi(a, S)$ where $S \subset A_{-a}$. Then,

$$\begin{aligned}
\phi(a, S) &= \sum_{k=0}^s (-1)^k \sum_{T \in \mathcal{F}(S, s-k)} p_{\overline{T}}(a) \\
&= \sum_{T \subset S} (-1)^{s-t} p_{\overline{T}}(a) \\
&= \sum_{\overline{T} \supset \overline{S}} (-1)^{(n-t)-(n-s)} p_{\overline{T}}(a).
\end{aligned}$$

By (3) in Theorem 1, $\sum_{T \supset \bar{S}} (-1)^{(n-t)-(n-s)} p_T(a) = \pi_{\bar{S}-a}^a$. ■

Hence, any Block-Marschak polynomial $\phi(a, S)$ is a Möbius coefficient of the type $\pi_{\bar{S}-a}^a$. The *duality* property follows from the fact that ϕ is associated with the choice set S and π with its complement $\bar{S}$.

Moreover, as shown below, the Möbius coefficients, hence the Block-Marschak polynomials, sum up to 1.

Lemma 2 : *Let a CPS $(A, \mathbf{P})$. Then, for any alternative $a \in A$, we have:*

$$\sum_{S \subset A-a} \pi_S^a = 1.$$

The proof follows immediately from (2) and (3) and is, therefore, omitted.

The subset S being arbitrary, Lemmas 1 and 2 imply that the nonnegativity of the Block-Marschak polynomials $\phi(a, S) = \pi_{\bar{S}-a}^a$ is equivalent to a condition requiring that the Möbius coefficients π^a define a probability distribution over 2_{-a} . Consequently, we have shown:

Theorem 3 (Falmagne revisited I) : *A CPS $(A, \mathbf{P})$ is stochastically rationalizable if and only if for each alternative $a \in A$, the corresponding set π^a of Möbius coefficients defines a probability distribution over 2_{-a} .*

3.2 The Block-Marschak polynomials when context matters

Choice theory has been criticized because it does not allow one to account for the specific context in which concrete choices are made, although arm-chair evidence shows that context matters (see Quandt, 1956, for an early criticism). In this section, we propose another interpretation of the Möbius coefficients, hence of the Block-Marschak polynomials, based on Billot and Thisse (1999). These authors have developed a model in which the choice set itself defines the “choice context” in which a particular alternative is to be selected. This is because a broader choice set allows for more diversity/flexibility in decision making. Such an approach agrees with a type of contextualization that has attracted a lot of attention in modern economics under the name of “preference for variety” (see, e.g. the new growth and trade theories). In this perspective, we will see that the Möbius coefficients can be interpreted as the probability that the individual has to add a new alternative to a given choice set. Given the duality relationship identified in Theorem 2 and the formal equivalence given by Lemma 1, *the Block-Marschak polynomials may then be viewed as the probability that the individual has to delete an alternative from the choice set she faces*. In other words, the Block-Marschak polynomials provide a simple and neat evaluation of the impact on individual behavior caused by a change in the choice context.

Let $(a, T) \rightarrow \pi_T^a$ be a mapping from the subset of $(a \times 2_{-a} : a \in A)$ containing all the (a, T) such that $a \notin T$, into the interval $[0, 1]$. Assume that π^a satisfies the axioms of a probability over 2_{-a} and then interpret π_T^a as the probability that an individual *wants to add* the alternative a to the choice set T in order to be in a broader choice context defined by T_{+a} .

Definition 3 : An adding probability system (APS) is a pair (A, Π) such that, for each $a \in A$, π^a satisfies the axioms of a probability over 2_{-a} .

Given Corollary 1 and Theorem 3, an APS yields probability distributions that are conceptually similar to those defining a stochastic rationalizable system. Indeed, the probability π_S^a for the alternative a to be added to a particular subset $S \subset A_{-a}$ does not seem to differ from the probability over the orderings in which a dominates all the alternatives of S . By contrast, whenever a is dominated by at least one alternative of S , a should not be added to S .

In order to check that this intuition is correct, we consider the MCS (A, Π) associated with a stochastically rationalizable CPS $(A, \mathbf{P})$ and show that it can be induced by $\mathbf{P}$. First, set $T = A_{-a}$. In this case, the probability π_T^a that the individual wants the alternative a to be added to $T = A_{-a}$ is identical to the probability that she wants to choose a in the context A , that is,

$$\pi_T^a = p_A(a).$$

Consider now $T = A_{-ab}$. Once a has been added to T , either a belongs to the subset A_{-b} or not. In either case, since $T = A_{-ab}$, the probability for a to be added to T without accounting for the fact that $a \in A$ by assumption, is therefore given by:

$$\pi_T^a = p_{A_{-b}}(a) - p_A(a). \quad (4)$$

If $T = A_{-abc}$, one might think that π_T^a is such that

$$\pi_T^a = p_{A_{-bc}}(a) - p_{A_{-b}}(a) - p_{A_{-c}}(a) - p_A(a).$$

However, this expression does not account for the fact that, when a belongs to A_{-b} (resp. A_{-c}), this may be because this alternative has been added to A_{-ab} (resp. A_{-ac}). Deleting these occurrences, we obtain:

$$\pi_T^a = p_{A_{-bc}}(A) - \left[p_{A_{-b}}(a) - \pi_{A_{-ab}}^a \right] - \left[p_{A_{-c}}(a) - \pi_{A_{-ac}}^a \right] - p_A(a).$$

Given (4), this may be rewritten as follows:

$$\pi_T^a = p_{A_{-bc}}(a) - p_{A_{-b}}(a) - p_{A_{-c}}(a) + p_A(a).$$

More generally, for any alternative $a \in A$ and all $S, T \subset A_{-a}$, the probability that the individual wants a to be added to T is given by:

$$\pi_T^a = \sum_{S \supset T} (-1)^{s-t} p_{S+a}(a)$$

which is identical to (3) in Theorem 2. Hence, Theorem 3 may be rewritten as follows:

Theorem 4 (Falmagne revisited II) : *A CPS $(A, \mathbf{P})$ is stochastically rationalizable if and only if the corresponding MCS (A, Π) is an APS.*

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