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IFAC PapersOnLine 55-26 (2022) 39–46

Adaptive output tracking for nonlinear infinite-dimensional systems ⋆

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Abstract: An adaptive model-free funnel controller is presented for the output tracking of a general class of input-output (nonlinear) systems, which may encompass systems with possibly infinite-dimensional internal dynamics. After describing the system class and the related assumptions, the main result states that funnel control is well-adapted for a general class of semilinear infinite-dimensional systems with globally Lipschitz nonlinearity, by using a decomposition of the state space based on the existing Byrnes-Isidori form. Standard assumptions are stated, and in particular the Bounded Input State Bounded Output (BISBO) stability of the nonlinear infinite-dimensional system. A way of getting this assumption is presented too. The theoretical results are applied on a damped sine-Gordon equation and illustrated by means of numerical simulations.

Keywords: Funnel Control – Nonlinear infinite-dimensional systems – Byrnes-Isidori form – sine-Gordon equation

1. INTRODUCTION

A funnel controller is an adaptive model-free output error feedback with a time-varying gain that adapts according to the output error in order to ensure a predetermined transient behavior for the tracking error with no necessary asymptotic tracking performance. This field of control has attracted attention in the last few years. It has been deeply developed in Ilchmann et al. (2002) for systems with relative degree one. Few years later, funnel control for MIMO nonlinear systems with known strict relative degree has been widely developed in Berger et al. (2018). A newly survey for funnel control for different types of systems can be found in Berger et al. (2021a). Funnel control has also been considered in several applications. For instance Ilchmann and Trenn (2004) developed a funnel controller to regulate the temperature in chemical reactors. Later, the position of a moving water tank has been controlled via a funnel controller in Berger et al. (2022). A linearized version of the Saint-Venant Exner infinite-dimensional dynamics has been used as internal dynamics. This shows that funnel control becomes more and more attractive for systems whose internal dynamics are infinite-dimensional. For instance, this topic has been considered in Berger et al. (2020) wherein it is proved that some class of infinite-dimensional linear systems fits the required assumptions for funnel control to be feasible. Moreover, they proved that linear infinite-dimensional systems that can be written in Byrnes-Isidori form, see Ilchmann et al. (2016) for more details about it, are encompassed in that new class. Unbounded control and observation operators have also been considered in Puche et al. (2019) where a class of distributed port-Hamiltonian systems is studied for funnel control. More recently, funnel control has also been applied to a nonlinear infinite-dimensional reaction-diffusion equation coupled with the nonlinear Fitzhugh-Nagumo model, which represent together defibrillation processes of the human heart, see Berger et al. (2021b). Note also that funnel control has been lately coupled to model-predictive-control (Funnel MPC) for nonlinear systems with relative degree one, see Berger et al. (2021c).

The main contribution in this paper consists in considering a class of nonlinear infinite-dimensional systems to which funnel control can be applied[1]. Based on the Byrnes-Isidori form for linear systems, we introduce a change of variables that aims at extracting the output dynamics of the system, which is assumed to be finite-dimensional. Based on this transformation, funnel control is shown to be feasible provided that the remaining part of the dynamics satisfies some BISBO stability assumption. Moreover, a way of getting this BISBO stability condition is presented for bounded nonlinearities. We illustrate our results on a nonlinear infinite-dimensional partial differential equation (PDE) modeling the dynamics of a damped sine-Gordon equation.

⋆ This research was conducted with the financial support of F.R.S-FNRS. Anthony Hastir is a FNRS Research Fellow under the grant FC 29535.

[1] This class is restricted to systems with relative degree one.

10.1016/j.ifacol.2022.10.374

The paper is organized as follows: the input-output representation of the system class for which funnel control is known to be feasible together with the control objective are presented in Section 2. In Section 3 a class of nonlinear controlled and observed infinite-dimensional systems with some appropriate assumptions is considered and is shown to fulfill the assumptions needed to apply funnel control. A way to get BISBO stability of the internal dynamics is presented. The application of the main results with some numerical simulations are given in Section 4 on a damped sine-Gordon equation with Dirichlet boundary conditions. Section 5 is dedicated to conclusions and perspectives.

2. INPUT-OUTPUT DIFFERENTIAL REPRESENTATION AND CONTROL OBJECTIVE

In this section, the considered class of systems for which funnel control is conducive is introduced and the control objective is presented. The required assumptions that enable the feasability of the approach are highlighted.

The scalar differential equation making the connection between the input and the output of a dynamical system whose internal dynamics are not necessarily known (model-free) are assumed to be given as

$$\begin{cases} \dot{y}(t) = N(d(t), T(y)(t)) + \Gamma(d(t), T(y)(t))u(t), \\ y(0) = y_0, \end{cases} \tag{1}$$

where the following conditions are assumed to hold.

Assumption 1. The disturbance $d \in L^\infty(\mathbb{R}^+, \mathbb{R})$, the nonlinear function N is in $C(\mathbb{R}^2, \mathbb{R})$ and the gain function $\Gamma \in C(\mathbb{R}^2, \mathbb{R})$ is such that $\Gamma(d, \varrho) > 0$ for all $(d, \varrho) \in \mathbb{R}^2$.

Assumption 2. The map $T : C(\mathbb{R}^+, \mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}^+, \mathbb{R})$ is a (possibly nonlinear) operator which satisfies the following conditions:

(1) Bounded trajectories are mapped into bounded trajectories (BIBO property), i.e. for all $k_1 > 0$, there exists $k_2 > 0$ such that for all $y \in C(\mathbb{R}^+, \mathbb{R})$,
$$\sup_{t\in\mathbb{R}^+} |y(t)| \leq k_1 \Rightarrow \sup_{t\in\mathbb{R}^+} |T(y)(t)| \leq k_2. \tag{2}$$
(2) The operator T is causal, i.e. for any $t \in \mathbb{R}^+$ and any $y, \hat{y} \in C(\mathbb{R}^+, \mathbb{R})$
$$y_{|[0,t)} = \hat{y}_{|[0,t)} \Rightarrow T(y)_{|[0,t)} = T(\hat{y})_{|[0,t)}, \tag{3}$$
where $f_{|I}$ denotes the restriction of a function f to the interval I.
(3) T is locally Lipschitz in the sense that for all $t \in \mathbb{R}^+$ and all $y \in C([0, t], \mathbb{R})$ there exist positive constants τ, δ and ρ such that for any $y_1, y_2 \in C(\mathbb{R}^+, \mathbb{R})$ with ${y_i}_{|[0,t]} = y, i = 1, 2$ and $|y_i(s) - y(t)| < \delta$ for all $s \in [t, t + \tau]$ and $i = 1, 2$ it holds that
$$\|(T(y_1) - T(y_2))_{|[t,t+\tau]}\|_\infty \leq \rho\|(y_1 - y_2)_{|[t,t+\tau]}\|_\infty, \tag{4}$$
where $\|f_{|[t,t+\tau]}\|_\infty := \sup_{s\in[t,t+\tau]} |f(s)|$.

The class of systems governed by (1) with Assumptions 1–2 is presented in (Berger et al., 2020, Section 1) for systems with (possible) memory and relative degree $r \in \mathbb{N}$. Here we consider systems with no memory and relative degree one, see (1). This class is quite general and encompasses systems with infinite-dimensional internal dynamics as shown in Berger et al. (2020) and Ilchmann et al. (2016) for instance. However it is still not clear which classes of distributed-parameter systems (DPS) may be written under the form of the input-output equation (1) as it is explicitely mentioned in Berger et al. (2020). The possible infinite-dimensional nature of the internal dynamics allowed with the representation (1) is encompassed in the map T, that aims at capturing this feature.

The considered control objective for (1) consists in developing an output error feedback $u(t) = \mathcal{G}(t, e(t))$ with $e(t) = y(t) - y_{\text{ref}}(t)$ for a given reference signal $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$, which, when connected to (1), yields a closed-loop system for which the error $e(t)$ evolves in the prescribed performance funnel

$$\mathcal{F}_\phi := \left\{(t, e) \in \mathbb{R}^+ \times \mathbb{R}, \phi(t)|e(t)| < 1\right\}, \tag{5}$$

where the function ϕ is assumed to belong to the class

$$\Phi := \Big\{\phi \in C(\mathbb{R}^+, \mathbb{R}),\ \phi, \dot{\phi} \in L^\infty(\mathbb{R}^+, \mathbb{R}), \\ \phi(t) > 0, \forall t \in \mathbb{R}^+ \text{ and } \liminf_{t\to\infty} \phi(t) > 0\Big\}. \tag{6}$$

This control objective is also considered in (Berger et al., 2020, Section 1) for systems with arbitrary relative degree $r \in \mathbb{N}$. As described in Berger et al. (2020), Ilchmann et al. (2016) or Berger et al. (2018), a controller that achieves the output tracking performance described above is expressed as

$$u(t) = \frac{-e(t)}{1 - \phi^2(t)e^2(t)}, \tag{7}$$

where $\phi \in \Phi$ and $\phi(0)|e(0)| < 1$. The controller (7) is called a funnel controller and can be viewed as the output error feedback $u(t) = -k(t)e(t)$ with a time-varying (adaptive) gain $k(t) = \frac{1}{1-\phi^2(t)e^2(t)}$. The following theorem, coming from Berger et al. (2018) with $r = 1$, characterizes the effectiveness of the controller (7) in terms of existence and uniqueness of solutions of the closed-loop system and also in terms of output tracking performance.

Theorem 3. Consider a system (1) with Assumptions 1–2. Let $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R}), \phi \in \Phi$ and $y_0 \in \mathbb{R}$ such that the condition $\phi(0)|e(0)| < 1$ holds true. Then the funnel controller (7) applied to (1) results in a closed-loop system whose solution $y : [0, \omega) \to \mathbb{R}, \omega \in (0, \infty]$, has the following properties:

(1) The solution is global, i.e. $\omega = \infty$;
(2) The input $u : \mathbb{R}^+ \to \mathbb{R}$, the gain function $k : \mathbb{R}^+ \to \mathbb{R}$ and the output $y : \mathbb{R}^+ \to \mathbb{R}$ are bounded;
(3) The tracking error $e : \mathbb{R}^+ \to \mathbb{R}$ evolves in the funnel $\mathcal{F}_\phi$ and is bounded away from the funnel boundaries in the sense that there exists $\epsilon > 0$ such that, for all $t \geq 0, |e(t)| \leq \frac{1}{\phi(t)} - \epsilon$.

Note that no asymptotic tracking is achieved with the controller (7) but the transient behavior of the error is managed thanks to the funnel boundaries, represented by the function $\frac{1}{\phi(t)}$, which captures $e(t)$ at any positive time. Moreover, provided that $\phi(t)$ belongs to Φ, one could imagine choosing such a function with the property $\lim_{t\to\infty} \frac{1}{\phi(t)} < \delta$ for some very small δ, ensuring that the error remains in the interval $(-\delta, \delta)$ for large times. Theorem 3 will be an important tool in the next section, in which output tracking control for a class of nonlinear distributed parameter systems is considered.

3. SYSTEMS DRIVEN BY NONLINEAR INFINITE-DIMENSIONAL INTERNAL DYNAMICS

This section contains the main contribution of the paper: it is shown that the differential link between the inputs and the outputs of a class of nonlinear infinite-dimensional systems admits the representation (1) with Assumptions 1-2 being satisfied. This consitutes also the difference with respect to e.g. Berger et al. (2020) wherein a class of operators T is introduced, which comes from systems modeled by linear infinite-dimensional internal dynamics. Our contribution enlarges the class of systems as it is specified in Hastir et al. (2021). However, examples in which nonlinear infinite-dimensional systems are considered have already been studied for funnel control in Berger et al. (2021b). Our work also extends the Byrnes Isidori form studied in Ilchmann et al. (2016) to nonlinear infinite-dimensional systems that satisfy some relatively standard assumptions.

3.1 Systems class

Let H be a real (separable) Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle_H$. The nonlinear systems that we consider are governed by the following controlled and perturbed abstract ordinary differential equation

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + b(u(t) + d(t)), \\ y(t) = \langle x(t), c\rangle_H, \\ x(0) = x_0 \in H, \end{cases} \tag{8}$$

for which we make the following three assumptions.

Assumption 4. The (unbounded) linear operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous (C_0) semigroup of bounded linear operators on H.

Assumption 5. The nonlinear operator $f : H \to H$ is uniformly Lipschitz continuous on H.

Assumption 6. The vectors b and c are in $D(A)$ and $D(A^*)$, respectively, where $D(A^*)$ is the domain of the adjoint of the operator A. Moreover, it is assumed that $\langle b, c\rangle_H > 0$. The disturbance [2] $d \in L^\infty(\mathbb{R}^+, \mathbb{R})$.

The scalar functions u and y stand for the input and the output, respectively. According to (Curtain and Zwart, 2020, Theorem 11.1.5), Assumptions 4 and 5 imply that the homogenenous [3] part of (8) has a unique mild solution on $[0, \infty)$ which can even be a classical one provided that $x_0 \in D(A)$. Note that, despite that Assumption 6 may be seen as restrictive, if the vectors b and c do not lie in $D(A)$ and $D(A^*)$, it is always possible to approximate them, as accurately as desired, by vectors of $D(A)$ and $D(A^*)$ since they are both dense subspaces of H. However, such an approximation may cause undesirable effects on the state and the output trajectories of (8). The analysis of the error made by an approximation is listed as perspective in Section 5.

3.2 Byrnes-Isidori decomposition and feasability of funnel control

It is shown in this subsection how the abstract differential equation (8) can be transformed into (1) by using a change of variables as it is made for linear systems in Ilchmann et al. (2016).

Let us therefore consider the following proposition that enables to decompose the state space H into the direct sum of two linear subspaces, see e.g. Ilchmann et al. (2016) and Byrnes et al. (1998).

Proposition 7. Let Assumption 6 hold. Then the state space H may be decomposed as

$$H = \operatorname{span}\{c\} \oplus \{b\}^\perp =: \mathcal{O} \oplus \mathcal{I}, \tag{9}$$

where $\{b\}^\perp := \{f \in H, \langle f, b\rangle_H = 0\}$.

Proof. See Hastir et al. (2021). □

Now let us introduce the operators cP and $I - cP =: P_{\mathcal{I}}$ where the operator $P : H \to \mathbb{R}$ is defined as $Pf = \frac{\langle f,b\rangle}{\langle c,b\rangle}$, for any $f \in H$. The following proposition characterizes the subspaces $\mathcal{O}$ and $\mathcal{I}$, in terms of these operators.

Proposition 8. The subspaces $\mathcal{O}$ and $\mathcal{I}$ satisfy $\mathcal{O} = cPH$ and $\mathcal{I} = P_{\mathcal{I}}H$, respectively.

Proof. See Hastir et al. (2021). □

According to Propositions 7 and 8 any element $f \in H$ can be decomposed uniquely in $\mathcal{O} \oplus \mathcal{I}$ as $f = cPf + P_{\mathcal{I}}f$. Moreover, observe that for any function $f \in H$ there holds $\langle P_{\mathcal{I}}f, b\rangle_H = 0$. Note that the operator $P_{\mathcal{I}}$ is not an orthogonal projector on $\mathcal{I}$. Such a projector will be denoted by $P^\perp : H \to \mathcal{I}$ and defined as $P^\perp f = f - b\frac{\langle f,b\rangle_H}{\langle b,b\rangle_H}$ for any $f \in H$. Now we shall introduce operators that will be of importance in order to transform system (8). Let us consider the operator $U : H \to \mathbb{R} \times \mathcal{I}$ defined as

$$Uf = \begin{pmatrix} Pf \\ P_{\mathcal{I}}f \end{pmatrix}. \tag{10}$$

This operator is boundedly invertible, see Ilchmann et al. (2016), with inverse $U^{-1} : \mathbb{R} \times \mathcal{I} \to H$ expressed as

$$U^{-1}\begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \alpha c + \eta. \tag{11}$$

The adjoint of U^{-1}, denoted by $U^{-*} : H \to \mathbb{R} \times \mathcal{I}$, reads as follows

$$U^{-*}f = \begin{pmatrix} \langle f, c\rangle_H \\ P^\perp f \end{pmatrix}. \tag{12}$$

Consequently, the operator $U^* : \mathbb{R} \times \mathcal{I} \to H$ is given by

$$U^*\begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \alpha\frac{b}{\langle c, b\rangle_H} + \eta - \langle c, \eta\rangle_H\frac{b}{\langle c, b\rangle_H}. \tag{13}$$

Like the operators U and U^{-1}, the operators U^* and U^{-*} are linear and bounded. This allows us to perform the change of variables for system (8) defined by $x = U^*\xi, \xi \in \mathbb{R} \times \mathcal{I}$. By the invertibility of U^*, one gets

$$\xi = U^{-*}x = \begin{pmatrix} \langle x, c\rangle_H \\ P^\perp x \end{pmatrix}. \tag{14}$$

By setting $\eta := P^\perp x$, one has that $\xi = \left(\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right)$. Now we take a look at the dynamics of the variable ξ. In the new variables (14), the system (8) reads as

$$\tilde{\Sigma} : \begin{cases} \dot{\xi}(t) = U^{-*}AU^*\xi(t) + U^{-*}f(U^*\xi(t)) \\ \qquad\qquad + U^{-*}bu(t) + U^{-*}bd(t), \\ y(t) = (1\ \ 0)\xi(t), \\ \xi(0) = U^{-*}x_0 =: \xi_0. \end{cases} \tag{15}$$

Note that the systems Σ and $\tilde{\Sigma}$ are equivalent according to the state transformation induced by U^{-*}. By using the

[2] By entering in the system (8) in this way, the disturbance d may be interpreted as modeling uncertainties on the input for instance.

[3] By "homogenenous" we mean uncontrolled and unperturbed.

definitions of U^{-*} and U^*, see (12) and (13), the linear operator $U^{-*}AU^*$ can be rewritten as

$$U^{-*}AU^*\xi = \begin{pmatrix} \langle AU^*\xi, c\rangle_H \\ P^\perp AU^*\xi \end{pmatrix} =: \begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}\xi,$$

where the operators P_0, S, R and Q are defined as follows

$$P_0 : \mathbb{R} \to \mathbb{R}, P_0 y = y\frac{\langle A^*c, b\rangle_H}{\langle c, b\rangle_H},$$
$$S : \mathcal{I} \to \mathbb{R}, S\eta = \langle \eta, P_{\mathcal{I}}A^*c\rangle_H,$$
$$R : \mathbb{R} \to \mathcal{I}, Ry = y\frac{P^\perp Ab}{\langle c, b\rangle_H},$$
$$Q : D(Q) \subset \mathcal{I} \to \mathcal{I}, Q\eta = P^\perp A\eta - \frac{\langle c, \eta\rangle_H}{\langle c, b\rangle_H}P^\perp Ab, \quad (16)$$

where $D(Q) = D(A) \cap \mathcal{I}$. According to Ilchmann et al. (2016), the operator Q is the infinitesimal generator of a C_0-semigroup[4] $(T_Q(t))_{t\geq 0}$ on $\mathcal{I}$. Moreover, since the operators P_0, S and R are bounded, the operator $\left(\begin{smallmatrix} P_0 & S \\ R & Q \end{smallmatrix}\right)$ is still the generator of a C_0-semigroup, see e.g. Curtain and Zwart (2020); Pazy (1983); Engel and Nagel (2006). By using (12) and (13), we can write the nonlinear part of (15) as

$$U^{-*}f(U^*\xi(t)) =: \begin{pmatrix} \langle \tilde{f}(y(t), \eta(t)), c\rangle_H \\ \tilde{f}(y(t), \eta(t)) - \frac{\langle \tilde{f}(y(t), \eta(t)), b\rangle_H}{\langle b, b\rangle_H}b \end{pmatrix},$$

where the nonlinear operator $\tilde{f} : \mathbb{R} \times \mathcal{I} \to H$ is defined as $\tilde{f}(y, \eta) = f(y\frac{b}{\langle c,b\rangle_H} + \eta - \langle c, \eta\rangle_H \frac{b}{\langle c,b\rangle_H})$. By using Assumption 5 and the fact that U^{-*} and U^* are linear bounded operators, the nonlinear operator $U^{-*}f(U^*\cdot)$ is still uniformly Lipschitz continuous from $\mathbb{R} \times \mathcal{I}$ into $\mathbb{R} \times \mathcal{I}$. This entails that the homogeneous part of (15) possesses a unique mild solution on $[0, \infty)$. Taking any initial condition in $\mathbb{R} \times D(Q)$ implies that this solution is classical. Now observe that the term $U^{-*}b$ is expressed as $\left(\begin{smallmatrix} \langle b,c\rangle_H & 0 \end{smallmatrix}\right)^T$. From these observations it follows that the dynamics of y and η may be written as

$$\dot{y}(t) = P_0 y(t) + S\eta(t) + \langle \tilde{f}(y(t), \eta(t)), c\rangle_H + \gamma u(t) + \gamma d(t) \quad (17)$$

and

$$\dot{\eta}(t) = Ry(t) + Q\eta(t) + P^\perp \tilde{f}(y(t), \eta(t)), \quad (18)$$

with initial conditions $y(0) = y_0$ and $\eta(0) = \eta_0$, respectively, where $\gamma := \langle b, c\rangle_H$.

The main result of this paper is the following: by considering some quite easily checkable assumptions, it is proved that the transformed equations (17)–(18) satisfy Assumptions 1–2 introduced above, which implies that funnel control is feasible for such a class of systems according to Theorem 3.

Observe that (17) admits the representation (1), where

(1) the gain function $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ is defined as $\Gamma(d, \varrho) = \gamma > 0$;
(2) the well-defined nonlinear operator $T : C(\mathbb{R}^+, \mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R}^+, \mathbb{R})$ has the form
$$T(y)(t) = P_0 y(t) + S\eta(t) + \langle \tilde{f}(y(t), \eta(t)), c\rangle_H, \quad (19)$$
where $\eta(t)$ is the mild solution of (18);
(3) the function $N : \mathbb{R}^2 \to \mathbb{R}$ reads as $N(d, \varrho) = \gamma d + \varrho$.

[4] Without loss of generality there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_Q(t)\| \leq Me^{\omega t}$.

This shows that Assumption 1 on system (1) is satisfied. It remains to show that the nonlinear operator T given by (19) possesses the three properties of Assumption 2. This constitutes the main result of this section. Before going into the details of this result, we shall denote by $\Sigma_{y\to\eta}$ the system which can be viewed as a system with input y and output η and whose dynamics are described by (18). We make the following assumption on that system.

Assumption 9. The system $\Sigma_{y\to\eta}$ whose dynamics are governed by (18) is BISBO stable in the following sense: for all $k > 0$ and all $\hat{k} > 0$, there exists $\tilde{k} > 0$ such that for all $y \in C(\mathbb{R}^+, \mathbb{R})$ and all $\eta_0 \in \mathcal{I}$,

$$\sup_{t\in\mathbb{R}^+} |y(t)| \leq k \text{ and } \|\eta_0\| \leq \hat{k} \Rightarrow \sup_{t\in\mathbb{R}^+} \|\eta(t)\| \leq \tilde{k}. \quad (20)$$

Theorem 10. The operator T defined in (19), which arises from the nonlinear system (8) via the change of variables (14), satisfies Assumption 2.

We shall give a short proof here. The complete proof may be found in (Hastir et al., 2021, Theorem 3.1).

Proof. It is divided into three steps, according to the three items of Assumption 2, respectively.
Step 1: Let us fix $k_1 > 0, k_* > 0, y \in C(\mathbb{R}^+, \mathbb{R})$ and $\eta_0 \in \mathcal{I}$ such that $\sup_{t\in\mathbb{R}^+} |y(t)| \leq k_1$ and $\|\eta_0\| \leq k_*$. There exists a positive $\tilde{k}$ such that, for this y, the mild solution of (18) with initial condition $\eta_0 \in \mathcal{I}$ satisfies $\sup_{t\in\mathbb{R}^+} \|\eta(t)\| \leq \tilde{k}$, according to Assumption 9. Thanks to the expression (19) of T, the boundedness of the operator S, the Cauchy-Schwarz inequality and Assumption 5, one may deduce the estimate

$$\sup_{t\in\mathbb{R}^+} |T(y)(t)| \leq |P_0|k_1 + \|S\|_{\mathcal{L}(\mathcal{I},\mathbb{R})}\tilde{k} + \tilde{\sigma}\|c\|_H =: k_2,$$

with $\tilde{\sigma} = \sigma + l_1 k_1 + l_2 \tilde{k}$, $l_1 > 0$ and $l_2 > 0$ denoting the Lipschitz constants of the operator $\tilde{f}$ associated with y and η, respectively, and $\sigma \geq 0$ being such that[5] $\|f(0)\|_H \leq \sigma$. Hence the BIBO condition required in Assumption 2 is satisfied.
Step 2: The causality can be easily established by noting that, for a fixed $y \in C(\mathbb{R}^+, \mathbb{R})$ the corresponding mild solution of (18) is unique. This entails that for $y, \hat{y} \in C(\mathbb{R}^+, \mathbb{R})$ such that $y_{|[0,t)} = \hat{y}_{|[0,t)}$, the corresponding mild solutions of (18), denoted by η and $\hat{\eta}$, respectively, satisfy $\eta_{|[0,t)} = \hat{\eta}_{|[0,t)}$. In view of the expression (19) of T, it follows that (3) holds.
Step 3: For the local Lipschitz continuity, let us consider $t \geq 0$ and $y \in C([0, t], \mathbb{R})$. Now let us take $y_1, y_2 \in C(\mathbb{R}^+, \mathbb{R})$ such that y_i coincides with y up to time t for $i = 1, 2$. The mild solutions of (18) with input y_i and starting at time t are given by $\eta_i(\tilde{t}) =$

$$T_Q(\tilde{t} - t)\eta_{i,t} + \int_t^{\tilde{t}} T_Q(\tilde{t} - s)[Ry_i(s) + P^\perp \tilde{f}(y_i(s), \eta_i(s))]ds$$

for any $\tilde{t} \in [t, t + \tau]$ with τ being an arbitrary positive constant. Note that the functions $\eta_{1,t}$ and $\eta_{2,t}$ correspond to $\eta_1(t)$ and $\eta_2(t)$, respectively. Since, by assumption, $y_1(t) = y(t) = y_2(t)$ and $\eta_1(0) = \eta_0 = \eta_2(0)$ and by

[5] This is valid since the nonlinear operator f maps the whole space H into itself, meaning that any point in H has a finite image by f in the $H-$norm.

the uniqueness of the mild solution of (18), the relation $\eta_{1,t} = \eta_{2,t}$ holds true. Consequently,

$$\|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| \leq \mathfrak{g} \int_t^{\tilde{t}} M e^{|\omega|(\tilde{t}-s)} |y_1(s) - y_2(s)| ds + 2l_2 \int_t^{\tilde{t}} M e^{|\omega|(\tilde{t}-s)} \|\eta_1(s) - \eta_2(s)\| ds,$$

where l_1 and l_2 are the positive constants introduced above in the proof and where the boundedness of the operator R together with Assumption 5 have been used. The notation $\|R\|_{\mathcal{L}(\mathbb{R},\mathcal{I})} + 2l_1 =: \mathfrak{g}$ has been adopted. Applying Gronwall's lemma to the function $e^{-|\omega|\tilde{t}}\|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\|$ and taking the supremum over all $\tilde{t}$ in $[t, t+\tau]$ on both sides yields the estimate

$$\sup_{\tilde{t}\in[t,t+\tau]} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| \leq \mathfrak{f}_\tau \sup_{\tilde{t}\in[t,t+\tau]} |y_1(\tilde{t}) - y_2(\tilde{t})|, \quad (21)$$

where $\mathfrak{f}_\tau := \mathfrak{g} M e^{(|\omega|+2Ml_2)\tau}\tau$. According to the definition (19) of T in combination with (21), there holds

$$\sup_{\tilde{t}\in[t,t+\tau]} |T(y_1)(\tilde{t}) - T(y_2)(\tilde{t})| \leq \rho \sup_{\tilde{t}\in[t,t+\tau]} |y_1(\tilde{t}) - y_2(\tilde{t})|,$$

where $\rho := |P_0| + \|S\|_{\mathcal{L}(\mathcal{I},\mathbb{R})}\mathfrak{f}_\tau + l_1\|c\|_H + l_2\|c\|_H\mathfrak{f}_\tau$. □

This means that funnel control is feasible for a nonlinear infinite-dimensional system of the form (8) which satisfies Assumptions 4, 5, 6 and 9. Moreover the closed-loop system which consists of the interconnection of (8), described by (17)–(18) (system $\tilde{\Sigma}$), with the funnel controller (7) has the properties described in Theorem 3. This system is depicted in Figure 1.

We state hereafter a useful criterion for checking Assumption 9.

Proposition 11. Assuming that the semigroup $(T_Q(t))_{t\geq 0}$ is exponentially stable on $\mathcal{I}$ and that the nonlinear operator f satisfies $\|f(x)\|_H \leq \hat{\sigma}$, for some constant $\hat{\sigma} > 0$ independent of x, for any $x \in H$, is sufficient to ensure that Assumption 9 is satisfied.

Proof. We refer to the proof of (Hastir et al., 2021, Proposition 3.1). □

4. EXAMPLE: A DAMPED SINE-GORDON EQUATION

Here we present an example of a damped nonlinear wave equation with Dirichlet boundary conditions, known as the sine-Gordon equation, for which a scalar output is asked to track some time-varying reference profile. The theory that is presented in the previous section is applied in order to show the feasability of funnel control for this nonlinear infinite-dimensional system. Numerical simulations are presented in order to reinforce the theoretical results.

The dynamics that are of interest here are expressed as

$$\begin{cases} \partial_{tt}x = \partial_{zz}x - \alpha\partial_t x + \nu\sin(x) + b(z)u(t) \\ x(0,t) = 0, x(1,t) = 0, \end{cases} \quad (22)$$

where the space variable $z \in [0,1]$ and $t \in \mathbb{R}^+$ denotes the time variable. The parameters ν and α are such that $\nu \in \mathbb{R}_0$ and $\alpha > 2\pi$. Equation (22) encompasses many phenomena in physics as the dynamics of a Josephson junction driven by a current source, see e.g. Temam (1997); Cuevas-Maraver et al. (2014), as well as the dynamics of mechanical transmission lines, see Cirillo et al. (1981) among others. The stability of the homogeneous dynamics of (22) has been investigated in Dickey (1976) and Callegari and Reiss (1973). For control problems related to (22), we refer to Dolgopolik et al. (2016) and Efimov et al. (2019) for instance, where boundary energy control and robust input-to-state stability are developed.

Clearly that model admits the abstract representation $\dot{\zeta}(t) = A\zeta(t) + F(\zeta(t)) + Bu(t), \zeta(0) = \zeta_0$, where the state vector $\zeta = \begin{pmatrix} x \\ \partial_t x \end{pmatrix}$ is considered on the state space $Z = D(A_0^{\frac{1}{2}}) \times X, X = L^2([0,1];\mathbb{R})$, while the operator A_0 is defined as $A_0 = -\frac{d^2}{d\zeta^2}$ on the domain

$$D(A_0) = \left\{x \in H^2([0,1];\mathbb{R}), x(0) = 0 = x(1)\right\}.$$

It can be seen that the operator A_0 is self-adjoint and coercive. In particular, thanks to Poincaré's inequality there holds $\langle A_0 x, x\rangle_X = \int_0^1 \left(\frac{dx}{dz}\right)^2 dz \geq \pi^2\|x\|_X^2$, which means that A_0 is coercive. Hence defining the square root of A_0 makes sense in this context, see (Curtain and Zwart, 2020, Lemma A.3.82). In particular, its domain is given by $D(A_0^{\frac{1}{2}}) = H_0^1([0,1];\mathbb{R})$. The operator $A : D(A) = D(A_0) \times D(A_0^{\frac{1}{2}}) \subset Z \to Z$ is given by $A = \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix}$. Thanks to (Curtain and Zwart, 2020, Example 2.3.5) the newly defined operator A is the infinitesimal generator of a C_0-semigroup on the Hilbert space Z. The nonlinear operator $F : Z \to Z$ is expressed as $F(\zeta_1, \zeta_2) = \begin{pmatrix} 0 \\ \nu\sin(\zeta_1) \end{pmatrix}$. The latter is uniformly Lipschitz continuous and satisfies $\|F(\zeta_1,\zeta_2)\|_Z \leq |\nu|$ for any $(\zeta_1\ \zeta_2)^T \in Z$, meaning that Assumptions 4 and 5 are satisfied here. As operator B, we consider the operator that sends the scalar input u into Z as $Bu = b(z)u = \begin{pmatrix} 0 \\ b_N(z) \end{pmatrix} u$ with b_N being expressed as $b_N(z) = \sum_{n=1, n \text{ odd}}^{N} \frac{4}{n\pi}\sin(n\pi z)$ for a fixed natural number N. Is is easy to see that this definition of b is such that[6] $b \in D(A)$. As function c we shall consider the following

$$c(z) = \begin{pmatrix} \frac{\alpha+\sqrt{\alpha^2-4\pi^2}}{2\pi^2} \\ 1 \end{pmatrix} \sin(\pi z),$$

which is such that the output trajectory is expressed as

$$y(t) = \langle c, \zeta(t)\rangle_Z = \frac{\alpha + \sqrt{\alpha^2 - 4\pi^2}}{2\pi} \int_0^1 \cos(\pi z)\partial_z x(z,t)dz + \int_0^1 \sin(\pi z)\partial_t x(z,t)dz. \quad (23)$$

This function c lies in $D(A^*)$ since both the first and the second components are in $H^2([0,1];\mathbb{R})$ and vanishes for $z = 0$ and $z = 1$. Moreover, it is such that the $Z-$inner product between b and c is given as $\langle b, c\rangle_Z = \frac{2}{\pi}$. This has the consequence that Assumption 6 is satisfied. Observe that the state space Z admits the decomposition $Z = \mathcal{O}\oplus\mathcal{I}$ where $\mathcal{I}$ is given by

$$\mathcal{I} = D(A_0^{\frac{1}{2}}) \times \{x \in X, \langle x, b_N\rangle_{L^2} = 0\}. \quad (24)$$

These facts imply that (22) may be written as (17)–(18). In order to show that funnel control is feasible for (22), it remains to prove that Assumption 9 is satisfied

[6] It can be seen that b_N constitutes an approximation of the indicator function $1_{[0,1]}(z)$ by means of the orthonormal basis $\{\phi_n(z)\}_{n\geq 1} := \{\sqrt{2}\sin(n\pi z))\}_{n\geq 1}$ of X. The series expansion of $1_{[0,1]}(z)$ is truncated up to order N as $\sum_{n=1}^{N}\langle 1_{[0,1]}, \phi_n\rangle\phi_n(z)$.

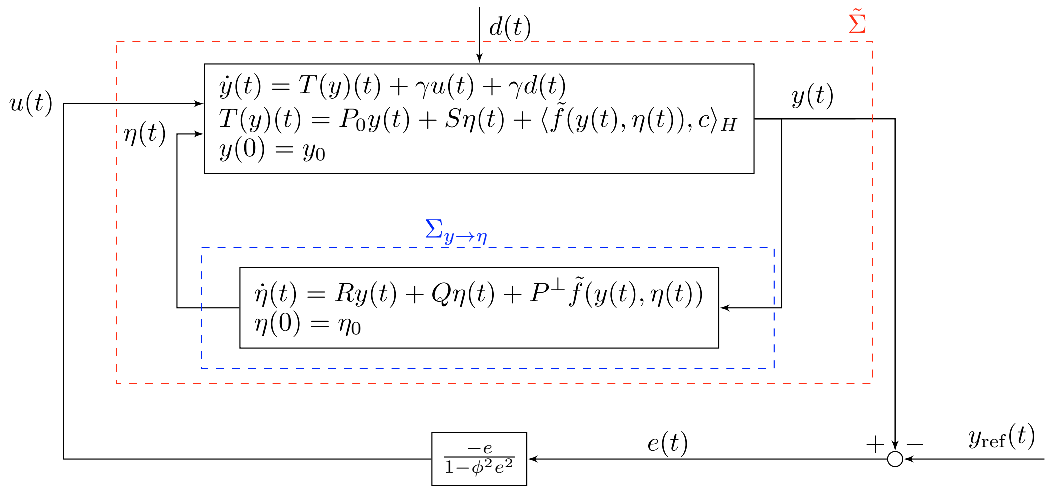

Fig. 1. Interconnection of $\tilde{\Sigma}$ and the funnel controller (7).

in the present example. According to the definition of the operator F, it holds that $\|F\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\|_Z \leq |\nu|$ for any $\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \in Z$. As a consequence, showing that the operator Q defined in (16) is the infinitesimal generator of an exponentially stable C_0-semigroup is a sufficient condition for BIBO stability to hold, see Proposition 11.

Therefore, let us expand the operator $\begin{pmatrix}P_0 & S\\ R & Q\end{pmatrix}$. Observe that the operator $P_0 : \mathbb{R} \to \mathbb{R}, P_0 y = p_0 y$ where $p_0 = \frac{\langle A^*c, b\rangle_Z}{\langle c, b\rangle_Z}$. By computing the explicit expression of p_0, one gets the following

$$p_0 = \frac{\langle A^*c, b\rangle_Z}{\langle c, b\rangle_Z} = -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4\pi^2}}{2},$$

which means that p_0 is the generator of the exponentially stable C_0-semigroup $(e^{p_0 t})_{t\geq 0}$. Going one step further, the computation of the operator $S : \mathcal{I} \to \mathbb{R}$ reveals that for any $\eta \in \mathcal{I}$ there holds

$$\begin{aligned} S\eta = \langle \eta, P_{\mathcal{I}} A^* c\rangle_Z &= \left\langle \eta, \begin{pmatrix} -1 \\ -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4\pi^2}}{2}\end{pmatrix} \sin(\pi z) - p_0 c \right\rangle_Z \\ &= \left\langle \eta, \sin(\pi z) \begin{pmatrix} -1 + \frac{1}{4\pi^2}(\alpha^2 - \alpha^2 + 4\pi^2) \\ 0 \end{pmatrix} \right\rangle_Z = 0, \end{aligned}$$

which means that $\begin{pmatrix}P_0 & S\\ R & Q\end{pmatrix} = \begin{pmatrix}p_0 & 0\\ R & Q\end{pmatrix}$, which is a triangular operator. As it is similar to the operator A, the corresponding semigroups are also similar, i.e. denoting by $(S(t))_{t\geq 0}$ and by $(\tilde{S}(t))_{t\geq 0}$ the C_0-semigroups generated by A and $\begin{pmatrix}P_0 & S\\ R & Q\end{pmatrix}$, respectively, the relation $\tilde{S}(t) = U^{-*}S(t)U^*$ holds for all $t \geq 0$. Consequently, $(S(t))_{t\geq 0}$ and $(\tilde{S}(t))_{t\geq 0}$ have the same growth bounds. In that way, let us have a look at the sign of the growth bound of the semigroup $(S(t))_{t\geq 0}$. By (Hastir, 2022, Proposition 2.1.7), the C_0-semigroup $(S(t))_{t\geq 0}$ is exponentially stable, meaning that its growth bound is negative. As the growth bound of the semigroup generated by P_0 is negative too, the growth bound of the semigroup generated by Q is also negative, showing that this semigroup is exponentially stable, which implies that funnel control is feasible for (22) according to Proposition 11.

Let us now illustrate the feasability of funnel control on (22) with numerical simulations. We consider the following set of parameters: $\alpha = 2\pi + \frac{1}{16}, \nu = -1$. Note that this choice of the damping parameter α entails that the condition $\alpha^2 - 4\pi^2 > 0$ is satisfied, which guarantees that the output is real. The initial conditions for the variables x and $\partial_t x$ have been chosen as $x(z,0) = 2z^3 - 3z^2 + z$ and $\partial_t x(z,0) = z^2 - z^4$, respectively, while the reference signal $y_{\text{ref}}(t) = \sin(4\pi t)e^{-2t} + 0.2$ which lies in $W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. The function $\phi(t)$, whose inverse determines the funnel boundaries, is fixed to $\phi(t) = \frac{4}{e^{-2t} + 0.005}$. Thanks to this definition of the function ϕ, it is easy to see that ϕ belongs to the class Φ introduced in (6). Furthermore, it can be shown easily that the vector of initial conditions $(x(z,0)\ \partial_t x(z,0))^T$ lies in $D(A) = D(A_0) \times D(A_0^{\frac{1}{2}})$. Moreover, the initial conditions together with the definition of the reference signal and the funnel boundaries are such that $\phi(0)|e(0)| = 0.4812 < 1$. Note also that the number of basis functions in the approximation of the indicator function, N, has been fixed to 50.

The output trajectory (23) with the reference signal $y_{\text{ref}}(t)$ are represented in Figure 2 whereas the corresponding funnel control input is given in Figure 3. The tracking error is depicted in Figure 4 wherein one observes that it remains within the funnel boundaries, as expected. The state trajectories corresponding to $x(z,t)$ and $\partial_t x(z,t)$ are shown in Figures 5 and 6, respectively.

5. CONCLUSION AND PERSPECTIVES

Adaptive output tracking for a class of nonlinear semilinear infinite-dimensional systems has been presented in this paper by means of funnel control. This controller is model-free and can be viewed as an error output feedback with a time-varying gain which adapts to the tracking error. The input-output relation on which is based the construction of the funnel controller together with the assumptions that ensure its feasability have been introduced in Section 2. The class of nonlinear infinite-dimensional systems of interest has been given in Section 3. There, a way of extracting the differential input-output relation has been presented and it has been proven that it fits the framework for which funnel control is conducive, under quite standard assumptions related to semigroup theory notably. An ex-

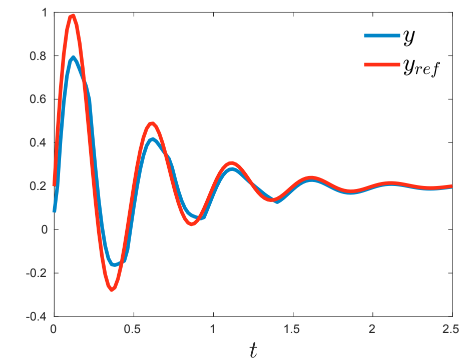

Fig. 2. Output trajectory (23) with the reference signal $y_{\text{ref}}(t)$.

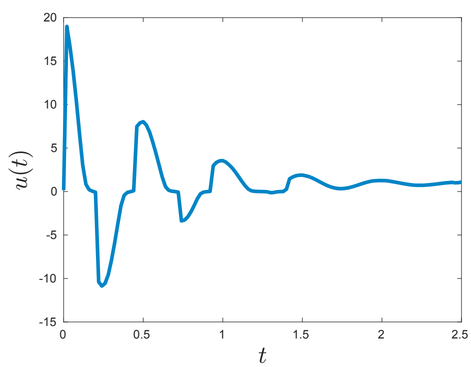

Fig. 3. Input trajectory.

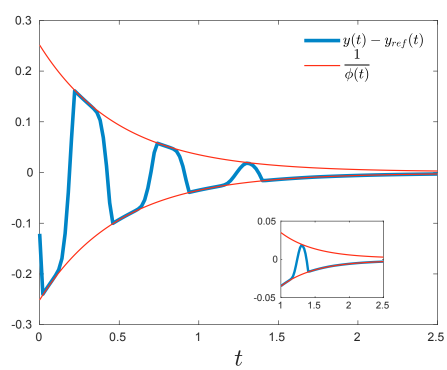

Fig. 4. Output error trajectory $y(t)-y_{\text{ref}}(t)$ with the funnel boundaries $\frac{1}{\phi(t)}$ and $-\frac{1}{\phi(t)}$.

ample involving a damped sine-Gordon equation has been studied in Section 4. The machinery of Section 3 has been developed in order to show the feasability of funnel control for that kind of wave system. Numerical simulations have also been presented.

As further research, we could imagine tackle the question of funnel control for nonlinear infinite-dimensional systems with relative degrees higher than one. The relaxation of the global Lipschitz continuity assumption asked for the nonlinearity could also be considered as a perspective. Other aspects like the feasability of funnel control for nonlin-

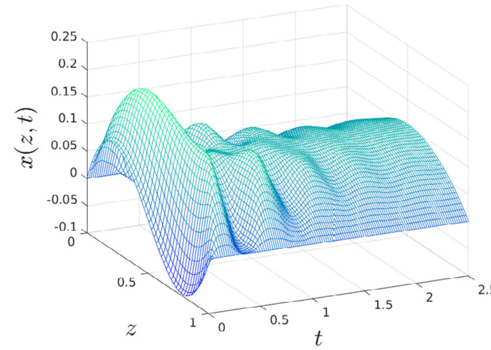

Fig. 5. Closed-loop state trajectory $x(t, z)$.

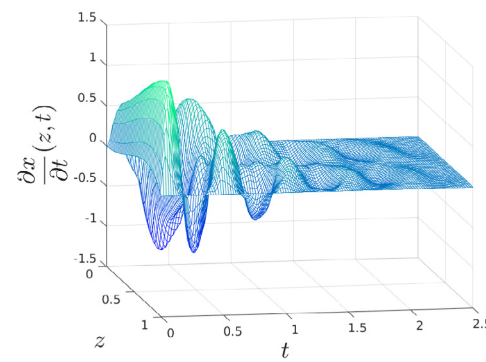

Fig. 6. Closed-loop state trajectory $\partial_t x(t, z)$.

ear infinite-dimensional systems with more general input and output operators including for instance unbounded control and observation operators could be studied too. This study should include the formalisation of the concept of nonlinear boundary controlled and observed systems. What is meant by well-posedness in this context should also be properly defined. Concerning this question, we refer e.g. to the preliminary works by Tucsnak and Weiss (2014), Hastir et al. (2019) and Schwenninger (2020). The error made by the approximation of the functions b and c by functions in $D(A)$ and $D(A^*)$, respectively, is also a possible question for further research.

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