THE LIMITATIONS OF FORMAL METHODS
FROM LADRIÈRE'S VIEWS TO PRAGMATIC FOUNDATIONS

I. INTRODUCTION

At the end of the nineteenth century, mathematics went through an existential crisis, suffering from a realistic threat of arbitrariness and paradoxes. Paradoxes, for example, were found in Cantor's calculus of transfinite numbers and in Frege's naive set theory, while arbitrariness seemed to be a danger for e.g. geometry, where the one and only Euclidean standard was replaced by a variety of coherent alternative geometries. Which one corresponded to the ultimate mathematical truth? Is there even an ultimate mathematical truth? And if we can no longer maintain the unjustified belief that there is a clear interpretation behind mathematics which makes its results evidently true, how do we know that what we are doing in mathematics is still consistent and meaningful?

It is in this context that philosophers and mathematicians, like Hilbert, Russell, Frege, Brouwer, and Heyting began to look for ways to provide a stronger foundation for the discipline. One promising means of achieving this was through *formal systems* that are supposed to structure the truths (or, more accurately, the theorems) of mathematical theories in a precise way.

What is a formal system? A formal system is the combination of a precise language and a collection of purely syntactic transformation rules that could be used to produce an exact set of sentences aligned in a very specific manner. In a context of mathematical proofs, the formal system will provide precise ways to construct the provable theorems of an *axiomatic theory*. An axiomatic theory is a theory that is based on axioms. The truths, or, more neutrally put, the theorems of an axiomatic theory are those sentences that are derivable from axioms by means of proofs. Moreover, the axioms of formal theories are precise strings of symbols and the proofs are entirely governed by rules for their exact transformations.

Thus, the purpose of formal axiomatic systems is to prove the theorems of the theory in a purely syntactical way. The advantage of a purely syntactic method for proving the theorems of a theory is that we obtain an exact and objective standard for verifying theoremhood. Informal proof methods are always to some extent subjective. As soon as the (not exactly specified) meaning of the terms becomes relevant for what a correct proof is, one introduces a subjective element into the justification of theoremhood. If theoremhood is reduced to what is obtainable by means of strict rules for the manipulation of strings of symbols, the theorems of the theory are less obscure and less uncertain. Once there is a verified proof, there is no longer any doubt as to whether the result is a theorem of the theory. In other words, a formalized theory is exactly determined. The theory itself becomes a well-defined mathematical object and the beliefs that one may obtain after proving things with the theory are maximally contextualized. One has specified exactly by which principles the proof depends and all proofs can be verified by both any mathematician and a computer.

Formal methods for certain mathematical theories have the strength to express some form of self-reference. While reference is a semantic notion and formal theories are precisely intended to avoid the influence of semantic aspects, some formal theories have enough strength to express sentences that behave as if they were self-referential. It is this unavoidable property that leads to the essential limitations of the theories. Which theories have this pseudo-self-referential strength? Gödel has proven that all formal theories that can capture recursive functions have this strength and that ordinary formal arithmetic (i.e. the most common formal theory of natural numbers; henceforth we will call it Peano arithmetic) does so, as well.

How can we understand this simulated self-reference? Every formal system is entirely based on strings of symbols, and manipulations between strings of symbols. David Hilbert was the first to notice that a formal system in this sense can be "arithmeticized", i.e. its formulas can be represented by numbers and its permitted manipulations by operations on those numbers. The whole system is thus reduced to purely arithmetical notions. It is then not surprising that theories that can do a certain amount of arithmetic have the strength to talk about the properties of formal theories and, more specifically, about their own properties. This indirect form of self-reference can be expressed by theories.

Self-referential strength brings along some fundamental limitations to formal systems. We will next present a succinct overview of the theorems that clarify these limitations: the so called limitative theorems.

II. LIMITATIVE THEOREMS

1. *Gödel's First Theorem: Incompleteness of Mathematical Theories*

Kurt Gödel[1] has proven that every consistent formal theory with a certain expressive strength is by definition incomplete. The expressive strength is present from at least the level where one can define recursive functions; the best known example is Peano arithmetic (the most common way to formalize arithmetic). Incomplete can mean two things: first, that the theory cannot prove all the truths of the domain. In other words: Gödel's first incompleteness theorem says that, for every theory of a certain strength, there is always a true formula that is not provable. The formula constructed by Gödel is currently called the *Gödel sentence*. This is a sentence that can be interpreted as showing (in some specific sense) that it is not provable.

Secondly, the theory can neither prove nor disprove certain formulas. They are in a certain sense undetermined from the point of view of the theory. In other words: Given this second reading, Gödel's first incompleteness theorem says that for each theory of certain strength, there is a formula *A* such that neither *A* nor not-*A* is provable in the theory. The Gödel sentence is an example of a sentence *A*.

While we have no exact way to determine the truth of non-empirical statements other than by means of formal theories, the one precise method we do have will thus never be sufficient to capture all truths.

2. *Gödel's Second Theorem: Non-Provability of Consistency*

This theorem states that a formal theory of certain strength (at least as strong as Peano arithmetic) cannot prove its own consistency if it is consistent.

Gödel has also suggested that his first incompleteness theorem can be proven inside Peano arithmetic. This was later formally proven by Hilbert and Bernays[2]. This means that, if we can prove Peano arithmetic's consistency within Peano arithmetic, we can also prove that the Gödel sentence is not provable. Yet, we can also prove that the non-provability of the Gödel sentence is equivalent to the Gödel sentence itself. Hence, if we can prove Peano arithmetic's consistency within Peano arithmetic, we can also prove the Gödel sentence within

1. K. GÖDEL, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, in *Monatshefte für Mathematik und Physik* 38 (1931) 173-198.

2. D. HILBERT – P. BERNAYS, *Grundlagen der Mathematik*, Berlin, Springer, [2]1968 [1934].

that system. The first incompleteness theorem, however, denies the possibility of the latter (unless Peano arithmetic is inconsistent). Hence, by *reductio ad absurdum*, it is impossible to prove the consistency of Peano arithmetic within the theory itself.

Thus, the most common formal system of arithmetic cannot prove its own consistency. Perhaps, *prima facie*, this does not appear to be problematic. After all, arithmetic is not primarily meant to prove its own consistency. And even if an arithmetical theory were to prove its consistency, would this give us any more faith in it? Consider that, because inconsistent theories prove everything, every inconsistent theory also proves its own consistency. If we took seriously the possibility that a given theory could be inconsistent, we would not think that we would have obtained a reason to put aside any scepticism about its consistency just because we received the information that it contains a proof of its own consistency. Compare this to the situation where someone you do not trust to tell the truth tells you that he tells the truth. Would this make you trust him? I do not think so.

Yet, the theorem is very important to help one understand the power and limitations of formal systems. Why? In order to explain this, I need to provide some information about David Hilbert's project to formalize mathematics. This project aimed to formalize all mathematics, especially arithmetic, in order to study the metatheory within the field as a precise mathematical notion. All important aspects of formal theories can be translated into facts about numbers. Sentences in a formal language and proofs of a formal system can be uniquely coded by natural numbers. The provability of a sentence is then just an arithmetical property of the number that corresponds to that sentence. Thus, the consistency of a formal system can be expressed as an arithmetical statement.

Hilbert wanted to achieve this arithmetisation of mathematics in order to prove the consistency of its formal systems. Formal mathematics is, by definition, abstract and devoid of any meaning. The upside is absolute exactness, but the downside is that it is uncertain if the formal theories can have any interpretation at all. In the worst case scenario, the formal theory may be inconsistent and therefore trivial. Trivial theories are uninteresting for mathematicians and any effort to prove theorems within them is a complete waste of time. This problem would be solved if we could find a reliable (finitistic) part of mathematics (which e.g. is less abstract and closer to the real world), in which we can prove the consistency of highly ideal and abstract mathematical theories. This reliable part would have an accessible enough interpretation so that we would not need to fear inconsistency.

Hilbert never specified what part of mathematics would be reliable enough, but there was little doubt that it should be a part of arithmetic. Now, if we cannot even prove the consistency of arithmetic (as the second incompleteness theorem teaches us), we definitely cannot prove the consistency of more complicated theories by means of a mere (reliable) part of arithmetic. Gödel's second incompleteness theorem thus proves that Hilbert's formalist project is no longer viable (or not in its most evident way).

On the other hand, the theorem has consequences for attempts at formulating general foundations of mathematics. It is natural to expect that a unified foundational theory of mathematics should be able to prove its own consistency. A foundational theory will often be assumed to provide a framework which supports all (or most) of mathematics. If it needs to be ready to express a lot of very different mathematical domains, it will inevitably be abstract in nature. The more abstract a theory is, the more it is susceptible to inconsistencies. Each classical foundational theory should be guaranteed to be free of inconsistencies (a foundational construction cannot risk collapsing, as each *classical* foundation showing inconsistencies inevitably does). If we want a precise consistency proof, it will need to be mathematical in nature and should be translatable into the foundational theory. Hence, the foundational theory needs to represent a proof for its own consistency. Given that, by Gödel's second theorem, a formal theory can never prove its own consistency, meaning that an ideal foundational theory cannot be formal in this sense. Yet, if it is not formal, how will we guarantee rigour and freedom from implicit presuppositions?

3. *Löwenheim-Skolem Theorem and the Existence of Unintended Models*

I will next present a version of the Löwenheim-Skolem theorem[3] that is slightly more general than its original version. The theorem states that every formal theory that has an interpretation with an infinite domain, has an interpretation with a domain of arbitrary infinite cardinality. This means that it is impossible for a formal theory to fix a certain interpretation. In standard interpretations of the relevant mathematical theories, for example, it is generally accepted and expected that there are $\aleph_0$ natural numbers and $2^{\aleph_0}$ real numbers. The precise meaning of $\aleph_0$ and $2^{\aleph_0}$ does not matter here, just that they are

3. J. VAN HEIJENOORT (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Cambridge, MA, Harvard University, 1967.

infinite numbers and that the second contains strictly more members than the first. If we want a formal theory which exactly describes the natural numbers, we would only want it to permit interpretations with a domain of size $\aleph_0$. However, this possibility is excluded by the Löwenheim-Skolem theorem. If the formal theory indeed has an interpretation with a domain of (infinite) size $\aleph_0$, it also has an interpretation with a domain of size $2^{\aleph_0}$ and one with a domain of any possible size. In the latter interpretations, there are definitely "intruding" objects, objects other than the natural numbers. This is due to the fact that these sizes are all strictly greater than $\aleph_0$.

The same holds true for a formal theory of the real numbers, but now also downwards: there necessarily exists an interpretation of a formal theory of the real numbers with a domain of size $\aleph_0$, i.e. the size of the set of natural numbers. Intuitively, there seems to be a serious shortcoming if a theory for the real numbers can be interpreted in such a way that there are far less objects in it than in the standard set of real numbers.

In the case of set theory, the problem is even more counter-intuitive. Every formal set theory has an infinitely-sized interpretation (every usual set theory at least proves the existence and non-identity of the empty set, its power set, the power set of this one, the power set of this one, etc.) and so should have an interpretation of every possible infinite size. Hence, there is also an interpretation of size $\aleph_0$ in view of the Löwenheim-Skolem theorem. Yet, a set theory usually allows the construction of an infinite set and of the power set of infinite sets. The size of the power set of each infinite set will always be strictly larger than $\aleph_0$. Thus, there is an interpretation with less objects than there are elements in the sets in this interpretation.

Counter-intuitively, set theory also defines cardinalities. Formally speaking, a cardinality only exists if one's set theory proves its existence. As such, no matter how dazzlingly large the cardinality is, if it exists it should be proven to exist in a set theory with only $\aleph_0$ sets in its domain. While the intuitive manner in which mathematicians look at set theory requires all sets to exist in a unique universe that contains all sets and all sizes, their rigorous formal theories allow for universes of sets of all possible infinite sizes, even the smallest ones. Moreover, if one has fixed a single idea of the universe of sets, no interpretation of formal set theory will be large enough to interpret this full universe (the universe is not a set and has by definition no cardinality).

This may seem rather difficult to grasp, but contemporary set theorists have little trouble with it. They have an ingenious methodology to look at interpretations of set theory. In a way, the set theory looks different from the inside. From the inside, it (formally) defines a huge universe with all possible sets of all cardinalities, but, from the outside, (the viewpoint of the interpretation) it may not be any larger than the set of natural numbers. The fact that the power set of the finite ordinals has more members than there are finite ordinals ($\aleph_0$), formally means that there is no bijection between the two sets. A bijection is a function, and a function is yet another set. Hence, just lacking this function set already makes it possible for the power set of the finite ordinals to internally appear as if it is larger than the finite ordinals. What was once known as Skolem's paradox (this counter-intuitive tension between the existence of unsuitably sized interpretations of mathematical theorems), is now just another curiosity of a relativized approach to abstract (formal) set theory.

4. *Tarski's Undefinability Theorem*

Tarski's theorem[4] indicates that a formal theory of certain strength cannot contain its own truth predicate. More specifically, theories of recursive function strength cannot define a predicate that is instantiated by a formula if and only if that formula is true. This means that we cannot expect to find one unified formal theory that can speak of both mathematics and semantics. A foundational theory of truth in mathematics needs to be kept apart from the foundational theory of actual mathematics. For empirical theories, this may not be problematic at all. The theory of biology is about biological entities, while the theory of truth of biology is about linguistic entities in the language of biology. A theory that mixes both risks confusing the two. For mathematics, the matter is not so obvious. The linguistic statements are themselves mathematical objects and the syntactic aspects of the field can be studied within mathematics.

In practice, we use formal tools (model theory) to study the semantics of mathematics (what else would one use?) with great success, but there is no hope that we can develop a single unifying and foundational formal theory of mathematics, which contains both useful mathematics (at least arithmetic) and the semantics of mathematics. This impossibility is formally demonstrated by Tarski in the form of his famous undefinability theorem.

5. *Non-Computability of Logic*

4. A. TARSKY, *Der Wahrheitsbegriff in den formalisierten Sprachen*, in *Studia Philosophica* 1 (1936) 261-405.

Turing and Church[5] have proven that first order logical consequence is not computable. This means that there cannot be a general algorithm that can tell us whether the conclusion for each set of premises and each candidate conclusion is indeed a valid consequence.

In the early days of the formalization of mathematical theories, the intuitive hope was that once a theory is fully formalized, the resulting mathematics would be a matter of simple calculation. The intuition here is that once one has exactly and fully defined a relation/set/function, determining the outcome of the relation/set/function for specific concrete arguments is a matter of mere automatable calculation, making it a matter of time and effort, not of luck or creativity.

The more mathematicians looked into the theory of computation, the more they found relations and functions that are completely well-defined, but that could not possess a general algorithm. A famous example is the fact that the halting problem is not computable. The problem is to decide whether a given algorithm will eventually stop for a given input. It is not surprising that this problem is perfectly well-defined, yet it cannot be calculated.

Thus, while formal methods have greatly decreased the mystery of mathematics, formalizability does not imply principled knowability. In other words, it turns out that the belief that, if one can describe the problem, he or she will inevitably be able to solve it is false.

For formal theories, this means that their theoremhood relations are generally not decidable. If a formal theory is complete, we can be certain that we have an algorithm for theoremhood. But we know from Gödel's first incompleteness theorem that (many) interesting mathematical theories are not complete. For these incomplete formal theories, we often only have a positive test. There is an algorithm that gives us the answer YES if the input is a theorem, but, if the input is not a theorem, the answer may elude our grasp forever.

6. *Independence Results in Set Theory*

Gödel's conclusion[6] that the continuum hypothesis is independent of the standard set theory, called ZFC, started a long series of formulas that were all proven to be independent of ZFC. The Cohen's forcing method greatly helped in proving such results. In each case, a set

5. G.S. BOOLOS – J.P. BURGESS – R.J. JEFFREY, *Computability and Logic*, Cambridge, Cambridge University Press, [5]2007.

6. K. GÖDEL, *The Consistency of the Continuum-Hypothesis*, Princeton, NJ, Princeton University Press, 1940.

theoretic formula and its negation can be consistently added to our best formal set theory. This makes it improbable that there is one universe of sets.

III. Ladrière's Views on the Limitation of Formal Methods

We will review Ladrière's more philosophical views on the limitative theorems as he expressed them in the last chapter "Suggestions philosophiques" of his postdoctoral thesis[7]. We do not aim to present a historically accurate picture of Ladrière's views on the limitative theorems, nor are we interested in the links between his views on the foundations of mathematics or those on other domains. Our only purpose is to provide an alternative contemporary interpretation of Ladrière's philosophical suggestions on the limitative theorems. It is this interpretation that will guide us to the alternative kind of foundations (so called *pragmatic foundations*) that we will introduce below.

We cite some crucial passages from that chapter, and provide some (free) translation, explanation, and comments.

1. *Ladrière Summarizes Gödel's Incompleteness Theorems*

> Or le théorème de Gödel nous a appris que dès le moment où une théorie possède une certaine puissance (et c'est déjà le cas pour l'arithmétique ordinaire), sa non-contradiction ne peut être établie que dans une métathéorie où interviennent des procédés de démonstration plus puissants que ceux de la théorie elle-même, ce qui entraîne en particulier pour conséquence que la métamathématique ne peut se contenter d'un cadre strictement finitiste[8].

Ladrière argues that Gödel's theorem has taught us that the consistency of theories with a certain strength (from ordinary arithmetic) can only be established in a metatheory, which defines more powerful proof methods than those of the theory. Ladrière claims that this implies that a strictly finitistic framework does not suffice for metamathematics.

7. J. Ladrière, *Les limitations internes des formalismes: Étude sur la signification du théorème de Gödel et des théorèmes apparentés dans la théorie des fondements des mathématiques*, Leuven, Nauwelaerts; Paris, Gauthier Villars, 1957.

8. *Ibid.*, p. 403.

Gödel's second incompleteness theorem could indeed be interpreted in such a way that it proves that, if sufficiently strong formal systems are consistent, they cannot demonstrate this consistency within a strictly finitistic framework. Yet, maintaining that metamathematics cannot therefore be done by means of strictly finitistic proof methods is a more contentious claim. If one is happy to do only (the most reliable) part of metamathematics, strict finitistic methods may suffice. An example is Gödel's theorem. This theorem can be perfectly proven by means of strictly finitistic methods. If we take the theorems as conditional metatheoretic statements (viz. for the first: if Peano arithmetic is consistent, then it is incomplete and the second: if Peano arithmetic is consistent, then this consistency cannot be proven within Peano arithmetic), finitistic methods suffice to prove them. Ladrière seems to presuppose that providing either absolute consistency proofs or other strong absolute results is a necessary requirement for metamathematics. A radical constructivist, for example, may disagree.

> Il nous a appris également qu'un système formel ne peut jamais être considéré comme une représentation adéquate de la théorie mathématique qu'il est censé exprimer, du moins en ce sens qu'il ne permet pas de décider effectivement de la validité de certains énoncés[9].

Ladrière summarizes Gödel's first theorem as follows: it proves that a formal system can never be considered an adequate representation of the mathematical theory it wants to express, at least in the sense that it cannot effectively decide the validity of certain statements.

It is important to remark that Ladrière seems to presuppose that all statements are either determinately true or false in the mathematical theories formal methods want to express, or at least that the Gödel sentence is determinately true. While it is unproblematic to acknowledge that there was already a mathematical conception before the formalization, calling it a theory with (pre-formal) valid statements is a heavier conclusion. Abstract mathematical theories like algebra and category theory do not necessitate validity for all statements, only for those for which the theory provides information. It is true that there is a strong intuition that the truth of all arithmetical statements already exists independently of how we formalize arithmetic. But is this really more than an intuition? We will later provide some arguments why we think it is indeed more than an intuition.

9. *Ibid*.

2. *Ladrière on the Usefulness of Formal Methods*

> Il y aurait ici deux erreurs d'interprétation à éviter. La première consisterait à donner au théorème de Gödel une portée qu'il n'a pas, en y voyant la manifestation d'une sorte d'échec de la méthode formelle[10].

Ladrière warns us against misinterpretations of Gödel's theorems. He mentions two inaccurate interpretations. The first gives it an impact it does not have. Ladrière remarks that Gödel's theorems do not imply a failure of the formal method.

> Mais il faut voir que les limitations dont il s'agit n'ont pas un caractère absolu. Elles ne signifient pas que la méthode formelle est vouée à la stérilité, elles en circonscrivent seulement le champ d'efficacité[11].

He notes that one should be aware that the limitations are not of an absolute nature. They do not mean that the formal method is destined to be sterile. The limitations only circumscribe the scope of its efficiency.

> […] elle rend apparentes les insuffisances du raisonnement naturel et des évidences trop immédiates et que, en ce sens, elle demeure l'instrument indispensable de l'accès à la vérité[12].

Ladrière remarks that the formal method shows the shortcomings of natural reasoning and hasty conclusions and that, in this sense, it remains the indispensable instrument of access to truth.

Here, Ladrière points out that the formal method has not become useless in view of its limitations and problems. We can make good use of this method in order to make human (mathematical) reasoning precise. Thanks to exact rules, we can criticize hasty pseudo-derivations and implicit unjustified assumptions. In our opinion, this is a very important remark. Delineating the strengths, the weaknesses, and the scope of a method does not make it useless. On the contrary, we know a system better and can apply it more effectively if we know its limits.

We now understand the positive aspects of Peano arithmetic better since Gödel. We know that it is strong enough to represent all recursive relations. Assuming Church's thesis, which states that all computable relations are recursive, every possible property that can be computed in all instances can also be proven to occur, or not occur, within Peano

10. *Ibid.*, p. 404.
11. *Ibid.*
12. *Ibid.*

arithmetic. This is extremely powerful for a theory that was supposed to formalize only the natural numbers, multiplication, and addition. We also know that the theory can even express a form of self-reference. For every arithmetical property *P* in the Peano's arithmetic language, we can find a statement for which we can explicitly prove that *P* holds for itself (e.g. we can prove of one specific sentence that it says that it is self-dividable by ten and we can prove of another sentence that it says not to be self-provable).

The more negative results concerning the formal theory have also turned out very useful. It is beneficial to know, for example, that one would waste time trying to prove or disprove certain statements that are provably undecidable (if the theory is consistent) in that theory. We know that there is no use in trying to prove the consistency of strong enough theories in it, to formulate truth predicates in Peano arithmetic, etc. Thanks to Gödel's incompleteness results, we also better understand how to interpret Peano arithmetic. Mathematicians now understand (consistent but) non-standard models of arithmetic that claim arithmetic to be provably inconsistent: it has an infinite non-standard object, which behaves like a proof for $0 = 1$, according to a criterion which correctly represents provability for finite candidate proofs, but fails to do this for those weird infinite objects. The existence of such infinite objects in some models is very useful to know what we could be talking about when we are using formal arithmetical language.

> […] la méthode formelle nous permet précisément de critiquer les pseudo-évidences du discours non formalisé et de mieux situer les frontières du contradictoire, en nous apprenant qu'elles ne se trouvent pas là où le discours non formalisé nous invite à les trouver[13].

Ladrière notices that the formal method allows us to criticize the pseudo-certainties of informal discourse in an exact way and to better situate the borders of the contradictory, teaching us that they do not lie where the informal discourse wants us to find them.

Contradiction is a great danger for mathematics. Classical mathematics becomes entirely useless in the presence of even one contradiction. Thanks to the formalisms, we know where exactly we can find contradictions. Furthermore, as a result of Gödel's careful (and formalizable) analysis, we know very well that we can (probably) do some careful consistent auto-reference in arithmetic (the sentence that says of itself that it is not provable), while auto-reference about truth

13. *Ibid*., p. 405.

(the sentence that says of itself that it is not true) is not possible, or at least not in a naive way. Similarly, formal set theory has shown us which transfinite sentences we can (probably) consistently formulate, and which others lead us, via paradoxes, into contradiction and triviality. Without formal methods, the only responsible action mathematicians could have taken in response to these paradoxes would have been to suspend all discussion related to infinite objects.

3. *Ladrière on the Limitations of Formal Methods*

> Les faits de limitation nous indiquent avec précision pourquoi le système total ne peut être qu'une idée-limite[14].

Ladrière points out that the limitation facts indicate with precision why the total can only be a limite idea. The total is an abstract concept here. It is the true maximally consistent extension of Peano arithmetic (and other formal theories). Finite proofs can never prove all of the statements in such a total extension. In principle, one could use some kind of infinite proof notion or infinite closure method (if the theory proves that it does not code a proof of the Gödel sentence for every finite number, it should prove the fact it does not prove it in general) to realize this total theory. But given that we are merely finite beings (and so presumably merely finite provers), the total (whatever that is) remains a limite idea.

> Ils nous montrent en effet que l'on ne peut jamais formaliser complètement le champ intuitif, que la dualité de l'intuitif et du formel demeure irréductible[15].

Ladrière concludes that the limitations show us that one can never completely formalize the intuitive field and that the duality of the intuitive and the formal will always stay irreducible.

Here, Ladrière assumes that our intuitions require that the real arithmetical field is total in the sense mentioned above. This is not implied by Gödel's theorem, but one is justified to claim so, as I will argue below.

> Il n'existe pas de système fermé qui serait comme le paradigme de tout discours, le canon suprême de la raison. Quel que soit le système que

14. *Ibid.*, p. 410.
15. *Ibid.*

> nous puissions considérer, il y a toujours des formes de raisonnement qui lui restent étrangères[16].

For Ladrière there is no closed system, which would be the paradigm of all discourse, the supreme canon of reasoning. Whatever system we choose, there are always forms of reasoning that it will not capture.

Here, Ladrière makes an interesting jump from the tension between the intuitive completeness of arithmetic and the practical incompleteness of our formal arithmetical tools to the tension between forms of reasoning in general and what a concrete formal system can capture. That we intuitively can think of an interpretation to make each formula either true or false does not imply that reasoning forms exist that can find out whether it is true or false. Perhaps an interpretation is a lot like a fictive universe as described in a story. We may often reasonably assume all relevant properties of all objects to be somehow fixed in the fictive universe, but if the story did not give us information about the truth of a certain property of a certain object, we have no method for determining it. That does not mean that we have means for finding out the truth beyond the story. The same happens for mathematics; even if we assume every formula to be ultimately true or false, there is no reason to assume the existence of reasoning methods that go beyond formal mathematical systems. From the fact that there is no formal system that can capture all truths (in an interpretation), one cannot conclude that no formal system can capture all reasoning forms.

If one believes that Gödel has proven (in an informal but convincing way) that the Gödel sentence is true, one will probably follow Ladrière in claiming that some reasoning forms cannot be captured for every formal system. Gödel has informally proven the Gödel sentence of Peano arithmetic, which is not provable within the system. Similarly, one can informally prove a formally improvable sentence in every richer system. Thus, a closed system will always miss at least one informally provable sentence. Adding that sentence as an axiom creates a new insufficient system missing another informally provable sentence.

Nonetheless, it is not at all obvious that Gödel has indeed informally proven the sentence named after him. One can formally prove that the Gödel sentence (for Peano arithmetic) is equivalent to the consistency of Peano arithmetic. If Gödel would have provided an informal argument in favour of the Gödel sentence, it would also suffice to prove the consistency of the formal theory. Most mathematicians believe that Peano arithmetic is consistent and were sufficiently convinced of this

16. *Ibid.*, p. 412.

belief e.g. by Gentzen's consistency proof, but we would find it very strange if some mathematicians were convinced of the consistency of Peano arithmetic after reading Gödel's proof (especially because the consistency of Peano arithmetic is a presupposition of Gödel's theorem).

We are also sceptical as to whether the informal reasoning process of always adding the missing formulas to prove another previously unproven formula of the new system cannot be captured in a formal theory. In formal set theory, we can construct this without much trouble. However, set theory has its own undecidable formulas.

To summarize: there are definitely reasonable informal arithmetical proofs that cannot be completed within Peano arithmetic. Yet, it is debatable as to whether there is a formal system that can adequately capture all correct informal arithmetical proofs. What we will call a pragmatic foundation in the remaining sections of this work will indeed discuss reasoning forms that provably go beyond what is possible in a formal system. The big difference, however, is that these mathematical reasoning forms are *non-deductive*. This means that they do not always give us more than provisional certainty. Most mathematicians will be very reluctant to call them "proofs", and would not want to capture them within a formal system. A formally proven statement should be undoubtedly true for everybody who accepts the proof system. This certainty is not warranted by non-deductive proofs.

> Ainsi la dualité du subjectif et de l'objectif ne peut être abolie, le formalisme ne peut constituer une sorte de modèle objectivé de la pensée, il ne peut enfermer la totalité de l'intelligible[17].

Ladrière concludes that the duality of the subjective and the objective cannot be abolished. The formalism cannot constitute an objectified model of thought, and it cannot capture (or enclose) the totality of the intelligible.

A similar comment is in order here. We agree that the formalism cannot constitute an objectified model of what is true within a subjective interpretation, but it is (as far as we know) far from obvious whether it is in principle impossible to capture in one formalism everything the human intellect can (informally) deductively prove.

If the meaning of "l'intelligible" in this passage is what can be demonstrated using informal but deductive (non-provisional) proof methods, we are sceptical about whether Ladrière's statement is sufficiently justified or even justifiable. If, however, "l'intelligible"

17. *Ibid.*

includes what may be *merely plausible* true (at any finite stage), our dialectical ideas introduced below fit very well with what Ladrière calls "le subjectif" or "l'intelligible". It will become clear that the ultimate results of the dialectical aspects of mathematical foundations mentioned below can clearly not be fully captured by one formal theory.

4. *Ladrière on Interpreting*

> C'est seulement au niveau du système formel que les concepts mathématiques prennent un sens rigoureux. Mais ce sens ne peut être considéré comme absolu. Il n'est d'ailleurs pas possible de construire un système d'une certaine puissance qui n'admette qu'un seul type de modèle[18].

Ladrière observes that it is only at the level of the formal system that mathematical concepts obtain a rigorous sense. This sense, however, cannot be considered absolute. Moreover, it is impossible to construct systems of certain strength that admit just one single type of interpretation.

Ladrière points at a very interesting tension. On the one hand, mathematical concepts only obtain a precise meaning once they are formalized. Yet, on the other hand, this meaning cannot be absolute (from a certain complexity on). In each different interpretation, the meaning of the concepts is slightly different. The meaning of addition only becomes precise in a theory of arithmetic, but its meaning cannot be completely fixed for all objects of the arithmetical domain. This can be seen as follows: for interpretations with non-standard numbers, the precise meaning of addition differs from interpretation to interpretation. The conclusion here seems to be that, for some complex concepts, their meaning can only partially be made completely rigorous.

> Comme on l'a vu, le formalisme ne peut recouvrir adéquatement le contenu de l'intuition et, en ce sens, l'idée d'une formalisation totale doit être considérée comme irréalisable[19].

Ladrière argues that a formalism cannot adequately cover the content of intuition. In that sense, the idea of a total formalization has to be considered unrealisable.

If the content of an intuition describes the collection of truths in an intuitive construction which interprets sufficiently strong theories, we

18. *Ibid.,* p. 414.
19. *Ibid.,* p. 438.

obviously agree with Ladrière. If an intuitive construction wants to be an interpretation, it will assume completeness or totality. Moreover, if the content of intuition is meant to be the provisional estimations mathematicians have in mind before really proving something, our pragmatic foundation will be nicely in line with Ladrière's observation. However, if the content of intuition is something which is really undeniably and non-fallibly established by intuition alone without formal proof, we tend to be sceptical: what would such an intuition be like and where would it get its forceful proving power?

5. *Ladrière on Formal Tools as Our Only Efficient Way to Realize Mathematical Thought*

> La pensée mathématique ne peut être efficace que dans la mesure où elle se lie à des conditions rigoureuses d'accomplissement. Et c'est dans le domaine du formel qu'elle rencontre ces conditions. Elle ne reconnaîtra dès lors d'existence mathématique qu'aux objets qui pourront être représentés dans un formalisme non-contradictoire. Mais le formalisme, c'est le domaine de la construction. C'est précisément ce qui en fait l'efficacité et en même temps ce qui en trace les limites[20].

Near the end of the chapter, Ladrière concludes that mathematical thinking can only be efficient to the extent that it is attached to rigorous conditions for its implementation. It is in the formal domain that these conditions can be found. It can therefore only recognize mathematical objects that are represented in a non-contradictory formalism. Yet, the formalism is the domain of the construction and that is exactly what makes mathematics so efficient while, at the same time, marking its limits.

> Si le système total est irréalisable, c'est parce que l'expérience mathématique se déploie sur le fond d'un horizon inépuisable. Cet horizon, étant condition de possibilité de l'expérience et du représentable, ne peut être pensé que comme condition de possibilité. (Encore ne le peut-il qu'au sein d'une réflexion qui, prenant l'expérience mathématique elle-même pour objet, se trouve nécessairement en discontinuité avec elle.) Il ne peut donc être représenté[21].

Ladrière remarks that, if the total system is unrealisable, it is because mathematical experience unfolds along an inexhaustible horizon. That

20. *Ibid.*, p. 439.
21. *Ibid.*, p. 443.

horizon, being the precondition of the possibility of experience and what is representable, can only be understood as a precondition (it can only exist in a reflection, with mathematical experience as its object, which is necessarily situated in discontinuity with this experience). It cannot be represented.

> Il est de la nature d'une existence finie de ne pouvoir atteindre les réalités-fondements que dans un acte de dévoilement qui n'en fait jamais apparaître que les manifestations. Et réciproquement les réalités-fondements ne peuvent se révéler à une existence finie qu'à travers des figures toujours inadéquates[22].

Finally, at the end of his book, Ladrière concludes that it is the nature of a finite being to only be able to reach the fundamental reality in an act of unveiling through manifestations. Reciprocally, the fundamental reality can only show itself to a finite being through inadequate forms.

In the latter two quotations, we get a picture which could suggest a platonistic position. It is hard not to read these passages describing mathematics as if there was a fundamental mathematical reality/realm and we, as finite beings with only finite proof methods, can only look as far as our horizon will allow. The limited horizon is typical for our finite formal methods, which stands in contrast to the infinity of the mathematical realm.

We believe this platonistic picture to be one of the many possible positions of mathematical truth. However, it will become clear that we see no reason to deny or confirm such a position. The mathematical realm offers truth to an interpretation, although which universe the objects of an interpretation live in is open to philosophical speculation. Nonetheless, we think it is perfectly possible to be agnostic about such questions and have a coherent view on the foundations of mathematics.

To conclude our section on Ladrière's views, we would like to focus on his very last words. He says that reality only reveals itself to us through inadequate forms. This is very interesting. It means that, even subjectively, the intuitions and reasoning forms of our thought are inadequate and are therefore insufficient or incorrect. This again could be read in such a way that would ultimately cause Ladrière to agree on the picture sketched below of the (foundational) interpretations of mathematics as a result of defeasible and dialectical provisional reasoning.

22. *Ibid.,* p. 444.

IV. BEYOND THE FORMAL SYSTEMS; CAN MATHEMATICS BE INTERPRETED COHERENTLY?

We would like to suggest that mathematics can indeed be coherently interpreted at a fundamental level, even without assuming a mathematical realm independent of human construction. We only have to content ourselves with a very imperfect way of interpreting fundamental mathematics.

Mathematics has always been a meaningful intellectual practice. The fact that we imagine mathematical statements to be about something may not be anything more than a concept we got used to by dealing with empirical statements. Nevertheless it is very difficult to conduct creative mathematical reasoning without knowing what one is reasoning about. Some may believe mathematics is about structural properties of spatiotemporal reality or of our intellectual capacities. Others may prefer mathematical statements to be about objects in a universe outside of the physical world, while some may prefer to talk about abstract human constructions, or about abstracted versions of other possible worlds or even of our own. Whatever the discourse, without interpretation it is virtually impossible for human reasoners to come up with ingenious proofs of complicated theorems, to construct previously unknown mathematical objects, to construct and criticize the formalization of mathematics, to engage in reverse mathematics, or to even propose new mathematical theories. These, however, are more discovery and metamathematics related that cannot be conceived in a language devoid of meaning. It is extremely unlikely that Gödel would have come up with any of his results if he had seen his own words as meaningless. It is formally provable that each formal theory able to capture recursive functions is necessarily incomplete. But, at least for the sake of argument, one needs to assume that there is something to which this statement refers. While formal theories and recursive functions do not exist in our spatio-temporal world, claiming that they do not exist is only possible from outside the metamathematical discipline.

Formal methods are extremely useful in making mathematical results completely exact and honest if it comes to the presuppositions of one's arguments, but it is unrealistic to expect mathematics to be reducible to what can be achieved by just these means. Even if mathematics is nothing more than a game mathematicians play, mathematics seems to also be a science *about* the game mathematicians play. Chess experts play a purely formal game, but they think/reason/talk/teach about something when they engage in their domain of expertise. Chess experts understand the possible concrete plays of the game as a mathematical object. The game of chess itself cannot be *about* a strategy, only a theory about possible games of chess can. If we consider its strategies as part of chess, chess (seen as the domain of expertise of chess players) is therefore ultimately also a science *about* chess games. In the same way, it seems reasonable to expect every purely formalistic mathematical idea to actually require mathematics to ultimately be a science *about* the properties of specific formal mathematical games. The latter is just as semantic as any other view on the philosophy of mathematics, i.e. also the formalist's mathematics is a science *about* something.

Thus, while a constructivist, for example, may object to the picture sketched by Ladrière of complete independent mathematical theories that cannot be understood by our finitary formal approximations of the theories, we think it makes a lot of sense to assume that mathematical theorems do have an interpretation. And although nothing forces us to completely understand this interpretation before making use of it, mathematicians who use classical logic, in principle, need to assume that each formula is either true or false in the interpretation. This interpretation need not be human independent, absolute, real or even convincing, but it must demarcate the true formulas from the false.

In this view, every theory is unavoidably interpreted. It is the interpretation that corresponds to "reality", which cannot be captured by our formal theories (according to Ladrière). While the interpretation may not have anything to do with reality as such, it is true that we can distinguish a domain of discourse from the formal methods used to make this domain precise. This interpretation is not just some problematic intuition, but an inescapable part of doing actual mathematics as a human. One speculative question that will likely never receive an answer is whether this interpretation is anything even remotely resembling Mathematical Reality or Mathematical Truth. Yet, we believe that it is the task of the metamathematician to clarify the formal aspects of both mathematics and its interpretations. The goal is to answer the following questions: how can a mathematician rationally both reason about mathematical objects (producing meaningful language) and engage in mathematics (following rules)?

It seems likely that even abstract mathematical theories like formal logic, category theory, and abstract algebra are fundamentally interpreted. Admittedly, it is useless to say that statements in e.g. topos theory (a part of category theory, which captures aspects of the structure of set theory) are to be interpreted by some concrete (elementary) topos. A concrete topos is, just like any other category, a collection of things with some binary relations between them (that respect a couple of principles). It is equally useless to say that statements in group theory (in abstract algebra) are to be interpreted by some concrete group (a set combined with an operation on elements of that set). Moreover, it does not make sense in formal logic to interpret concrete sets of formulas by using a concrete model. All of these theories are proposed as abstract ideas that move away from the concrete. In category theory, we are interested in what categories with certain properties have in common. In algebra, we are interested in what algebras with certain properties have in common. In logic, we are interested in what is true across all possible models (of a set of sentences). Thus, it may seem essential that these mathematical theories are not to be interpreted, but studied through their formal properties.

We think this is a misunderstanding similar to believing that chess expertise is not really an "expertise". The interesting and unavoidable interpretations of such abstract theories are not the concrete domains they conceptualize, but rather the collection of all such domains. Category theory is not about the elements of a category, but concern (the collection/universe of all) *categories*. Abstract algebra is not about the objects in a concrete algebra, but about (the universe of) algebras. Formal logic is not about the objects in a concrete domain of a model, but about all possible formulas and theories, their relations, and their models. More generally, abstract mathematical theories are about structures (shared by many concrete mathematical collections), not about the concrete objects within a structure. Moreover, we claim that such theories have to be about these structures and cannot exist in the completely formal void of pure string manipulation. For why would we engage in such a project or think that there is any chance that these manipulations could be in any way applicable to empirical and applied sciences (if they are just empty string manipulations)?

Even formalists are fundamentally engaged in a semantic practice. Their practice of formalizing theories does not concern the concrete objects that they refer to by these theories (if there even are objects like that), but the formalisms themselves as pure formal games (and so as mathematical objects themselves). The best evidence for this is that formalists are at least concerned about the well-defined nature of formalism and its relative consistency in relation to other formalisms. One cannot be authentically concerned about these things if one is agnostic about their very existence. The formalistic sentence "system T is consistent if system S is consistent" has a meaning: it talks about the relation between two mathematical objects.

As such, there is some evidence for the following normative position: A formal system can only be interesting (for mathematics) if it is relevant to the truth of sentences in an interpreted language.

As we saw, the interpreted language does not need to be the language of the formal system (and often is not). But if one accepts this position, one also needs to clarify the definition of an interpretation, its ontological and epistemological status, and how human beings deal with it. It surely is something we cannot access by simple observation, because abstract objects, like mathematical objects, cannot be observed through our human senses. Nor can we obtain it by deduction within a formal system. A part can be formalized inside a formal system (think of set theoretic formalizations of the standard model of arithmetic), but it seems dubious that pure deduction can provide us with a full interpretation. The interpretation is usually present prior to its formalization. Formal methods of dealing with interpretations are usually the result of a process of specification of an intuitive and tentative idea.

V. WHAT IS A FOUNDATION OF MATHEMATICS?

Identifying a foundation of mathematics can be a rather mysterious process. It is a concept that can give the mathematician working on local mathematical problems the certainty that his work is (ideally) a (small) contribution to the construction of a unified, solid/reliable, and meaningful mathematical building. The clearest form in which one can recognize a foundation is the naive logistic project of reducing all mathematics to a pure and exact logic. The classical view is that logical truth is absolutely true and independent of the truth in our world. Once one accepts such a position, and the logistic project succeeds, we have a mathematics which is coherent (logical truth is by definition coherent), unified (all of mathematics is reduced to the one discipline of logic), reliable (logical truth is taken to be the maximal level of reliability) and meaningful (if logic is the theory of reasoning and language, then so is mathematics for the logicist; the mathematical construction is thus part of the large project of understanding human reasoning).

A formalistic foundation uses formal systems to construct a mathematical building. The formal rules could be seen as forming the cement that holds the building blocks together. Yet, they can only warrant the solidity/reliability-aspects of the mathematical building. For the “being unified”-aspect, we also need reduction or translation methods to reformulate all kinds of mathematical systems into the formal system's language. More surprisingly, it seems unavoidable that a formalistic foundation interprets at least the axioms of the unifying formal system in order to guarantee meaningfulness or usefulness. The formalist need not decide exactly what meaning is given to the axioms, but she should at least have some guarantee that there could be a meaning or use. In the case of Euclid's geometry, it was clear for Euclid that geometry was about the space of the physical world and that the axioms were evident truths about space. A set theoretic formalistic foundation will ultimately have to claim that the basic axioms of set theory are evident truths concerning what the things we call sets look like (in the world) or how mathematics constructs collections to collect mathematical objects (sets as human constructions).

One may ask why a mathematical foundation should warrant usefulness or meaningfulness. Admittedly, this is, to a large extent, a philosophical choice. Nonetheless, it seems hard to escape that, if interesting mathematics is to make a difference for the truth of interpreted language, the very basic concepts to which other mathematical concepts are reduced should make a difference. Moreover, both mathematicians and philosophers of mathematics are interested in answers to questions on what mathematical knowledge is and why it seems so certain (more certain than empirical sciences). If the foundation to which mathematics is reduced is nothing but empty or useless formal machinery, one may wonder how the rest of mathematics can be interpreted. And again, the ultimate objects of (the intended interpretation of) a mathematical foundation do not necessarily have to be referents of the constants that occur within the axioms of the foundation. They may be the possible structures to which the axioms could refer. In other words, they may be possible universes rather than one specific universe, or may even be the possible syntactic constructions and relations between them. Yet, we endorse that a foundation without objects or any interpretation whatsoever is nothing more than a machine producing an unintelligible stream of symbols.

Should a foundation really be unifying? Not necessarily, but in order to count as a foundation of mathematics (rather than of a specific mathematical field), it should be a foundation of multiple mathematical disciplines/theories/fields. It is probably too much to ask for a single foundation that represents all mathematics. Mathematics is likely too large, vague and open-ended as a discipline for such aims. It is not clear what can be considered mathematics. The term 'mathematics' ranges from the very abstract to the more concrete. Is logic part of mathematics? Is formalized metaphysics part of mathematics? Is string theory mathematics? Is computer science part of mathematics? Is applied statistics part of mathematics? More fundamentally, there seems to be no restriction on the human capacity to construct (via metalinguistic considerations) ever more complicated mathematical structures. Once we fix a certain practice as mathematical, we can always come up with the study about that practice and, if it happens rigorously, there is no reason to not also call that study a part of mathematical practice. As such, unifying all of mathematics seems to be out of the picture. Yet, if one only aims to establish a foundation of, e.g., local class field theory, there is no reason why it would teach us anything valid outside of local class field theory. There is no reason why such a foundation would bring us any closer to understanding mathematical knowledge and practice (outside local class field theory). If, on the other hand, something is a foundation of several diverse disciplines (set theory and category theory can e.g. be the foundation for number theory, analysis, algebra, recursion theory, and many more), there is a reasonable probability that other mathematical practices could also be captured by the foundation, therefore also allowing the philosophical content of the foundation to be applicable to these mathematical practices.

Let us summarize.

1. Formal methods are by far the best tool we have to make mathematical ideas and practices precise, reliable, clear, and open to critique
2. a formal theory is only interesting if it has an effect on meaningful statements (formulated in an interpreted language)
3. the limitations of formal methods teach us that, for most reasonable interpretations, the formal methods can never fix the entirety of the truths within the interpretation
4. a foundation should be solid and (useful or meaningful)
5. formal methods can provide solidity
6. interpretations can provide meaning
7. interpretations are not necessarily referring to an external human independent reality
8. a foundation based on meaning and formal methods should explain how one should deal with the gap between the two
9. a foundation should explain how its components relate to mathematical practice in a precise manner.

VI. PRAGMATIC FOUNDATIONS

1. *General Aspects*

In this section, we will present a short introduction to a novel proposal to deal with foundational questions and problems[23] (for a preliminary implementation of this proposal). We will suggest some steps towards an approach to the foundation of mathematics that takes both formal methods and their limitations seriously, while simultaneously attempting to go beyond them in a maximally precise way. The original foundational projects in the philosophy of mathematics first started around the end of the 19th century. The technical and symbolic tools needed to explicate reasoning, proofs, algorithms, models, theories, etc. has boomed since that time. There are now not only all kinds of classical logic tools, but also non-classical ones that are useful for better understanding the foundation of mathematics. Paraconsistent logics, relevance logics, intuitionistic logics, game theory, forcing methods, non-standard mathematics, non-monotonic logics: all of these are potentially relevant to the foundation of mathematics and could potentially enable solutions of problems that had hitherto seemed unsolvable.

Considering the fact that a foundation has to be meaningful or useful, it is not easy to avoid an image of mathematics in which its universe is a mysterious realm outside of our spatio-temporal world to which only skilled mathematicians have some privileged access. Despite lacking a mathematical sixth sense, and because pure deduction does not suffice, it is tempting to conclude that this privileged access must happen via mathematical intuition. Although it is not at all evident (or useful for that matter) to deny the existence of this mathematical realm, such considerations are of little practical use. Mysterious notions do not bring us closer to an explanation of the success of mathematical practice.

Thus, we aim to propose a more explanatory foundational approach, which we call *pragmatic foundation*. An exact definition of what it means for a foundation to be pragmatic will not be given, but, after this introduction, it should be clear for the reader that the approach takes (or aims to take) pragmatic aspects of mathematics relatively seriously, certainly more so than existing foundational projects, such as Brouwer's intuitionism, Frege's logicism, Bourbaki's set theoretic foundation, and Hilbert's formalism.

23. P. VERDÉE, *Non-Monotonic Set Theory as a Pragmatic Foundation of Mathematics*, in *Foundations of Science* 18 (2012), no. 4, 655-680.

The general aim of our approach is to be maximally relevant and understandable. More concretely, we aim for a maximally precise and rigorous approach that could ideally found any field of mathematics. At the same time, we want to be metaphysically parsimonious and avoid presupposing heavy platonistic (or other, e.g. psychological, sociological, religious) ideas, while also stopping short of suggesting anti-platonism.

We would like to investigate how modern logical and mathematical tools make it possible to obtain all of these requirements. Our research so far is quite promising.

2. *Ingredients of Our Pragmatic Foundation Project*

The pragmatic foundational project we propose has the following key ingredients: a) a pluralistic formal core; b) a dialectical interpretation; c) precise dialectical methods; d) ultimate interpretations. Let us elucidate these ingredients one by one.

a) *Pluralistic Formal Core*

The formal axiomatic method has made it possible for mathematics to become clearer, more solid, more self-critical, and less susceptible to paradoxical constructions. At some point in their development, all "grown up" mathematical theories are successfully formalized. Those formalizations make the informal theories (which are by themselves not much more than a body of quite diverse definitions, assumptions, and informal proofs provided by a group of people using a similar vocabulary) into unified bodies of knowledge that hang closely together. We should therefore recognize how essential formal methods are for the foundations of mathematics. Yet, regardless of the importance of some formal theories, aiming for a single, all-encompassing one represents a failed project. Set theory or topos theory can capture a lot of mathematics, but not all of it. There are always notions that escape a unified formal theory, for reasons of paradoxes (think of the category of all categories, set of all sets, ordinal of all ordinals etc.) and for reasons of mathematical creativity (as soon as we have a mathematical formally fixed construction, we can always come up with more complex notions constructed on the basis of the fixed theory). On the other hand, because of Gödel's incompleteness, the one unifying formal theory will always miss some true sentences.

Thus, it is important that one understands the foundation of mathematics as containing a multitude of formal theories, at least one for each mathematical field. Within each field, however, there could be multiple formal theories all capturing different ways to disambiguate an informal concept. ZF captures a whole different notion of set than Quine's NF. Both, however, are acceptable mathematical notions that are worth studying. The fact that they seem to be competing theories results from their use of the same informal word "set". Nonetheless, having a multitude of set-like notions is not harmful to mathematics, and one often just uses a different vocabulary. A pragmatic foundation aims to contain all of these formal theories (if they are studied carefully in mathematics). Even more importantly, it aims to contain (all) interesting relations between theories, plus the proofs of the validity of these relations. Which theory can interpret (or, equivalently, prove the consistency of) another theory? Which theories can be either reduced or extended to other theories? Which theories are (in)compatible with which theories? Incidentally, note that these relations are actually ternary: theory X proves that theory Y stands in a certain relation with theory Z. Pure logic does not suffice for such proofs, for in pure logic we cannot even speak of theories.

To sum up: we consider a set of formal theories and proven relations between these theories as the formal core of the foundation. The relations ensure the fact that the combined theories form a unified and strongly connected core. Through this, the unification does not create a single theory of everything, but one dense gigantic network of highly interconnected mathematical theories.

b) *Dialectical Interpretations*

The formal core is completely deductive in nature. Formal theories are defined by closing axioms under a deductive logic, and the relations between the theories are formally (and deductively) proven in one of the other formal theories. Mathematics, however, is much more than a deductive practice. Mathematical theories and proofs are constructed or discovered. The context of discovery (in contrast to the context of justification)[24] is strongly linked to interpretations. The context of discovery does not proceed deductively. Mathematicians use, in this context, non-deductive defeasible reasoning methods very similar to the ones used by empirical scientists. One uses both inductive and abductive methods, ceteris paribus assumptions, counterfactual reasoning, disambiguation, analogical reasoning, and inconsistency

24. I use "context of justification" in a very narrow sense; not the context in which one is trying to come up with a justification, but the one in which one communicates the justification of an already justified result.

handling. Illustrating this in detail requires a deep study of mathematical practice. Here, I will give some idea of how such non-deductive methods are used: Induction: finding a proof by first jumping to the provisional conclusion that, because one finds a couple of positive instances of a property (and no negative instances), the property holds for all instances; Abduction: one conjectures and wants to prove B, while knowing that if A then B, then one conjectures A and tries to prove that one; Ceteris Paribus assumptions: one will often assume consistency and decidability as long as it is not proven that we are dealing with an unusual case (more particularly, e.g. mathematicians using arithmetic will assume that the conjecture they are working on is not an undecidable sentence of formal arithmetic by either proving the theorem or finding a counterexample; in the undecidable case neither will work); Counterfactual reasoning: new theories are often developed by asking what would be true if we were to change some fundamental properties of the objects under consideration (think e.g. of non-standard analysis, which is obtained by asking what would happen if one gave up the Archimedian property of standard analysis); Disambiguation: when developing a field of mathematics, one often starts with a formally ambiguous notion and solves the related problems by disambiguating it into two more precise notions (think e.g. of the distinction between set and class in NBG or ZF set theory).

Interpreting mathematical theories requires very similar methods: analogical reasoning, reasoning with consistency assumptions, inductive reasoning, reasoning with imprecise definitions, etc. Admittedly, not all interpretations can likely even be linguistically expressed. Yet, the interpretations that make a difference for the foundation of mathematics are those that are communicable. A completely private interpretation of a mathematician is of little importance for mathematics as a discipline that needs interpretation. All communicable semantic reasoning happens in a language. Although it does not happen deductively, this reasoning may follow certain syntactical rules. Moreover, discussing the results of this reasoning is possible. Let us take the example of interpreting Peano arithmetic by an intuitive version of the standard model of arithmetic. If one wants to claim that the Gödel sentence is true in the standard model, one could use the analogy of a finite set of natural numbers up to some (very) high number. One reasons that it is impossible for them to code the proof of the Gödel sentence. In the finite case, we can thus conclude that the Gödel sentence is not provable by (the proof encoded by) any of those numbers. Next, we notice that we need to prove something about an infinite set of natural numbers. To do that we need an inductive generalization: if it holds for every similar set of arbitrary size, it will also hold for the infinite set of all natural numbers. This is not a deductive logical rule, but a contentious claim about the generalizability of finite estimations to the infinite case. We can only take this step by assuming that everything behaves normally. The rule is thus only valid Ceteris Paribus and is part of defeasible and non-deductive reasoning.

We also conjecture that the non-deductive reasoning patterns found in mathematical discovery, creativity, and interpretation practices can be precisely and symbolically explained and do obey certain form-related rules. But, unlike deductive reasoning methods, the methods found here are fundamentally dialectical and dynamic in nature. As conclusions are often only provisionally valid, one can challenge them by trying to rebut them. It is these games of provisional conclusions, their rebuttals, and the rebuttals of the rebuttals that constitute the rules of rationality for the interpretation aspect of reasoning. We do not wish to determine whether the dialectical aspects are realized internally (by self-critical beings) or externally (by a social group of mathematicians). Both seem valid and reasonable.

It is this aspect that is supposed to give finite intellectual beings the capacity to move towards establishing the meaningfulness of mathematical theories.

c) *Precise Dialectical Methods*

In order to make the creative and provisional aspects of the practice of interpretation communicable and universally criticisable, we need to define form based rules for making provisional conclusions, and for criticizing both these conclusions and their critics. There is no reason why we should not aim to make this part of our foundation as precise as the formal part. Even though the results maintain a lower degree of reliability and certainty, these techniques should not be vague.

We do not want to claim that working mathematicians discussing interpretations are guided by standards of reasoning that can be formulated in a perfectly exact way. We only want to propose a way of constructing foundations of mathematics that (at least) aims at making these rules precise. Foundations have always been normative or explanatory in nature. Nobody claims that actual mathematicians were fundamentally always reasoning along the lines of foundational structuralism, logicism, intuitionism, or formalism. All of these foundations have consistently been rational reconstructions that have significantly departed from actual reasoning. The fact that we want to stay closer, and more relevant, to actual mathematical practice does not change the fact that our foundation is bound to be a reconstruction that deviates from actual mathematical thought.

This aspect warrants the reliability and the unifiability of the interpretative methods.

d) *Ultimate Interpretation*

The last ingredient is that which corresponds to the ultimate interpretation. It aims to produce complete versions of mathematical theories. It aims to represent how mathematics is ideally supposed to converge.

For reasons related to Gödel's first incompleteness theorem, the ultimate interpretation cannot be semi-recursive (like a formal theory) in case of theories of certain expressivity. Non-semi-recursive theories are not susceptible to incompleteness in a Gödelian sense.

So how do we obtain such conclusive interpretations? Can they even be thought? Are they in any way related to our rather earthly pragmatic views of foundations?

The answer is positive. Given both the correct formal tools and the right rules of the dialectics of interpretation, the dynamic reasoning process should ultimately converge into a conclusive interpretation. All things considered, our correctly adjusted estimations of how to interpret formal theories should stabilize into what can ultimately count as a good interpretation after possibly transfinite number of adjustments.

Now, one may wonder how it can be assured that the dialectical steps converge into a real interpretation that corresponds to Mathematical Reality. It cannot. After all, Mathematical Reality may be so inaccessible to us that any of our methods will inevitably go in the wrong direction. Yet, this is of no concern to pragmatic foundations. If the discrepancies with Real Mathematics do not affect actual practice, it is harmless. If they do affect actual practices, the dialectical self-critical nature of the foundational interpretation practices will make sure that the result is long adjusted before the ultimate interpretation is reached (in an abstract and infinite way, of course).

3. Implementing Pragmatic Foundations: Adaptive Logics

Adaptive logics[25] are very suitable for the implementation of the dialectical and ultimate aspects of the aforementioned foundational

25. D. BATENS, *The Need for Adaptive Logics in Epistemology*, in D. GABBAY – S. RAHMAN – J. SYMONS – J.P. VAN BENDEGEM (eds.), *Logic, Epistemology and the Unity*

enterprise. They were designed by Batens and his students to formalize all kinds of defeasible reasoning, mostly in the empirical sciences.

The reason why logics in the adaptive tradition are particularly suitable for the kind of pragmatic foundation introduced here is that they are precise ways to capture creativity and uncertainty, and, moreover, they combine provisional reasoning with a sort of ultimate standard. For each adaptive logic, there is a dynamic proof system that symbolically captures dialectical reasoning processes. These dynamic proofs contain provisional conclusions. Yet, specific provisional conclusions become ultimately stable (often the prover is not aware of this at a finite stage of the proof process). These are the conclusions that constitute the ultimate adaptive consequences of the premises of the provisional proof. These consequences are typically not semi-recursive for recursive premises[26].

Consider an adaptive logic and a set of mathematical premises (e.g. the axioms of a foundational formal theory plus some meta-statements about the theory). The (ultimate) adaptive logic consequences of the premises will generally be a non-semi-recursive theory. Such theories go beyond the scope of the limitations of formal systems. They can be complete and prove their own consistency.

Another reason why adaptive logics are good candidates for the specification of our pragmatic foundation is that they are very flexible. One can choose to base them on both classical and non-classical logics. The best known adaptive logics are even paraconsistent in nature, which means that they are inconsistency tolerant. Given how susceptible intuitive mathematics is to paradoxes, it may be safer to start interpreting this field in a way in which inconsistencies do not trivialize the whole theory. Moreover, they are also flexible in the sense that they can easily be combined with more complex adaptive logics that contain a dialectical structure of arbitrary complexity.

I will not go into the technicalities of the adaptive framework, but we will list some successes that will (likely) be both achievable by an adaptive logic and interesting for the (pragmatic) foundations of mathematics.

- a logic for the omega-rule (and other forms of transfinite proof techniques)
- a logic for assuming consistency unless, and until, proven otherwise

of Science, Dordrecht, Kluwer Academic Publishers, 2004, 459-485; D. BATENS, *A Universal Logic Approach to Adaptive Logics*, in *Logica Universalis* 1 (2007) 221-242.

26. D. BATENS – K. DE CLERCQ – P. VERDÉE – J. MEHEUS, *Yes Fellows, Most Human Reasoning is Complex*, in *Synthese* 166 (2009) 113-131.

- an inconsistency tolerant safety net for inconsistencies (but also assuming the law of non-contradiction by default)
- intentionally including inconsistent parts of mathematics (Russel set, self-reference, one's own semantics)
- semantic considerations as conditional assumptions: information originating from the interpretation one has in mind can be explicitly assumed (until proven that there are other elements pointing at another interpretation)
- capture mathematical abduction and counterfactual reasoning to discover mathematical results and to grasp the links between formal theories.

CONCLUSION

We discussed several limitations of formal systems. We saw that a formal system cannot prove everything that is true in the fundamental interpretation of mathematics (if there is such a thing). Repeated in an imprecise way, formal theories cannot simultaneously deal with both mathematics and the semantics of mathematics.

Next, we looked at Ladrière's comments on these limitations. He neatly expresses how formal systems are the right (and the only) way of making mathematics precise and free of contradictions. However, according to Ladrière, it is precisely their finite and exact nature that explains why they can never suffice to make all informal reasoning precise. The totality of mathematics cannot be captured in one formal system.

Inspired by a non-platonistic reading of Ladrière's ideas, we developed our own proposal for the foundation of mathematics, viz. what we call a pragmatic foundation. Through this foundation, we want to bring together and confront intuition and formalism in a way that is guided by, and that wants to be useful for, actual mathematical practice.

In mathematical thinking, human creativity plays an important role. Neither the capacity of stepping outside of theories to construct ever more complicated (meta-)results nor the ability to link seemingly unrelated theories are informal in the strict sense of the word. The methods are not pure syntactic manipulations inside a pre-defined system. Still, every attempt at making them precise results in a formal system. We propose our new foundational process to mediate between the formal and informal systems.

In our approach, we choose symbolic and precise methods for both the formal and creative/interpretative aspects of mathematics. We believe that it is more explanatory to not rely on intuitions, ambiguous notions, or sloppy language. Advances in non-classical logic research and, more specifically, logics within the adaptive logic family, have made it clear that non-deductive reasoning can be understood by rules that are as precise as the ones for deductive logics. If these are used for the foundation of mathematics, we hope to obtain a foundation that not only clarifies the strong axiomatic part, but also the underlying creative and interpretative aspects.

Université catholique de Louvain Peter VERDÉE