Degree, quaternions and periodic solutions

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Abstract

The paper computes the Brouwer degree of some classes of homogeneous polynomials defined on quaternions and applies the results, together with a continuation theorem of coincidence degree theory, to the existence and multiplicity of periodic solutions of a class of systems of quaternionic valued ordinary differential equations.

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1 Introduction

The study of periodic solutions of quaternionic valued ordinary differential equations using topological degree techniques has been initiated in [3], and was followed by the interesting contributions of Zoladek [23] and Wilczynsky [20, 21]. Of interest are also the recent studies of linear equations by Cai and Kou [1, 2], Cheng, Kit and Xia [4], and Kou and Xia [9], of autonomous equations by Gasull, Llibre and Zhang [7, 22], and the applications to neural networks by Li and his coworkers [8, 10, 11, 12, 13, 14].

In this paper, after recalling in Section 2 the required basic concepts and notations about quaternions, whose set is denoted by $\mathbb{H}$, we describe in Section 3 some results on the Jacobian of the powers of q given in [18], which are used in Section 4 to compute the Brouwer degree of powers of a quaternion q or of its conjugate $\overline{q}$. In Section 5, those formulas allow the determination of the Brouwer degree of some classes of homogeneous polynomials in q and $\overline{q}$ (Theorem 5.1). This result is applied to the obtention of existence theorems for some classes of algebraic quaternionic equations, and in particular to an extension of a theorem of Eilenberg and Niven [6] (Theorem 5.2). The case of systems of quaternionic mappings and equations is considered in Section 6 (Theorem 6.1).

Those formulas are then used in Section 7, together with a continuation theorem of coincidence degree theory ([15, 16]), to prove the existence of T-periodic solutions of systems of quaternionic valued ordinary differential equations of the form

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{a}_k \overline{\mathbf{q}}_k^{n_k} + \mathbf{g}_k(t, \mathbf{q}_1, \ldots, \mathbf{q}_m) \;\; (1 \le k \le m),$$

where each $\mathbf{q}_k$ is a function from $[0,T]$ into $\mathbb{H}$, and the perturbations $\mathbf{g}_k(t,\cdot)$ verify a suitable growth condition at infinity (Theorem 7.1).

A single quaternionic valued differential equation corresponds to a four-dimensional system of differential equations. The presence of a multiplication for the quaternions often suggests and simplifies the obtention of the a priori bounds requested by the topological approach, which would remain unaccessible or very cumbersone in the equivalent four-dimensional formulation. On the other hand, the non-commutativity of the multiplication of quaternions complicates the obtention of those a priori bounds with respect to the case of similar real or complex valued differential equations.

In Section 8, when $\mathbf{g}_k(t,0) = 0$ for all $t \in [0,T]$ and all $k \in \{1,\ldots,m\}$, we add conditions on the behavior of $\mathbf{g}_k$ near $\mathbf{q} = \mathbf{0}$ ensuring the existence of non trivial periodic solutions for the same class of systems (Theorem 8.1).

2 Quaternions

We denote as usual a (real) quaternion $q = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ by

$$q = q_0 e + q_1 i + q_2 j + q_3 k,$$

where q_0, q_1, q_2, q_3 are real numbers, e is the unit element and i, j, k satisfy the multiplication table formed by

$$i^2 = j^2 = k^2 = ijk = -e,\ ij = -ji = k,$$

and the other products are derived by cyclic interchange of i, j, k [5]. This gives, if $r = r_0 e + r_1 i + r_2 j + r_3 k$,

$$\begin{array}{rcl} qr & := & q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 + (q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2) i \\ & + & (q_0 r_2 - q_1 r_3 + q_2 r_0 + q_3 r_1) j + (q_0 r_3 + q_1 r_2 - q_2 r_1 + q_3 r_0) k. \end{array}$$

The inner product, the modulus and the conjugate are respectively defined by

$$\begin{array}{rcl} \langle q, r \rangle & = & q_0 r_0 + q_1 r_1 + q_2 r_2 + q_3 r_3, \\ \|q\| & = & \langle q, q \rangle^{1/2} = \left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right)^{1/2}, \\ \overline{q} & = & q_0 e - q_1 i - q_2 j - q_3 k, \end{array}$$

so that

$$\overline{qr} = \overline{r}\,\overline{q},\ q\overline{q} = \overline{q}q = \|q\|^2 e.$$

Moreover, q commutes with any quaternion r if and only if $q \in \mathbb{R}e$. We often identify $\mathbb{R}e$ with $\mathbb{R}$ and write q_0 instead of $q_0 e$. Consequently,

$$\|qr\|^2 = qr\overline{qr} = qr\overline{r}\,\overline{q} = q\|r\|^2\overline{q} = q\overline{q}\|r\|^2 = \|q\|^2\|r\|^2,$$

so that

$$\|qr\| = \|q\|\|r\| = \|rq\|.$$

The above inner product makes $\mathbb{H}$ a four-dimensional real Hilbert space with orthonormal basis $\{e, i, j, k\}$. The real part operator is defined by

$$\Re : \mathbb{H} \to \mathbb{R}, \quad q = q_0 + q_1 i + q_2 j + q_3 k \mapsto q_0,$$

and the vectorial or imaginary part of $q \in \mathbb{H}$ by

$$\Im q := \widetilde{q} = q_1 i + q_2 j + q_3 k.$$

An easy computation shows that

$$\widetilde{q}^2 = -\|\widetilde{q}\|^2,$$

so that any $I \in \mathbb{H}$ such that $I = \widetilde{I}$ and $\|I\| = 1$ verifies the relation $I^2 = -1$. In this way, we can think in $\mathbb{H}$ as a bundle of complex planes with the real line as a common axis.

3 The Jacobian of powers of a quaternion or of its conjugate

If $n \geq 1$, if we identify $\mathbb{H}$ with $\mathbb{R}^4$, the mapping

$$p_n : \mathbb{R}^4 \to \mathbb{R}^4, \ q \mapsto q^n$$

is a homogeneous mapping vanishing only at 0. The following computation of the Jacobian J_{p_n} of the n^{th} power of a quaternion, taken from [18], is reproduced here for the reader convenience.

Lemma 3.1 *Let $n \geq 1$ be an integer and $p_n : \mathbb{H} \to \mathbb{H}$, $q \mapsto q^n$ considered as a mapping from $\mathbb{R}^4$ into $\mathbb{R}^4$. Then*

$$J_{p_n}(q) = \begin{cases} n^4 \|q\|^{4(n-1)} & \text{if} \quad \Im q = 0 \\ n^2 \|q\|^{2(n-1)} \left[\frac{\Im(q^n)}{\Im q}\right]^2 & \text{if} \quad \Im q \neq 0. \end{cases} \tag{1}$$

Proof. An elementary induction shows that the value at $h \in \mathbb{H}$ of the total derivative $Dp_n(q)$ of p_n at $q \in \mathbb{H}$ is given by

$$Dp_n(q)h = \sum_{m=0}^{n-1} q^m h q^{n-1-m}. \tag{2}$$

We prove the lemma by choosing an appropriate basis for the quaternions. If $\Im q = 0$, then by the commutativity of q and h, we get $Dp_n(q)h = nq^{n-1}h$ and hence

$$J_{p_n}(q) = n^4 \|q\|^{4(n-1)}.$$

If $\Im q \neq 0$, q generates a 2-dimensional subfield of $\mathbb{H}$ which can be identified to $\mathbb{C}$, namely $\mathbb{C} = \{a + bI : a, b \in \mathbb{R}\}$, where $I = \widetilde{q}/\|\widetilde{q}\|$. From (2), it is immediate to check that $\mathbb{C}$ is invariant under the action of $Dp_n(q)$. The same is true for a fixed complementary subspace $\mathbb{D}$ of $\mathbb{H}$ spanned by quaternions J and K such that I, J, K verify the usual relations. By analogy with the complex case, one has $Dp_n(q)c = nq^{n-1}c$ for $c \in \mathbb{C}$, and the Jacobian of the restriction of $Dp_n(q)$ on $\mathbb{C}$ is given by $n^2\|q\|^{2(n-1)}$. On the other hand, for $c \in \mathbb{C}$ and $d \in \mathbb{D}$, we have $dc = \overline{c}d$, where $\overline{c}$ is the complex conjugate of c in $\mathbb{C}$, and hence

$$Dp_n(q)d = \left(\sum_{m=0}^{n-1} q^m \overline{q}^{n-1-m}\right) d = \frac{q^n - \overline{q}^n}{q - \overline{q}} d = \frac{\Im(q^n)}{\Im q} d.$$

So the Jacobian of the restriction of $Dp_n(q)$ on $\mathbb{D}$ is $\left[\frac{\Im(q^n)}{\Im q}\right]^2$. Consequently, for $\Im q \neq 0$, one has

$$J_{p_n}(q) = n^2\|q\|^{2(n-1)} \left[\frac{\Im(q^n)}{\Im q}\right]^2.$$

■

Corollary 3.1 *With the notations of Lemma 3.1, $J_{p_n}(q) \geq 0$ for all $q \in \mathbb{H}$, and $J_{p_n}(q) > 0$ outside of the set $\mathcal{N}$ of measure zero defined by*

$$\left\{ q \in \mathbb{H} : \left(\sin^2 \frac{k\pi}{n}\right) q_0^2 = \left(\cos^2 \frac{k\pi}{n}\right) \left(q_1^2 + q_2^2 + q_3^2\right) \quad (1 \leq k \leq n-1) \right\}.$$

Proof. If $\Im q = 0$, it follows from (1) that $J_{p_n}(q) > 0$ for $q \neq 0$. If $\Im q \neq 0$, one can write $q = \|q\|(\cos\theta + I \sin\theta)$ with

$$I = \frac{\widetilde{q}}{\|\widetilde{q}\|}, \quad \cos\theta = \frac{q_0}{\|q\|}, \quad \sin\theta = \frac{\|\widetilde{q}\|}{\|q\|},$$

so that $q^n = \|q\|^n(\cos n\theta + I \sin n\theta)$. By (1), $J_{p_n}(q) = 0$ if and only if $\Im q^n = 0$, i.e. if and only if $\theta = \frac{k\pi}{n}$ $(1 \leq k \leq 2n-1)$. The set of those zeros is contained in the $n-1$ cones

$$\left\{ q \in \mathbb{H} : \left(\sin^2 \frac{k\pi}{n}\right) q_0^2 = \left(\cos^2 \frac{k\pi}{n}\right) \left(q_1^2 + q_2^2 + q_3^2\right) \quad (1 \leq k \leq n-1) \right\}.$$

So $\mathcal{N}$ has measure zero. ■

4 The Brouwer degree of powers of a quaternion or of its conjugate

Let $B(R)$ denote the open ball of center 0 and radius $R > 0$ in $\mathbb{R}^4$. The Brouwer degree $d[p_n, B(R), 0]$ of f with respect to $B(R)$ and 0 is well defined and independent of $R > 0$ (see. e.g. [17]). Recall [17] that if $f : \overline{B}(R) \to \mathbb{R}^4$

is of class C^1 and such that its Jacobian $J_f(q) \neq 0$ whenever $f(q) = 0$, then $f^{-1}(\{0\})$ is finite and

$$d[f, B(R), 0] = \sum_{q \in f^{-1}(\{0\})} \operatorname{sgn} J_f(q). \tag{3}$$

We now compute $d[p_n, B(R), 0]$ and $d[\overline{p_n}, B(R), 0]$.

Lemma 4.1 *For each $R > 0$, one has*

$$d[p_n, B(R), 0] = n. \tag{4}$$

Proof. By the homotopy invariance property of the Brouwer degree [17], we have, for all $\varepsilon > 0$ sufficiently small

$$d[p_n, B(R), 0] = d[p_n, B(R), \varepsilon i].$$

Now, the equation $q^n = \varepsilon i$ has n roots (see e.g. [5], p. 204-205), which are the n complex roots of εi, namely

$$q_k = \varepsilon^{1/n} \left[\cos \frac{(4k+1)\pi}{2n} + i \sin \frac{(4k+1)\pi}{2n} \right] \quad (0 \leq k \leq n-1). \tag{5}$$

None of those q_k is real, and hence, using formula (1), we obtain

$$J_{p_n}(q_k) = \frac{n^2 \varepsilon^{4\left(1-\frac{1}{n}\right)}}{\sin^2 \frac{(4k+1)\pi}{2n}} > 0 \quad (0 \leq k \leq n-1),$$

so that formula (4) follows from formula (3). ■

For the case of the n^{th} power of the conjugate of q, we notice that $\overline{q}^n = \overline{q^n} = \overline{p}_n(q)$.

Lemma 4.2 *For each $R > 0$, one has*

$$d[\overline{p}_n, B(R), 0] = -n. \tag{6}$$

Proof. In terms of the components (q_0, q_1, q_2, q_3) of q,

$$\begin{aligned} \overline{p}_n : (q_0, q_1, q_2, q_3) &\mapsto (q_0^n, -q_1^n, -q_2^n, -q_3^n) \\ &= \operatorname{diag}(1, -1, -1, -1)(q_0^n, q_1^n, q_2^n, q_3^n), \end{aligned}$$

and hence $J_{\overline{p}_n}(q) = -J_{p_n}(q)$. Formula (4) follows from formula (3). ■

5 The Brouwer degree of homogeneous polynomials of a quaternion and its conjugate

Let $J_1 \cup J_2$ be a partition of $\{1, 2, \ldots, n\}$ (one of the J_i can be empty), $n_1 = \# J_1$, $n_2 = \# J_2$, and, for each $1 \leq j \leq n$, let $c_j : \mathbb{H} \to \mathbb{H}$ be defined by

$$c_j(q) = q \quad \text{if } \ j \in J_1, \quad c_j(q) = \overline{q} \quad \text{if } \quad j \in J_2. \tag{7}$$

Given $\mathbf{a}_j \in \mathbb{H} \backslash \{0\}$, $(0 \leq j \leq n)$, let $h_n : \mathbb{H} \to \mathbb{H}$ be the homogeneous polynomial in q and $\overline{q}$ of degree n defined by

$$h_n(q) = a_0 c_1(q) a_1 c_2(q) a_2 \ldots a_{n-1} c_n(q) a_n. \tag{8}$$

If we identify $\mathbb{H}$ with $\mathbb{R}^4$, $h_n : \mathbb{R}^4 \to \mathbb{R}^4$ is a homogeneous polynomial mapping of degree n vanishing only at 0, and hence $d[h_n, B(R), 0]$ is well defined and independent of $R > 0$.

Theorem 5.1 *If the c_j are defined by formula (7), $a_j \in \mathbb{H} \setminus \{0\}$ $(0 \leq j \leq n)$, and if h_n is defined by formula (8), then, for each $R > 0$, one has*

$$d[h_n, B(R), 0] = n_1 - n_2.$$

Proof. Let $A_j : [0, 1] \to \mathbb{H} \setminus \{0\}$ be continuous and such that

$$A_j(0) = 1, A_j(1) = a_j \quad (0 \leq j \leq n)$$

and let $g_n : \mathbb{H} \to \mathbb{H}$ be defined by

$$g_n(q) = c_1(q) c_2(q) \ldots c_n(q).$$

By the homotopy invariance of the Brouwer degree, we have

$$d[h_n, B(R), 0] = d[g_n, B(R), 0].$$

As $q\overline{q} = \overline{q}q = \|q\|^2$, we have $c_j(q) c_k(q) = c_k(q) c_j(q)$ for all $1 \leq j, k \leq n$. Hence,

$$g_n(q) = \begin{cases} q^{n_1 - n_2} \|q\|^{2n_2} & \text{if} \quad n_1 > n_2 \\ \|q\|^n & \text{if} \quad n_1 = n_2 \\ \overline{q}^{n_2 - n_1} \|q\|^{2n_1} & \text{if} \quad n_1 < n_2. \end{cases}$$

Noticing that, on $\partial B(R)$, one has

$$g_n(q) = \begin{cases} q^{n_1 - n_2} R^{2n_2} & \text{if} \quad n_1 > n_2 \\ R^n & \text{if} \quad n_1 = n_2 \\ \overline{q}^{n_2 - n_1} R^{2n_1} & \text{if} \quad n_1 < n_2, \end{cases}$$

and using the boundary dependence property of Brouwer degree [17], we obtain

$$d[g_n, B(R), 0] = \begin{cases} d[p_{n_1 - n_2}, B(R), 0] & \text{if} \quad n_1 > n_2 \\ 0 & \text{if} \quad n_1 = n_2 \\ d[\overline{p_{n_2 - n_1}}, B(R), 0] & \text{if} \quad n_1 < n_2, \end{cases}$$

and the result follows from Lemmas 4.1 and 4.2. ∎

Corollary 5.1 *If the c_j $(1 \leq j \leq n)$ are defined by formula (7), $a_j \in \mathbb{H} \setminus \{0\}$ $(0 \leq j \leq n)$, h_n is defined by formula (8), $g : \mathbb{H} \to \mathbb{H}$ is continuous and if there exists $R > 0$ such that*

$$\|g(q)\| < \left(\prod_{k=0}^{n} \|a_j\| \right) R^n \text{ for all } q \in \partial B(R),$$

then

$$d_B[h_n + g, B(R), 0] = n_1 - n_2. \tag{9}$$

Proof. For each $\lambda \in [0,1]$ and each $q \in \partial B(R)$, we have, by assumption

$$\|h_n(q) + \lambda g(q)\| \geq \|h_n(q)\| - \|g(q)\| = \left(\prod_{k=0}^{n} \|a_k\| \right) R^n - \|g(q)\| > 0,$$

and the result follows from the homotopy invariance of the Brouwer degree and Theorem 5.1. ∎

A direct application of Corollary 5.1 is a slight extension of the fundamental theorem of algebra for quaternions proved by Eilenberg and Niven [6] using Brouwer degree arguments. A more analytical proof of Eilenberg-Niven theorem can be found in [19], and a direct proof based only upon differential forms is given in [18].

Theorem 5.2 *With the notations of (7), if $a_j \in \mathbb{H} \setminus \{0\}$ $(0 \leq j \leq n)$, $n_1 \neq n_2$, and $g : \mathbb{H} \to \mathbb{H}$ is a continuous function such that*

$$\lim_{\|q\| \to \infty} \frac{\|g(q)\|}{\|q\|^n} = 0, \tag{10}$$

then the equation

$$a_0 c_1(q) a_1 c_2(q) a_2 \ldots c_n(q) a_n + g(q) = 0$$

has at least one solution in $\mathbb{H}$.

Proof. By condition (10), there exists $R > 0$ such that, for $q \in \partial B(R)$,

$$\|g(q)\| \leq \frac{1}{2} \left(\prod_{k=0}^{n} \|a_k\| \right) \|q\|^n = \frac{1}{2} \left(\prod_{k=0}^{n} \|a_k\| \right) R^n < \left(\prod_{k=0}^{n} \|a_k\| \right) R^n.$$

Consequently, by Corollary 5.1, $d[h_n + g, B(R), 0] = n_1 - n_2 \neq 0$, and the result follows from the existence property of the Brouwer degree [17]. ∎

Corollary 5.2 *If $n \geq 1$, $a_j \in \mathbb{H} \setminus \{0\}$ $(0 \leq j \leq n)$, and if g verifies condition (10), then the equation*

$$a_0 q a_1 q a_2 \ldots q a_n + g(q) = 0$$

has at least one solution in $\mathbb{H}$. This is in particular the case if $g(q)$ is a polynomial in q of degree smaller or equal to $n - 1$.

An immediate consequence of Corollary 5.2 is a fundamental theorem of algebra for some classes of quaternionic polynomials.

Corollary 5.3 *If $n \geq 1$, $a_j \in \mathbb{H}$ $(0 \leq j \leq n)$, $b_j \in \mathbb{H}$ $(0 \leq j \leq n-1)$, and $a_0, b_0 \neq 0$, the quaternionic polynomial*

$$a_0 q^n b_0 + + a_1 q^{n-1} b_1 + \ldots + a_{n-1} q b_{n-1} + a_n$$

has at least one zero in $\mathbb{H}$.

6 A generalization to systems of quaternionic equations

Let $m \geq 2$ be an integer, $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m) \in \mathbb{H}^m$, and using the notations of Section 5, for each $k \in \{1, 2, \ldots, m\}$, let $n_k \geq 1$ be an integer, $J_{k,1} \cup J_{k,2}$ be a partition of $\{1, 2, \ldots, n_k\}$ (one of the $J_{k,i}$ can be empty), $n_{k,1} = \# J_{k,1}$, $n_{k,2} = \# J_{k,2}$, and, for each $1 \leq j \leq n_k$, define $c_{k,j} : \mathbb{H} \to \mathbb{H}$ by

$$c_{k,j}(q) = q \quad \text{if} \;\; j \in J_{k,1}, \quad c_{k,j}(q) = \overline{q} \quad \text{if} \quad j \in J_{k,2}.$$

Given $a_{k,j} \in \mathbb{H} \setminus \{0\}$ $(0 \leq j \leq n_k)$, let $h_{k,n_k} : \mathbb{H} \to \mathbb{H}$ be the homogeneous polynomial mapping of q and $\overline{q}$ of degree n defined by

$$h_{k,n_k}(q) = a_{k,0} c_{k,1}(q) a_{k,1} c_{k,2}(q) a_{k,2} \ldots a_{k,n_k-1} c_{k,n_k}(q) a_{k,n_k},$$

and $\mathbf{h} : \mathbb{H}^m \to \mathbb{H}^m$ be the mapping defined by

$$\mathbf{h}(\mathbf{q}) = (h_{1,n_1}(\mathbf{q}_1), \ldots, h_{m,n_m}(\mathbf{q}_m)).$$

Clearly, $\mathbf{h}(\mathbf{q}) = \mathbf{0}$ if and only if $\mathbf{q} = \mathbf{0}$, and the Brouwer degree $d[\mathbf{h}, B(R_1) \times \ldots \times B(R_m), \mathbf{0}]$ exists and is independent of $R_k > 0$ $(1 \leq k \leq m)$.

Theorem 6.1 *With the notations above,*

$$d[\mathbf{h}, B(R_1) \times \ldots \times B(R_m), \mathbf{0}] = \prod_{k=1}^{m} (n_{k,1} - n_{k,2}). \tag{11}$$

Proof. It follows from the cartesian product property of the Brouwer degree [17] and from Theorem 5.1 that

$$d[\mathbf{h}, B(R_1) \times \ldots \times B(R_m), \mathbf{0}] = \prod_{k=1}^{m} d[h_{k,n_k}, B(R_k), 0] = \prod_{k=1}^{m} (n_{k,1} - n_{k,2}).$$

■

Remark 6.1 Because of the excision property of the Brouwer degree [17] and the fact that $\mathbf{h}(\mathbf{q})$ only vanishes at $\mathbf{q} = \mathbf{0}$, formula (11) holds with $B(R_1) \times \ldots \times B(R_m)$ replaced by any open bounded neighborhood of the origin.

Let $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_m) : \mathbb{H}^m \to \mathbb{H}^m$ be a continuous mapping.

Corollary 6.1 *With the notations and assumptions of Theorem 6.1, assume that, for each $k \in \{1, \ldots, m\}$ there exists $R_k > 0$ such that*

$$\|\mathbf{g}_k(\mathbf{q})\| < \left(\prod_{j=0}^{m} \|a_{k,j}\| \right) R_k^{n_k} \quad \textit{for all} \ \ \mathbf{q} \in \partial[B(R_1) \times \ldots \times B(R_m)]. \tag{12}$$

Then

$$d[\mathbf{h} + \mathbf{g}, B(R_1) \times \ldots \times B(R_m), \mathbf{0}] = \prod_{k=1}^{m} (n_{k,1} - n_{k,2}).$$

Proof. We show that for each $\lambda \in [0,1]$ and each $\mathbf{q} \in \partial[B(R_1) \times \ldots \times B(R_m)]$, we have $\mathbf{h}(\mathbf{q}) + \lambda \mathbf{g}(\mathbf{q}) \neq \mathbf{0}$. If it is not the case, there is some $\lambda \in [0,1]$ and some $\mathbf{q} \in \partial[B(R_1) \times \ldots \times B(R_m)]$ such that, for each $k \in \{1, \ldots, m\}$, one has $\mathbf{h}_k(\mathbf{q}_k) + \mathbf{g}_k(\mathbf{q}) = 0$. As $\mathbf{q} \in \partial[B(R_1) \times \ldots \times B(R_m)]$, there is some $p \in \{1, \ldots, m\}$ such that $\|\mathbf{q}_p\| = R_p$, and hence, using (12),

$$\begin{aligned} 0 &= \|\mathbf{h}_p(\mathbf{q}_p) + \lambda \mathbf{g}_p(\mathbf{q})\| \geq \|\mathbf{h}_p(\mathbf{q}_p)\| - \|\mathbf{g}_p(\mathbf{q})\| \\ &= \left(\prod_{j=0}^{m} \|a_{p,j}\| \right) R_p^{n_p} - \|\mathbf{g}_p(\mathbf{q})\| > 0, \end{aligned}$$

a contradiction. The result then follows from the homotopy invariance property of the Brouwer degree and formula (11). ∎

7 Periodic solutions of quaternionic systems of differential equations

Let $n_k, \ m \geq 1$ be integers, $T > 0$, $\mathbf{a}_k \in \mathbb{H} \setminus \{0\}$ $(1 \leq k \leq m)$, $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_m) : [0,T] \times \mathbb{H}^m \to \mathbb{H}^m$ be continuous, and let us consider the T-periodic problem

$$\begin{aligned} \frac{d\mathbf{q}_k}{dt} &= \mathbf{a}_k \overline{\mathbf{q}}_k^{n_k} + \mathbf{g}_k(t, \mathbf{q}_1, \ldots, \mathbf{q}_m) \\ \mathbf{q}_k(0) &= \mathbf{q}_k(T) \quad (1 \leq k \leq m). \end{aligned} \tag{13}$$

We write, like in Section 6, $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m)$ and $\|\mathbf{q}\|_m = \max_{1 \leq k \leq m} \|\mathbf{q}_k\|$. Set

$$n_0 := \min\{n_1, n_2, \ldots, n_m\}.$$

The following existence result is not contained in the theorems given in [3].

Theorem 7.1 *If the condition*

$$\lim_{\|\mathbf{q}\|_m \to \infty} \frac{\mathbf{g}_k(t, \mathbf{q})}{\|\mathbf{q}_1\|^{n_0} + \ldots + \|\mathbf{q}_m\|^{n_0}} = 0 \tag{14}$$

holds uniformly in $t \in [0,T]$, then the problem (13) has at least one solution.

Proof. We use the continuation theorem of coincidence degree theory ([16], Theorem 2.4) in the space $C([0,T],\mathbb{H}^m)$ with the uniform norm

$$\|\mathbf{q}\|_C := \max_{1\le k\le m}\max_{t\in[0,T]}\|\mathbf{q}_k(t)\|.$$

Defining $L : D(L)\subset C([0,T],\mathbb{H}^m)\to C([0,T],\mathbb{H}^m)$ by

$$D(L)=\{\mathbf{q}\in C([0,T],\mathbb{H}^m) : \mathbf{q} \text{ is of class } C^1 \text{ and } \mathbf{q}(0)=\mathbf{q}(T)\},\ \ L\mathbf{q}=\frac{d\mathbf{q}}{dt},$$

and $N : C([0,T],\mathbb{H}^m)\to C([0,T],\mathbb{H}^m)$ by

$$N(\mathbf{q}) = \left(\mathbf{a}_1\mathbf{q}_1^{n_1}(\cdot)+\mathbf{g}_1(\cdot,\mathbf{q}(\cdot)),\dots,\mathbf{a}_m\mathbf{q}_m^{n_m}(\cdot)+\mathbf{g}_m(\cdot,\mathbf{q}(\cdot))\right),$$

it is easy to show that L is Fredholm of index zero, N is L-compact on bounded sets of $C([0,T],\mathbb{H}^m)$ [15, 16], and that the kernel $N(L)$ and range $R(L)$ of L are respectively given by

$$N(L)=\{\mathbf{q}\in C([0,T],\mathbb{H}^m) : \mathbf{q}=P\mathbf{q}\},\ R(L)=\{\mathbf{q}\in C([0,T],\mathbb{H}^m) : P\mathbf{q}=0\}$$

where P is the mean value operator

$$P\mathbf{q} := \frac{1}{T}\int_0^T \mathbf{q}(t)\,dt.$$

Let $\mathbf{q}$ be a possible solution of the differential system

$$\begin{aligned}\frac{d\mathbf{q}_k}{dt} &= \lambda\left[\mathbf{a}_k\overline{\mathbf{q}}_k^{n_k}+\mathbf{g_k}(t,\mathbf{q}_1,\dots,\mathbf{q}_m)\right],\ \ (\lambda\in(0,1])\\ \mathbf{q}_k(0) &= \mathbf{q}_k(T),\ \ (1\le k\le m).\end{aligned}\tag{15}$$

for some $\lambda\in(0,1]$. We have, for each $k\in\{1,\dots,m\}$,

$$\frac{d(\overline{\mathbf{a}}_k\mathbf{q}_k^{n_k+1})}{dt}=\overline{\mathbf{a}}_k\frac{d(\mathbf{q}_k^{n_k+1})}{dt}=\sum_{j=0}^{n_k}\overline{\mathbf{a}}_k\mathbf{q}_k^j\frac{d\mathbf{q}_k}{dt}\mathbf{q}_k^{n_k-j},$$

which, combined with (15), gives

$$\frac{d(\overline{\mathbf{a}}_k\mathbf{q}_k^{n_k+1})}{dt}=\lambda\sum_{j=0}^{n_k}\overline{\mathbf{a}}_k\mathbf{q}_k^j\mathbf{a}_k\overline{\mathbf{q}}_k^{n_k}\mathbf{q}_k^{n_k-j}+\lambda\sum_{j=0}^{n_k}\overline{\mathbf{a}}_k\mathbf{q}_k^j\mathbf{g}_k(t,\mathbf{q})\mathbf{q}_k^{n_k-j}.$$

Now,

$$\begin{aligned}\sum_{j=0}^{n_k}\overline{\mathbf{a}}_k\mathbf{q}_k^j\mathbf{a}_k\overline{\mathbf{q}}_k^{n_k}\mathbf{q}_k^{n_k-j} &= \sum_{j=0}^{n_k}\overline{\mathbf{a}}_k\mathbf{q}_k^j\mathbf{a}_k\overline{\mathbf{q}}_k^j\|\mathbf{q}_k\|^{2n_k-2j}\\ &= \sum_{j=0}^{n_k}\|\overline{\mathbf{a}}_k\mathbf{q}_k^j\|^2\|\mathbf{q}_k\|^{2n_k-2j}=(n_k+1)\|\mathbf{a}_k\|^2\|\mathbf{q}_k\|^{2n_k},\end{aligned}$$

so that

$$\frac{d(\overline{\mathbf{a}}_k \mathbf{q}_k^{n_k+1})}{dt} = \lambda(n_k+1)\|\mathbf{a}_k\|^2\|\mathbf{q}_k\|^{2n_k} + \lambda \sum_{j=0}^{n_k} \overline{\mathbf{a}}_k \mathbf{q}_k^j \mathbf{g}_k(t,\mathbf{q})\mathbf{q}_k^{n_k-j}. \tag{16}$$

Integrating both members of (16), using the T-periodicity and letting

$$Q_k := \int_0^T \|\mathbf{q}_k(t)\|^{n_k}\, dt,$$

we obtain

$$\begin{aligned} 0 &= (n_k+1)\|\mathbf{a}_k\|^2 \int_0^T \|\mathbf{q}_k(t)\|^{2n_k} dt \\ &+ \int_0^T \left(\sum_{j=0}^{n_k} \overline{\mathbf{a}}_k \mathbf{q}_k^j \mathbf{g}_k(t,\mathbf{q}(t))\mathbf{q}_k(t)^{n_k-j} \right) dt. \end{aligned}$$

Consequently,

$$\begin{aligned} (n_k+1)\|\mathbf{a}_k\|^2 Q_k &\leq \int_0^T \left(\sum_{j=0}^{n_k} \|\mathbf{a}_k\|\|\mathbf{g}_k(t,\mathbf{q}(t))\|\|\mathbf{q}_k(t)\|^{n_k} \right) dt \\ &= (n_k+1)\|\mathbf{a}_k\| \int_0^T \|\mathbf{g}_k(t,\mathbf{q}(t))\|\|\mathbf{q}_k(t)\|^{n_k} dt, \end{aligned}$$

and hence, for each $k \in \{1,\ldots,m\}$,

$$Q_k \leq \|\mathbf{a}_k\|^{-1} \int_0^T \|\mathbf{g}_k(t,\mathbf{q}(t))\|\|\mathbf{q}_k(t)\|^{n_k} dt. \tag{17}$$

Using assumption (14), given $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that

$$\|\mathbf{g}_k(t,\mathbf{q})\| \leq \varepsilon \left(\sum_{j=1}^{m} \|\mathbf{q}_j\|^{n_0} \right) + M(\varepsilon), \tag{18}$$

which, introduced in (17), gives, with

$$I_1 = \{t \in [0,T] : \|\mathbf{q}_j(t)\| < 1\},\ I_2 = \{t \in [0,T] : \|\mathbf{q}_j(t)\| \geq 1\},$$

$$\begin{aligned} Q_k &\leq \|\mathbf{a}_k\|^{-1} \left(\varepsilon \sum_{j=1}^{m} \int_0^T \|\mathbf{q}_j(t)\|^{n_0} \|\mathbf{q}_k(t)\|^{n_k} dt + M(\varepsilon) \int_0^T \|\mathbf{q}_k(t)\|^{n_k} dt \right) \\ &\leq \|\mathbf{a}_k\|^{-1} \varepsilon \left[\sum_{j=1}^{m} \int_0^T \frac{1}{2} \left(\|\mathbf{q}_j(t)\|^{2n_0} + \|\mathbf{q}_k(t)\|^{2n_k} \right)\, dt \right] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \|\mathbf{a}_k\|^{-1} M(\varepsilon) T^{1/2} \left(\int_0^T \|\mathbf{q}_k(t)\|^{2n_k} dt \right)^{1/2} \\
&= \quad \|\mathbf{a}_k\|^{-1} \varepsilon \left[\sum_{j=1}^m \frac{1}{2} \left(\int_{I_1} \|\mathbf{q}_j(t)\|^{2n_0} + \int_{I_2} \|\mathbf{q}_j(t)\|^{2n_0} \right) dt \right] \\
&+ \quad \frac{m}{2} \|\mathbf{a}_k\|^{-1} \varepsilon Q_k + \|\mathbf{a}_k\|^{-1} M(\varepsilon) T^{1/2} Q_k^{1/2} \\
&\leq \quad \frac{\|\mathbf{a}_k\|^{-1} \varepsilon}{2} \left[\sum_{j=1}^m (T + Q_j) + m Q_k \right] + \|\mathbf{a}_k\|^{-1} M(\varepsilon) T^{1/2} Q_k^{1/2}.
\end{aligned}
$$

Summing the last inequalities over k, we obtain

$$
\begin{aligned}
\sum_{k=1}^m Q_k \quad &\leq \quad \frac{mT\varepsilon}{2} \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1} \right) + \frac{\varepsilon}{2} \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1} \right) \left(\sum_{j=1}^m Q_j \right) \\
&+ \quad \frac{\varepsilon m}{2} \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1} Q_k \right) + M(\varepsilon) T^{1/2} \sum_{k=1}^m \left(\|\mathbf{a}_k\|^{-1} Q_k^{1/2} \right) \\
&\leq \quad \frac{\varepsilon}{2} \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1} + m \max_{1 \leq k \leq m} (\|\mathbf{a}_k\|^{-1}) \right) \left(\sum_{k=1}^m Q_k \right) \\
&+ \quad M(\varepsilon) T^{1/2} m \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-2} \right)^{1/2} \left(\sum_{k=1}^m Q_k \right)^{1/2} \\
&+ \quad \frac{mT\varepsilon}{2} \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1} \right).
\end{aligned}
$$

If we take now

$$
\varepsilon = \varepsilon_0 := \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1} + m \max_{1 \leq k \leq m} (\|\mathbf{a}_k\|^{-1}) \right)^{-1},
$$

we obtain

$$
\begin{aligned}
\sum_{k=1}^m Q_k \quad &\leq \quad 2M(\varepsilon_0) T^{1/2} m \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-2} \right)^{1/2} \left(\sum_{k=1}^m Q_k \right)^{1/2} \\
&+ \quad mT\varepsilon_0 \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1} \right),
\end{aligned}
$$

and hence

$$
\sum_{k=1}^m Q_k = \sum_{k=1}^m \int_0^T \|\mathbf{q}_k(t)\|^{2n_k} dt \leq R_0^2, \tag{19}
$$

where R_0 is the unique positive solution of the algebraic equation

$$x^2 - 2M(\varepsilon_0)T^{1/2}m\left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-2}\right)^{1/2} x - mT\varepsilon_0\left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1}\right) = 0.$$

Now, it follows from (15) and from (18) with, say, $\varepsilon = 1$, that, for each $k \in \{1,\ldots,m\}$,

$$\left\|\frac{d\mathbf{q}_k}{dt}(t)\right\| \leq \|\mathbf{a}_k\|\|\mathbf{q}_k(t)\|^{n_k} + \sum_{j=1}^m \|\mathbf{q}_j(t)\|^{n_0} + M(1),$$

and hence, using (19) and the following direct consequence of Hölder inequality

$$\left(\frac{1}{T}\int_0^T \|\mathbf{q}_k(t)\|^{\mu}\,dt\right)^{1/\mu} \leq \left(\frac{1}{T}\int_0^T \|\mathbf{q}_k(t)\|^{\nu}\,dt\right)^{1/\nu} \quad (1 \leq \mu \leq \nu),$$

we obtain, for each $k \in \{1,\ldots,m\}$,

$$\begin{aligned}
\int_0^T \left\|\frac{d\mathbf{q}_k}{dt}(t)\right\| dt &\leq \|\mathbf{a}_k\|\int_0^T \|\mathbf{q}_k(t)\|^{n_k}\,dt + \sum_{j=1}^m \int_0^T \|\mathbf{q}_j(t)\|^{n_0}\,dt + TM(1)\\
&\leq \|\mathbf{a}_k\|T^{1/2}Q_k^{1/2} + \sum_{j=1}^m T^{1-\frac{n_0}{2n_j}} Q_j^{\frac{n_0}{2n_j}} + TM(1)\\
&\leq \|\mathbf{a}_k\|T^{1/2}R_0 + \sum_{j=1}^m T^{1-\frac{n_0}{2n_j}} R_0^{\frac{n_0}{n_j}} + TM(1).
\end{aligned}$$

Those conditions, together with (19) easily imply the existence of $R > 0$ such that, for each $\lambda \in (0,1]$ and possible solution of (15), one has

$$\max_{t\in[0,T]} \|\mathbf{q}_k(t)\| < R \ \ (1 \leq k \leq m).$$

We have now to find a priori estimates for the possible solutions $\mathbf{c} \in \mathbb{H}^m$ of the system $PN(\mathbf{c}) = 0$, explicitely

$$\mathbf{a}_k\mathbf{c}_k^{n_k} + \frac{1}{T}\int_0^T \mathbf{g}_k(t,\mathbf{c})\,dt = 0 \quad (1 \leq k \leq m). \tag{20}$$

If $\mathbf{c}$ is a possible solution of (20), then, for each $k \in \{1,\ldots,m\}$,

$$\|\mathbf{a}_k\|\|\mathbf{c}_k\|^{n_k} \leq \varepsilon\left(\sum_{j=1}^m \|\mathbf{c}_j\|^{n_0}\right) + M(\epsilon) \leq \varepsilon\left[\sum_{j=1}^m (1 + \|\mathbf{c}_j\|^{n_j})\right] + M(\varepsilon),$$

so that, by adding those inequalities after division of both members by $\|\mathbf{a}_k\|$,

$$\sum_{k=1}^m \|\mathbf{c}_k\|^{n_k} \leq \left(\sum_{k=1}^m \|\mathbf{a}_k\|^{-1}\right)\left[\varepsilon\left(\sum_{j=1}^m \|\mathbf{c}_j\|^{n_j} + m\right) + M(\varepsilon)\right].$$

Choosing $\varepsilon = \varepsilon_1 = \frac{1}{2}\left(\sum_{k=1}^{m} \|\mathbf{a}_k\|^{-1}\right)^{-1}$, we obtain

$$\sum_{k=1}^{m} \|\mathbf{c}_k\|^{n_k} \leq 2\left(\sum_{k=1}^{m} \|\mathbf{a}_k\|^{-1}\right)[\varepsilon_1 m + M(\varepsilon_1)],$$

and hence there exists $r > 0$ such that each possible solution of $PN(\mathbf{c}) = 0$ is such that

$$\|\mathbf{c}_k\| < r \quad (1 \leq k \leq m).$$

Increasing the smallest of the numbers r and R, we can assume without loss of generality that $r = R$.

It remains to compute $d[PN|_{N(L)}, B(R)^m, \mathbf{0}]$. The estimates obtained for the possible solutions of the system (20) hold as well for the family of equations

$$\mathbf{f}_k(\mathbf{c}, \lambda) := \mathbf{a}_k \mathbf{c}_k^{n_k} + \frac{\lambda}{T}\int_0^T \mathbf{g}_k(t, \mathbf{c})\, dt = 0 \quad (1 \leq k \leq m) \quad (\lambda \in [0,1]).$$

Hence, by the homotopy invariance property of the Brouwer degree and Theorem 6.1, we have

$$\begin{aligned} d[PN|_{N(L)}, B(R)^m, \mathbf{0}] &= d[\mathbf{f}(\cdot,1), B(R)^m, \mathbf{0}] = d[\mathbf{f}(\cdot,0), B(R)^m, \mathbf{0}] \\ &= (-1)^m \prod_{k=1}^{m} n_k \neq 0. \end{aligned}$$

All conditions of Theorem 2.4 in [16] are satisfied and the proof is complete. ■

Remark 7.1 The proof of Theorem 2.4 in [16] also implies that, for the coincidence degree $d_L[L - N, b(R)^m]$ of $L - N$ in $b(R)^m$, where $b(R)$ denotes the open ball centered at the origine and of radius R in $C([0,T], \mathbb{H})$, one has

$$|d_L(L - N, b(R)^m]| = |d[PN|_{N(L)}, B(R)^m, 0]| = \prod_{k=1}^{m} n_k. \tag{21}$$

Remark 7.2 As shown by a counterexample of the form

$$\frac{dq}{dt} = q^2 + h(t), \quad q(0) = q(T)$$

given in [3], the conclusion of Theorem 7.1 does not hold if the $\overline{\mathbf{q}}_k$ in the first terms of the right-hand member are replaced by $\mathbf{q}_k$.

For $n_k = 1 \ (1 \leq k \leq m)$, we have the following special case of Theorem 7.1.

Corollary 7.1 *If* $\mathbf{a}_k \in \mathbb{H} \setminus \{0\}$ $(1 \leq k \leq m)$ *and the condition*

$$\lim_{\|\mathbf{q}\|_m \to \infty} \frac{\mathbf{g}_k(t, \mathbf{q})}{\|\mathbf{q}_1\| + \ldots + \|\mathbf{q}_m\|} = 0 \tag{22}$$

holds uniformly in $t \in [0,T]$, then the problem

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{a}_k\overline{\mathbf{q}}_k + \mathbf{g}_k(t,\mathbf{q}_1,\ldots,\mathbf{q}_m), \quad \mathbf{q}_k(0) = \mathbf{q}_k(T) \quad (1 \le k \le m)$$

has at least one solution.

Remark 7.3 It follows from Corollary 7.1 that the non-homogeneous linear problem

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{a}_k\overline{\mathbf{q}}_k + \mathbf{g}_k(t), \quad \mathbf{q}_k(0) = \mathbf{q}_k(T) \quad (1 \le k \le m)$$

has a solution for each forcing term $\mathbf{g}(t)$. This implies that the corresponding homogeneous linear problem

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{a}_k\overline{\mathbf{q}}_k, \quad \mathbf{q}_k(0) = \mathbf{q}_k(T) \quad (1 \le k \le m)$$

only admits the trivial solution.

Corollary 7.2 *If the $\mathbf{p}_k$ are polynomials in $\mathbf{q}_1,\overline{\mathbf{q}}_1,\ldots,\mathbf{q}_m,\overline{\mathbf{q}}_m$ of total degree smaller or equal to $n_0 - 1$ whose coefficients are continuous functions of $[0,T]$ into $\mathbb{H}$ $(1 \le k \le m)$, then the problem*

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{a}_k\overline{\mathbf{q}}_k^{n_k} + \mathbf{p}_k(t,\mathbf{q}_1,\overline{\mathbf{q}}_1,\ldots,\mathbf{q}_m,\overline{\mathbf{q}}_m), \quad \mathbf{q}_k(0) = \mathbf{q}_k(T) \quad (1 \le k \le m)$$

has at least one solution.

Example 7.1 If $a \in \mathbb{H}\setminus\{0\}$ and $b_1, b_2, c_1, c_2, f : [0,T] \to \mathbb{H}$ are continuous, the problem

$$\frac{dq}{dt} = a\overline{q}^2 + b_1(t)qb_2(t) + c_1(t)\overline{q}c_2(t) + f(t), \quad q(0) = q(T)$$

has at least one solution.

8 Nontrivial periodic solutions of quaternionic valued differential systems

Let us now assume that

$$\mathbf{g}(t,0) = \mathbf{0} \ \text{ for all } \ t \in [0,T], \tag{23}$$

so that $\mathbf{0}$ is a solution of problem (13). Supplementary assumptions on the behavior of $\mathbf{g}$ near $\mathbf{q} = \mathbf{0}$ imply the existence of a nontrivial solution.

Theorem 8.1 *Assume that the assumptions of Theorem 7.1 and condition (23) hold, and that there exists a continuous mapping* $\mathbf{A} : [0,T] \times \mathbb{H}^m \to \mathcal{L}(\mathbb{H}^m, \mathbb{H}^m)$ *such that*

$$\lim_{\|\mathbf{q}\|_m \to 0} \frac{\mathbf{g}(t,\mathbf{q}) - \mathbf{A}(t)\mathbf{q}}{\|\mathbf{q}\|} = 0.$$

and such that the linear problem

$$\frac{d\mathbf{q}}{dt} = \mathbf{A}(t)\mathbf{q}, \quad \mathbf{q}(0) = \mathbf{q}(T) \tag{24}$$

has only the trivial solution. Then, if $n_k \geq 2$ *for at least one* $k \in \{1, \ldots, m\}$, *the problem (13) has at least one non trivial solution.*

Proof. Of course, the a priori bounds obtained in the proof of Theorem 7.1 are still valid. It follows from the assumptions and Proposition VIII.3 of [15] that there exists $r \in (0, R)$ sufficiently small such that 0 is the unique zero of $L - N$ in $\overline{b}(r)^m \subset C([0,T], \mathbb{H}^m)$ and that

$$d_L[L - N, b(r)^m] = d_L[L - A, b(r)^m] = \pm 1, \tag{25}$$

where $A : C([0,T], \mathbb{H}^m) \to C([0,T], \mathbb{H}^m)$ is the linear operator defined by $A\mathbf{q}(\cdot) = \mathbf{A}(\cdot)\mathbf{q}(\cdot)$. Now, from the additivity property of coincidence degree [15, 16], we have

$$d_L[L - N, b(R)^m \setminus \overline{b}(r)^m] = d_L[L - N, b(R)^m] - d_L[L - N, b(r)^m]$$

so that, using (21) and (25),

$$\begin{aligned} |d_L[L - N, b(R)^m \setminus \overline{b}(r)^m]| &\geq |d_L[L - N, b(R)^m]| - |d_L[L - N, b(r)^m]| \\ &= \prod_{j=1}^m n_j - 1 \geq 1, \end{aligned}$$

The existence property of the coincidence degree [15, 16] implies that $L - N$ has a zero in $b(R)^m \setminus \overline{b}(r)^m$, and hence problem (13) has a nontrivial T-periodic solution. ■

Remark 8.1 A class of linear problems (24) having only the trivial solution is given by Remark 7.3. Other examples can be found in [3].

Example 8.1 If $a, f \in \mathbb{H} \setminus \{0\}$ and $b, c : [0,T] \to \mathbb{H}$ are continuous, the problem

$$\frac{dq}{dt} = a\overline{q}^3 + b(t)q^2 + c(t)\overline{q}^2 + f\overline{q}, \quad q(0) = q(T)$$

has a nontrivial solution.

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