COMMENT ON "HOW TO SHARE WHEN CONTEXT MATTERS: THE MÖBIUS VALUE AS A GENERALIZED SOLUTION FOR COOPERATIVE GAMES" BY A. BILLOT AND J. THISSE (2005)

Pierre Dehez

LIDAM Discussion Paper CORE
2026 / 05

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Voie du Roman Pays 34, L1.03.01

B-1348 Louvain-la-Neuve

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Comment on "How to share when context matters: the Möbius value as a generalized solution for cooperative games" by A. Billot and J. Thisse (2005).

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Abstract

Billot and Thisse's article is placed in the context of past and recent literature on Harsanyi's dividend distribution. A number of shortcomings and errors are highlighted.

JEL Classification: C71

Keywords: Harsanyi dividends, value

1. Harsanyi dividends

Billot and Thisse (2005) introduced the concept of "Möbius value" based on the concept of *dividend* introduced by Harsanyi in 1959 within the theory of cooperative games with transferable utility. His idea was to associate with each coalition a dividend that could then be distributed among its members to define an allocation of the social surplus. The set of allocations that results from *all possible* distributions of the dividends is an object, known today as the *Harsanyi set*. It was studied in the 70's, independently by Vasil'ev, in papers published in Russian, and by Hammer, Peled and Sorensen in a paper published in a Belgian operations research journal.[1] Derks, Haller and Peters later popularized the concept in a paper published in *International Journal of Game Theory* in 2000. At that time, they did not know about the contributions of Vasil'ev which became known with a paper of Vasil'ev and van der Laan published in 2002. Billot and Thisse did clearly ignore that literature: their Möbius value is nothing but the Harsanyi set.

Consider a game (N,v). A distribution of the Harsanyi dividends can be summarized by a matrix λ of dimensions $n\times(2^n-1)$ whose columns are the non-negative vectors λ^T $(T\subset N, T\neq\varnothing)$ satisfying

$$\sum_{i\in N}\lambda_i^T=1 \text{ and } \lambda_i^T=0 \text{ for all } i\notin T.$$

The vector λ^T specifies how the dividend α_T is allocated within coalition *T*. In particular, $\lambda_i^{\{i\}}=1$ for all *i* and λ^N can be any vector in the unit simplex Δ_n. The Harsanyi payoff vector derived from a distribution matrix λ is then given by the inner product $h(N,v,\lambda)=\lambda\cdot\alpha(N,v)$:

$$h_i(N,v,\lambda)=\sum_{T\in\mathcal{C}(N)}\lambda_i^T\alpha_T(N,v)=v(i)+\sum_{T\in\mathcal{C}(N)\setminus\{i\}}\lambda_i^T\alpha_T(N,v)\qquad(i=1,\ldots,n).$$

In their Theorem 1, Billot and Thisse state three axioms that actually lead to a much larger set, the distribution coefficients being allowed to be non-negative. What is missing is a positivity axiom requiring payoffs to be non-negative when applied to positive games (games whose dividends are non-negative). Moreover, the two conditions used to define the concept of "sharing rule" do not appear in the proof of Theorem 1. Concerning their "φ-Null player" axiom, it is equivalent to the "Null player" axiom used in Shapley's axiomatization of the value. Concerning the formulation of the Shapley value as resulting from the uniform distribution of dividends (Theorem 2), it can be found in Harsanyi 1963's paper.

2. Restrictions on dividend distributions

The idea of placing restrictions on dividend distributions is an interesting one, though it is definitely not new. The first set of restrictions that Billot and Thisse introduce is part of their definition of a sharing rule and corresponds to the concept of "monotonic sharing systems" introduced by par Derks et al. (2000). The second one concerns "negligible players". It is actually a consequence of the first.

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[1] While the latter used the name "selectope", here we retain the term "Harsanyi set" that is now accepted.

Vasil'ev (1988) and later Derks et al. (2006) have characterized the Weber set – the set of random order values defined by Weber (1988) – with a stronger monotonicity condition that is equivalent to the "stochastic rationalizable sharing rule" axiom introduced by Billot and Thisse in their Theorem 3.

Concerning the relation between Harsanyi-payoffs and weighted Shapley values, Derks et al. (2000) have proved that dividend distributions that are Bayesian consistent produce allocations that are weighted Shapley values. That condition is equivalent to the "Luce axiom of choice" introduced by Billot and Thisse in their Theorem 4.

Derks et al. (2000) also proved that, without restrictions on dividend distributions, the Harsanyi set and the core coincident *if and only if* the game is (almost) positive.[2] Billot and Thisse claim that, under monotonicity of dividend distribution, convexity is a *necessary and sufficient* condition for the coincidence of the core and the Harsanyi set (Theorem 5 and 6). This is actually true only for 2- and 3-player games, as the following example shows. The Harsanyi dividends associated with the 4-player convex game defined by $v(S) = |S| - 1$ are given by:

$$\alpha_{\{i\}} = 0$$
$$\alpha_{\{i,j\}} = \alpha_{\{1,2,3,4\}} = 1$$
$$\alpha_{\{i,j,k\}} = -1$$

The following distribution matrix is monotonic: the share of a player never increases when enlarging the coalition to which he belongs.

	12	**13**	**14**	**23**	**24**	**34**	**123**	**124**	**134**	**234**	**1234**
1	0.5	0.4	0.3	0	0	0	0.4	0.3	0.3	0	0.2
2	0.5	0	0	0.5	0.3	0	0.3	0.3	0	0.2	0.2
3	0	0.6	0	0.5	0	0.3	0.3	0	0.3	0.2	0.2
4	0	0	0.7	0	0.7	0.7	0	0.4	0.4	0.6	0.4

The resulting Harsanyi payoff vector (0.4, 0.7, 0.8, 1.1) does not belong to the core: the coalition {1,2,3} only obtains 1.9. There is actually not much hope: convexity ensures individual rationality up to 4-player games but not beyond, as another example shows (see Dehez, 2017).

3. Further developments: graph structures

Since 2000, there have been a number of papers – most of them listed among the references – in particular recent papers proposing restrictions on dividend distribution, resulting from a graph structure on the players. van den Brink et al. (2014) model the ordering of players by a directed graph, a useful approach for dealing for instance with river games (Ambec and Sprumont, 2002)

[2] Almost positive games are games whose multi-player coalitions have non-negative worth.

or liability games (Dehez and Ferey, 2013). Another instance of graph-driven restrictions on dividend distributions is given by van den Brink et al. (2011) who consider games with communication graphs à la Myerson (1977). There the idea is to link dividend distributions to the "power" of players, as measured for instance by the size of their neighborhood.

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