

NONEXISTENCE AND OPTIMAL DECAY OF SUPERSOLUTIONS TO CHOQUARD EQUATIONS IN EXTERIOR DOMAINS

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ABSTRACT. We consider a semilinear elliptic problem with a nonlinear term which is the product of a power and the Riesz potential of a power. This family of equations includes the Choquard or nonlinear Schrödinger–Newton equation. We show that for some values of the parameters the equation does not have nontrivial nonnegative supersolutions in exterior domains. The same techniques yield optimal decay rates when supersolutions exists.

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1. INTRODUCTION.

We study the nonlocal nonlinear equation

$$(C) \quad -\Delta u + Vu = (I_\alpha * u^p)u^q \quad \text{in } \Omega,$$

where $\Omega \subseteq \mathbb{R}^N$ is a domain, for given exponents $p > 0$, $q \in \mathbb{R}$ and potential $V : \Omega \rightarrow \mathbb{R}$. Here, $I_\alpha : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}$ denotes the Riesz potential, which is defined for $0 < \alpha < N$ and $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}},$$

where

$$A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}$$

and Γ is the Gamma function [35, p.19].

The nonlocal equation (C) has several physical origins. When $\Omega = \mathbb{R}^3$, $\alpha = 2$, $p = 2$, $q = 1$ and V is constant, the equation writes as

$$(1.1) \quad -\Delta u + Vu = (I_2 * u^2)u \quad \text{in } \mathbb{R}^3,$$

or, equivalently,

$$(1.2) \quad \begin{cases} -\Delta u + Vu = Wu & \text{in } \mathbb{R}^3, \\ -\Delta W = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

If u solves (1.1) then the function ψ defined by $\psi(t, x) = e^{it\lambda}u(x)$ is a solitary wave of the focusing Hartree equation

$$i\psi_t = -\Delta\psi - (I_2 * \psi^2)\psi.$$

Equation (1.1) first appeared at least as early as in 1954, in a work by S.I. Pekar [31] describing the quantum mechanics of a polaron at rest (see discussion in [20]). In 1976 P. Choquard used (1.1) to describe an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one component plasma, see [18]. In 1996 R. Penrose proposed (1.2) as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon [32].

When the potential V is constant, E.H. Lieb [18] established existence and uniqueness of a positive radial ground state solution of (1.1) using variational methods. P.-L. Lions extended Lieb’s results by replacing I_2 with a wider class of convolution kernels and established existence of infinitely many radial (changing-sign) solutions with increasing energy [21; 22, Chapter III]. G. P. Menzala established further existence and nonexistence results for equations of type (1.1) with a variable potential $V(x)$ and general convolution kernels [25, 26].

I. Moroz, R. Penrose and P. Tod have studied independently the existence, uniqueness and decay properties of the positive ground state and changing-sign radial solutions of (1.1) numerically and via ODE methods [28, 37] (see also [17, Appendix A]). An ODE based proof of the existence and uniqueness of the radial ground state of (C) with $V = 1$, $\alpha = 2$, $p = 2$ and $q = 1$ in dimensions $N \geq 1$ was recently obtained by P. Choquard, J. Stubbe and M. Vuffray [8]. L. Ma and L. Zhao have

studied the symmetry of positive radial ground state of $(\mathcal{C})$ with constant V in higher dimensions using the moving-plane method [24].

J. Wei and M. Winter [39] have considered the singular perturbation problem

$$(1.3) \quad -\varepsilon^2 \Delta u + Vu = (I_2 * u^2)u \quad \text{in } \mathbb{R}^3.$$

Assuming that $\inf V > 0$ they have proved that in the semi-classical limit $\varepsilon \rightarrow 0$ there exist multibump positive solutions of (1.3) which concentrate as $\varepsilon \rightarrow 0$ to critical points of the potential V . S. Secchi [36] studied the existence of positive solutions of (1.3) which concentrate to critical points of V under the assumption that V does not decay too fast at infinity.

In the context of local semilinear equations of the type

$$(1.4) \quad -\Delta u + Vu = Wu^q$$

it is well known that the existence of positive solutions and supersolutions in $\mathbb{R}^N$ or in exterior domains of $\mathbb{R}^N$ requires a careful apriori balance between the value of the nonlinear exponent q and the decay rate at infinity of the potentials V and W [4, 12]. Such results are often called nonlinear Liouville theorems, see [34, Section 1.8] and further references therein.

The main purpose of this work is to establish sharp Liouville type nonexistence results for positive supersolutions of nonlocal Choquard equation $(\mathcal{C})$ in exterior domains. For instance, for the classical Choquard equation (1.1) we obtain the following result as a particular case of Theorem 3.

Theorem. *Let $\lambda > 0$. The problem*

$$(\mathcal{C}_{\gamma,\lambda}) \quad -\Delta u(x) + \frac{\lambda^2}{|x|^\gamma} u(x) \geq (I_2 * u^2)(x) u(x) \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_1$$

admits a positive supersolution if and only if $\lambda > 0$ and $-\infty < \gamma \leq 1$.

Moreover, if $\gamma < 1$ and if $u \geq 0$ is a nontrivial nonnegative supersolution of $(\mathcal{C}_{\gamma,\lambda})$, then there exists $m > 0$ such that ,

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{1-\frac{\gamma}{4}} \exp \int_m^{|x|} \sqrt{\frac{\lambda^2}{s^\gamma} - \frac{m}{s}} ds > 0;$$

if $\gamma = 1$ and if $u \geq 0$ is a nontrivial nonnegative supersolution of $(\mathcal{C}_{\gamma,\lambda})$, then there exists $m \in (0, \lambda)$ such that,

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{3}{4}} \exp(2(\lambda - m)\sqrt{|x|}) > 0.$$

The above lower bounds are optimal.

In particular, this gives a negative answer to a question posed by S. Secchi [36, p. 3855].

When $\gamma < 1$, the integral in the asymptotics is an incomplete Beta function

$$\int_m^{|x|} \frac{\lambda}{s^{\frac{\gamma}{2}}} \sqrt{1 - \frac{\rho^{1-\gamma}}{s^{1-\gamma}}} ds = \frac{1}{1-\gamma} B_{1-\frac{m}{|x|^{1-\gamma}}} \left(\frac{3}{2}, -\frac{2+\gamma}{2(1-\gamma)} \right),$$

which can also be represented in terms of the hypergeometric function. The asymptotic can be made explicit by taking the Taylor expansion of the square root:

— if $\gamma < 0$, then

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{1-\frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{\frac{2-\gamma}{2}}\right) > 0,$$

— if $\gamma = 0$, then there exists $\rho > 0$ such that

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{1-\frac{\rho}{2\lambda}} \exp(\lambda|x|) > 0,$$

— if $0 < \gamma < \frac{2}{3}$, then there exists $\rho > 0$ such that

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{1-\frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{\frac{2-\gamma}{2}} - \frac{m}{\lambda\gamma} |x|^{\frac{\gamma}{2}}\right) > 0;$$

for $\frac{2}{3} \leq \gamma < 1$, see Remark 6.1.

In particular if u is the unique radial positive ground state solution of

$$-\Delta u + u = (I_2 * u^2)u \quad \text{in } \mathbb{R}^3$$

(see for example [8, 17, 18, 24] for proofs of existence and uniqueness), then there exists $\rho > 0$ such that

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{1-\frac{\rho}{2}} \exp(|x|) > 0.$$

Thus the ground state u decays slower than the fundamental solution of $-\Delta + 1$ in $\mathbb{R}^3$. (Note that in [39, Theorem I.1 (1.7)], the correction to the exponent ρ seems missing.) One has in fact

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{1-\frac{\rho}{2}} \exp(|x|) \in (0, \infty),$$

where $\rho > 0$ is characterized by the groundstate energy [30].

In the study of the general Choquard equation $(\mathcal{C})$ with its multiple parameters, we classify the cases with respect to the decay rate of the potential V and with respect to the type of the nonlinearity.

We distinguish between four different types of potentials:

(i) *unperturbed Laplacian*

$$V(x) = 0,$$

(ii) *fast decay potentials*

$$V(x) = \frac{\lambda}{|x|^\gamma}, \quad \text{with } \lambda \in \mathbb{R} \text{ and } \gamma > 2,$$

(iii) *Hardy potentials*

$$V(x) = \frac{\nu^2 - \left(\frac{N-2}{2}\right)^2}{|x|^2} \quad \text{with } \nu > 0,$$

(iv) *slow decay potentials*

$$V(x) = \frac{\lambda^2}{|x|^\gamma} \quad \text{with } \lambda > 0 \text{ and } \gamma < 2.$$

The classification of potentials is motivated by the decay rate of the fundamental solution of the linear operator $-\Delta + V$: for fast decay and Hardy potentials it decays polynomially, while for slow decay potentials it has exponential decay. This difference is essential for our considerations.

The above radial potentials could be replaced by wider classes of nonradial potentials with equivalent decay rate of the fundamental solution of $-\Delta + V$. We restrict ourself to the explicit power-like potentials in order to simplify the exposition.

The other distinction is made with respect to the types of the nonlinearity. In the context of the local equations (1.4) one usually distinguishes between the superlinear case $q > 1$ and sublinear case $q < 1$. The corresponding classification for Choquard's equation $(\mathcal{C})$ is more complex. According to the order of homogeneity of its right-hand side, we distinguish

(i) the *superlinear case* $q > 1$,

(ii) the *locally sublinear case* $p + q > 1$ and $q < 1$,

(iii) the *globally sublinear case* $p + q < 1$.

The superlinear and the globally sublinear cases correspond to the superlinear and the sublinear cases for the local equation (1.4); the locally sublinear case is a transitional region which has no analogue in the local equation. The transitional *locally linear case* ($q = 1$) and *globally linear case* ($p + q = 1$) require particularly careful consideration.

The above classifications of potentials and nonlinearities produces a large variety of different cases in our analysis of $(\mathcal{C})$ requiring specific consideration. Nevertheless, for all classes of potentials and types nonlinearities we use the same unified approach which is based on two main tools:

- lower and upper Phragmen–Lindelöf type estimates on the decay at infinity of positive supersolutions of the linear operator $-\Delta + V$; see Proposition 5.1 and Proposition 6.1;
- a nonlocal nonlinear extension of the Agmon–Allegretto–Piepenbrink positivity principle [1, Theorem 3.1], which relates the existence of a positive supersolution to $(\mathcal{C})$ to an integral inequality; see Proposition 3.2.

Combining linear estimates for $-\Delta + V$ with the positivity principle either leads to a contradiction which implies nonexistence of positive supersolutions of $(\mathcal{C})$, or provides a bound on the admissible rate of the decay of a solution. Explicit construction of appropriate supersolutions shows that these bounds are optimal.

We point out that our nonexistence and decay results are optimal for *supersolutions*. We have observed that these bounds give nonetheless good insight in the decay of minimal energy *solutions* of $(\mathcal{C})$ on $\mathbb{R}^N$ in the variational case $q = p - 1$ [30].

2. STATEMENT OF THE RESULTS.

2.1. Notion of a supersolution. Let $N \geq 1$ and $\Omega \subseteq \mathbb{R}^N$ be an open set and $V \in L^1_{\text{loc}}(\mathbb{R}^N)$ be a generic potential. In order to formulate our results we shall clarify the notion of supersolution to Choquard equation $(\mathcal{C})$ which we adopt in this work.

Definition 2.1. A nonnegative function $u \in L^1_{\text{loc}}(\Omega)$ is a (*distributional*) *supersolution* of $(\mathcal{C})$ if

$$(2.1) \quad \int_{\Omega} \frac{u(x)^p}{1 + |x|^{N-\alpha}} dx < \infty,$$

$Vu \in L^1_{\text{loc}}(\Omega)$, $(I_{\alpha} * u^p)u^q \in L^1_{\text{loc}}(\Omega)$, and for every nonnegative test function $\varphi \in C_c^{\infty}(\Omega)$ one has

$$-\int_{\Omega} u \Delta \varphi + \int_{\Omega} V u \varphi \geq \int_{\Omega} (I_{\alpha} * u^p) u^q \varphi.$$

Here we extend the usual definition of the convolution product by setting, for $x \in \Omega$,

$$(I_{\alpha} * u^p)(x) = \int_{\Omega} I_{\alpha}(x - y) u(y)^p dy.$$

This coincides with the standard convolution product of I_{α} with the extension of u by 0 to $\mathbb{R}^N$.

If $\omega \subset \Omega$ is open, then in general $I_{\alpha} * (u|_{\omega})^p < I_{\alpha} * u^p$. Hence if u is a supersolution of $(\mathcal{C})$ in Ω , then u is a supersolution of $(\mathcal{C})$ in ω but our notion of supersolution is not local, as u can be a distributional supersolution of $(\mathcal{C})$ in the open sets $\Omega_1 \subset \mathbb{R}^N$ and $\Omega_2 \subset \mathbb{R}^N$ but not in $\Omega_1 \cap \Omega_2$.

2.2. Equation with the unperturbed Laplacian. Consider the Choquard equation $(\mathcal{C})$ with the potential $V \equiv 0$, that is

$$(\mathcal{C}_0) \quad -\Delta u = (I_\alpha * u^p)u^q.$$

It is an easy consequence of (2.1) and standard lower bounds on superharmonic functions that $(\mathcal{C}_0)$ has no positive supersolutions in exterior domains of $\mathbb{R}^N$ in dimensions $N = 1, 2$ (Proposition 4.3). In higher dimensions the existence of nontrivial nonnegative supersolutions of $(\mathcal{C}_0)$ is more complex.

Theorem 1. *Let $N \geq 3$, $0 < \alpha < N$, $p > 0$, $q \in \mathbb{R}$ and $\rho > 0$. Equation $(\mathcal{C}_0)$ has a nonnegative nontrivial supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$ if and only if the following assumptions hold simultaneously:*

$$(2.2a) \quad p > \frac{\alpha}{N-2},$$

$$(2.2b) \quad p + q > \frac{N+\alpha}{N-2},$$

$$(2.2c) \quad q > \frac{\alpha}{N-2} \quad \text{if } \alpha > N-2,$$

$$(2.2d) \quad q \geq 1 \quad \text{if } \alpha = N-2,$$

$$(2.2e) \quad q > 1 - \frac{N-\alpha-2}{N}p \quad \text{if } \alpha < N-2.$$

Moreover, if $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then

$$(2.3a) \quad \liminf_{|x| \rightarrow \infty} u(x)|x|^{N-2} > 0 \quad \text{if } q > \frac{\alpha}{N-2},$$

$$(2.3b) \quad \exists m > 0 : \quad \liminf_{|x| \rightarrow \infty} u(x)|x|^{N-2-m} > 0 \quad \text{if } q = \frac{\alpha}{N-2} = 1,$$

$$(2.3c) \quad \liminf_{|x| \rightarrow \infty} u(x)|x|^{N-2}(\log|x|)^{-\frac{N-2}{N-\alpha-2}} > 0 \quad \text{if } q = \frac{\alpha}{N-2} < 1,$$

$$(2.3d) \quad \liminf_{|x| \rightarrow \infty} u(x)|x|^{\frac{N-\alpha-2}{1-q}} > 0 \quad \text{if } q < \frac{\alpha}{N-2} < 1.$$

The above lower bounds are optimal.

The optimality of lower bounds (2.3) is understood in the sense that

- if $p > \frac{\alpha}{N-2}$, $p + q > \frac{N+\alpha}{N-2}$ and $q > \frac{\alpha}{N-2}$, then there exists a nontrivial supersolution $u \geq 0$ such that

$$\limsup_{|x| \rightarrow \infty} u(x)|x|^{N-2} < \infty,$$

- if $p > \frac{N}{N-2}$ and $q = \frac{\alpha}{N-2} = 1$, then for every $m > 0$ there exists a nontrivial supersolution $u \geq 0$ such that

$$\limsup_{|x| \rightarrow \infty} u(x)|x|^{N-2-m} < \infty,$$

- if $p > \frac{N}{N-2}$ and $q = \frac{\alpha}{N-2} = 1$, then there exists a nontrivial supersolution $u \geq 0$ such that

$$\limsup_{|x| \rightarrow \infty} u(x)|x|^{N-2}(\log|x|)^{-\frac{N-2}{N-\alpha-2}} < \infty,$$

- if $1 - \frac{N-\alpha-2}{N}p < q < \frac{\alpha}{N-2} < 1$, then there exists a nontrivial supersolution $u \geq 0$ such that

$$\limsup_{|x| \rightarrow \infty} u(x)|x|^{\frac{N-\alpha-2}{1-q}} < \infty.$$

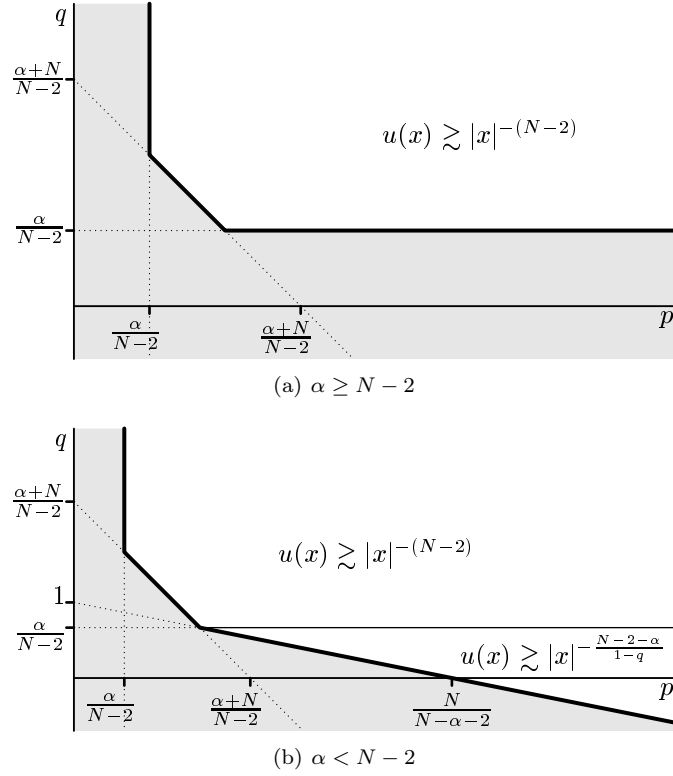


FIGURE 1. Existence, decay and nonexistence regions for $(\mathcal{C}_0)$ in the (p, q) -plane

In view of the bounds (2.3) in what follows we refer to the region $\{q > \frac{\alpha}{N-2}\}$ of the (p, q) -plane as the *Green decay* region, while we call $\{q < \frac{\alpha}{N-2}\}$ the *sublinear decay* region.

When $q = p - 1$, equation $(\mathcal{C}_0)$ has a variational structure, with the energy formally defined by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N \setminus B_\rho} |\nabla u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N \setminus B_\rho} (I_\alpha * u^p) u^p.$$

The existence conditions (2.2) then transform into

$$\begin{aligned} (2.2b)' \quad & p > \frac{1}{2} \frac{2N-2+\alpha}{N-2} && \text{if } 0 < \alpha \leq 2, \\ (2.2e)' \quad & p > \frac{2N}{2N-\alpha-2} && \text{if } 2 < \alpha < N-2, \\ (2.2d)' \quad & p \geq 2 && \text{if } 2 < \alpha = N-2, \\ (2.2c)' \quad & p > 1 + \frac{\alpha}{N-2} && \text{if } \max\{2, N-2\} < \alpha < N. \end{aligned}$$

When $q = 0$, equation $(\mathcal{C}_0)$ is written as $-\Delta u = I_\alpha * u^p$. If $\alpha < N-2$, this is equivalent to $u = I_{\alpha+2} * u^p$. Existence and nonexistence of nontrivial nonnegative supersolutions in exterior domains for such equations were recently studied in [7, 27].

As $\alpha \rightarrow 0$ one has $\lim_{\alpha \rightarrow 0} I_\alpha * \varphi = \varphi$ for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ and $-\Delta u = u^{p+q}$ becomes a limiting equation for $(\mathcal{C}_0)$ when $\alpha \rightarrow 0$. Such local equation admits nontrivial nonnegative supersolutions in exterior domains if and only if $p+q > \frac{N}{N-2}$.

(see [12; 34, Section 1.8]). Similarly, as $\alpha \rightarrow N$, $-\Delta u = (4\pi)^{-\frac{N}{2}} (\int_{\mathbb{R}^N} |u|^p) u^q$ is a limiting equation which has a supersolution in an exterior domains if and only if $q > \frac{N}{N-2}$ and $p > \frac{N}{N-2}$. Our results are formally consistent with these limiting cases.

2.3. Equations with fast decay and Hardy potentials. Consider Choquard equation $(\mathcal{C})$ with the *fast decay* potential, that is

$$(C_{\lambda,\gamma}^F) \quad -\Delta u(x) + \frac{\lambda}{|x|^\gamma} u(x) = (I_\alpha * u^p)(x) u(x)^q,$$

where $\lambda \in \mathbb{R}$ and $\gamma > 2$. In Theorem 8 we show that all the nonexistence, existence and optimal decay results of Theorem 1 remain stable with respect to the perturbations of $-\Delta$ by the fast decay potentials and do not depend on particular values of λ and γ . This is a consequence of the well known fact that the fundamental solution of the Schrödinger operator $-\Delta + V$ with a fast decay potential V decays at infinity as $|x|^{-(N-2)}$, that is as the Green function of $-\Delta$ on $\mathbb{R}^N$. It turns out that the values of all critical exponents and decay rates of Theorem 1 are controlled by the decay rate of the fundamental solution of $-\Delta + V$. See Section 4.4 for details.

In Section 5 we study Choquard equation $(\mathcal{C})$ with the *Hardy* potential, that is

$$(C_\nu^H) \quad -\Delta u(x) + \left(\nu^2 - \left(\frac{N-2}{2}\right)^2\right) \frac{1}{|x|^2} u(x) = (I_\alpha * u^p)(x) u(x)^q,$$

where $\nu > 0$. Hardy potential provides an important example of a perturbation where the decay rate of the fundamental solution of $-\Delta + V$ remains polynomial but depends explicitly on the value of the constant ν . In Theorem 9 we show that as a consequence, some of the critical exponents and decay rates of Theorem 1 become sensitive to the constant ν , although the qualitative picture remains essentially similar to the case of the unperturbed equation $(\mathcal{C})$. Full statements and sketches of the proofs of relevant results are given in Section 5.

2.4. Equation with slow decay potentials. Consider the Choquard equation $(\mathcal{C})$ with the *slow decay* potential, that is

$$(C_{\lambda,\gamma}^S) \quad -\Delta u(x) + \frac{\lambda^2}{|x|^\gamma} u(x) = (I_\alpha * u^p)(x) u(x)^q,$$

where $\lambda > 0$ and $-\infty < \gamma < 2$. It is well known that if V is a slow decay potential then the fundamental solution of $-\Delta + V$ decays exponentially at infinity. As a consequence, the qualitative picture of the existence and nonexistence of positive supersolutions of $(C_{\lambda,\gamma}^S)$ changes compared to equations with fast decay or Hardy potentials.

For local equations of type (1.4) with superlinear $q > 1$ one usually expects to find a *fast decay* positive solution, which decays at infinity at the same rate as the fundamental solution of $-\Delta + V$. We will see that this is indeed the case for the $(C_{\lambda,\gamma}^S)$ when $q > 1$, while for $q < 1$ positive supersolutions of $(C_{\lambda,\gamma}^S)$ decay at most polynomially. The decay of solutions in the borderline region $q = 1$ remains exponential but the detailed picture becomes particularly complex. Note however that for Choquard equation $(C_{\lambda,\gamma}^S)$ the natural threshold between sub and super-linear homogeneity is $p + q = 1$ rather than $q = 1$, so the polynomial behavior of supersolutions to $(C_{\lambda,\gamma}^S)$ in the superlinear region seems to be a new phenomenon.

For equation $(C_{\lambda,\gamma}^S)$ we shall consider separately the *exponential decay region* $q \geq 1$ and the *polynomial decay region* $q < 1$, because different mechanisms are responsible for the decay and nonexistence properties of positive solutions in these two regions.

2.4.1. *Exponential decay region $q > 1$.* Our first result regarding equations with slow decaying potentials states that in the globally superlinear case $q > 1$ then $(\mathcal{C}_{\lambda,\gamma}^S)$ always admits a positive solution which decay at infinity at the same rate as the fundamental solution of $-\Delta + V$.

Theorem 2. *Let $N \geq 1$, $\lambda > 0$, $\gamma < 2$, $0 < \alpha < N$, $p > 0$, $q > 1$ and $\rho > 0$. Then equation $(\mathcal{C}_{\lambda,\gamma}^S)$ has a nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$. Moreover, if $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then*

$$(2.4) \quad \liminf_{|x| \rightarrow \infty} u(x)|x|^{\frac{N-1}{2}-\frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma}|x|^{\frac{2-\gamma}{2}}\right) > 0.$$

The above lower bound is optimal.

2.4.2. *Locally linear region $q = 1$.* In the borderline case $q = 1$ the behavior of positive solutions to $(\mathcal{C}_{\lambda,\gamma}^S)$ is more complex. It turns out that the existence and decay properties of nontrivial nonnegative supersolutions of $(\mathcal{C}_{\lambda,\gamma}^S)$ with $q = 1$ and arbitrary $p > 0$ are controlled by the relevant properties of positive solutions of the linear equation

$$(2.5) \quad -\Delta u(x) + \frac{\lambda^2}{|x|^\gamma} u(x) = \frac{m}{|x|^{N-\alpha}} u(x),$$

where $\gamma < 2$, $\lambda > 0$ and $m > 0$. The detailed analysis of the decay rate of positive supersolutions of equation (2.5) is given in Section 6.1. The corresponding result for $(\mathcal{C}_{\lambda,\gamma}^S)$ reads as follows.

Theorem 3. *Let $N \geq 1$, $\lambda > 0$, $\gamma < 2$, $0 < \alpha < N$, $p > 0$, $q = 1$ and $\rho > 0$. Then equation $(\mathcal{C}_{\lambda,\gamma}^S)$ has a nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$ if and only if*

$$(2.6) \quad \gamma \leq N - \alpha.$$

Moreover, if $\gamma < N - \alpha$ and $u \geq 0$ is a nontrivial nonnegative supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then there exists $m > 0$ such that

$$(2.7) \quad \liminf_{|x| \rightarrow \infty} u(x)|x|^{\frac{N-1}{2}-\frac{\gamma}{4}} \exp \int_{m^{\frac{1}{N-\alpha-\gamma}}}^{|x|} \sqrt{\frac{\lambda^2}{s^\gamma} - \frac{m}{s^{N-\alpha}}} ds > 0.$$

and if $\gamma = N - \alpha$ and $u \geq 0$ is a nontrivial nonnegative supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then there exists $m \in (0, \lambda)$ such that

$$(2.8) \quad \liminf_{|x| \rightarrow \infty} u(x)|x|^{\frac{N-1}{2}-\frac{\gamma}{4}} \exp\left(\frac{2(\lambda-m)}{2-\gamma}\right)|x|^{\frac{2-\gamma}{2}} > 0.$$

The above lower bounds are optimal.

The value of the constant $m > 0$ in (2.7) and in (2.8) depends on the supersolution u . Hence, optimality of (2.7) should be understood in the sense that given any $m > 0$ there is a nontrivial supersolution $u \geq 0$ of $(\mathcal{C}_{\lambda,\gamma}^S)$ in an exterior domain such that

$$\limsup_{|x| \rightarrow \infty} u(x)|x|^{\frac{N-1}{2}-\frac{\gamma}{4}} \exp \int_{m^{\frac{1}{N-\alpha-\gamma}}}^{|x|} \sqrt{\frac{\lambda^2}{s^\gamma} - \frac{m}{s^{N-\alpha}}} ds < \infty.$$

2.4.3. *Sublinear region $q < 1$.* When $q < 1$ solutions of $(\mathcal{C}_{\lambda,\gamma}^S)$ start to decay polynomially. The values $\gamma = N - \alpha$ and $\gamma = -\alpha$ should be distinguished as two critical thresholds separating different qualitative properties of positive supersolutions to $(\mathcal{C}_{\lambda,\gamma}^S)$. We set apart our results depending on the value of γ . First we consider the case when $\alpha > N - 2$ and V decays at a moderately slow rate $N - \alpha \leq \gamma < 2$.

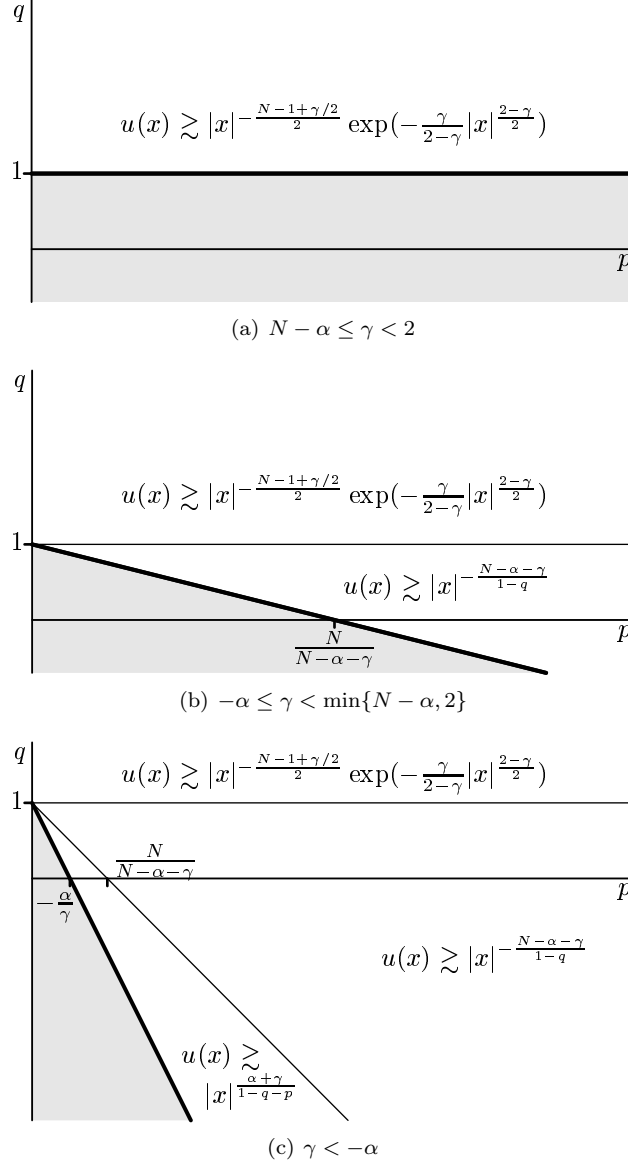


FIGURE 2. Existence, decay and nonexistence regions for $(\mathcal{C}_{\lambda, \gamma}^S)$ in the (p, q) -plane

Theorem 4. Let $N \geq 1$, $\lambda > 0$, $\gamma < 2$, $N - 2 < \alpha < N$, $p > 0$, $q < 1$ and $\rho > 0$. If

$$N - \alpha \leq \gamma < 2,$$

then equation $(\mathcal{C}_{\lambda, \gamma}^S)$ has no nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$.

Next we look at the intermediate slow decay or slow growth régime $-\alpha < \gamma < \min\{N - \alpha, 2\}$, which includes in particular the autonomous case $\gamma = 0$.

Theorem 5. Let $N \geq 1$, $\lambda > 0$, $\gamma < 2$, $0 < \alpha < N$, $p > 0$, $q < 1$, and $\rho > 0$. If $p + q \neq 1$ and

$$-\alpha < \gamma < \min\{N - \alpha, 2\}.$$

then equation $(\mathcal{C}_{\lambda,\gamma}^S)$ has a nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$ if and only if

$$(2.9) \quad q > 1 - \frac{N - \alpha - \gamma}{N}p.$$

Moreover, if $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then

$$(2.10) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-\gamma}{1-q}} > 0.$$

The above lower bound is optimal.

The most complex picture occurs in the fast growth regime $\gamma < -\alpha$.

Theorem 6. *Let $N \geq 1$, $\lambda > 0$, $\gamma < 2$, $0 < \alpha < N$, $p > 0$, $q < 1$ and $\rho > 0$. If $p + q \neq 1$ and*

$$\gamma \leq -\alpha.$$

then equation $(\mathcal{C}_{\lambda,\gamma}^S)$ has a nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$ if and only if

$$(2.11) \quad q > 1 + \frac{\gamma}{\alpha}p.$$

Moreover, if $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then

$$(2.12a) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-\gamma}{1-q}} > 0 \quad \text{if } 1 + \frac{N-\alpha-\gamma}{N}p < q < 1,$$

$$(2.12b) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N}{p}} (\log|x|)^{-\frac{1}{1-q-p}} > 0 \quad \text{if } q = 1 + \frac{N-\alpha-\gamma}{N}p,$$

$$(2.12c) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{-\frac{\alpha+\gamma}{1-q-p}} > 0 \quad \text{if } 1 + \frac{\gamma}{\alpha}p < q < \frac{\alpha}{N-2}.$$

The above lower bounds are optimal.

In the homogeneous case $p+q = 1$ we obtain results similar to those of Theorem 5, with exception that, unlike in all previous results, the existence becomes sensitive to the choice of radius $\rho > 0$. In addition, in the fully homogeneous case $\gamma = -\alpha$ and $p+q = 1$, the existence and nonexistence becomes sensitive to the value of λ . To ensure the existence of a positive solution, λ has to be sufficiently large so that the potential V can compensate for the loss of positivity due to the nonlocal right hand side. To formulate the result, denote

$$\lambda^* := 2^{-\frac{\alpha}{2}} \frac{\Gamma(\frac{N-\alpha}{4})}{\Gamma(\frac{N+\alpha}{4})}.$$

This quantity is related to the optimal constant in a weighted Hardy-Littlewood-Sobolev inequality of Stein and Weiss [38], see Section 7.4.

Theorem 7. *Let $N \geq 2$, $0 < \alpha < N$, $\lambda > 0$ and $p > 0$. If*

$$p + q = 1,$$

then

- (i) *if $\alpha > -\gamma$ and $\lambda > 0$, or if $\alpha = -\gamma$ and $\lambda < \lambda^*$, then for every $\rho > 0$, $(\mathcal{C}_{\lambda,\gamma}^S)$ has no nontrivial nonnegative supersolutions in $\mathbb{R}^N \setminus \bar{B}_\rho$;*
- (ii) *if $\alpha < -\gamma$ and $\lambda > 0$, or if $\alpha = -\gamma$ and $\lambda > \lambda^*$, then there exists $\rho_0 > 0$ such that if $\rho > \rho_0$, $(\mathcal{C}_{\lambda,\gamma}^S)$ has a positive supersolutions in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

Moreover, if $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N}{1-q}} > 0.$$

The above lower bound is optimal if $\alpha < -\gamma$.

Theorem 7 gives only partial results in the borderline case $p + q = 1$. The accurate description of the existence, nonexistence and optimal decay properties of positive supersolutions of the equation

$$-\Delta u(x) + \frac{\lambda^2}{|x|^\gamma} u = (I_\alpha * u^{1-q})(x) u^q(x)$$

is an interesting open problem which is however lies beyond the scope of the present work.

2.5. Outline. The rest of the paper is organized as follows. In Section 3 we prove general local and nonlocal versions of positivity principles, in the spirit of the Agmon–Allegretto–Piepenbrink positivity principle (see [1, 2]). These positivity principles are fundamental in our considerations both for nonexistence as well as for optimal decay estimates. In Section 4 we prove Theorem 1 and discuss briefly equation with fast decay potentials. In Section 5 we consider equation with Hardy potentials. Finally, in Sections 6 and 7 we study equation with slow decay potentials. Appendix A contains various estimates of the Riesz potentials which are extensively used in the paper. In Appendix B we prove suitable versions of a comparison principle and a weak Harnack inequality for distributional supersolutions.

3. LOCAL AND NONLOCAL POSITIVITY PRINCIPLES.

According to the classical Agmon–Allegretto–Piepenbrink positivity principle (see [1, Theorem 3.1]), the linear equation $-\Delta u + Vu = 0$ admits a nontrivial non-negative weak supersolution $u \in H_{\text{loc}}^1(\Omega)$ if and only if the corresponding quadratic form $\int_\Omega |\nabla \varphi|^2 + V\varphi^2$ is nonnegative for every $\varphi \in C_c^\infty(\Omega)$. An extension of such positivity principle to distributional supersolutions can be found in [9, Theorem 2.12] (see also [10, Lemma B.1]).

We formulate a version of the positivity principle adapted to distributional supersolutions of the nonlinear equation

$$-\Delta u + Vu = Wu^q.$$

Proposition 3.1. *Let $N \geq 1$, $\Omega \subseteq \mathbb{R}^N$ be an open connected set and $q \in \mathbb{R}$. Let $V \in L_{\text{loc}}^1(\Omega)$, $W : \Omega \rightarrow [0, \infty)$ be measurable and let $u \in L_{\text{loc}}^1(\Omega)$. Assume that $Vu, Wu^q \in L_{\text{loc}}^1(\Omega)$. If $u \geq 0$ and*

$$-\Delta u + Vu \geq Wu^q \quad \text{in } \Omega,$$

in the sense of distributions, then either $u = 0$ almost everywhere or $u > 0$ almost everywhere in Ω , $Wu^{q-1} \in L_{\text{loc}}^1(\Omega)$ and for every $\varphi \in C_c^\infty(\Omega)$ one has

$$\int_\Omega |\nabla \varphi|^2 + V\varphi^2 \geq \int_\Omega Wu^{q-1} \varphi^2.$$

The conclusion $u > 0$ is crucial to interpret the u^{q-1} when $q \leq 1$. If $q < 0$ and $W > 0$, then this conclusion is already contained implicitly in the assumption $Wu^q \in L_{\text{loc}}^1(\Omega)$.

Proof of Proposition 3.1. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be such that $\text{supp } \eta \subset B_1$, $\int_{\mathbb{R}^N} \eta = 1$ and $\eta \geq 0$. For $\delta > 0$ and $x \in \mathbb{R}^N$, let $\eta_\delta(x) = \delta^{-N} \eta(x/\delta)$ and let $\tilde{\eta}_\delta(x) = \eta_\delta(-x)$. Let $\varphi \in C_c^\infty(\Omega)$. Since the support of φ is compact, there exists $\delta_0 > 0$ such that for every $x \in \Omega$, $\varphi(x) > 0$ implies $B_{\delta_0}(x) \subset \Omega$. Given $\varepsilon > 0$ and $\delta \in (0, \delta_0]$, we can

thus take $\tilde{\eta}_\delta * \frac{\varphi^2}{\eta_\delta * u + \varepsilon} \in C_c^\infty(\Omega)$ as a test function. We compute

$$\begin{aligned} - \int_{\Omega} u \Delta \left(\tilde{\eta}_\delta * \frac{\varphi^2}{\eta_\delta * u + \varepsilon} \right) &= - \int_{\Omega} (\eta_\delta * u) \Delta \left(\frac{\varphi^2}{\eta_\delta * u + \varepsilon} \right) \\ &= \int_{\Omega} \nabla(\eta_\delta * u) \cdot \nabla \left(\frac{\varphi^2}{\eta_\delta * u + \varepsilon} \right) \end{aligned}$$

and

$$|\nabla \varphi|^2 = \nabla(\eta_\delta * u) \cdot \nabla \left(\frac{\varphi^2}{\eta_\delta * u + \varepsilon} \right) + \left| \nabla \left(\frac{\varphi}{\eta_\delta * u + \varepsilon} \right) \right|^2 (\eta_\delta * u + \varepsilon)^2.$$

Therefore,

$$(3.1) \quad \int_{\Omega} |\nabla \varphi|^2 + \frac{\eta_\delta * (Vu)}{\eta_\delta * u + \varepsilon} \varphi^2 \geq \int_{\Omega} \frac{\eta_\delta * (Wu^q)}{\eta_\delta * u + \varepsilon} \varphi^2 + \int_{\Omega} \left| \nabla \left(\frac{\varphi}{\eta_\delta * u + \varepsilon} \right) \right|^2 (\eta_\delta * u + \varepsilon)^2.$$

Since $Vu \in L_{\text{loc}}^1(\Omega)$ and $Wu^q \in L_{\text{loc}}^1(\Omega)$, one has $\eta_\delta * Vu \rightarrow Vu$ and $\eta_\delta * Wu^q \rightarrow Wu^q$ in $L_{\text{loc}}^1(\Omega)$ as $\delta \rightarrow 0$. Since $u \in L^1(\Omega)$, $\eta_\delta * u \rightarrow u$ almost everywhere in Ω as $\delta \rightarrow 0$. Hence, letting $\delta \rightarrow 0$ we obtain by Lebesgue's dominated convergence theorem

$$\int_{\Omega} |\nabla \varphi|^2 + \frac{Vu}{u + \varepsilon} \varphi^2 \geq \int_{\Omega} \frac{Wu^q}{u + \varepsilon} \varphi^2.$$

Letting now $\varepsilon \rightarrow 0$, by Lebesgue's dominated convergence theorem again and by Lebesgue's monotone convergence theorem we conclude that

$$\int_{\Omega} |\nabla \varphi|^2 + V\varphi^2 \geq \int_{\Omega \setminus u^{-1}(\{0\})} Wu^{q-1} \varphi^2.$$

Let us now prove that $u > 0$ almost everywhere, following H. Brezis and A. C. Ponce [5]. Let $a \in \Omega$ and $\rho > 0$ such that $B_{2\rho}(a) \subset \Omega$ and take $\varphi \in C_c^\infty(B_{2\rho}(a))$ such that $\varphi = 1$ on $B_\rho(a)$. By the Poincaré–Wirtinger inequality and the triangle inequality we have

$$\int_{B_\rho(a)} \int_{B_\rho(a)} \left| \log \frac{\eta_\delta * u(x) + \varepsilon}{\eta_\delta * u(y) + \varepsilon} \right| dx dy \leq C \int_{B_\rho(a)} \left| \nabla \frac{1}{\eta_\delta * u + \delta} \right|^2 |\eta_\delta * u + \delta|^2,$$

whence by (3.1)

$$\int_{B_\rho(a)} \int_{B_\rho(a)} \left| \log \frac{\eta_\delta * u(x) + \varepsilon}{\eta_\delta * u(y) + \varepsilon} \right| dx dy \leq \int_{\Omega} |\nabla \varphi|^2 + \frac{(\eta_\delta * Vu)}{\eta_\delta * u + \varepsilon} \varphi^2 - \frac{\eta_\delta * Wu^q}{\eta_\delta * u + \varepsilon} \varphi^2.$$

Letting now $\delta \rightarrow 0$, we have as before

$$\int_{B_\rho(a)} \int_{B_\rho(a)} \left| \log \frac{u(x) + \varepsilon}{u(y) + \varepsilon} \right| dx dy \leq \int_{\Omega} |\nabla \varphi|^2 + \frac{Vu}{u + \varepsilon} \varphi^2.$$

Now note that for every $x, y \in B_\rho(a) \times B_\rho(a)$ such that $u(x) = 0$ and $u(y) > 0$,

$$\lim_{\varepsilon \rightarrow 0} \left| \log \frac{u(x) + \varepsilon}{u(y) + \varepsilon} \right| = \infty,$$

this allows to conclude that either $u = 0$ or $u > 0$ almost everywhere in $B_\rho(a)$. Since $a \in \Omega$ is arbitrary and Ω is connected, we conclude that either $u > 0$ almost everywhere or $u = 0$ almost everywhere in Ω . $\square$

If one assumed $u \in H_{\text{loc}}^1(\Omega)$, one could have taken directly $\delta = 0$ in the above proof. If $u^{-1} \in L_{\text{loc}}^\infty(\Omega)$, one could take directly $\varepsilon = 0$. The latter would follow from the weak Harnack inequality if for example $V_+ \in L_{\text{loc}}^s(\Omega)$ for some $s > \frac{N}{2}$ (see for example [13, Theorem 8.18]), but uniform positivity of the solution fails for more singular potentials $V \in L_{\text{loc}}^1(\Omega)$.

In the context of Choquard's equation $(\mathcal{C})$ we prove the following nonlocal version of the Agmon–Allegretto–Piepenbrink positivity principle for distributional solutions in the sense of Definition 2.1.

Proposition 3.2. *Let $N \geq 1$, $\Omega \subseteq \mathbb{R}^N$ be an open and connected set, $p > 0$, $q \in \mathbb{R}$, $0 < \alpha < N$, $V \in L^1_{\text{loc}}(\Omega)$ and $u \in L^1_{\text{loc}}(\mathbb{R}^N)$. If $u \geq 0$ and*

$$-\Delta u + Vu \geq (I_\alpha * u^p)u^q \quad \text{in } \Omega,$$

in the sense of distribution, then either $u = 0$ almost everywhere or $u > 0$ almost everywhere in Ω , $u^{q-1} \in L^1_{\text{loc}}(\Omega)$ and for every $R > 0$ and $\varphi \in C_c^\infty(\Omega \cap B_R)$ one has

$$\int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} V \varphi^2 \geq \frac{A_\alpha}{2^{N-\alpha} R^{N-\alpha}} \left(\int_{\Omega \cap B_R} u^p \right) \left(\int_{\Omega} u^{q-1} \varphi^2 \right).$$

Proof. By Proposition 3.1 with $W = I_\alpha * u^p \in L^1_{\text{loc}}(\Omega)$, either $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ or $u > 0$ almost everywhere in Ω , $(I_\alpha * u^p)u^{q-1} \in L^1_{\text{loc}}(\Omega)$ and for every $\varphi \in C_c^\infty(\Omega)$

$$\int_{\Omega} |\nabla \varphi|^2 + V \varphi^2 \geq \int_{\Omega} (I_\alpha * u^p) u^{q-1} \varphi^2.$$

Since for every $x, y \in B_R$,

$$I_\alpha(x - y) \geq \frac{A_\alpha}{2^{N-\alpha} R^{N-\alpha}},$$

we have $u^{q-1} \in L^1_{\text{loc}}(\Omega)$,

$$\frac{A_\alpha}{2^{N-\alpha} R^{N-\alpha}} \left(\int_{\Omega \cap B_R} u^p \right) \left(\int_{\Omega} u^{q-1} \varphi^2 \right) \leq \int_{\Omega} (I_\alpha * u^p) u^{q-1} \varphi^2,$$

and the conclusion follows. $\square$

4. EQUATION WITH THE UNPERTURBED LAPLACIAN: PROOF OF THEOREM 1.

4.1. Nonexistence. Our essential tools in the analysis of nonexistence of nontrivial nonnegative supersolutions to equation $(\mathcal{C}_0)$ are the nonlocal positivity principle of Proposition 3.2 and the following quantitative integral estimate, which can be viewed as an integral version of the comparison principle for the Laplacian in exterior domains. The result is a particular case of its generalization to Hardy potentials in Proposition 5.1.

Proposition 4.1. *Let $R, \rho > 0$ be such that $R > 2\rho$, $u \in L^1_{\text{loc}}(B_{4R} \setminus B_{\rho/4})$ and $f \in L^1_{\text{loc}}(B_{4R} \setminus B_{\rho/4})$. If $u \geq 0$, $f \geq 0$ in $B_{4R} \setminus \bar{B}_{\rho/4}$ and*

$$-\Delta u \geq f \quad \text{in } B_{4R} \setminus \bar{B}_{\rho/4}$$

in the sense of distributions, then

$$\frac{1}{\rho^2} \int_{B_\rho \setminus B_{\rho/2}} u + \int_{B_R \setminus B_\rho} f \leq \frac{C}{R^2} \int_{B_{2R} \setminus B_R} u.$$

Taking $f = 0$ and applying the weak Harnack inequality for superharmonic functions (see [19, Theorem 9.10] or Lemma B.3 below), we immediately derive from Proposition 4.1 the standard Green decay pointwise lower bounds on nontrivial nonnegative supersolutions to $(\mathcal{C}_0)$ in exterior domains.

Lemma 4.2. *Let $\rho > 0$ and $u \in L^1_{\text{loc}}(\mathbb{R}^N \setminus B_\rho)$. If $u \geq 0$ and*

$$-\Delta u \geq 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_\rho$$

in the sense of distributions, then either $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, or

- (i) *if $N = 1, 2$, then $\liminf_{|x| \rightarrow \infty} u(x) > 0$,*
- (ii) *if $N \geq 3$, then $\liminf_{|x| \rightarrow \infty} u(x)|x|^{N-2} > 0$.*

Comparing Riesz potential blowup upper bound of (2.1) with the Green decay bounds of Lemma 4.2 we immediately establish our first nonexistence results.

Proposition 4.3. *Let $N = 1, 2$ and $\rho > 0$. If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

Proof. Simply note that Lemma 4.2 (i) and (2.1) are incompatible for all $p > 0$. $\square$

Proposition 4.4. *Let $N \geq 3$ and $\rho > 0$. Assume that $p \leq \frac{\alpha}{N-2}$. If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

Proof. Simply note that Lemma 4.2 (ii) and (2.1) are incompatible for $p \leq \frac{\alpha}{N-2}$. $\square$

Our next step is to explore nonlocal positivity principle of Proposition 3.2 in order to obtain an upper bound on nontrivial nonnegative supersolutions of $(\mathcal{C}_0)$ in the superlinear region $p + q \geq 1$ more accurate than (2.1).

Lemma 4.5. *Let $N \geq 1$, $0 < \alpha < N$, $p > 0$, $q \in \mathbb{R}$ and $\rho > 0$. There exists $C > 0$ such that if $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then either $u = 0$ almost everywhere or $u > 0$ almost everywhere, $u^{q-1} \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \bar{B}_\rho)$ and for every $R \geq 2\rho$,*

$$\left(\int_{B_{2R} \setminus B_\rho} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \leq CR^{2N-\alpha-2}.$$

Proof. Choose $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\text{supp } \varphi \subset B_4 \setminus \bar{B}_{1/2}$, $\varphi = 1$ on $B_2 \setminus \bar{B}_1$ and $\varphi \leq 1$. For $R > 0$ and $x \in \mathbb{R}^N$ set

$$(4.1) \quad \varphi_R(x) = \varphi\left(\frac{x}{R}\right).$$

If $R \geq 2\rho$, we have $\text{supp } \varphi_R \subset B_{4R} \setminus \bar{B}_{R/2} \subset \mathbb{R}^N \setminus \bar{B}_\rho$ and

$$\int_{\mathbb{R}^N \setminus B_\rho} |\nabla \varphi_R|^2 = R^{N-2} \int_{\mathbb{R}^N} |\nabla \varphi|^2.$$

Therefore, by Proposition 3.2,

$$\left(\int_{B_{2R} \setminus B_\rho} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \leq R^{2N-\alpha-2} \left(\frac{4^{N-\alpha}}{A_\alpha} \int_{\mathbb{R}^N} |\nabla \varphi|^2 \right). \quad \square$$

A consequence of the upper bound of Lemma 4.5 is the following nonexistence result.

Proposition 4.6. *Let $N \geq 3$ and $\rho > 0$. Assume that*

$$1 \leq p + q \leq \frac{N + \alpha}{N - 2}.$$

If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.

Proof. In the *subcritical case* $1 \leq p + q < \frac{N+\alpha}{N-2}$, we observe that by the Cauchy-Schwarz inequality and Lemma 4.2 (ii) we obtain

$$\left(\int_{B_{2R} \setminus B_\rho} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \geq \left(\int_{B_{2R} \setminus B_R} u^{\frac{p+q-1}{2}} \right)^2 \geq R^{2N-(N-2)(p+q-1)}.$$

This is not compatible with the upper bound of Lemma 4.5.

In the *critical case* $p + q = \frac{N+\alpha}{N-2}$, we need to improve the lower bound of Lemma 4.2 (ii). To do this, assume by contradiction that $u > 0$ on a set of positive

measure of $\mathbb{R}^N \setminus \bar{B}_\rho$. Using the Cauchy–Schwarz inequality and the lower bound of Lemma 4.2 (ii), since $\frac{p+q}{2} = \frac{N+\alpha}{2(N-2)} > 0$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_\rho} (I_\alpha * u^p) u^q &\geq \int_{\mathbb{R}^N \setminus B_\rho} \int_{\mathbb{R}^N \setminus B_\rho} u(x)^{\frac{p+q}{2}} I_\alpha(x-y) u(y)^{\frac{p+q}{2}} dx dy \\ &\geq \int_{\mathbb{R}^N \setminus B_\rho} \int_{\mathbb{R}^N \setminus B_\rho} \frac{1}{|x|^{\frac{N+\alpha}{2}}} I_\alpha(x-y) \frac{1}{|y|^{\frac{N+\alpha}{2}}} dx dy = \infty. \end{aligned}$$

Then by Proposition 4.1 we conclude that

$$\liminf_{R \rightarrow \infty} \frac{1}{R^2} \int_{B_{2R} \setminus B_R} u = \infty.$$

Applying Hölder’s inequality if $p+q \geq 3$ and the weak Harnack inequality (Lemma B.3) if $1 \leq p+q < 3$, we obtain

$$\left(\frac{1}{R^2} \int_{B_{2R} \setminus B_R} u \right)^{\frac{p+q-1}{2}} \leq \frac{C}{R^{N-\frac{\alpha+2}{2}}} \int_{B_{2R} \setminus B_R} u^{\frac{p+q-1}{2}},$$

which brings a contradiction with the upper bound of Lemma 4.5 combined with the Cauchy–Schwarz inequality. $\square$

If $\alpha \geq N-2$ we give precise lower bounds on $\int_{B_{2R} \setminus B_R} u^{q-1}$ to obtain a further nonexistence result.

Proposition 4.7. *Let $N \geq 3$ and $\rho > 0$. Assume that $\alpha \geq N-2$ and $1 < q \leq \frac{\alpha}{N-2}$. If $u \geq 0$ is a supersolution of (C_0) in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

Proof. Assume that $u > 0$ on a set of positive measure. From Lemma 4.2 (ii) we obtain

$$(4.2) \quad \int_{B_{2R} \setminus B_R} u^{q-1} \geq c R^{N-(N-2)(q-1)}.$$

On the other hand, by Lemma 4.5

$$\int_{B_{2R} \setminus B_R} u^{q-1} \leq \frac{C R^{2N-\alpha-2}}{\int_{B_{2R} \setminus B_\rho} u^p} \leq C' R^{2N-\alpha-2}.$$

This brings a contradiction if $q < \frac{\alpha}{N-2}$.

In the case $q = \frac{\alpha}{N-2}$ using Lemma 4.2 (ii) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_{2\rho}} (I_\alpha * u^p) u^q &\geq \frac{A_\alpha}{2^{N-\alpha}} \int_{B_{2\rho} \setminus B_\rho} u^p \int_{\mathbb{R}^N \setminus B_\rho} \frac{u(x)^q}{|x|^{N-\alpha}} \\ &\geq c \int_{\mathbb{R}^N \setminus B_\rho} \frac{1}{|x|^{N-\alpha+(N-2)q}} dx = \infty, \end{aligned}$$

for some $c > 0$, since $(N-2)q = \alpha$. By Proposition 4.1 and the weak Harnack inequality,

$$\liminf_{R \rightarrow \infty} \frac{1}{R^{N-(N-2)(q-1)}} \int_{B_{2R} \setminus B_R} u^{q-1} = \infty,$$

which leads to a contradiction with (4.2). $\square$

An alternative proof of Proposition 4.7 is obtained by noting that if $|x| \geq 2\rho$,

$$I_\alpha * u(x) \geq \frac{A_\alpha}{2^{N-\alpha}|x|^{N-\alpha}} \int_{B_{2\rho} \setminus B_\rho} u^p,$$

and exploring the fact that

$$(4.3) \quad -\Delta u(x) \geq \frac{c}{|x|^{N-\alpha}} u(x)^q$$

does not have positive solutions in exterior domains if $1 < q \leq \frac{\alpha}{N-2}$ [6; 16, Theorem 1.2; 23, Theorem 2.2 and Lemma 6.3].

The transitional locally linear case $q = 1$ requires a special consideration.

Proposition 4.8. *Let $N \geq 3$ and $\rho > 0$. Assume that $\alpha > N - 2$ and $q = 1$. If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

Proof. Since $q = 1$, by Lemma 4.5 for any $R > \rho$ we have

$$R^{2N-\alpha-2} \geq c \left(\int_{B_R \setminus B_\rho} u^p \right) \left(\int_{B_{2R} \setminus B_R} 1 \right).$$

We conclude that $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ when $N - 2 < \alpha$. $\square$

The next result shows that in the superlinear case $p + q \geq 1$ different mechanisms are responsible for the nonexistence and decay of nontrivial nonnegative supersolutions of $(\mathcal{C}_0)$ in the Green decay region $q \geq 1$ and *sublinear decay region* $q < 1$.

Proposition 4.9. *Let $N \geq 3$ and $\rho > 0$. Assume that $p + q \geq 1$, $q < 1$ and*

$$q \leq 1 - \frac{N - \alpha - 2}{N} p.$$

If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.

Proof. We observe that by Hölder's inequality since $q < 1$,

$$\begin{aligned} \int_{B_{2R} \setminus B_R} 1 &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right)^{\frac{p}{p+1-q}} \\ &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{p+1-q}} \left(\left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \right)^{\frac{p+q-1}{p+1-q}}. \end{aligned}$$

By Lemma 4.5, on the one hand

$$\left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \leq C R^{2N-\alpha-2},$$

and on the other hand

$$\int_{B_{2R} \setminus B_R} u^{q-1} \leq C \frac{R^{2N-\alpha-2}}{\int_{B_{2R} \setminus B_\rho} u^p} \leq \frac{C}{\int_{B_{2\rho} \setminus B_\rho} u^p} R^{2N-\alpha-2}.$$

In the *subcritical case* $q < 1 - \frac{N-\alpha-2}{N} p$, this brings a contradiction since $p + q \geq 1$.

Otherwise, in the *critical case* $q = 1 - \frac{N-\alpha-2}{N} p$, we have by the previous inequalities

$$\int_{B_{2R} \setminus B_R} u^p \geq \frac{\left(\int_{B_{2R} \setminus B_R} 1 \right)^{1+\frac{p}{1-q}}}{\left(\int_{B_{2R} \setminus B_R} u^{1-q} \right)^{\frac{p}{1-q}}} \geq c R^{N-\frac{p}{1-q}(N-\alpha-2)} = c$$

for some $c > 0$. Therefore,

$$\lim_{R \rightarrow \infty} \int_{B_{2R} \setminus B_\rho} u^p = \infty$$

and

$$\lim_{R \rightarrow \infty} R^{\alpha+2-2N} \int_{B_{2R} \setminus B_R} u^{q-1} = 0,$$

so we can conclude as previously. $\square$

Comparing Riesz potential blowup upper bound of (2.1) with Lemma 4.5 we obtain the next nonexistence statement.

Proposition 4.10. *Let $N \geq 3$ and $\rho > 0$. Assume that $p + q < 1$. If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

Proof. Since $p + q < 1$, by the Hölder inequality we obtain

$$\begin{aligned} \int_{B_{2R} \setminus B_R} 1 &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right)^{\frac{p}{p+1-q}} \\ &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q-p}{p+1-q}} \left(\left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \right)^{\frac{p}{p+1-q}}. \end{aligned}$$

By (2.1) and by Lebesgue's dominated convergence theorem

$$\lim_{R \rightarrow \infty} R^{N-\alpha} \int_{B_{2R} \setminus B_R} u^p = 0.$$

This brings a contradiction with Lemma 4.5. $\square$

4.2. Pointwise decay bounds. In the sublinear decay region $q < 1$ the integral estimate of Lemma 4.5 could be used to prove that the Green decay bounds of Lemma 4.2 are no longer accurate if applied to $(\mathcal{C}_0)$. In fact, if $q < 1$ then nontrivial nonnegative supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ decay at the same polynomial rate as positive supersolutions to the sublinear local equation (4.3).

Proposition 4.11. *Let $N \geq 3$ and $\rho > 0$. Assume that $q < 1$. If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then either $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ or*

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-2}{1-q}} > 0.$$

In view of Proposition 4.7, Proposition 4.11 is trivially true when $\alpha > N - 2$. Proposition 4.11 only gives an improvement over Lemma 4.2 if $q < \frac{\alpha}{N-2}$.

The lower bound of Proposition 4.11 for the local equation (4.3) is well-known [23, Lemma 6.1]. We present the proof here for completeness and to illustrate the use of Lemma 4.5.

Proof of Proposition 4.11. By Lemma 4.5 we have

$$\int_{B_{2R} \setminus B_R} u^{q-1} \leq C \frac{R^{2N-\alpha-2}}{\int_{B_{2R} \setminus B_\rho} u^p} \leq C' R^{2N-\alpha-2}.$$

By the weak Harnack inequality of Lemma B.3, there exists $c > 0$ such that

$$\inf_{B_{5R/3} \setminus B_{4R/3}} u \geq c \left(R^{-N} \int_{B_{2R} \setminus B_R} u^{q-1} \right)^{\frac{1}{q-1}},$$

so the assertion follows. $\square$

In the limiting case $q = \frac{\alpha}{N-2}$ the bounds of Lemma 4.2 and Proposition 4.11 coincide but can be improved.

Proposition 4.12. *Let $N \geq 3$ and $\rho > 0$. Assume that*

$$q = \frac{\alpha}{N-2} < 1.$$

If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then either $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ or

$$(4.4) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{N-2} (\log |x|)^{-\frac{N-2}{N-\alpha-2}} > 0.$$

Proof. By Proposition 4.1, there exists $c > 0$ such that for every $r > 8\rho$,

$$\frac{1}{r^2} \int_{B_{2r} \setminus B_r} u \geq c \int_{B_r \setminus B_{4\rho}} (I_\alpha * u^p) u^q.$$

Since for every $x \in \mathbb{R}^N \setminus B_{4\rho}$,

$$(4.5) \quad (I_\alpha * u^p)(x) \geq \frac{A_\alpha}{2^{N-\alpha}|x|^{N-\alpha}} \int_{B_{4\rho} \setminus B_\rho} u^p,$$

we have

$$(4.6) \quad \frac{1}{r^2} \int_{B_{2r} \setminus B_r} u \geq \frac{A_\alpha}{4^{N-\alpha}} \int_{8\rho}^r \left(\int_{B_t \setminus B_{t/2}} u^q \right) \frac{1}{t^{1+N-\alpha}} dt.$$

On the other hand, by Harnack's inequality, there exists $C > 0$ such that

$$\begin{aligned} \frac{1}{r^2} \int_{B_{2r} \setminus B_r} u &\leq \frac{1}{r^2} \int_{B_{2r} \setminus B_{r/4}} u \leq C r^{N-2} \inf_{B_r \setminus B_{r/2}} u \\ &\leq C' \left(\frac{1}{r^{N-q(N-2)}} \int_{B_r \setminus B_{r/2}} u^q \right)^{\frac{1}{q}}. \end{aligned}$$

We have thus for some $c' > 0$,

$$(4.7) \quad \frac{\frac{1}{t^{1+N-\alpha}} \int_{B_r \setminus B_{r/2}} u^q}{\left(\int_{8\rho}^r \left(\int_{B_t \setminus B_{t/2}} u^q \right) \frac{1}{t^{1+N-\alpha}} dt \right)^{1-q}} \geq \frac{c'}{r^{1+q(N-2)-\alpha}}.$$

If $q = \frac{\alpha}{N-2} < 1$, the integration of this inequality with respect to r from 8ρ to $R > 8\rho$ yields

$$\left(\int_{8\rho}^R \left(\int_{B_t \setminus B_{t/2}} u^q \right) \frac{1}{t^{1+N-\alpha}} dt \right)^{\frac{N-\alpha-2}{N-2}} \geq c' \frac{N-2}{N-\alpha-2} \log \frac{R}{8\rho}.$$

Recalling (4.6), we have for some c'' ,

$$\frac{1}{r^2} \int_{B_{2r} \setminus B_r} u \geq c'' \left(\log \frac{R}{8\rho} \right)^{\frac{N-2}{N-\alpha-2}}.$$

We conclude by Harnack's inequality. $\square$

The proof of Proposition 4.12 shows in fact that any supersolution of (4.3) with $q = \frac{\alpha}{N-2}$ has the same lower bound at infinity. The result seems to be new also for the local inequality

$$-\Delta u(x) \geq \frac{c}{|x|^{N-\alpha}} u(x)^{\frac{\alpha}{N-2}} \quad \text{for } x \in \mathbb{R}^N \setminus \bar{B}_\rho,$$

where the lower bound (4.4) on positive supersolutions is established by the same arguments as above.

In the transitional locally linear case $\alpha = N - 2$ and $q = 1$, the Green decay bounds of Lemma 4.2 can be improved.

Proposition 4.13. *Let $N \geq 3$ and $\rho > 0$. Assume that*

$$q = \frac{\alpha}{N-2} = 1.$$

If $u \geq 0$ is a supersolution of $(\mathcal{C}_0)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then either $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ or there is $m > 0$ such that

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{N-2-m} > 0.$$

Proof. One follows the line of the proof of Proposition 4.12 until (4.7), whose integration now yields

$$\log\left(\frac{1}{r^2} \int_{B_{2r} \setminus B_r} u\right) \geq m \log \frac{R}{8\rho},$$

for some $m > 0$, allowing to conclude by the weak Harnack inequality. $\square$

An alternative proof consists in inferring from (4.5) that

$$-\Delta u(x) \geq \frac{A_\alpha \int_{B_{2\rho} \setminus B_\rho} u^p}{2^{N-\alpha} |x|^2} u(x) \quad \text{for } x \in \mathbb{R}^N \setminus \bar{B}_{2\rho}.$$

and deducing the assertion from the lower bound of Lemma 5.2.

4.3. Optimal decay. We are going to show that the above nonexistence results are sharp by constructing explicit nontrivial nonnegative supersolutions. First we prove that in the Green decay region $(\mathcal{C}_0)$ admits nontrivial nonnegative supersolutions which decay at infinity at the same rate as the Green function of $-\Delta$.

Proposition 4.14. *Let $N \geq 3$ and $\rho > 0$. If*

$$p > \frac{\alpha}{N-2}, \quad p+q > \frac{N+\alpha}{N-2} \quad \text{and} \quad q > \frac{\alpha}{N-2},$$

then $(\mathcal{C}_0)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{N-2} < \infty.$$

Proof. Fix $\beta > 0$. For $x \in \mathbb{R}^N \setminus \bar{B}_\rho$ and $\mu > 0$, set

$$u_\mu(x) = \frac{\mu}{|x|^{N-2}} \left(1 - \frac{1}{(1 + \log \frac{|x|}{\rho})^\beta} \right).$$

Then for $x \in \mathbb{R}^N \setminus \bar{B}_\rho$ we compute

$$\begin{aligned} -\Delta u_\mu(x) &= \mu \frac{\beta}{|x|^N (1 + \log \frac{|x|}{\rho})^{\beta+1}} \left((N-2) + \frac{\beta+1}{1 + \log \frac{|x|}{\rho}} \right) \\ &\geq \mu \frac{\beta(N-2)}{|x|^N (1 + \log \frac{|x|}{\rho})^{\beta+1}}. \end{aligned}$$

On the other hand, since $\frac{\alpha}{N-2} < p < \frac{N}{N-2}$, by Lemma A.1 there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$(4.8) \quad (I_\alpha * u_\mu^p)(x) \leq \frac{C\mu^p}{|x|^{p(N-2)-\alpha}}.$$

Therefore for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$(I_\alpha * u_\mu^p)(x) u_\mu(x)^q \leq \frac{C\mu^{p+q}}{|x|^{(p+q)(N-2)-\alpha}}.$$

Since $p+q > 1$ and $p+q > \frac{N+\alpha}{N-2}$, we conclude that u_μ is the required supersolution for all sufficiently small $\mu > 0$.

If $p = \frac{N}{N-2}$ then instead of (4.8) by Lemma A.1 there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$(I_\alpha * u_\mu^p)(x) \leq \frac{C\mu^p (1 + \log \frac{|x|}{\rho})}{|x|^{N-\alpha}},$$

from which we conclude as before.

Finally, if $p > \frac{N}{N-2}$ then by Lemma A.1 there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$(I_\alpha * u_\mu^p)(x) \leq \frac{C}{|x|^{N-\alpha}}.$$

from which we conclude since $q > \frac{\alpha}{N-2}$. $\square$

Next we construct a supersolution matching decay estimate (2.3c) in the transitional region $q = \frac{\alpha}{N-2} < 1$.

Proposition 4.15. *Let $N \geq 3$ and $\rho > 0$. If*

$$p > \frac{N}{N-2} \quad \text{and} \quad q = \frac{\alpha}{N-2} < 1,$$

then $(\mathcal{C}_0)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{N-2} (\log |x|)^{-\frac{N-2}{N-\alpha-2}} < \infty.$$

Proof. Given $\mu > 0$, for $x \in \mathbb{R}^N \setminus \bar{B}_\rho$ we set

$$u_\mu(x) = \mu \frac{\left(\frac{1}{N-\alpha-2} + \log \frac{|x|}{\rho}\right)^{\frac{N-2}{N-\alpha-2}}}{|x|^{N-2}}.$$

One has for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$\begin{aligned} -\Delta u_\mu(x) &= \mu \frac{\left(\frac{1}{N-\alpha-2} + \log \frac{|x|}{\rho}\right)^{\frac{N-\alpha}{N-\alpha-2}}}{|x|^N} \\ &\quad \times \left(\frac{(N-2)^2}{N-\alpha-2} - \frac{\alpha(N-2)}{(N-\alpha-2)^2 \left(\frac{1}{N-\alpha-2} + \log \frac{|x|}{\rho}\right)} \right) \\ &\geq \mu(N-2) \frac{\left(\frac{1}{N-\alpha-2} + \log \frac{|x|}{\rho}\right)^{\frac{N-\alpha}{N-\alpha-2}}}{|x|^N}. \end{aligned}$$

Since $\frac{\alpha}{N-2} < p < \frac{N}{N-2}$, by Lemma A.1 we obtain (4.8) for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$. Since $p + q > 1$, u_μ is a supersolution when μ is small enough. $\square$

Next we construct a supersolution matching decay estimate (2.3b) in the transitional locally linear régime $\alpha = N - 2$ and $q = 1$, when the critical line $q = \frac{\alpha}{N-2}$ belongs to the existence region.

Proposition 4.16. *Let $N \geq 3$ and $\rho > 0$. If*

$$\alpha = N - 2, \quad p > \frac{N}{N-2} \quad \text{and} \quad q = 1.$$

then for every $m > 0$ equation $(\mathcal{C}_0)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{N-2-m} < \infty.$$

Proof. Without loss of generality, we can assume that $m < N - 2 - \frac{N}{p}$. Given $\mu > 0$, we set for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$u_\mu(x) = \frac{\mu}{|x|^{N-2-m}}.$$

Then we compute for every $x \geq 0$, if $m \in (0, N - 2)$,

$$-\Delta u_\mu(x) = \mu \frac{m(N-2-m)}{|x|^2} u_\mu(x) > 0.$$

On the other hand, if $\mu \leq N - 2 - \frac{N}{p}$ we obtain by Lemma A.1 since $N - \alpha = 2$ for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$(I_\alpha * u_\mu^p)(x) \leq \frac{C\mu^p}{|x|^2}.$$

Since $p > 0$, we conclude that u_μ is the required supersolution for all sufficiently small $\mu > 0$. $\square$

Finally we construct a supersolution in the sublinear decay region which matches the decay estimate (2.3d).

Proposition 4.17. *Let $N \geq 3$ and $\rho > 0$. If*

$$1 - \frac{N - \alpha - 2}{N}p < q < \frac{\alpha}{N - 2} < 1,$$

then $(\mathcal{C}_0)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$ which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x)|x|^{\frac{N-\alpha-2}{1-q}} < \infty.$$

Proof. Set for $\mu > 0$ and $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$u_\mu(x) = \frac{\mu}{|x|^{\frac{N-\alpha-2}{1-q}}}.$$

We compute

$$-\Delta u_\mu(x) = \mu \frac{(N - \alpha - 2)(\alpha - q(N - 2))}{(1 - q)^2} \frac{1}{|x|^{\frac{N-2q-\alpha}{1-q}}}.$$

Since $q < \frac{\alpha}{N-2} < 1$, we have $\frac{(N-\alpha-2)(\alpha-q(N-2))}{(1-q)^2} > 0$. On the other hand, $p \frac{N-\alpha-2}{1-q} > N$. Hence by Lemma A.1 we obtain for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$(I_\alpha * u_\mu^p)(x) \leq \frac{C\mu^p}{|x|^{N-\alpha}}$$

and thus

$$(I_\alpha * u_\mu^p)(x)u_\mu(x)^q \leq \frac{C\mu^{p+q}}{|x|^{\frac{N-2q-\alpha}{1-q}}}.$$

Note that $p+q > 1 + \frac{\alpha+2}{N}p > 1$, so we conclude that u_μ is the required supersolution for all sufficiently small $\mu > 0$. $\square$

Propositions 4.14, 4.16 and 4.17 confirm sharpness of the nonexistence statements (2.2) and optimality of the decay estimates (2.3). This completes the proof of Theorem 1.

4.4. Equation with fast decay potentials. Theorem 1 could be easily extended to the perturbed Choquard equation $(\mathcal{C})$ with *fast decay* potentials

$$V(x) = \frac{\lambda}{|x|^\gamma}$$

where $\lambda \in \mathbb{R}$ and $\gamma > 2$. It is well known that nontrivial nonnegative supersolutions to the linear Schrödinger operator $-\Delta + V$ with fast decay potential V have the same minimal decay rate at infinity as the fundamental solution of the unperturbed operator $-\Delta$, (cf. [16], [33, Section 3] or [29, Lemma 3.4]). As a consequence, we can establish a complete analogue of Theorem 1.

Theorem 8. *Let $N \geq 3$, $\gamma > 2$, $\lambda \in \mathbb{R}$, $0 < \alpha < N$, $p > 0$, $q \in \mathbb{R}$ and $\rho > 0$. Then $(\mathcal{C}_{\lambda,\gamma}^F)$ has a nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$ if and only if the assumptions (2.2) hold simultaneously. Moreover, if $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^F)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then the lower bounds (2.3) hold and these bounds are optimal.*

The proof of Theorem 8 follows closely the proof of Theorem 1. Note only that if V is a fast decay potential then complete analogues of Proposition 4.1 and Lemma 4.2 could be established following the arguments in the proof of Proposition 5.1. In addition, if φ_R is defined by (4.1), then

$$\lim_{R \rightarrow \infty} \frac{1}{R^{N-2}} \int_{\mathbb{R}^N \setminus B_\rho} V |\varphi_R|^2 = 0.$$

The estimate of Lemma 4.5 remains thus stable after a perturbation of $(\mathcal{C})$ by a fast decay potential. We omit further details.

5. EQUATION WITH HARDY POTENTIALS.

In this section we consider perturbed equation $(\mathcal{C})$ with *Hardy potential*

$$V(x) = \frac{\nu^2 - \left(\frac{N-2}{2}\right)^2}{|x|^2} \quad (\nu > 0).$$

It is well known that if V is a Hardy potential then nontrivial nonnegative supersolutions to the linear Schrödinger operator $-\Delta + V$ decay polynomially at infinity, however the exact rate of decay depends explicitly on the value of the constant ν . We will show that all the results of Theorem 1 could be extended with minimal suitable modifications to Choquard's equations $(\mathcal{C}_{\lambda,\gamma}^S)$ with Hardy potentials.

5.1. Equation with Hardy potentials. Using decay estimate for supersolutions to linear equations with Hardy's potential we deduce the following extension of Theorem 1.

Theorem 9. *Let $N \geq 2$, $0 < \alpha < N$, $p > 0$, $q \in \mathbb{R}$, $\nu > 0$ and $\rho > 0$. Then $(\mathcal{C}_\nu^H)$ has a nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_\rho$ if and only if the following assumptions hold simultaneously:*

$$(5.1a) \quad p > \frac{\alpha}{\frac{N-2}{2} + \nu},$$

$$(5.1b) \quad p + q > 1 + \frac{\alpha + 2}{\frac{N-2}{2} + \nu},$$

$$(5.1c) \quad q > 1 + \frac{\alpha - (N-2)}{\frac{N-2}{2} + \nu} \quad \text{if } \alpha > N-2,$$

$$(5.1d) \quad q \geq 1, \quad \text{if } \alpha = N-2,$$

$$(5.1e) \quad q > 1 - \frac{N - \alpha - 2}{N} p \quad \text{if } \alpha < N-2,$$

$$(5.1f) \quad q > 1 - \frac{N - \alpha - 2}{\frac{N-2}{2} - \nu} \quad \text{if } \alpha < N-2 \text{ and } 0 < \nu < \frac{N-2}{2}.$$

Moreover, if $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_\nu^H)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then

$$(5.2a) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-2}{2} + \nu} > 0 \quad \text{if } q > 1 + \frac{\alpha - (N-2)}{\frac{N-2}{2} + \nu} > 1,$$

$$(5.2b) \quad \exists m > 0 : \liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-2}{2} + \nu - m} > 0 \quad \text{if } q = 1 \text{ and } \alpha = N-2,$$

$$(5.2c) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-2}{2} + \nu} (\log|x|)^{-\frac{\frac{N-2}{2} + \nu}{N-\alpha-2}} > 0 \quad \text{if } q = 1 - \frac{N-\alpha-2}{\frac{N-2}{2} + \nu} < 1,$$

$$(5.2d) \quad \liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-2}{1-q}} > 0 \quad \text{if } q < 1 - \frac{N-\alpha-2}{\frac{N-2}{2} + \nu} < 1.$$

The above lower bounds are optimal.

The optimality of lower bounds (5.2) is understood in the sense similar to that of Theorem 1. In particular, if $q = 1$, $\alpha = N - 2$ and $p > \frac{N}{\frac{N-2}{2} + \nu}$ then for every $m > 0$ there exists a positive radial supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$ such that

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-2}{2} + \nu - m} < \infty.$$

The nonexistence region (5.1e) as well as lower bound (5.2b) are stable with respect to the variation of ν . The nonexistence region (5.1f), which is nonempty only for $0 < \nu < \frac{N-2}{2}$, is a new phenomenon compared to the free Laplacian.

5.2. Estimates for linear equations with Hardy potentials. We derive several decay estimate for the auxiliary linear equations with Hardy potential. Our first result is an integral version of the Phragmen-Lindelöf type estimates.

Proposition 5.1. *Let $N \geq 2$, $R, r > 0$ be such that $R > 2r$, $u \in L^1_{\text{loc}}(B_{4R} \setminus B_{r/4})$ and $f \in L^1_{\text{loc}}(B_{2R} \setminus B_{r/4})$. If $u \geq 0$, $f \geq 0$ and*

$$-\Delta u(x) + \frac{\nu^2 - (\frac{N-2}{2})^2}{|x|^2} u(x) \geq f(x) \quad \text{in } \mathbb{R}^N \setminus \bar{B}_{r/4}$$

in the sense of distributions, then

$$\begin{aligned} \frac{1}{R^{\frac{N+2}{2} + \nu}} \int_{B_{2R} \setminus B_R} u + \int_{B_R \setminus B_r} \frac{f(x)}{|x|^{\frac{N-2}{2} + \nu}} dx &\leq \frac{C}{r^{\frac{N+2}{2} + \nu}} \int_{B_r \setminus B_{r/2}} u, \\ \frac{1}{r^{\frac{N+2}{2} - \nu}} \int_{B_r \setminus B_{r/2}} u + \int_{B_R \setminus B_r} \frac{f(x)}{|x|^{\frac{N-2}{2} - \nu}} dx &\leq \frac{C}{R^{\frac{N+2}{2} - \nu}} \int_{B_{2R} \setminus B_R} u. \end{aligned}$$

Proof. To prove the first inequality, choose $\eta \in C^\infty((0, \infty))$ such that $\eta \geq 0$, $\eta = 1$ on $[1, 2]$ and $\text{supp } \eta \subset (1/2, 4)$. Let $\theta \in C^\infty((0, \infty))$ be the solution of the Cauchy problem

$$\begin{cases} -\theta''(s) - \frac{N-1}{s} \theta'(s) + \frac{\nu^2 - (\frac{N-2}{2})^2}{s^2} \theta(s) = -\eta(s) & \text{for } s \in (0, \infty), \\ \theta(4) = 0, \\ \theta'(4) = 0. \end{cases}$$

By the variation of parameters formula θ can be represented for $s \in (0, \infty)$ by

$$\theta(s) = \frac{1}{2\nu} \int_s^4 \left(\left(\frac{\sigma}{s} \right)^{\frac{N-2}{2} + \nu} - \left(\frac{\sigma}{s} \right)^{\frac{N-2}{2} - \nu} \right) \sigma \eta(\sigma) d\sigma.$$

In particular, for $s \in (0, 1)$,

$$\theta(s) = \frac{1}{2\nu s^{\frac{N-2}{2} + \nu}} \int_1^4 \eta(\sigma) \sigma^{\frac{N}{2} + \nu} d\sigma - \frac{1}{2\nu s^{\frac{N-2}{2} - \nu}} \int_1^4 \eta(\sigma) \sigma^{\frac{N}{2} - \nu} d\sigma,$$

so we conclude that $\lim_{s \rightarrow 0} \theta(s) = +\infty$ and for every $\theta \geq 0$ in $(0, \infty)$. Choose $\psi \in C^\infty(\mathbb{R}^N)$ such that $\psi = 0$ on $B_{1/2}$ and $\psi = 1$ on $\mathbb{R}^N \setminus \bar{B}_1$ and define for $x \in B_{4R} \setminus \bar{B}_{r/4}$

$$\varphi(x) = \theta\left(\frac{|x|}{R}\right) \psi\left(\frac{x}{r}\right).$$

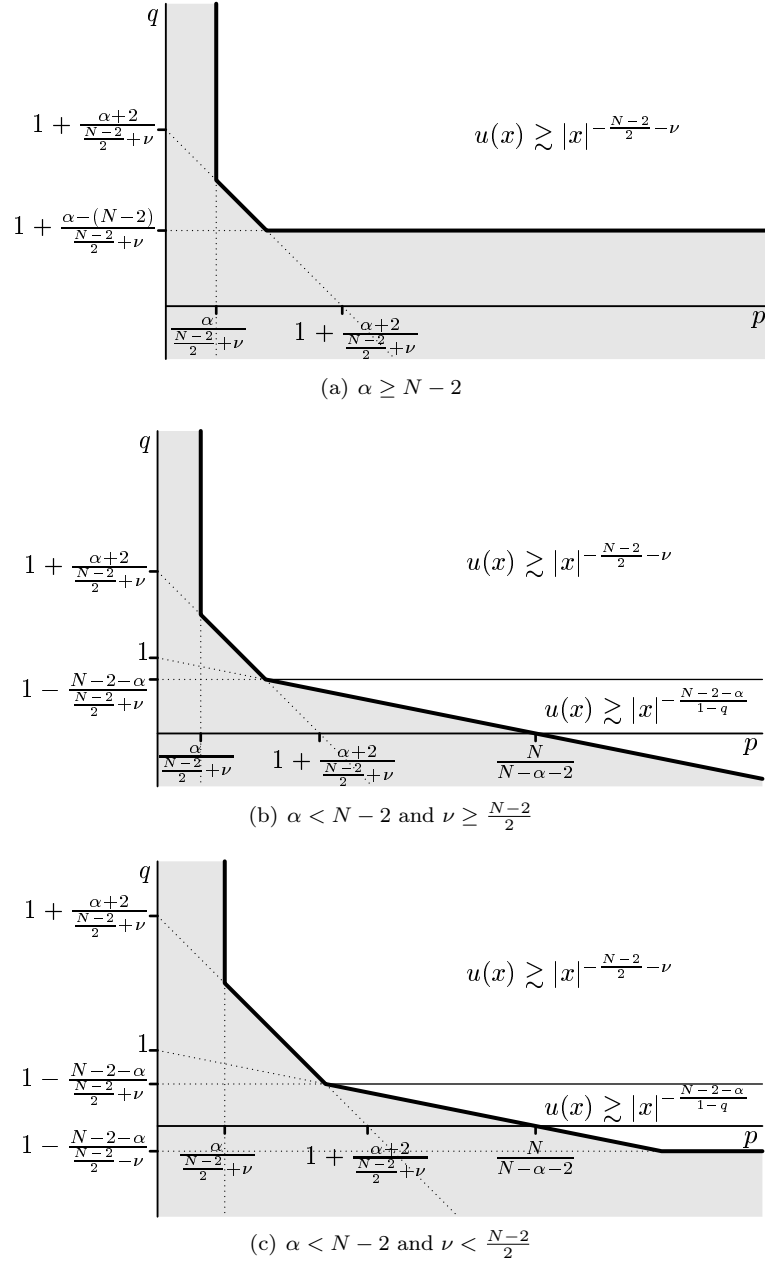


FIGURE 3. Existence, decay and nonexistence regions for $(\mathcal{C}^H_\nu)$ in the (p, q) -plane

Since $\varphi \in C_c^\infty(B_{4R} \setminus \bar{B}_{r/4})$ and $\varphi \geq 0$, we have

$$\int_{B_{4R} \setminus B_{r/2}} u(x) \left(-\Delta \varphi(x) + \frac{\nu^2 - (\frac{N-2}{2})^2}{|x|^2} \varphi(x) \right) dx \geq \int_{B_{4R} \setminus B_{r/2}} f \varphi.$$

Noting that

$$R^{\frac{N-2}{2}+\nu} \int_{B_R \setminus B_r} \frac{f(x)}{|x|^{\frac{N-2}{2}+\nu}} dx \leq \int_{B_{4R} \setminus B_{r/2}} f \varphi,$$

$$\begin{aligned}
\int_{B_{4R} \setminus B_r} u(x) \left(-\Delta \varphi(x) + \frac{\nu^2 - (\frac{N-2}{2})^2}{|x|^2} \varphi(x) \right) dx &= -\frac{1}{R^2} \int_{B_{4R} \setminus B_r} u(x) \eta\left(\frac{x}{R}\right) dx \\
&\leq -\frac{1}{R^2} \int_{B_{2R} \setminus B_R} u, \\
\int_{B_r \setminus B_{r/2}} u(x) \left(-\Delta \varphi(x) + \frac{\nu^2 - (\frac{N-2}{2})^2}{|x|^2} \varphi(x) \right) dx &\leq C \frac{R^{\frac{N-2}{2} + \nu}}{r^{\frac{N+2}{2} + \nu}} \int_{B_r \setminus B_{r/2}} u.
\end{aligned}$$

we complete the proof of the first inequality.

To obtain the second inequality, choose $\eta \in C^\infty((0, \infty))$ such that $\eta = 1$ on $[1/2, 1]$ and $\text{supp } \eta \subset (1/4, 2)$, define $\theta \in C^\infty((0, \infty))$ as the solution of the Cauchy problem

$$\begin{cases} -\theta''(s) - \frac{N-1}{s} \theta'(s) + \frac{\nu^2 - (\frac{N-2}{2})^2}{s^2} \theta(s) = \eta(s) & \text{for every } s \in (0, \infty), \\ \theta(\frac{1}{4}) = 0, \\ \theta'(\frac{1}{4}) = 0, \end{cases}$$

choose $\psi \in C^\infty(\mathbb{R}^N)$ such that $\psi = 1$ on $B_{1/2}$ and $\psi = 0$ on $\mathbb{R}^N \setminus \bar{B}_1$, set for $x \in B_{4R} \setminus \bar{B}_{r/4}$,

$$\varphi(x) = \theta\left(\frac{|x|}{r}\right) \psi\left(\frac{x}{R}\right).$$

and conclude similarly to the above argument. $\square$

Taking $f = 0$ and employing the weak Harnack inequality (see [19, Theorem 9.10] or Lemma B.3 below), we immediately derive from Proposition 4.1 the usual pointwise lower bound and a Phragmen–Lindelöf type integral upper bound on nontrivial nonnegative supersolutions to linear equations with Hardy’s potential in exterior domains (see [23, Lemma 4.7] for relevant results).

Lemma 5.2. *Let $N \geq 2$, $\nu > 0$, $r > 0$ and $u \in L^1_{\text{loc}}(\mathbb{R}^N \setminus B_r)$. If $u \geq 0$ and*

$$-\Delta u(x) + \frac{\nu^2 - (\frac{N-2}{2})^2}{|x|^2} u(x) \geq 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_r$$

in the sense of distributions, then

$$\limsup_{R \rightarrow \infty} R^{-\frac{N+2}{2} - \nu} \int_{B_{2R} \setminus \bar{B}_R} u < \infty$$

and either $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, or

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-2}{2} + \nu} > 0.$$

5.3. Sketch of the proof of the nonexistence.

Sketch of the proof of Theorem 9. The proof follows closely the proof of Theorem 1, with Lemma 5.2 being used instead of Lemma 4.2. Note that when V is the Hardy potential, and φ_R is defined as in (4.1), then there exists $C > 0$ such that for every $R \geq \rho$,

$$\int_{\mathbb{R}^N \setminus B_\rho} V |\varphi_R|^2 \leq C R^{N-2}$$

growth at the same rate as (4.1). Hence, the estimates of Lemma 4.5 are not affected by perturbation of (C) by the Hardy potential. Then the proofs of nonexistence in the cases (5.1a), (5.1b), (5.1c), (5.1d), (5.1e) are carried over following the same arguments as in the proof of Theorem 1.

The nonexistence region (5.1f), which is nonempty only for $0 < \nu < \frac{N-2}{2}$, is a new phenomenon compared to the free Laplacian. This is the case when the

upper bound of Lemma 5.2 becomes relevant. The complete proof of the nonexistence of nontrivial nonnegative supersolutions below the line (5.1f) is given in Proposition 5.3 below. $\square$

Proposition 5.3. *Let $N \geq 3$, $N - 2 < \alpha < N$, $p > 0$, $\nu > 0$ and $\rho > 0$. If $q < 1$,*

$$\left(\frac{N-2}{2} - \nu\right)(1-q) \geq N - \alpha - 2.$$

and $u \geq 0$ is a supersolution of $(\mathcal{C}_\nu^H)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.

Proof. In the subcritical case $(\frac{N-2}{2} - \nu)(1-q) > N - \alpha - 2$, by Lemma 5.1,

$$(5.3) \quad \int_{B_{2R} \setminus B_R} u \leq CR^{\frac{N+2}{2} + \nu},$$

by the counterpart of Lemma 4.5,

$$(5.4) \quad \int_{B_{2R} \setminus B_R} u^{q-1} \leq \frac{C}{\int_{B_{2\rho} \setminus \rho} u^p} R^{2N-\alpha-2}.$$

Now by Hölder's inequality, since $q < 1$,

$$\left(\int_{B_{2R} \setminus B_R} 1\right)^{2-q} \leq \left(\int_{B_{2R} \setminus B_R} u\right)^{1-q} \int_{B_{2R} \setminus B_R} u^{q-1},$$

this brings a contradiction in the subcritical case $(\frac{N-2}{2} - \nu)(1-q) > N - \alpha - 2$.

In the critical case $(\frac{N-2}{2} - \nu)(1-q) = N - \alpha - 2$ we first note that

$$\int_{\mathbb{R}^N \setminus B_\rho} \frac{(I_\alpha * u^p)(x)u(x)^q}{|x|^{\frac{N-2}{2} + \nu}} dx \geq c \int_{\mathbb{R}^N \setminus B_\rho} \frac{u(x)^q}{|x|^{N-\alpha + \frac{N-2}{2} + \nu}} dx$$

If we can prove that the integral on the right hand side diverges, then using Proposition 5.1 we can improve the upper bound (5.3) and reach the contradiction as before. This is clearly the case when $q = 0$.

If $0 < q < 1$ then by Hölder's inequality

$$\int_{B_{2R} \setminus B_R} u^q \geq \frac{\left(\int_{B_{2R} \setminus B_R} 1\right)^{\frac{1}{1-q}}}{\left(\int_{B_{2R} \setminus B_R} u^{q-1}\right)^{\frac{q}{1-q}}}.$$

This implies by (5.3) that

$$\int_{B_{2R} \setminus B_R} u^q \geq cR^{N - \frac{(N-\alpha-2)q}{1-q}}.$$

Since $(\frac{N-2}{2} - \nu)(1-q) = N - \alpha - 2$ we conclude that

$$\int_{\mathbb{R}^N \setminus B_\rho} \frac{u(x)^q}{|x|^{N-\alpha + \frac{N-2}{2} + \nu}} dx = \infty.$$

If $q < 0$ then by Hölder's inequality

$$\int_{B_{2R} \setminus B_R} u^q \geq \frac{\left(\int_{B_{2R} \setminus B_R} 1\right)^{1-q}}{\left(\int_{B_{2R} \setminus B_R} u\right)^{-q}},$$

and thus by (5.4)

$$\int_{B_{2R} \setminus B_R} u^q \geq cR^{N - (\frac{N-2}{2} - \nu)q},$$

so we conclude as previously. $\square$

An alternative proof of Proposition 5.3 is obtained by noting that u solves

$$-\Delta u(x) + V(x)u(x) \geq \frac{A_\alpha \int_{B_{2\rho} \setminus B_\rho} u^p}{2^{N-\alpha} |x|^{N-\alpha}} u(x)^q,$$

which does not have positive solutions in exterior domains if $q \leq 1 - \frac{N-\alpha-2}{\frac{N-2}{2}-\nu}$, $\alpha > N-2$ and $0 < \nu < \frac{N-2}{2}$ (see [23, Theorem 2.2]).

5.4. Pointwise decay bounds and optimal decay. In the Green decay region $q \geq 1 - \frac{N-\alpha-2}{\frac{N-2}{2}+\nu}$, the polynomial lower bound (5.2a) follows directly from Lemma 5.2. In the sublinear decay region $q < 1 - \frac{N-\alpha-2}{\frac{N-2}{2}+\nu}$, the polynomial lower bound (5.2d) is derived similarly to the proof of Proposition 4.11, using the integral estimate (5.4) and the weak Harnack inequality of Lemma B.3.

The constructions of explicit nontrivial nonnegative supersolutions showing the optimality (5.2a), (5.2b) and (5.2c) follow very closely the proofs of Proposition 4.14, 4.16 and 4.15. The same applies to the existence of explicit nontrivial nonnegative supersolutions with the decay rate of (5.2d) in the sublinear decay region in the case when $\nu \geq \frac{N-2}{2}$, where the arguments repeat Proposition 4.17. Below we construct a radial supersolution with the decay rate of (5.2d) in the sublinear decay region under the assumption $0 < \nu < \frac{N-2}{2}$, when the upper bound of Lemma 5.2 becomes relevant.

Proposition 5.4. *Let $N \geq 3$ and $\rho > 0$. If $\alpha < N-2$, $0 < \nu < \frac{N-2}{2}$,*

$$1 - \frac{N-\alpha-2}{N} p < q,$$

and

$$1 - \frac{N-\alpha-2}{\frac{N-2}{2}-\nu} < q < 1 - \frac{N-\alpha-2}{\frac{N-2}{2}+\nu}.$$

then (C_ν^H) admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, which satisfies

$$\lim_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-2}{1-q}} < \infty.$$

Proof. For $\mu > 0$ and $x \in \mathbb{R}^N \setminus \bar{B}_\rho$, set

$$u_\mu(x) = \frac{\mu}{|x|^{\frac{N-\alpha-2}{1-q}}}.$$

We compute

$$-\Delta u_\mu(x) + \frac{\nu^2 - (\frac{N-2}{2})^2}{|x|^2} u_\mu(x) = \frac{\mu \left(\nu - \frac{N-2}{2} + \frac{N-\alpha-2}{1-q} \right) \left(\nu + \frac{N-2}{2} - \frac{N-\alpha-2}{1-q} \right)}{|x|^{\frac{N-2q-\alpha}{1-q}}}$$

and we observe that if $1 - \frac{N-\alpha-2}{\frac{N-2}{2}-\nu} < q < 1 - \frac{N-\alpha-2}{\frac{N-2}{2}+\nu} < 1$, $\left(\nu - \frac{N-2}{2} + \frac{N-\alpha-2}{1-q} \right) \left(\nu + \frac{N-2}{2} - \frac{N-\alpha-2}{1-q} \right) > 0$. Since $p \frac{N-\alpha-2}{1-q} > N$, Lemma A.1 gives $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$(I_\alpha * u_\mu^p)(x) u_\mu(x)^q \leq \frac{C \mu^{p+q}}{|x|^{\frac{N-2q-\alpha}{1-q}}}.$$

Noting that $p+q > 1 + \frac{\alpha+2}{N} p > 1$, we conclude that u_μ is the required supersolution for all sufficiently small $\mu > 0$. $\square$

6. EQUATION WITH SLOW DECAY POTENTIALS: CASE $q \geq 1$.

In this and subsequent sections we consider perturbed Choquard equation $(\mathcal{C})$ with *slow decay* potential

$$V(x) = \frac{\lambda^2}{|x|^\gamma}.$$

for some fixed $\lambda > 0$ and $-\infty < \gamma < 2$. It is well known that nontrivial nonnegative supersolutions to the linear Schrödinger operator $-\Delta + V$ decay exponentially at infinity. The decay rates for the nonlocal Choquard equation $(\mathcal{C})$ are more complex. We will distinguish between the exponential decay region $q \geq 1$ and polynomial decay region $q < 1$. Within the exponential decay region we consider separately the case $q > 1$ and the borderline *locally linear* case $q = 1$. Before doing this, we shall consider a related class of linear equation.

6.1. Estimates for linear equations with slow decay potentials. Here we establish sharp decay estimates for the minimal positive solutions at infinity (see Definition B.1 in the Appendix B) of the linear Schrödinger equations with slow decay potentials. The following result extends a fine decay estimate by S. Agmon [2, Theorem 3.3].

Proposition 6.1. *Let $N \geq 1$, $\gamma < 2$, $\rho \geq 0$ and $W \in C^1([\rho, \infty))$ be a nonnegative function. If*

$$\lim_{s \rightarrow \infty} W(s) > 0,$$

and for some $\beta > 0$,

$$\lim_{s \rightarrow \infty} W'(s)s^{1+\beta} = 0,$$

then there exists a nonnegative radial function $H : \mathbb{R}^N \setminus \bar{B}_\rho \rightarrow \mathbb{R}$ such that

$$(6.1) \quad -\Delta H(x) + \frac{W(|x|)^2}{|x|^\gamma} H(x) = 0 \quad \text{for every } x \in \mathbb{R}^N \setminus \bar{B}_\rho,$$

and

$$(6.2) \quad \lim_{|x| \rightarrow \infty} H(x)|x|^{\frac{N-1}{2}-\frac{\gamma}{4}} \exp \int_\rho^{|x|} \frac{W(s)}{s^{\frac{\gamma}{2}}} ds = 1.$$

When $W(s) = \lambda > 0$ the result is standard (see for example [3, p.332]). When $\gamma = 1$ and $\beta > \frac{1}{2}$ the result was proved by Agmon [2, Theorem 3.3]. Relevant estimates also could be found in [15, Theorem 2, 3].

It follows immediately that H is a minimal positive solution at infinity of (6.1). Indeed, the function $U(x) = 1$ is a supersolution to (6.3) and the pair $U(x) = 1$ and $H(x)$ satisfy condition (B.2).

Proof of Proposition 6.1. Since $\gamma < 2$, we can assume without loss of generality that $\beta < 1 - \frac{\gamma}{2}$. Similarly to the construction in the proof of S. Agmon [2, Theorem 3.3], for $s > \rho$ and $\tau \in \mathbb{R}$ we define

$$\phi_\tau(s) = -\frac{N-1-\frac{\gamma}{2}}{2s} - \frac{W(s)}{s^{\frac{\gamma}{2}}} + \frac{\tau}{s^{1+\beta}},$$

and for $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$\Phi_\tau(x) = \exp \left(\int_\rho^{|x|} \phi_\tau(s) ds \right).$$

By the chain rule,

$$\Delta \Phi_\tau(x) = \left(\phi'_\tau(|x|) + \frac{N-1}{|x|} \phi_\tau(|x|) + \phi_\tau(|x|)^2 \right) \Phi_\tau(x),$$

hence

$$\begin{aligned} -\Delta \Phi_\tau(x) &= \left(\frac{(N-2)(N-1-\frac{\gamma}{2})}{2|x|^2} + \frac{W'(|x|)}{|x|^{\frac{\gamma}{2}}} + \frac{(N-1-\frac{\gamma}{2})W(|x|)}{|x|^{\frac{\gamma}{2}+1}} \right. \\ &\quad \left. - \frac{(N-2-\beta)\tau}{|x|^{2+\beta}} - \left(\frac{N-1-\frac{\gamma}{2}}{2|x|} + \frac{W(|x|)}{|x|^{\frac{\gamma}{2}}} - \frac{\tau}{|x|^{1+\beta}} \right)^2 \right) \Phi_\tau(x) \\ &= \left(-\frac{W(|x|)^2}{|x|^\gamma} + \frac{2\tau W(|x|) + \omega_\tau(x)}{|x|^{1+\frac{\gamma}{2}+\beta}} \right) \Phi_\tau(x), \end{aligned}$$

where $\omega_\tau : \mathbb{R}^N \setminus \bar{B}_\rho \rightarrow \mathbb{R}$ is given by

$$\omega_\tau(x) = W'(|x|)|x|^{1+\beta} + \frac{(N-1-\frac{\gamma}{2})(N-3+\frac{\gamma}{2})}{4|x|^{1-\frac{\gamma}{2}-\beta}} + \frac{\tau(1-\frac{\gamma}{2}+\beta)}{|x|^{1-\frac{\gamma}{2}}} - \frac{\tau^2}{|x|^{1-\frac{\gamma}{2}+\beta}}.$$

Observe that $\lim_{|x| \rightarrow \infty} \omega_\tau(x) = 0$.

Choose $\bar{\tau} > 0$ and $\bar{\tau} < 0$. A direct computation verifies that $\Phi_{\bar{\tau}}$ is a subsolution and $\Phi_{\bar{\tau}}$ is a supersolution to equation (6.1) in the exterior of a ball B_R , for a sufficiently large $R > \rho$. Applying the classical sub and supersolutions principle, we conclude that (6.1) admits a radial solution H in $\mathbb{R}^N \setminus \bar{B}_R$, such that $\Phi_{\bar{\tau}} < H < \Phi_{\bar{\tau}}$. Since $\limsup_{|x| \rightarrow \infty} \frac{\Phi_{\bar{\tau}}(x)}{\Phi_{\bar{\tau}}(x)} < \infty$, we deduce that up to multiplication by a constant H has the required asymptotic.

Since H is radial, it can be extended to a positive solution on $\mathbb{R}^N \setminus \bar{B}_\rho$. Indeed, otherwise H would vanish on a sphere ∂B_r with $r > \rho$. Since $\lim_{|x| \rightarrow \infty} H(x) = 0$ and $V \geq 0$, this would imply by the maximum principle that $H = 0$ on $\mathbb{R}^N \setminus \bar{B}_r$. $\square$

Remark 6.1. We apply Proposition 6.1 in order to understand the rate of decay of positive solutions of the linear equation

$$(6.3) \quad -\Delta u(x) + \frac{\lambda^2}{|x|^\gamma} u(x) = \frac{m}{|x|^\sigma} u(x) \quad \text{for every } x \in \mathbb{R}^N \setminus \bar{B}_\rho.$$

where $-\infty < \gamma < 2$, $\lambda > 0$, $\sigma > \gamma$, $m \leq \lambda^2 \rho^{\gamma-\sigma}$ (see [2, Theorem 3.3] for the case $\gamma = 0$). Equation (6.3) appears as a localization of Choquard's equation ($\mathcal{C}_{\lambda,\gamma}^S$) in the half-linear case $q = 1$. Let $H(x)$ be a minimal positive solution at infinity of (6.3), as constructed in Proposition 6.1. The asymptotics of H are related to the asymptotics as $r \rightarrow \infty$ of the function

$$\psi(r) := \int_{\rho_0}^r \sqrt{\frac{\lambda^2}{s^\gamma} - \frac{m}{s^\sigma}} ds = \frac{\lambda}{1-\gamma/2} \int_{\rho_0^{1-\gamma/2}}^{r^{1-\gamma/2}} \sqrt{1 - \frac{m}{\lambda^2 t^{\frac{\sigma-\gamma}{1-\gamma/2}}}} ds,$$

where $\rho_0 > 0$ is chosen so that $\frac{\lambda^2}{\rho_0^\gamma} - \frac{m}{\rho_0^\sigma} \geq 0$. By the Taylor expansion of the square root, we have for every $k \in \mathbb{N}$,

$$\sqrt{1 - \frac{m}{\lambda^2 t^{\frac{\sigma-\gamma}{1-\gamma/2}}}} = 1 - \sum_{j=1}^k \frac{1}{2j-1} \binom{2j}{j} \left(\frac{m}{(\lambda^2)^{2j} t^{\frac{\sigma-\gamma}{1-\gamma/2}}} \right)^j + O\left(\frac{1}{t^{(k+1)\frac{\sigma-\gamma}{1-\gamma/2}}} \right).$$

If $k < \frac{1-\gamma/2}{\sigma-\gamma} < k+1$, then

$$\psi(r) = \lambda \left(\frac{r^{1-\gamma/2}}{1-\gamma/2} - \sum_{j=1}^k \frac{r^{1-\gamma/2-j(\sigma-\gamma)}}{(1-\gamma/2-j(\sigma-\gamma))(2j-1)} \binom{2j}{j} \frac{m^j}{(\lambda^2)^{2j}} \right) + O(1).$$

whereas if $k = \frac{1-\gamma/2}{\sigma-\gamma}$,

$$\psi(r) = \lambda \left(\frac{r^{1-\gamma/2}}{1-\gamma/2} - \sum_{j=1}^k \frac{r^{1-\gamma/2-j(\sigma-\gamma)}}{(1-\gamma/2-j(\sigma-\gamma))(2j-1)} \binom{2j}{j} \frac{m^j}{(\lambda^2)^{2j}} \right)$$

$$- \frac{1}{2k-1} \binom{2k}{k} \frac{m^k}{(2\lambda)^{2k}} \log r \Big) + O(1).$$

In particular, if $\sigma > 1 + \frac{\gamma}{2}$, then

$$\lim_{|x| \rightarrow \infty} H(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{1-\frac{\gamma}{2}}\right) \in (0, \infty);$$

if $\sigma = 1 + \frac{\gamma}{2}$, then

$$\lim_{|x| \rightarrow \infty} H(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4} - \frac{m}{2\lambda}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{1-\frac{\gamma}{2}}\right) \in (0, \infty);$$

if $\frac{1}{2} + \frac{3\gamma}{4} < \sigma < 1 + \frac{\gamma}{2}$, then

$$\lim_{|x| \rightarrow \infty} H(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{1-\frac{\gamma}{2}} - \frac{m}{\lambda(2+\gamma-2\sigma)} |x|^{1+\frac{\gamma}{2}-\sigma}\right) \in (0, \infty);$$

and if $\sigma = \frac{1}{2} + \frac{3\gamma}{4}$, then

$$\lim_{|x| \rightarrow \infty} H(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4} - \frac{m^2}{8\lambda^3}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{1-\frac{\gamma}{2}} - \frac{m}{\lambda(2+\gamma-2\sigma)} |x|^{1+\frac{\gamma}{2}-\sigma}\right) \in (0, \infty).$$

6.2. Exponential decay region $q > 1$: proof of Theorem 2. First we establish the lower bound (2.4) of Theorem 2.

Proposition 6.2. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $q > 1$ and $\rho > 0$. If $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then*

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{1-\frac{\gamma}{2}}\right) > 0.$$

Proof. Simply note that u is a supersolution to the linear equation

$$-\Delta u(x) + \frac{\lambda^2}{|x|^\gamma} u(x) = 0 \quad \text{for every } x \in \mathbb{R}^N \setminus \bar{B}_\rho,$$

so the assertion follows by the comparison principle of Proposition B.2 and the standard decay estimates for the linear equation, see [3, p. 332], or Proposition 6.1 with $W(s) = \lambda$ above. $\square$

Next we construct a supersolution to $(\mathcal{C}_{\lambda, \gamma}^S)$ which justifies optimality of the lower bound of Proposition 6.2 and thus completes the proof of Theorem 2.

Proposition 6.3. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $q > 1$ and $\rho > 0$. Then $(\mathcal{C}_{\lambda, \gamma}^S)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, such that*

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{1-\frac{\gamma}{2}}\right) < \infty.$$

Proof. Choose $m > 0$ such that $m < \lambda^2 \rho^{2-\gamma}$. Let $H \in C^2(\mathbb{R}^N \setminus \bar{B}_\rho)$ be the positive solution of the equation

$$-\Delta H(x) + \frac{\lambda^2}{|x|^\gamma} H(x) = \frac{m}{|x|^2} H(x) \quad \text{for every } x \in \mathbb{R}^N \setminus \bar{B}_\rho$$

such that

$$\lim_{|x| \rightarrow \infty} H(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4}} \exp\left(\frac{2\lambda}{2-\gamma} |x|^{1-\frac{\gamma}{2}}\right) = 1$$

given by Proposition 6.1. Set $u_\mu = \mu H$ for $\mu > 0$. By Lemma A.1 there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$(I_\alpha * u_\mu^p)(x) \leq \frac{C\mu^p}{|x|^{N-\alpha}}.$$

Since $q > 1$, there exists a sufficiently small $\bar{\mu} > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$\frac{C\bar{\mu}^{p+q-1}}{|x|^{N-\alpha}} H(x)^{q-1} \leq \frac{m}{|x|^2}.$$

Hence, if $\mu < \bar{\mu}$,

$$\begin{aligned} -\Delta u_\mu(x) + \frac{\lambda^2}{|x|^\gamma} u_\mu(x) &= \frac{m}{|x|^2} u_\mu(x) \\ &\geq \frac{C\mu^{p+q-1}}{|x|^{N-\alpha}} H(x)^{q-1} u_\mu(x) \geq (I_\alpha * u_\mu^p(x)) u_\mu^q(x), \end{aligned}$$

that is, u_μ is the required supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$. $\square$

6.3. Borderline region $q = 1$: proof of Theorem 3. Our next step is to explore nonlocal positivity principle of Proposition 3.2 in order to obtain a slow decay counterpart of Lemma 4.5.

Lemma 6.4. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $q \in \mathbb{R}$ and $\rho > 0$. If $u \geq 0$ is a supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then either $u = 0$ almost everywhere in $\mathbb{R}^N \setminus \bar{B}_\rho$ or $u = 0$ almost everywhere in $\mathbb{R}^N \setminus \bar{B}_\rho$, $u^{q-1} \in L_{\text{loc}}^1(\mathbb{R}^N \setminus \bar{B}_\rho)$ and for every $R > 2\rho$,*

$$\left(\int_{B_{2R} \setminus B_\rho} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \leq CR^{2N-\alpha-\gamma}.$$

Proof. Apply Lemma 3.2 to the family of test functions (4.1) and note that if V is a slow decay potential then

$$\int_{\mathbb{R}^N \setminus B_\rho} V |\varphi_R|^2 = O(R^{N-\gamma}) \quad \text{as } R \rightarrow \infty.$$

Then proceed as in the proof of Lemma 4.5. $\square$

As an immediate consequence of the upper bound of Lemma 6.4 we prove the nonexistence statement (2.6) of Theorem 3.

Proposition 6.5. *Let $N \geq 1$, $\alpha \in (N-2, N)$, $N-\alpha < \gamma < 2$, $\lambda > 0$, $p > 0$, $q = 1$ and $\rho > 0$. If $u \geq 0$ is a supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

Proof. Assume that $u > 0$ almost everywhere. Since $q = 1$, by Lemma 6.4 for any $R > \rho$ we have

$$R^{2N-\alpha-\gamma} \geq C \left(\int_{B_R \setminus B_\rho} u^p \right) \left(\int_{B_{2R} \setminus B_R} 1 \right).$$

This brings a contradiction if $N - \gamma < \alpha$. $\square$

To understand the existence and asymptotic properties of positive supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ when $q = 1$ and $\gamma \leq N - \alpha$, we consider the local equation

$$(6.4) \quad -\Delta u(x) + \frac{\lambda^2}{|x|^\gamma} u(x) = \frac{m}{|x|^{N-\alpha}} u(x) \quad \text{for } x \in \mathbb{R}^N \setminus \bar{B}_\rho.$$

Clearly, if $\gamma = N - \alpha$ and $m > \lambda^2$ then (6.4) has no positive supersolutions, by the local positivity principle of Proposition 3.1. If $\gamma = N - \alpha$ and $m = \lambda^2$ then (6.4) simplifies to $-\Delta u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$. In all other cases, that is if $\gamma = N - \alpha$ and $m < \lambda^2$ or if $\gamma < N - \alpha$ and $m \in \mathbb{R}$, equation (6.4) in $\mathbb{R}^N \setminus \bar{B}_\rho$ with a sufficiently large $\rho > 0$ admits by Proposition 6.1 an exponentially decaying minimal positive solution at infinity with the decay rate given by (6.2). The result below shows that the decay of positive supersolutions of Choquard's equation $(\mathcal{C}_{\lambda,\gamma}^S)$ is controlled by the decay rate of minimal positive solution at infinity of (6.4).

Proposition 6.6. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $q = 1$ and $\rho > 0$. If $\gamma \leq N - \alpha$ and if $u \geq 0$ is a supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then either $u = 0$ almost everywhere or there exists $\rho_m > 0$ and $m > 0$ such that $m < \lambda^2 \rho_m^{N-\alpha-\gamma}$ and*

$$\liminf_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4}} \exp \int_{\rho_m}^{|x|} \sqrt{\frac{\lambda^2}{s^\gamma} - \frac{m}{s^{N-\alpha}}} ds > 0.$$

Proof. Let $u \geq 0$ be a nontrivial supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$. One has then for every $x \in \mathbb{R}^N \setminus \bar{B}_{2\rho}$,

$$(I_\alpha * u^p)(x) \geq \frac{A_\alpha}{2^{N-\alpha} |x|^{N-\alpha}} \int_{B_{2\rho} \setminus B_\rho} u^p,$$

Hence, for every $x \in \mathbb{R}^N \setminus \bar{B}_{2\rho}$,

$$(I_\alpha * u^p)(x) \geq \frac{m}{|x|^{N-\alpha}},$$

with $m = \frac{A_\alpha}{2^{N-\alpha}} \int_{B_{2\rho} \setminus B_\rho} u^p > 0$. By the comparison principle of Proposition B.2 and decay estimate (6.2) we conclude that u satisfies the announced asymptotics. $\square$

The presence of the correction term related to the size of the constants m in the asymptotic is essential. We prove that for any admissible $m > 0$ there is a supersolution to $(\mathcal{C}_{\lambda, \gamma}^S)$ for which the lower bound cannot be improved. This justifies optimality of the lower bound and thus completes the proof of Theorem 3.

Proposition 6.7. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $q = 1$ and $\rho > 0$. If $\gamma \leq N - \alpha$, then for every $\rho_m > 0$ and $0 < m < \lambda^2 \rho_m^{N-\alpha-\gamma}$, there exists a radial nontrivial nonnegative supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_{\rho_m})$ such that*

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-1}{2} - \frac{\gamma}{4}} \exp \int_{\rho_m}^{|x|} \sqrt{\frac{\lambda^2}{s^\gamma} - \frac{m}{s^{N-\alpha}}} ds < \infty.$$

Proof. Without loss of generality, we can assume that $m < \lambda^2 \rho^{N-\alpha-\gamma}$. Let H_m be the solution of (6.4) given by Proposition 6.1. By Lemma A.1, there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$(I_\alpha * H_m^p)(x) \leq \frac{C}{|x|^{N-\alpha}},$$

for some $C > 0$. Set $u_\mu = \mu H_m$. Then if $0 < \mu < \left(\frac{m}{C}\right)^{1/p}$, one has for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$-\Delta u_\mu(x) + \frac{\lambda^2}{|x|^\gamma} u_\mu(x) = \frac{m}{|x|^{N-\alpha}} u_\mu(x) \geq (I_\alpha * u_\mu^p(x)) u_\mu(x),$$

that is, u_μ is the required solution of $(\mathcal{C}_{\lambda, \gamma}^S)$. $\square$

7. EQUATION WITH SLOW DECAY POTENTIALS: CASE $q < 1$.

7.1. Nonexistence. We shall establish two qualitatively different nonexistence results, first in the region where $q < 1$ and $p + q \geq 1$, and second in the sublinear region $p + q < 1$. We will see that the values $\gamma = N - \alpha$ and $\gamma = -\alpha$ represent the critical decay rate thresholds where different mechanisms are responsible for the existence and nonexistence of positive solutions of $(\mathcal{C}_{\lambda, \gamma}^S)$. First we prove nonexistence statements of Theorems 4 and 5.

Proposition 7.1. *Let $N \geq 1$, $0 < \alpha < N$, $\gamma < 2$, $\lambda > 0$, $q < 1$, $p > 0$ and $\rho > 0$. If*

$$p + q > 1 \quad \text{and} \quad q \leq 1 - \frac{N - \alpha - \gamma}{N}p,$$

or

$$p + q = 1 \quad \text{and} \quad \gamma > -\alpha,$$

and $u \geq 0$ is a supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.

The statement simplifies for some values of γ : if $\gamma \geq N - \alpha$, then there is no nontrivial solution for $q < 1$ whereas if $\gamma \leq -\alpha$ there is no nontrivial solution for $q < 1$ and $p + q > 1$.

Proof of Proposition 7.1. Let $R > \rho$. Since $q < 1$, $p > 0$, by Hölder's inequality we have

$$(7.1) \quad \begin{aligned} \int_{B_{2R} \setminus B_R} 1 &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right)^{\frac{p}{p+1-q}} \\ &\leq \left(\left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \right)^{\frac{1-q}{p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right)^{\frac{p+q-1}{p+1-q}}. \end{aligned}$$

By Lemma 6.4, on the one hand

$$(7.2) \quad \left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \leq CR^{2N-\alpha-\gamma}$$

and on the other hand

$$(7.3) \quad \int_{B_{2R} \setminus B_R} u^{q-1} \leq C \frac{R^{2N-\alpha-\gamma}}{\int_{B_{2R} \setminus B_\rho} u^p} \leq C \frac{R^{2N-\alpha-\gamma}}{\int_{B_{2\rho} \setminus B_\rho} u^p}.$$

This brings a contradiction when $p + q \geq 1$ and $q < 1 - \frac{N-\alpha-\gamma}{N}p$.

In the *critical case* $q = 1 - \frac{N-\alpha-\gamma}{N}p$, by (7.1), (7.2) and (7.3), there exists $c > 0$ such that

$$\int_{B_{2R} \setminus B_R} u^p \geq \frac{\left(\int_{B_{2R} \setminus B_R} 1 \right)^{1+\frac{p}{1-q}}}{\left(\int_{B_{2R} \setminus B_R} u^{1-q} \right)^{\frac{p}{1-q}}} \geq cR^{N-\frac{p}{1-q}(N+\alpha+\gamma)} = c.$$

Therefore,

$$\lim_{R \rightarrow \infty} \int_{B_{2R} \setminus B_\rho} u^p = \infty$$

and (7.3) can be improved to

$$\lim_{R \rightarrow \infty} R^{\alpha+\gamma-2N} \int_{B_{2R} \setminus B_R} u^{q-1} = 0,$$

so that we can conclude as previously. $\square$

In the sublinear region $p + q < 1$, described in Theorem 6, the nonexistence régime is different.

Proposition 7.2. *Let $N \geq 1$, $0 < \alpha < N$, $\gamma < 2$, $\lambda > 0$, $p > 0$, $q < 1$ and $\rho > 0$. If $p + q < 1$,*

$$q \leq 1 + \frac{\gamma}{\alpha}p,$$

and $u \geq 0$ is a supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.

If $\gamma \geq -\alpha$, this proposition merely states that there is no nontrivial supersolution for $p + q < 1$.

Proof of Proposition 7.2. Let $R > \rho$. Since $p + q < 1$, by the Hölder inequality we have

$$(7.4) \quad \begin{aligned} \int_{B_{2R} \setminus B_R} 1 &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right)^{\frac{p}{p+1-q}} \\ &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q-p}{p+1-q}} \left(\left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) \right)^{\frac{p}{p+1-q}}. \end{aligned}$$

By (2.1) and by Lebesgue's dominated convergence theorem, one has

$$\lim_{R \rightarrow \infty} R^{N-\alpha} \int_{B_{2R} \setminus B_R} u^p = 0,$$

hence by (7.4)

$$\lim_{R \rightarrow \infty} R^{-(2N-\alpha)-\alpha \frac{1-q}{p}} \left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{q-1} \right) = \infty.$$

This brings a contradiction with Lemma 6.4 when $q \leq 1 + \frac{\gamma}{\alpha}p$. $\square$

7.2. Pointwise decay bounds. In the sublinear decay region $q < 1$ the exponential decay estimates of Proposition 6.1 are no longer relevant. In fact, if $q < 1$ then nontrivial nonnegative supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ decay at a polynomial rate. We prove this in several steps. We first observe that the decay of u is related to the behavior of the integral of u^p on large balls.

Lemma 7.3. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $q < 1$ and $\rho > 0$. If $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda, \gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then*

$$\liminf_{|x| \rightarrow \infty} \frac{|x|^{N-\alpha-\gamma}}{\int_{B_{2|x|} \setminus B_\rho} u^p} u(x)^{1-q} > 0.$$

Proof. Choose a function $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi = 1$ on $B_{1/2}$ and $\text{supp } \varphi \subset B_1$. For $x \in \mathbb{R}^N \setminus B_r$ define

$$\psi_x(y) = \varphi\left(\frac{y-x}{|x|^{\frac{\gamma}{2}}}\right).$$

If $|x|$ is large enough, one has $\text{supp } \psi_x \cap B_\rho = \emptyset$. Note that $\varphi \geq 1$ on $B_{|x|^{\gamma/2}}(x)$ and that

$$\int_{\mathbb{R}^N \setminus B_r} |\nabla \psi_x(y)|^2 + \frac{1}{|x|^\gamma} |\psi_x(y)|^2 dy \leq C|x|^{\gamma \frac{N-2}{2}}.$$

By Proposition 3.2, one has

$$\frac{1}{|x|^{\gamma \frac{N-2}{2}}} \int_{B_{|x|^{\gamma/2}}(x)} u^{q-1} \leq C \frac{|x|^{N-\alpha-\gamma}}{\int_{B_{2|x|} \setminus B_r} u^p}.$$

Since $q < 1$, by the weak Harnack inequality of Proposition B.3, applied in the ball $B_{|x|^{\gamma/2}}(x)$, we conclude that

$$u(x)^{1-q} \geq \frac{c}{|x|^{N-\alpha-\gamma}} \int_{B_{2|x|} \setminus B_r} u^p,$$

where the constant $c > 0$ does not depend on $x \in \mathbb{R}^N \setminus B_r$. $\square$

As first consequence, we have the following asymptotics:

Proposition 7.4. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $q < 1$ and $\rho > 0$. If $u \geq 0$ is a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$, then*

$$\liminf_{|x| \rightarrow \infty} |x|^{\frac{N-\alpha-\gamma}{1-q}} u(x) > 0.$$

Proof. Apply Lemma 7.3 and note that $\int_{B_{2|x|} \setminus B_\rho} u^p \geq \int_{B_{2\rho} \setminus B_\rho} u^p > 0$ when $|x| \geq \rho$. $\square$

In the fully sublinear case $p + q < 1$ the lower bound of Proposition 7.4 can be further improved.

Lemma 7.5. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $p + q < 1$ and $\rho > 0$. Let $u \geq 0$ be a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

(i) *If $q < 1 - \frac{N-\alpha-\gamma}{N}p$ then for every $R \geq 2\rho$*

$$\int_{B_R \setminus B_\rho} u^p \geq cR^{N+p\frac{\alpha+\gamma}{1-p-q}}.$$

(ii) *If $q = 1 - \frac{N-\alpha-\gamma}{N}p$ then for every $R \geq 2\rho$*

$$\int_{B_R \setminus B_\rho} u^p \geq c \left(\log \frac{R}{\rho} \right)^{\frac{1-q}{1-p-q}}.$$

Proof. From Lemma 6.4, for $R \geq 2\rho$ it holds

$$\int_{B_{2R} \setminus B_R} u^p \int_{B_{2R} \setminus B_R} u^{q-1} \leq CR^{2N-\alpha-\gamma}.$$

On the other hand, since $q < 1$ by Hölder's inequality we have

$$\int_{B_{2R} \setminus B_R} u^{q-1} \geq \frac{\left(\int_{B_{2R} \setminus B_R} 1 \right)^{1+\frac{1-q}{p}}}{\left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{p}}}.$$

Since $2N - \alpha - \gamma \leq N(1 + \frac{1-q}{p})$ we obtain

$$\int_{B_{2R} \setminus B_R} u^p \leq CR^{N-\alpha-\gamma-N\frac{1-q}{p}} \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{p}}.$$

We deduce that there exists $c > 0$ such that for every $R \geq 2\rho$,

$$\int_{B_{2R} \setminus B_R} u^p \geq cR^{N+p\frac{\alpha+\gamma}{1-p-q}}.$$

Then the conclusion is immediate when $q < 1 - \frac{N-\alpha-\gamma}{N}p$, and follows by summation over dyadic annuli when $q = 1 - \frac{N-\alpha-\gamma}{N}p$. $\square$

Proposition 7.6. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $p + q < 1$ and $\rho > 0$. Let $u \geq 0$ be a nontrivial supersolution of $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.*

(1) *If $q < 1 - \frac{N-\alpha-\gamma}{N}p$ then*

$$\liminf_{|x| \rightarrow \infty} |x|^{-\frac{\gamma+\alpha}{1-q-p}} u(x) > 0.$$

(2) *If $q = 1 - \frac{N-\alpha-\gamma}{N}p$ then*

$$\liminf_{|x| \rightarrow \infty} |x|^{-\frac{\gamma+\alpha}{1-q-p}} (\log |x|)^{\frac{-1}{1-p-q}} u(x) > 0.$$

Proof. Writing

$$u(x)^{q-1} \geq \frac{u(x)^{q-1}}{\int_{B_{2|x|} \setminus B_\rho} u^p} \int_{B_{2|x|} \setminus B_\rho} u^p,$$

the conclusion follows from Lemmas 7.3 and 7.5. $\square$

7.3. Optimal decay. We complete the proof of Theorem 5 by constructing explicit supersolutions to $(\mathcal{C}_{\lambda,\gamma}^S)$, which show the optimality of nonexistence and decay results of Propositions 7.1 and 7.4.

Proposition 7.7. *Let $N \geq 1$, $\gamma < 2$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$ and $\rho > 0$. If*

$$1 - \frac{N - \alpha - \gamma}{N} p < q < 1,$$

and $p + q \neq 1$, then $(\mathcal{C}_{\lambda,\gamma}^S)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, such that

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-\gamma}{1-q}} < \infty.$$

Note that the assumption implies that $\gamma < N - \alpha$.

Proof of Proposition 7.7. Set for $\nu > 0$ and $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$v_\nu(x) = \frac{1}{(|x|^2 + \nu^2)^{\frac{N-\alpha-\gamma}{2(1-q)}}}.$$

Compute for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$\begin{aligned} & -\Delta v_\nu(x) + \frac{\lambda^2}{|x|^\gamma} v_\nu(x) \\ &= \left(\frac{N - \alpha - \gamma}{(1-q)(|x|^2 + \nu^2)} \left(N - \left(2 + \frac{N - \alpha - \gamma}{1-q} \right) \frac{|x|^2}{|x|^2 + \nu^2} \right) + \frac{\lambda^2}{|x|^\gamma} \right) v_\nu(x). \end{aligned}$$

Since $\gamma < 2$, there exists $R > 0$ such that for every $\nu > 0$ and $x \in \mathbb{R}^N \setminus B_R$,

$$-\Delta v_\nu(x) + \frac{\lambda^2}{|x|^\gamma} v_\nu(x) \geq \frac{\lambda^2}{2|x|^\gamma} v_\nu(x).$$

If $\nu > 0$ is sufficiently large, for every $x \in B_R \setminus \bar{B}_\rho$,

$$-\Delta v_\nu(x) + \frac{\lambda^2}{|x|^\gamma} v_\nu(x) > 0.$$

Then there exists $c > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$-\Delta v_\nu(x) + \frac{\lambda^2}{|x|^\gamma} v_\nu(x) \geq \frac{c}{|x|^{\frac{N-\alpha-\gamma}{1-q} + \gamma}}.$$

For $\mu > 0$, set $u_\mu = \mu v_\nu$. Since $p^{\frac{N-\alpha-\gamma}{1-q}} > N$, by Lemma A.1 there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$(I_\alpha * u_\mu^p)(x) \leq \frac{C \mu^p}{|x|^{N-\alpha}}.$$

One concludes that u is a supersolution to $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ for all sufficiently small $\mu > 0$ if $p + q > 1$, or for all large μ if $p + q < 1$. $\square$

Now we complete the proof of Theorem 6 by establishing the optimality of Propositions 7.1 and 7.4.

Proposition 7.8. *Let $N \geq 1$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $\gamma < -\alpha$ and $\rho > 0$. If*

$$q = 1 - \frac{N - \alpha - \gamma}{N}p,$$

then $(\mathcal{C}_{\lambda,\gamma}^S)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, such that

$$\lim_{|x| \rightarrow \infty} u(x)|x|^{\frac{N}{p}}(\log|x|)^{\frac{1}{1-q-p}} = c > 0.$$

Proof. Set for $\nu > 1$ and $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$v_\nu(x) = \frac{(\log(|x|^2 + |\nu|^2))^{\frac{1}{1-q-p}}}{(|x|^2 + \nu^2)^{\frac{N}{2p}}}.$$

One has

$$\begin{aligned} -\Delta v_\nu(x) &= \left(\frac{N^2}{p} - \frac{2N}{(1-p-q)\log(|x|^2 + \nu^2)} \right. \\ &\quad + \left(-\frac{N(N+2p)}{p^2} + \frac{4(N+p)}{p(1-p-q)\log(|x|^2 + \nu^2)} \right. \\ &\quad \left. \left. - \frac{4(p+q)}{(1-p-q)^2(\log(|x|^2 + \nu^2))^2} \right) \frac{|x|^2}{|x|^2 + \nu^2} \right) v_\nu(x). \end{aligned}$$

One observes that as $\gamma < 2$, if ν is large enough, then there exists $c > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$-\Delta v_\nu(x) \geq \frac{c}{|x|^{-\frac{\alpha+\gamma}{1-q-p}+\gamma}}.$$

Set now $u_\mu = \mu v_\nu$. By Lemma A.2, there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$(I_\alpha * u^p)(x) \leq \frac{C\mu^p(1 + \log \frac{|x|}{\rho})^{\frac{p}{1-q-p}+1}}{|x|^{N-\alpha}}.$$

Since $q = 1 - \frac{N-\alpha-\gamma}{N}p < 1-p$, u_μ is a supersolution to $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ for all sufficiently large $\mu > 0$. $\square$

Proposition 7.9. *Let $N \geq 1$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$, $\gamma < -\alpha$ and $\rho > 0$. If*

$$1 + \frac{\gamma}{\alpha}p < q < 1 - \frac{N - \alpha - \gamma}{N}p,$$

then $(\mathcal{C}_{\lambda,\gamma}^S)$ admits a nontrivial nonnegative supersolution u in $\mathbb{R}^N \setminus \bar{B}_\rho$ such that

$$\lim_{|x| \rightarrow \infty} u(x)|x|^{-\frac{\alpha+\gamma}{1-q-p}} = c > 0.$$

Proof. Set

$$u_\mu(x) = \frac{\mu}{(|x|^2 + \nu^2)^{-\frac{\alpha+\gamma}{2(1-q-p)}}},$$

where ν is chosen as in the proof of Proposition 7.7 so that there exists $c > 0$ such that for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$-\Delta u_\mu(x) + \frac{\lambda^2}{|x|^\gamma} u_\mu(x) \geq \frac{c\mu}{|x|^{-\frac{\alpha+\gamma}{1-q-p}+\gamma}}.$$

By our assumptions, we have $\alpha < -\frac{p(\alpha+\gamma)}{1-q-p} < N$. Therefore, by Lemma A.1 we obtain for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$(I_\alpha * u_\mu^p)(x) \leq \frac{C\mu^p}{|x|^{-\frac{p(\alpha+\gamma)}{1-q-p}-\alpha}}.$$

Since $p + q < 1$, we conclude that u a supersolution to $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ for all sufficiently large $\mu > 0$. $\square$

7.4. Homogeneous regime $p + q = 1$: proof of Theorem 7. We now consider the homogeneous case of equation $(\mathcal{C}_{\lambda,\gamma}^S)$, that is the equation

$$(7.5) \quad -\Delta u + \frac{\lambda^2}{|x|^\gamma} u = (I_\alpha * u^p) u^{1-p}.$$

In order to study (7.5) we will need a modified version of the nonlocal positivity principle of Proposition 3.2 which allows a more accurate control of constants in the integral inequality.

Lemma 7.10. *Let $N \geq 1$, $0 < \alpha < N$, $\lambda > 0$, $\gamma < 2$, $p > 0$ and $\rho > 0$. If $u \in L_{\text{loc}}^1(\mathbb{R}^N \setminus \bar{B}_\rho)$ be a nonnegative supersolution of (7.5) in $\mathbb{R}^N \setminus \bar{B}_\rho$, then either $u = 0$ almost everywhere in $\mathbb{R}^N \setminus \bar{B}_\rho$, or $u > 0$ almost everywhere in $\mathbb{R}^N \setminus \bar{B}_\rho$ and for every $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$, one has*

$$\int_{\mathbb{R}^N \setminus \bar{B}_\rho} |\nabla \varphi|^2 + \int_{\mathbb{R}^N \setminus \bar{B}_\rho} \frac{\lambda^2}{|x|^\gamma} |\varphi|^2 \geq \int_{\mathbb{R}^N \setminus \bar{B}_\rho} \int_{\mathbb{R}^N \setminus \bar{B}_\rho} \varphi(x) I_\alpha(x-y) \varphi(y) dx dy.$$

Proof. By Proposition 3.1 with $W = I_\alpha * u^p \in L_{\text{loc}}^1(\Omega)$, either $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ or $u > 0$ almost everywhere in $\mathbb{R}^N \setminus \bar{B}_\rho$, $u^{-p} \in L_{\text{loc}}^1(\Omega)$ and for every $\varphi \in C_c^\infty(\Omega)$

$$\int_{\Omega} |\nabla \varphi|^2 + \frac{\lambda^2}{|x|^\gamma} \varphi^2 \geq \int_{\Omega} (I_\alpha * u^p) u^{-p} \varphi^2.$$

Now by the Cauchy–Schwarz inequality we derive

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} \varphi(x) I_\alpha(x-y) \varphi(y) dx dy \\ & \leq \left(\int_{\Omega} \int_{\Omega} u(x)^{-p} \varphi(x)^2 I_\alpha(x-y) u(y)^p dx dy \right)^{\frac{1}{2}} \\ & \quad \left(\int_{\Omega} \int_{\Omega} u(x)^p I_\alpha(x-y) u(y)^{-p} \varphi(y)^2 dx dy \right)^{\frac{1}{2}} \\ & = \int_{\Omega} \int_{\Omega} u(x)^{-p} \varphi(x)^2 I_\alpha(x-y) u(y)^p dx dy, \end{aligned}$$

and the conclusion follows. $\square$

7.4.1. Case $\gamma \neq -\alpha$ and $p + q = 1$. If $\alpha > -\gamma$ the nonexistence follows from Proposition 7.1, while for $\alpha < -\gamma$ we can construct a solution outside a sufficiently large ball.

Proposition 7.11. *Let $N \geq 1$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$ and $q = 1 - p$. If $\gamma < -\alpha$, then there exists $\rho_0 > 0$ such that $(\mathcal{C}_{\lambda,\gamma}^S)$ admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_{\rho_0})$, such that*

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-\gamma}{1-q}} < \infty.$$

Proof. Define $u_\rho : \mathbb{R}^N \setminus \bar{B}_\rho \rightarrow \mathbb{R}$ for $x \in \mathbb{R}^N \setminus \bar{B}_\rho$ by

$$u_\rho(x) = \frac{1}{|x|^{\frac{N-\alpha-\gamma}{1-q}}}.$$

One has for every $x \in \mathbb{R}^N \setminus \bar{B}_\rho$

$$-\Delta u_\rho(x) + \frac{\lambda^2}{|x|^\gamma} u_\rho(x) = \frac{N-\alpha-\gamma}{1-q} \left(N-2 - \frac{N-\alpha-\gamma}{1-q} \right) \frac{1}{|x|^{\frac{N-\alpha-\gamma}{1-q}+2}} + \frac{\lambda^2}{|x|^{\frac{N-\alpha-\gamma}{1-q}+\gamma}}.$$

Since $\gamma < 2$, there exists ρ_0 such that if $\rho \geq \rho_0$,

$$-\Delta u_\rho(x) + \frac{\lambda^2}{|x|^\gamma} u_\rho(x) \geq \frac{\lambda^2}{2|x|^{\frac{N-\alpha-\gamma}{1-q}+\gamma}}.$$

By a change of variable, by the assumption $\gamma + \alpha < 0$ and by Lemma A.1, there exists $C > 0$ such that for every $x \in \mathbb{R}^N \setminus B_\rho$,

$$\begin{aligned} (I_\alpha * u_\rho^p)(x) &= \int_{\mathbb{R}^N \setminus \bar{B}_\rho} I_\alpha(x-y) \frac{1}{|y|^{N-\alpha-\gamma}} dy \\ &= \rho^{\gamma+\alpha-(N-\alpha)} \int_{\mathbb{R}^N \setminus \bar{B}_1} I_\alpha\left(\frac{x}{\rho} - y\right) \frac{1}{|y|^{N-\alpha-\gamma}} dy \leq \frac{C\rho^{\gamma+\alpha}}{|x|^{N-\alpha}}; \end{aligned}$$

hence,

$$(I_\alpha * u_\rho^p)(x) u(x)^q \leq \frac{C\rho^{\gamma+\alpha}}{|x|^{\frac{N-\alpha-\gamma}{1-q}+\gamma}}.$$

Since $\gamma + \alpha < 0$, we conclude that u_ρ is a supersolution to $(\mathcal{C}_{\lambda,\gamma}^S)$ in $\mathbb{R}^N \setminus \bar{B}_\rho$ for all sufficiently large $\rho \geq \rho_0$. $\square$

The restriction on the radius $\rho_0 > 0$ is essential.

Proposition 7.12. *Let $N \geq 1$, $\lambda > 0$, $0 < \alpha < N$, $p > 0$ and $q = 1 - p$. If $\gamma < -\alpha$, then there exists $\rho^* > 0$ such that $(\mathcal{C}_{\lambda,\gamma}^S)$ has no nontrivial nonnegative supersolution in $\mathbb{R}^N \setminus \bar{B}_{\rho^*}$,*

Proof. Choose $\varphi \in C_c^\infty(\mathbb{R}^N \setminus B_1)$ and let for $R > 0$ and $x \in \mathbb{R}^N$, $\varphi_R(x) = \varphi(x/R)$. One has

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * \varphi_R) \varphi_R &= R^{N+\alpha} \int_{\mathbb{R}^N} (I_\alpha * \varphi) \varphi, \\ \int_{\mathbb{R}^N} |\nabla \varphi_R|^2 &= R^{N-2} \int_{\mathbb{R}^N} |\nabla \varphi|^2, \\ \int_{\mathbb{R}^N} \frac{|\varphi_R(x)|^2}{|x|^\gamma} dx &= R^{N-\gamma} \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^\gamma} dx. \end{aligned}$$

Since $-\gamma < \alpha < 2$, this brings a contradiction with the positivity principle of Lemma 7.10 if R is small enough. $\square$

7.4.2. Case $\gamma = -\alpha$ and $p + q = 1$. We now consider the most delicate case of equation $(\mathcal{C}_{\lambda,\gamma}^S)$ when $\gamma = -\alpha$, that is the equation

$$(7.6) \quad -\Delta u + \lambda^2 |x|^\alpha u = (I_\alpha * u^p) u^{1-p}.$$

In order to study our problem, we review relevant inequalities. The weighted version of the Hardy–Littlewood–Sobolev inequality of Stein and Weiss [38],

$$(7.7) \quad \int_{\mathbb{R}^N} |I_{\alpha/2} * \varphi|^2 \leq \lambda^2 \int_{\mathbb{R}^N} |x|^\alpha |\varphi(x)|^2 dx,$$

holds if and only if

$$\lambda^2 \geq \sigma_\alpha^* = 2^{-\alpha} \left(\frac{\Gamma(\frac{N-\alpha}{4})}{\Gamma(\frac{N+\alpha}{4})} \right)^2$$

see [14, Theorem 2.5].

The constant σ_α is also related to convolution of Riesz kernels. Indeed, according to the semigroup property of the Riesz kernels [35, p.20], for $0 < \alpha < \beta < N$ it holds

$$\int_{\mathbb{R}^N} I_\alpha(x-y) \frac{1}{|y|^\beta} dy = \frac{\sigma_\alpha(\beta)}{|x|^{\beta-\alpha}},$$

where

$$\sigma_\alpha(\beta) = \frac{A_{N-\beta+\alpha}}{A_{N-\beta}} = 2^{-\alpha} \frac{\Gamma\left(\frac{\beta-\alpha}{2}\right)\Gamma\left(\frac{N-\beta}{2}\right)}{\Gamma\left(\frac{N-\beta+\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}.$$

One can verify (see [11, Lemma 2.1]) that $\sigma_\alpha : (\alpha, N) \rightarrow \mathbb{R}$ is an even function with respect to $\beta = (N + \alpha)/2$. Moreover,

$$\lim_{\beta \rightarrow \alpha} \sigma_\alpha(\beta) = \lim_{\beta \rightarrow N} \sigma_\alpha(\beta) = +\infty,$$

σ_α is strictly decreasing on $(\alpha, (N + \alpha)/2)$, strictly increasing on $((N + \alpha)/2, N)$, and attains its minimum at $\beta = (N + \alpha)/2$, with

$$\sigma_\alpha^* = \min_{\alpha < \beta < N} \sigma_\alpha(\beta) = \sigma_\alpha\left(\frac{N + \alpha}{2}\right).$$

Lemma 7.13. *Let $N \geq 1$, $0 < \alpha < N$, $\lambda > 0$ and $\rho > 0$. One has for every $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$*

$$\int_{\mathbb{R}^N} (I_\alpha * \varphi) \varphi \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 + \lambda^2 \int_{\mathbb{R}^N} |x|^\alpha |\varphi(x)|^2 dx$$

if and only if $\lambda^2 \geq \sigma_\alpha^$.*

Proof. It is clear by (7.7) and the semigroup property of the Riesz potential that $\lambda^2 \geq \sigma_\alpha^*$ implies the required inequality.

Now assume that the inequality holds. Choose $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Let $\varphi_R(x) = \varphi(x/R)$. For R large enough, $\text{supp } \varphi_R \subset \mathbb{R}^N \setminus \bar{B}_\rho$, and by assumption

$$\begin{aligned} R^{N+\alpha} \int_{\mathbb{R}^N} (I_\alpha * \varphi) \varphi &= \int_{\mathbb{R}^N} (I_\alpha * \varphi_R) \varphi_R \\ &\leq \int_{\mathbb{R}^N} |\nabla \varphi_R|^2 + \lambda^2 \int_{\mathbb{R}^N} |x|^\alpha |\varphi_R(x)|^2 dx \\ &= R^{N-2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 + \lambda^2 R^{N+\alpha} \int_{\mathbb{R}^N} |x|^\alpha |\varphi(x)|^2 dx \end{aligned}$$

Letting now $R \rightarrow \infty$, we deduce that

$$\int_{\mathbb{R}^N} (I_\alpha * \varphi) \varphi \leq \lambda^2 \int_{\mathbb{R}^N} |x|^\alpha |\varphi(x)|^2 dx.$$

Since $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is arbitrary and by the optimality condition in (7.7), we conclude that $\lambda^2 \geq \sigma_\alpha^*$. $\square$

Using the above modified nonlocal positivity principle we prove the following.

Proposition 7.14. *Let $N \geq 2$, $0 < \alpha < N$, $p > 0$ and $\rho > 0$. If*

$$0 < \lambda^2 < \sigma_\alpha^*,$$

and $u \geq 0$ is a supersolution of (7.6) in $\mathbb{R}^N \setminus \bar{B}_\rho$, then $u = 0$ in $\mathbb{R}^N \setminus \bar{B}_\rho$.

Proof. This follows from Lemma 7.10 and Lemma 7.13. $\square$

Using semigroup property of the Riesz kernels it is easy to construct explicit supersolutions to (7.6).

Proposition 7.15. *Let $N \geq 1$, $0 < \alpha < N$ and $p > 0$. If*

$$\lambda^2 > \sigma_\alpha^*,$$

then there exists $\rho > 0$ such that (7.6) admits a radial nontrivial nonnegative supersolution $u \in C^\infty(\mathbb{R}^N \setminus \bar{B}_\rho)$ satisfying

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{\frac{N}{p}} < +\infty.$$

Proof. Set for $x \in \mathbb{R}^N \setminus \bar{B}_\rho$,

$$u_\rho(x) = \frac{1}{|x|^{\frac{N+\alpha}{2p}}}.$$

Then we compute

$$\begin{aligned} -\Delta u_\rho(x) + \frac{\lambda^2}{|x|^\alpha} u_\rho(x) &= \frac{N+\alpha}{2p} \left(N-2 - \frac{N+\alpha}{2p} \right) |x|^{-\frac{N+\alpha}{2p}-2} + \lambda^2 |x|^{\alpha-\frac{N+\alpha}{2p}} \\ &= \lambda^2 |x|^{\alpha-\frac{N+\alpha}{2p}} (1 + o(1)), \end{aligned}$$

as $|x| \rightarrow \infty$. On the other hand,

$$-(I_\alpha * u_\rho^p)(x) u_\rho^{1-p}(x) \leq \sigma_\alpha \left(\frac{N+\alpha}{2} \right) |x|^{\alpha-\frac{N+\alpha}{2p}} = \sigma_\alpha^* |x|^{\alpha-\frac{N+\alpha}{2p}}$$

We conclude that u_ρ is a supersolution of (7.6) if $\rho > 0$ is large enough. $\square$

We do not make any claim about the existence or nonexistence of nontrivial nonnegative supersolutions of (7.6) at the threshold value $\lambda = \sqrt{\sigma_\alpha^*}$. The above proof shows that supersolutions exist if $\alpha < N-4$ and $\frac{N+\alpha}{2(N-2)} < p < 1$.

Proof of Theorem 7. If $\alpha \neq -\gamma$ then Theorem 7 is a consequence of Propositions 7.1, 7.11 and the decay estimate of Propositions 7.4. When $\alpha = -\gamma$ the conclusion of Theorem 7 follows our results for equation (7.6) (Propositions 7.14 and 7.15) and Proposition 7.4. $\square$

APPENDIX A. RIESZ POTENTIAL ESTIMATES.

Here we collect some estimates of the Riesz potentials which were extensively used in the main part of the paper. Most of the estimates are standard, however we sketch some of the proofs for the readers convenience.

Lemma A.1. *Let $v \in L^1_{\text{loc}}(\mathbb{R}^N)$, $\alpha \in (0, N)$ and $\beta > \alpha$. If*

$$\limsup_{|x| \rightarrow \infty} v(x) |x|^\beta < \infty,$$

then

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} (I_\alpha * v)(x) |x|^{\beta-\alpha} &< \infty && \text{if } \alpha < \beta < N, \\ \limsup_{|x| \rightarrow \infty} (I_\alpha * v)(x) \frac{|x|^{N-\alpha}}{\log|x|} &< \infty && \text{if } \beta = N, \\ \limsup_{|x| \rightarrow \infty} (I_\alpha * v)(x) |x|^{N-\alpha} &< \infty && \text{if } \beta > N. \end{aligned}$$

Proof. Without loss of generality, we can assume that $v(x) \leq \frac{1}{|x|^\beta}$ if $|x| \geq 1$. Then for every $x \in \mathbb{R}^N \setminus B_2$ it holds

$$(I_\alpha * v)(x) \leq A_\alpha \int_{\mathbb{R}^N \setminus B_1} \frac{1}{|y|^\beta} \frac{1}{|x-y|^{N-\alpha}} dy + \frac{2^{N-\alpha} A_\alpha}{|x|^{N-\alpha}} \int_{B_1} |v|.$$

Observe that if $|x-y| \leq \frac{|x|}{2}$, then $|y| \geq \frac{|x|}{2}$. Hence, since $\alpha < N$,

$$\int_{B_{|x|/2}(x)} \frac{1}{|y|^\beta} \frac{1}{|x-y|^{N-\alpha}} dy \leq 2^\beta \int_{B_{|x|/2}(x)} \frac{1}{|x|^\beta} \frac{1}{|x-y|^{N-\alpha}} dy = \frac{C}{|x|^{\beta-\alpha}}.$$

Next, if $|y| \geq 2|x|$, then $|x-y| \geq \frac{|y|}{2}$. Therefore, since $\beta > \alpha$,

$$\int_{\mathbb{R}^N \setminus B_{2|x|}} \frac{1}{|y|^\beta} \frac{1}{|x-y|^{N-\alpha}} dy \leq 2^{N-\alpha} \int_{\mathbb{R}^N \setminus B_{2|x|}} \frac{1}{|y|^{\beta+N-\alpha}} dy = \frac{C'}{|x|^{\beta-\alpha}}.$$

Finally, we obtain

$$\begin{aligned} \int_{B_{2|x|} \setminus (B_1 \cup B_{|x|/2}(x))} \frac{1}{|y|^\beta} \frac{1}{|x-y|^{N-\alpha}} dy &\leq \int_{B_{2|x|} \setminus (B_1 \cup B_{|x|/2}(x))} \frac{1}{|y|^\beta} \frac{2^{N-\alpha}}{|x|^{N-\alpha}} dy \\ &\leq \frac{2^{N-\alpha}}{|x|^{N-\alpha}} \int_{B_{2|x|} \setminus B_1} \frac{1}{|y|^\beta} dy, \end{aligned}$$

where

$$\int_{B_{2|x|} \setminus B_1} \frac{1}{|y|^\beta} dy \leq \begin{cases} C''|x|^{N-\beta} & \text{if } \beta < N, \\ C'' \log(2|x|) & \text{if } \beta = N, \\ C'' & \text{if } \beta > N, \end{cases}$$

so the assertion follows. $\square$

Lemma A.2. *Let $v \in L^1_{\text{loc}}(\mathbb{R}^N)$, $\alpha \in (0, N)$ and $\sigma \in \mathbb{R}$. If*

$$\limsup_{|x| \rightarrow \infty} v(x) \frac{|x|^N}{(\log|x|)^\sigma} < \infty.$$

then

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} (I_\alpha * v)(x) |x|^{N-\alpha} &< \infty && \text{if } \sigma < -1, \\ \limsup_{|x| \rightarrow \infty} (I_\alpha * v)(x) \frac{|x|^{N-\alpha}}{(\log(\log|x|))} &< \infty && \text{if } \sigma = -1, \\ \limsup_{|x| \rightarrow \infty} (I_\alpha * v)(x) \frac{|x|^{N-\alpha}}{(\log|x|)^{\sigma+1}} &< \infty && \text{if } \sigma > -1. \end{aligned}$$

Proof. Without loss of generality, we can assume that $v(x) \leq \frac{|\log|x||^\sigma}{|x|^N}$ for $|x| \geq 2$. Then for $x \in \mathbb{R}^N \setminus B_4$ it holds

$$(I_\alpha * v)(x) \leq C \int_{\mathbb{R}^N \setminus B_2} \frac{|\log|y||^\sigma}{|y|^N} \frac{1}{|x-y|^{N-\alpha}} dy + \frac{C}{|x|^{N-\alpha}} \int_{B_2} |v|.$$

Observe also that if $|x-y| \leq \frac{|x|}{2}$ then $|y| \geq \frac{|x|}{2}$. Since $r \mapsto (\log r)^\sigma / r^N$ is nonincreasing for $r \geq e^{\frac{\sigma}{N}}$, if $|x| \geq 2e^{\frac{\sigma}{N}}$

$$\begin{aligned} \int_{B_{|x|/2}(x)} \frac{|\log|y||^\sigma}{|y|^N} \frac{1}{|x-y|^{N-\alpha}} dy &\leq 2^N \int_{B_{|x|/2}(x)} \frac{|\log \frac{|x|}{2}|^\sigma}{|x|^N} \frac{1}{|x-y|^{N-\alpha}} dy \\ &= \frac{C |\log \frac{|x|}{2}|^\sigma}{|x|^{N-\alpha}}. \end{aligned}$$

Next, if $|y| \geq 2|x|$ then $|x-y| \geq \frac{|y|}{2}$. Hence, since $N > \alpha$,

$$\int_{\mathbb{R}^N \setminus B_{2|x|}} \frac{|\log|y||^\sigma}{|y|^N} \frac{1}{|x-y|^{N-\alpha}} dy \leq C \int_{\mathbb{R}^N \setminus B_{2|x|}} \frac{|\log|y||^\sigma}{|y|^{2N-\alpha}} dy \leq C' \frac{|\log|x||^\sigma}{|x|^{N-\alpha}}.$$

Finally, we obtain

$$\begin{aligned} \int_{B_{2|x|} \setminus (B_2 \cup B_{|x|/2}(x))} \frac{|\log|y||^\sigma}{|y|^N} \frac{1}{|x-y|^{N-\alpha}} dy &\leq \int_{B_{2|x|} \setminus (B_2 \cup B_{|x|/2}(x))} \frac{|\log|y||^\sigma}{|y|^N} \frac{2^{N-\alpha}}{|x|^{N-\alpha}} dy \\ &\leq \frac{2^{N-\alpha}}{|x|^{N-\alpha}} \int_{B_{2|x|} \setminus B_2} \frac{|\log|y||^\sigma}{|y|^N} dy, \end{aligned}$$

where

$$\int_{B_{2|x|} \setminus B_1} \frac{1}{|y|^N} = \begin{cases} C'' & \text{if } \sigma < -1, \\ C'' \log(\log(2|x|)) & \text{if } \sigma = -1, \\ C''(\log|x|)^{\sigma+1} & \text{if } \sigma > -1, \end{cases}$$

which completes the proof. $\square$

APPENDIX B. TOOLS FOR DISTRIBUTIONAL SOLUTIONS.

B.1. Truncation of supersolutions. The following lemma provides a powerful tool of approximation of distributional supersolutions by weak supersolutions. It is essentially a reformulation of two truncation results by H. Brezis and A. Ponce [5, Lemma 1 and 2].

Lemma B.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open connected set, $u \in L^1_{\text{loc}}(\Omega)$ and $V : \Omega \rightarrow \mathbb{R}$ be measurable. If $Vu \in L^1_{\text{loc}}(\Omega)$, $u \geq 0$ and*

$$-\Delta u + Vu \geq 0 \quad \text{in } \Omega$$

in the sense of distributions, then for every $k \in \mathbb{R}$ one has

$$T_k(u) \in H^1_{\text{loc}}(\Omega)$$

and

$$-\Delta T_k(u) + VT_k(u) \geq 0 \quad \text{in } \Omega$$

in the weak sense, where $T_k(s) = \min(s, k)$.

The first part of the lemma is proved by taking $\tilde{\eta}_\delta * (\varphi(k - T_k(\eta_\delta * u)))$ as a test function in the inequality, for a suitable family of mollifiers η_δ , see [5, Lemma 1]. The second part is a consequence of a variant on Kato's inequality [5, Lemma 2].

B.2. Minimal positive solutions at infinity. Consider the linear Schrödinger equation

$$(B.1) \quad -\Delta u + Vu = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_\rho,$$

where $\rho > 0$ and $V \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \bar{B}_\rho)$.

Definition B.1. We say that $H \in C^1(\mathbb{R}^N \setminus \bar{B}_\rho)$ is a *minimal positive solution at infinity* of (B.1) if H is a weak positive solution of (B.1) and there exists a weak positive supersolution $U \in H^1(\mathbb{R}^N \setminus \bar{B}_\rho)$ of (B.1) such that

$$(B.2) \quad \liminf_{|x| \rightarrow \infty} \frac{U(x)}{H(x)} = +\infty.$$

For example, the fundamental solution of (B.1) in $\mathbb{R}^N$ (if it exists) is a minimal positive solution of (B.1) at infinity. A minimal positive solution of (B.1) might however not decay at infinity may not decay to zero at infinity. For instance, constants are minimal positive solutions at infinity for $-\Delta$ in $\mathbb{R}^2 \setminus \bar{B}_\rho$.

Proposition B.2. *Assume that $V \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \bar{B}_\rho)$ is nonnegative. Let H be a minimal positive solution at infinity of (B.1). If $u \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \bar{B}_\rho)$ satisfies*

$$-\Delta u + Vu \geq 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_\rho$$

in the sense of distributions and

$$\inf_{B_{3\rho} \setminus \bar{B}_\rho} u > 0,$$

then

$$u \geq \frac{\inf_{B_{3\rho} \setminus \bar{B}_\rho} u}{\sup_{\partial B_{2\rho}} H} H \quad \text{in } \mathbb{R}^N \setminus \bar{B}_{2\rho}.$$

Proof. Define $v = \min(u, \inf_{B_{3\rho} \setminus \bar{B}_\rho} u)$. By Lemma B.1 we conclude that $v \in H_{\text{loc}}^1(\mathbb{R}^N \setminus \bar{B}_\rho)$ and

$$-\Delta v + Vv \geq 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_\rho.$$

Moreover, $v = \inf_{B_{3\rho} \setminus \bar{B}_\rho} u$ on $B_{3\rho} \setminus \bar{B}_\rho$.

Let $U > 0$ be a supersolution of (B.1) given by (B.2). By the comparison principle for weak sub and supersolutions (cf. [1, Theorem 2.7]), we obtain

$$v \geq \frac{\inf_{B_{3\rho} \setminus \bar{B}_\rho} u}{\sup_{\partial B_{2\rho}} H} (H - \varepsilon U) \quad \text{in } \mathbb{R}^N \setminus \bar{B}_{2\rho},$$

for every sufficiently small $\varepsilon > 0$. Since $u \geq v$, the assertion follows. $\square$

B.3. Weak Harnack inequality. The weak Harnack inequality is usually formulated in the literature for classical or weak supersolutions of elliptic equations [13, Theorem 8.18]. The proposition below shows that the result remains valid for distributional supersolutions (see [19, Theorem 9.10] for the case $p = 1$).

Proposition B.3. *Let $N \geq 0$, $\lambda \geq 0$ and $p \in (-\infty, \frac{N}{N-2})$ and $p \neq 0$. There exists $C > 0$ such that for every $u \in L^1(B_{3\rho})$, if $u \geq 0$,*

$$-\Delta u + \lambda u \geq 0 \quad \text{in } B_{3\rho}$$

and $u^p \in L^1(B_{2\rho})$, then

$$\inf_{B_\rho} u \geq C \left(\frac{1}{\rho^N} \int_{B_{2\rho}} u^p \right)^{\frac{1}{p}}.$$

Proof. Define $u_k = T_k(u)$, where $T_k(s) = \min(s, k)$. By Lemma B.1, $u_k \in H_{\text{loc}}^1(\mathbb{R}^N \setminus \bar{B}_{3\rho})$ and

$$-\Delta u_k + \lambda u_k \geq 0 \quad \text{in } B_{3\rho}.$$

Then by the weak Harnack inequality for weak supersolutions (cf. [13, Theorem 8.18]),

$$\inf_{B_\rho} u_k \geq C \left(\frac{1}{\rho^N} \int_{B_{2\rho}} (u_k)^p \right)^{\frac{1}{p}}.$$

We conclude by Lebesgue's monotone convergence theorem if $p > 0$, or by Fatou's lemma if $p < 0$. $\square$

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