

Fixed-points Method in Robert Maillart's Analysis of Rigid Frame Bridges

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Abstract: This paper explains Robert Maillart's analysis approach – using the fixed-points method – for the Weissensteinstrasse Overpass that he designed in 1938. A brief history and a technical explanation of this method are also presented as background information.

The structure of this reinforced-concrete bridge consists of two identical plane rigid frames, each possesses five degrees of static indeterminacy. This bridge was one of the few continuous bridges designed by Maillart. Its long-forgotten analysis method, the fixed-points method, is a geometrical method for the elastic analysis of continuous beams and frames, which addresses the static indeterminacy by means of geometric constraint of fixed points on bending moment diagrams. This method was invented by Christian Otto Mohr for analyzing continuous beams, which was based on the work of Carl Culmann, and then refined and extended by Karl Wilhelm Ritter for the analysis of frames. Ernst Suter then generalized and analytically formulated this method for complex cases. In view of the historical context, this method is of special significance for the history of graphic statics. The history and core techniques of this method, including fixed points, crossing-lines, distribution factor, and virtual lateral constraint, are all outlined. The techniques are illustrated with Maillart's analysis of Weissensteinstrasse Overpass.

Keywords: the Weissensteinstrasse Overpass, Graphic Statics, the Fixed-Points Method, Continuous Structures, Statically Indeterminate.

1 Introduction

1.1 Background

The Weissensteinstrasse Overpass (WO, Überführung der Weissensteinstrasse) is a reinforced concrete road bridge near the Fischermätteli station in Bern Switzerland. The deck is supported by a rigid frame and spans over the railway tracks. This bridge was designed and engineered by the renowned engineer Robert Maillart in 1938 and erected in the same year. This bridge was praised as one of the best works of Maillart for its simplicity of form [1] (Fig. 1).

Maillart analyzed this continuous, statically indeterminate bridge using graphic statics, a geometric-based approach, as in the design of his better-studied articulated statically determinate arch bridges. The notes on the original drawings of WO stated that the analysis of this bridge was based on the fixed-points method (FPM, German: Methode der Festpunkte) presented by Ernst Suter [2]. FPM is a method of graphic statics for the elastic analysis of the continuous structure. The continuous structure is normally statically indeterminate (or hyperstatic) due to its redundant constraints. Unlike the analysis of the statically determinate structure, its reactions and inner forces cannot be solved only with the static equilibrium of moment and vectorial forces. Deformation has to be involved, sometimes indirectly, in its analysis.

Graphic statics is a classic geometric-based approach for the analysis and design of structures, which comprises a variety of graphical methods for various structure types [3]. It was extensively studied and applied around the turn of the 19th and 20th century, but was almost totally superseded by algebraical and numerical methods by the mid of 20th century. In recent years mainly owing to its unparalleled visual clarity, graphics statics has been rediscovered as an insightful method for structure design with enhanced efficiency through computerization [4][5]. The reintroduction of graphic statics into the design was accompanied and inspired by the investigation of historical designs and writings [6][7].



Fig. 1. the views of the Weissensteinstrasse Overpass from the east and below (photographed respectively by Maryam Jafari and Shuyuan Han in 2022)

1.2 Research aim and significance

The investigation of the analysis of WO may facilitate the study and understanding of Maillart's continuous bridges, which have been relatively overlooked and partly misunderstood. The original drawings of WO reveal that this bridge was actually designed as a rigid frame with columns engaged in moment-resisting of vertical loading. However, it has long been associated to continuous beams simply supported by pillars [1] [8]. Such a misunderstanding may not be limited to this bridge. Max Bill's text creates confusion by referring to a "continuous beam" to describe the majority of these bridge types, as he also uses it to describe the Schwandbach Bridge and the Rhine Bridge project in Schaffhausen, which are

stiffened arch-bridges. However, his words do not mention the “continuous beam” as a static system.

In the calculation note for the Liesberg bridge, as for the Châtelard aqueduct, the structural interaction between beams and columns is not considered, so that in the model the column does not take up any bending forces associated with the loads on the deck. (For the bridge over the Lütschine near Grundlischwand, a beam-column interaction would be very limited since the columns are not long enough to be really effective in bending in an interaction with the deck.) In the geometry of the connection curves between the columns and the supporting beam, the bridges at Liesberg (1935), Grundlischwand (1937), Huttwil (1936), and Altendorf (1938) exhibit a striking resemblance. While there are indications that their static systems may be identical, the lack of available archives precludes a definitive confirmation of this hypothesis. In contrast, the WO (1938) is geometrically much more assertive regarding a continuity between beams and columns. A fair amount of knowledge of FPM is a prerequisite for the investigation of Mailart’s analysis of WO. The lack of knowledge of FPM may have contributed to the longstanding misunderstanding. This work can be typical in terms of its structural working and design method, as it is possible also that this method could have shaped the design of the later similar bridges in Huttwil (1936) and Altendorf (1938), re-classifying these latter bridges differently.

The study of FPM could potentially benefit the historical research of the works by other engineers. Though all but forgotten nowadays, FPM used to allegedly enjoy popularity among engineers. In 1923 Ernst Suter reported in his book that: *“Because of its clarity, the method of fixed points is rightly very popular among engineers.”* [9]. A case in point, Maurice Koechlin, a prominent engineer famous for his work of the Eiffel Tower, expounded this method to French readers and applied it to multiple continuous girders in his book [10]. However, there seems to be a lack of detailed technical information on FPM in contemporary literature, notwithstanding reviews of other methods of graphic statics for continuous structure. Thomas Fidler’s characteristic points method for continuous beams (1883) was concisely reviewed by [11]. The trial closing-string method for portal frames by William Wolfe (1921) method has recently been reviewed by the authors [12]. Besides, the semi-graphical method of Du Bois and Greene for continuous girders was covered by T. Boothby [13]. The trial closing-string method for continuous beams has been lucidly covered by E. Saliklis in a more pedagogical approach [14]. Despite sharing parts of the theoretical basis with FPM, these alternative methods are essentially different from FPM.

The heyday of graphic statics (later 19th Century to mid-20th Century) saw the proliferation of continuous bridges and subsequently of rigid frames. Graphical methods for continuous structures represented a vital yet unsuccessful endeavor to extend graphic statics to these structures. This was why many classic treatises of graphic statics dedicated chapters to the methods for continuous structures [3][15][16]. During the recent “renaissance” of graphic statics, there have been quite a few profound studies on historical graphical methods for articulated structures [17][18][19][20][21]. By contrast, so scarce are the historical reviews of the

methods for continuous structures [12] that graphic statics is frequently considered only convenient to statically determinate structures. Nonetheless, exploration emerges on the analysis of hyperstatic structures with graphic statics in recent years [22][23][24][25].

1.3 The structure of this paper

The following part comprises four sections: Section 2 presents a brief history of FPM. Section 3 briefly explains the techniques of this method, providing the basic knowledge for the next part. Section 4 explained Maillart's application of FPM on WO, along with the original drawing. Section 5 summarizes this paper and proposes directions for future work.

2 A brief history of fixed-points method

When graphic statics emerged and rapidly diffused in the second half of the 19th Century, continuous-girder bridges proliferated in continental Europe as a result of the railway boom and the increasing application of iron in construction. Railways bridges required higher bearing capacity. Iron, while meeting such need, called for structural efficiency due to its relatively high cost. Continuous bridges were deemed more efficient than the simply-supported bridge for they could lessen the maximum bending moment for given spans. So, the problem of its analysis becomes a theoretical focus among engineering theorists.

Carl Culmann discussed in-depth the analysis of continuous beams in his groundbreaking treatise on graphic statics. However, he did not manage to present a graphical solution to this structural type [3]. This task was accomplished by Otto Mohr who devised FPM for solving continuous beams based on Culmann's work [26]. Mohr's method was then refined and extended by Karl Wilhelm Ritter [27], who applied FPM to rigid frames of constant section members.

In the foreword of his treatise on continuous beams published at the turn of the century, Ritter claimed that engineers' preference for the continuous bridge had decreased as its downsides had also been examined more closely [16]. However, the impetus for the theoretical exploration of analysis methods of continuous structures was succeeded by the multiplication of rigid frames.

Ernst Suter, a chief engineer of the leading RC construction company Wayss & Freytag AG, profoundly pushed forward the application of FPM to RC frames. Suter gave special attention to RC frames with members of varying second area moment over the length, and his method is advantageous in addressing such cases. He perfected and generalized the FPM in such a way that it was comprehensive for the calculation of almost all statically indeterminate frames, including rigid frames with complex loading, arbitrarily oriented bars, multiple spans and stories, and arched bars. As an alternative to sometimes time-consuming drawing procedures, Suter also formulated the full set of algebraic expressions of FPM [9].

3 A brief technical explanation of fixed-points method

This section summarizes the rationale and core techniques of FPM, which include fixed points (*festpunkte*), crossing-lines (*kreuzlinien*), distribution factor (*verteilungsmaß*), and virtual lateral constraint. Demonstrations and technical details are contained in Mohr [26], Ritter [16], and especially Suter's writing [9]. Besides, English readers can refer to [28], and French readers [10]. It is important to note that in FPM, the bending moment diagram serves as a tool for analysis rather than merely a graph presenting the results.

FPM is a geometrical method for the elastic analysis of continuous beams and frames, which addresses the static indeterminacy through geometric constraint of fixed points on the bending moment diagram. It is based on the fact that on unloaded spans the contraflexure points of the bending moment diagram (the moment is induced from loaded spans) are fixed points, meaning that the coordinates of the zero-moment points of unloaded spans are independent of loading. This could be readily proved by the principle of superposition of linear elastic structures: when the inducing moment from the adjacent span increases by n times, the moment of contraflexure point shall remain zero. The coordinate of fixed points is an inherent value of a frame, which depends on the composition of bending stiffness EI (Young's modulus times the second moment of area), end constraints (connected to pinned or fixed supports or to other members), and member length. There are two fixed points on each member because there exists one contraflexure point for the moment propagating from either side. However, if one extreme end is on a pin support, the fixed point on this side coincides with this support.

The fixed points can be located with either the analytical or graphical approach, both of which are based on the fact that the contraflexure points (fixed points) share the abscissae with the inflection points on the tentative deflection curve induced by an arbitrary end moment. For the analytical approach, the abscissae of inflection points are deduced from the rotation angles induced by one unit of end bending moment. It is more convenient to derive the inflection point from end rotation rather than deflection, especially if the second moment area varies along the length. In the graphical approach, to simplify the drawing, the elastic curve is represented as envelopes tangent to the elastic curve at joints. The essences of the two approaches are the same as the slope of the envelopes equals end rotation.

The next step in FPM is to find the moment at loaded span(s). The bending moment diagram of FPM at a loaded span is composed of a funicular polygon and a closing string. The funicular polygon represents the statically-determinate bending moment due to the loading, as if the span in question was simply-support at ends. it is plotted using Culmann's force diagram [3]. However, since the ends are actually continuous, additional moment reaction will be induced by the continuity and propagated to adjacent spans. The reaction moment is represented by the closing string, which is located using crossing-lines. As the statically-determinate moment and reaction moment have opposite signs, the vertical distance between

the funicular and the closing string represents the difference of the two moments, which is the actual moment.

With the moment reaction at the ends of loaded spans found, one can plot the induced bending moment diagram in of all unloaded spans, by drawing straight lines connecting the ends of the loaded spans to the fixed points, linking from the closest fixed point to the farthest in sequence. Once the bending moment is determined, the shear forces and reactions can then be derived from it. For frames under loads of multiple spans, the total bending moment can be obtained by adding up the propagated moment of each load.

For frames with joint(s) connecting more than two members, it is necessary to determine moment distribution factors. The distribution factor represents the ratio of the moment induced at a joint to the moment inducing it at the same joint. This will be exemplified with the case of WO in 4.2.

For frames with unsymmetrical loading or stiffness, side sway will occur and induces additional stress. To take its effect into consideration, a virtual lateral constraint is added on an arbitrary upper joint beforehand to retain the frame in place. After determining the reaction at this constraint, the impact of the sway can be obtained by applying the counter-force of this “retaining force” as the sole load on the same frame.

4 Robert Maillart’s analysis of the Weissensteinstrasse Overpass with fixed-points method

4.1 Description of the structure

The reinforced concrete structure of WO consists of two parallel, identical continuous frames, with a traffic way built-in between them on transverse beams, and sidewalks cantilevered from the transverse beams. The bridge width between the railings is 10.8m, with 6.6m is for the road and 2.1 m*2 for the sidewalks. The roadway slab spans directly between the main girders in the end spans, while resting on transverse joists in the middle span. The slab atop this structure was replaced in 1996 while the original frame was maintained.

Each of the continuous frames is supported by rollers at two ends, *A* and *D*, and fixed at column bases, *E* and *F*, which in total provide eight freedom constraints and lead to five degrees of external static indeterminacy. The depth of the girder varies along the length and increased towards the continuous joints (Fig. 1). As demonstrated in Fig. 2, all the five members were numbered for more concise denotation.

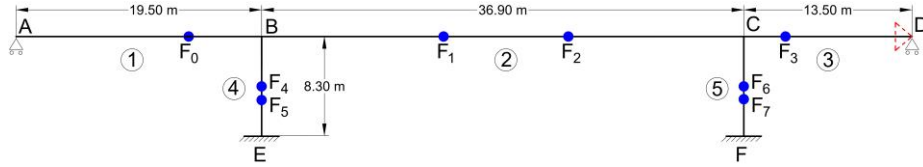


Fig. 2. the structure of WO with fixed points F_0 – F_7 marked with blue dots and the virtual additional constraint on roller D represented with red triangle (source: redrawn from [2]).

4.2 Determination of fixed points and distribution factors

Taking member AB (member 1) as an example. AB , was divided into five segments by four sections (a – d), as demonstrated in Fig. 3. The bending stiffness of each segment was determined using the stiffness of its left end. For instance, the segment between b and c used the stiffness of section b . The bending stiffness was calculated without considering reinforcing bars, as their effect on the second area moment is inconsequential at this stage of analysis [9].

With the bending stiffness determined, the rotation of the five segments induced by a virtual unite moment M at end A was then calculated. The location of fixed points F_0 and the moment distribution factor from member 2 to member 1 (μ_{2-1}) were thus determined. F_0 marked the contraflexure point under moment M , and it was 5.4m from B . The distribution factor μ_{2-1} was calculated to be 0.52, indicating that one unit of moment at the left end of member 2 (BC) would induce a moment of 0.52 unites on the right end of member 1 (AB) through the joint B . The remaining 0.48 unites would be balanced by column BE .

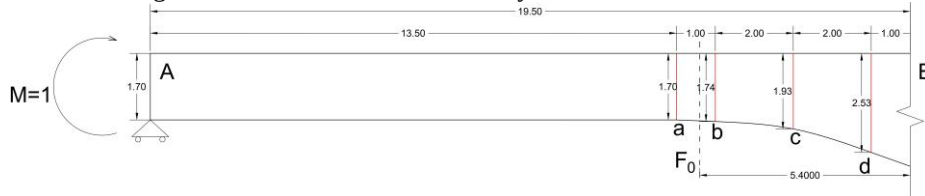


Fig. 3. the location of sections a – d (red lines) and fixed-points F_0 under moment M of member AB (source: redrawn from [2]).

4.3 Determination of bending moment of vertical load

The present analysis concerns the determination of the bending moment induced by the vertical load, which was decomposed into the contributions of the dead load and the traffic load. The traffic load is mainly composed of vehicle load on the road and the live load on the cantilevered sideways. The bending moments of dead and traffic load were evaluated separately using the same approach, and then added to obtain the overall moment. The analysis procedure presented below is limited to the determination of the bending moment caused by the dead load only. Moreover, since this structure is unsymmetrical, a virtual lateral constraint was

added to support D beforehand to prevent lateral displacement (see Fig. 2). The value of this “retaining force” was derived from the total moment in the end and found to be negligible.

The bending moment induced by the dead load of each span on all girders needs to be determined separately and then combined to obtain the total moment. Taking the dead load on span BC for example (Fig. 4), the diagram of its resulting bending moment on this span consists of the funicular polygon joining BC representing statically-determinate moment, a preliminary closing string $B''C''$ (red continuous line) representing reaction moment. The reaction moment would propagate into side spans, and induced moment (red continuous lines). For convenience of drawing, the magnitude of propagated moment on side spans was determined graphically with the moment distribution angles above joints B and C . The sine of the angles equals corresponding distribution factor. The propagated moment will equal the length of the side opposite the angle, taking the inducing moment as the length of the hypotenuse.

The funicular polygon for BC was plotted by means of force diagram 2. Due to the variation in the dead load with the girder height, the load was divided into discrete parts 9-21. Methods of the construction of the polygon can be found in [12].

The preliminary closing string $B''C''$ was determined with the fixed points F_1 , F_2 and the crossing-lines BC' , CB' . To draw these crossing-lines, point B' was first located using Suter's semi-graphical method [9], with the length of BB' represented the reaction moment at point B if span BC was cut free at point C . Similarly, the length of CC' represented the moment at point C if the span was severed at B . The crossing-lines BC' and CB' were then drawn by jointing B' to C and C' to B , intersecting the verticals passing through fixed points F_1 , F_2 at points F_1' and F_2' respectively. Through these intersects, preliminary closing string $B''C''$ was drawn, cutting BB' at B'' and CC' at C'' .

The rationale for crossing-lines is as follows: when the imaginary “disconnect” right end of span BC “reconnects” node C , the “connecting” moment at C (CC'') will not only propagate into span CD , but also to node B through fixed points F_2 . The propagated moment $B'B''$ is determined by joining C and C'' to the point F_2' under F_2 and extending to the vertical passing through B . Note that it makes no difference to the length of $B'B''$ whether the length CC'' is projected through F_2' or F_2 . The propagated moment $B'B''$ offset part of moment BB' , leaving the moment BB'' . Likewise, the moment CC'' can be perceived as moment CC' less moment $C'C''$, which is propagated from BB'' through points F_1' . To conform to this reciprocal relationship, BB'' and CC'' must equal the actual reaction bending moment. Further detailed proof can be found in the books listed in the first paragraph of Section 3.

Finally, the closing string for the total moment reaction was constructed by adding the ordinate of all the preliminary closing strings of each span. The moment at column top equalled the moment difference of the two girder-ends that connect to it. The oblique lines for the moment diagrams of columns were then constructed through the fixed points of columns. as shown in Fig. 4, the overall

bending moment was represented by the shaded area between the closing strings and the funicular lines.

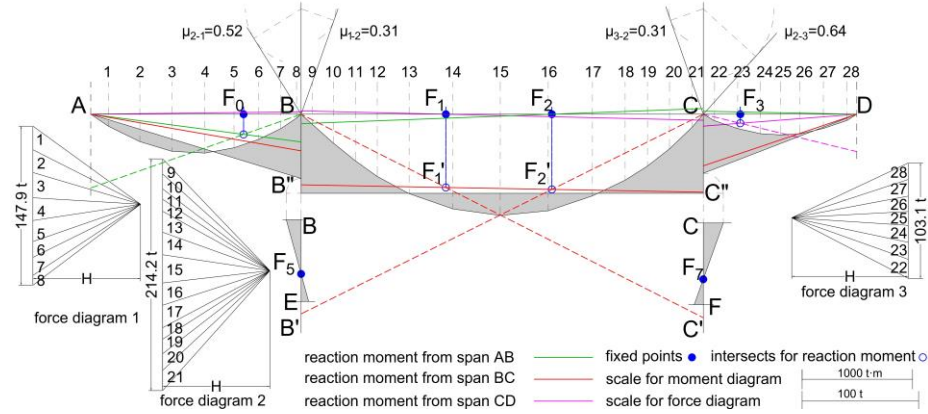


Fig. 4. The bending moment diagram of dead load (shaded) with angles above for moment distribution, and auxiliary force diagrams 1–3 below. The dashed crossing lines correspond to the preliminary closing strings with the same color (source: redrawn from [2]).

5 Conclusion and future developments

This paper described the bases of the fixed-points method, a geometric-based method for the elastic analysis of continuous beams and frames in the begin of the 20th century. Invented by Otto Mohr and mainly developed by Ritter and Suter, the method provided a way to analyze continuous bridges and rigid frames that became very common at that time. FPM addresses the static indeterminacy by means of constraints points on the bending moment diagram. These points characterize a structural system and are defined either by analytical or geometrical methods on the basis of the geometrical definition of frames and respective stiffness EI of their constitutive members.

Fixed point and distribution factors are defined on the basis of evaluating rotations under unite moment at ends of the bars. Then crossing-lines and virtual lateral load are used to define respective moments in the frame. Robert Maillart followed this method for analyzing of the Weissensteinstrasse Overpass, which evolving geometry reflects the balance of bending forces along the frame. However, some questions remain to be answered, as the actual interaction of the analysis and the design process of the geometry of the bridge.

The FPM bases of a set of indicators that characterize a frame and that can potentially inform the design of a frame under bending forces. Its potential for design purposes remains to be investigated, as does its possible use in the design of other historic structures.

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