

Establishing the consistency of the asymptotic formula of Deleersnijder (2002) with the exact one-dimensional dam break flow solution of Ritter (1892)

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Motivation

The two-dimensional, non-linear, shallow water equations are considered. No dissipative processes are taken into account. The lower boundary of the water column is assumed to be a horizontal plane and no forcing is applied at the upper boundary. For this configuration, I carried out in 2002 an asymptotic analysis of the behaviour of the velocity and water height near the free boundary separating the wet region (i.e. the region where water is present) and the dry one (i.e. the region where there is no water). These results may be found in Deleersnijder (2018).

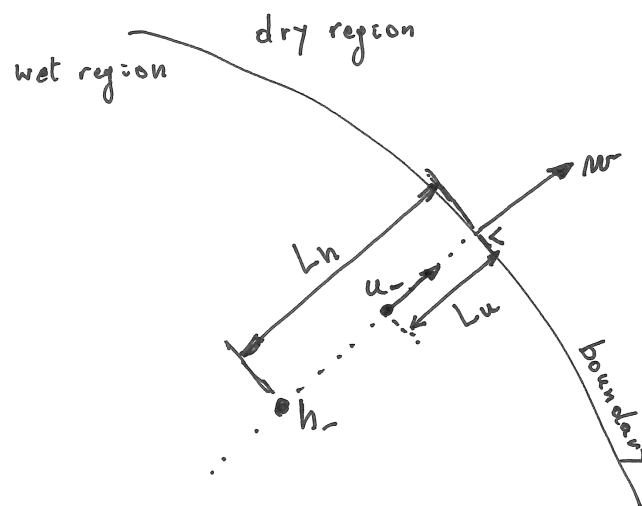
The main result of the abovementioned study is as follows:

$$w \sim u_- + \frac{2L_u}{L_h} \sqrt{gh_-}, \quad L_u, L_h \rightarrow 0 \quad (1)$$

where:

- g is the gravitational acceleration;
- w is the velocity of the boundary (in the direction normal to the boundary);
- u_- is the depth-averaged velocity component that is normal to the boundary at distance L_u to the boundary;
- h_- is the water height at distance L_h to the boundary.

The signification of these variables and parameters is also illustrated in the figure below.



The reasoning that led to (1) was inspired by Deleersnijder and Roland (1993). It was hoped that this formula could be instrumental in improving wetting-drying schemes. Soares-Frazao (personal communication, 2002) made a cursory attempt to implement it in her dam break numerical model (Soares-Frazao 2002), obtaining disappointing results. Whether this failure was due to an inappropriate algorithmic implementation or the intrinsic

inappropriateness of this asymptotic formula is as yet unknown.

Recently, in an exchange of electronic mails with Adam Candy (TUDelft), I claimed that the consistency of asymptotic formula (1) with the well-known dam break solution of Ritter (1892) had been verified, but, in my archives, I found no evidence that this had actually been achieved, either because I never did the job or I have lost the relevant documents. This is why the objective of the present working note is to fill this gap.

The Ritter (1892) dam break solution

The one-dimensional, dam break solution of Ritter is recalled in Castro-Orgaz and Chanson (2017). In the unbounded domain $-\infty < x < \infty$, the continuity and momentum equations to be solved read

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \quad (2)$$

and

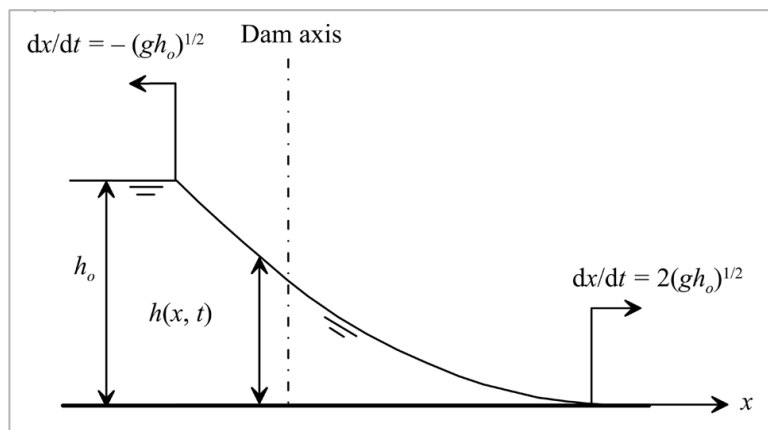
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad (3)$$

where t and x are the time and the space coordinate; the velocity and water column height are denoted $u(t,x)$ and $h(t,x)$, respectively. The initial conditions are

$$h(0,x) = \begin{cases} 0, & 0 < x \\ h_0, & x < 0 \end{cases} \quad (4)$$

and

$$u(0,x) = 0. \quad (5)$$



The solution of this partial differential problem reads

$$h(t,x) = \begin{cases} 0, & 2t\sqrt{gh_0} < x \\ \frac{1}{9g} [2\sqrt{gh_0} - x/t]^2, & -t\sqrt{gh_0} < x < 2t\sqrt{gh_0} \\ h_0, & x < -t\sqrt{gh_0} \end{cases} \quad (6)$$

and

$$u(t,x) = \begin{cases} 0, & 2t\sqrt{gh_0} < x \\ \frac{2}{3}[\sqrt{gh_0} + x/t], & -t\sqrt{gh_0} < x < 2t\sqrt{gh_0} \\ 0, & x < -t\sqrt{gh_0} \end{cases} \quad (7)$$

The water column height is neatly illustrated by Figure (2a) of Castro-Orgaz and Chanson (2017) (shown above).

It is readily seen that the free boundary separating the wet and dry zones is located at $x = 2t\sqrt{gh_0}$. Thus, the velocity at which the boundary moves in the direction of increasing x is

$$w = 2\sqrt{gh_0} = u(t, 2t\sqrt{gh_0}) \quad (8)$$

Consistency of the asymptotic solution

In the Ritter solution, water height h_- used in (1) is

$$h_- = h(t, 2t\sqrt{gh_0} - L_h) = \frac{1}{9g} \left[2\sqrt{gh_0} - \frac{2t\sqrt{gh_0} - L_h}{t} \right]^2 = \frac{L_h^2}{9gt^2} \quad (10)$$

Similarly, velocity u_- is

$$u_- = u(t, 2t\sqrt{gh_0} - L_u) = \frac{2}{3} \left[\sqrt{gh_0} + \frac{2t\sqrt{gh_0} - L_u}{t} \right] = 2\sqrt{gh_0} - \frac{2L_u}{3t} \quad (11)$$

Substituting (10)-(11) into the right-hand side of (1) yields

$$\begin{aligned} u_- + \frac{2L_u}{L_h} \sqrt{gh_-} &= \overbrace{2\sqrt{gh_0} - \frac{2L_u}{3t}}^{=u_-} + \frac{2L_u}{L_h} \sqrt{g \overbrace{\frac{L_h^2}{9gt^2}}^{=\sqrt{h_-}}} \\ &= 2\sqrt{gh_0} - \frac{2L_u}{3t} + \frac{2L_u}{3t} = 2\sqrt{gh_0} \end{aligned} \quad (12)$$

which, with the help of (8), may be rewritten as follows

$$w = u_- + \frac{2L_u}{L_h} \sqrt{gh_-}, \quad 0 < L_u, L_h < 3t\sqrt{gh_0} \quad (13)$$

Thus, asymptotic expression (1) is consistent with the Ritter (1892) dam break solution. In fact, it is more than that (see also the Appendix). This relation holds true for any pair of velocity and water height defined in the region impacted by the dam failure, whose length is $3t\sqrt{gh_0}$. This does not prove that the asymptotic approach reported in Deleersnijder (2018) is valid, but it lends credence to it.

Whether or not the asymptotic approach of Deleersnijder (2018) could help in improving wetting-drying schemes is not clear as yet. In this regard, it must be borne in mind that generalising the seemingly simple and successful method of Delersnijder and Roland (1993) is not that straightforward (Roland 1993, Deleersnijder 1994).

Appendix

Deleersnijder (2018) suggested an asymptotic approach, which, to be applied to the Ritter solution, requires the introduction of space variable

$$\tilde{x} = x - 2t\sqrt{gh_0} \quad , \quad (\text{A.1})$$

with $\tilde{x} < 0$ upstream of the edge of the region impacted by the dam failure. Clearly, $|\tilde{x}|$ is the distance to the edge.

Substituting (A.1) into (6)-(7) yields

$$h = \frac{\tilde{x}^2}{9gt^2} \quad , \quad -3t\sqrt{gh_0} < \tilde{x} < 0 \quad (\text{A.2})$$

and

$$u = 2\sqrt{gh_0} + \frac{2\tilde{x}}{3t} \quad , \quad -3t\sqrt{gh_0} < \tilde{x} < 0 \quad . \quad (\text{A.3})$$

Expressions (A.2) and (A.3) obviously are similar to asymptotic relations (14) and (15) of Deleersnijder (2018).

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