

RIGIDITY AND PERSISTENCE FOR ENSURING SHAPE MAINTENANCE OF MULTIAGENT META FORMATIONS (EXT'D VERSION)

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ABSTRACT. This paper treats the problem of the merging of formations, where the underlying model of a formation is graphical. We first analyze the rigidity and persistence of meta-formations, which are formations obtained by connecting several rigid or persistent formations. Persistence is a generalization to directed graphs of the undirected notion of rigidity. In the context of moving autonomous agent formations, persistence characterizes the efficacy of a directed structure of unilateral distance constraints seeking to preserve a formation shape. We derive then, for agents evolving in a two- or three-dimensional space, the conditions under which a set of persistent formations can be merged into a persistent meta-formation, and give the minimal number of interconnections needed for such a merging. We also give conditions for a meta-formation obtained by merging several persistent formations to be persistent.

Keywords: Formations, Meta-formations, Rigidity, Persistence, Autonomous Agents

1. INTRODUCTION

Recently, significant interest has been shown on the behavior of autonomous agent formations (groups of autonomous agents interacting with each other) [2, 4, 7, 9, 19], and more recently on meta-formations, which is the name ascribed to an interconnection of formations, generally with the individual formations being separate [1, 25]. By autonomous agent, we mean here any human-controlled or unmanned vehicle moving by itself and having a local intelligence or computing capacity, such as ground robots, air vehicles or underwater vehicles. Many reasons such as obstacle avoidance and dealing with a predator can indeed lead a (meta-)formation to be split into smaller formations which are later re-merged. Those smaller formations need to be organized in such a way that they can behave autonomously when the formation is split. Conversely, some formations may need to be temporarily merged into a meta-formation to accomplish a certain task, this meta-formation being split afterwards.

The particular property of formations and meta-formations which we analyze here is *persistence*. This graph-theoretical notion which generalizes the notion of rigidity to directed graphs was introduced in [9] to analyze the behavior of autonomous agent formations governed by unilateral distance constraints: Many applications require the shape of a multi-agent formation

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to be preserved during a continuous move. For example, target localization by a group of unmanned airborne vehicles (UAVs) using either angle of arrival data or time difference of arrival information appears to be best achieved (in the sense of minimizing localization error) when the UAVs are located at the vertices of a regular polygon [5]. Other examples of optimal placements for groups of moving sensors can be found in [17]. This objective can be achieved by explicitly keeping *some* inter-agent distances constant. In other words, some inter-agent distances are explicitly maintained constant so that all the inter-agent distances remain constant. The information structure arising from such a system can be efficiently modelled by a graph, where agents are abstracted by vertices and actively constrained inter-agent distances by edges.

We assume here that those constraints are unilateral, i.e., that the responsibility for maintaining a distance is not shared by the two concerned agents but relies on only one of them. This unilateral character can be a consequence of the technological limitations of the autonomous agents. Some UAV's can for example not efficiently sense objects that are behind them or have an angular sensing range smaller than 360° [3, 8, 20]. Also, some of the authors of this paper are working with agents in which optical sensors have blind three dimensional cones. It can also be desired to ease the trajectory control of the formation, as it allows so-called leader-follower formations [2, 6, 21]. In such a formation, one agent (leader) is free of inter-agent distance constraints and is only constrained by the desired trajectory of the formation, and a second agent (first follower) is responsible for only one distance constraint and can set the relative orientation of the formation. The other agents have no decision power and are forced by their distance constraints to follow the two first agents.

This asymmetry is modelled using directed edges in the graph. Intuitively, an information structure is persistent if, provided that each agent is trying to satisfy all the distance constraints for which it is responsible, it can do so, with all the inter-agent distances then remaining constant, and as a result the formation shape is preserved. A necessary but not sufficient condition for persistence is *rigidity* [9], which intuitively means that, provided that all the prescribed distance constraints are satisfied during a continuous displacement, all the inter-agent distances remain constant (These concepts of persistence and rigidity are more formally reviewed in the next section). The above notion of rigidity can also be applied to structural frameworks where the vertices correspond to joints and the edges to bars. The main difference between rigidity and persistence is that rigidity assumes all the constraints to be satisfied, as if they were enforced by an external agency or through some mechanical properties, while persistence considers each constraint to be the responsibility of a single agent. As explained in [9], persistence implies rigidity, but it also implies that the responsibilities imposed on each agent are not inconsistent, for there can indeed be situations where this is so, and they must be avoided. Rigidity is thus an undirected notion (not depending on the edge directions), while persistence is a directed one. Both rigidity and persistence can be analyzed from a graph-theoretical point of view, and it can be proved [9, 22, 28] that if a formation is rigid (resp. persistent), then almost all formations represented by the same graph are rigid (resp. persistent).

As stated in [1], the problem of merging rigid formations into a rigid meta-formation has been considered in a number of places. In [18, 23], the rigidity of a multi-graph (a graph in which some vertices are abstractions of smaller graphs) is analyzed. In two dimensions, the vertices of a multi-graph can be thought as two dimensional solid bodies at the boundary of which some bars can be attached; two vertices are then connected by an edge if the corresponding bodies are attached to the same bar. Of course, the idea extends obviously to three dimensions. Operational ways to merge two rigid formations into a larger rigid formation can also be found in [7, 26].

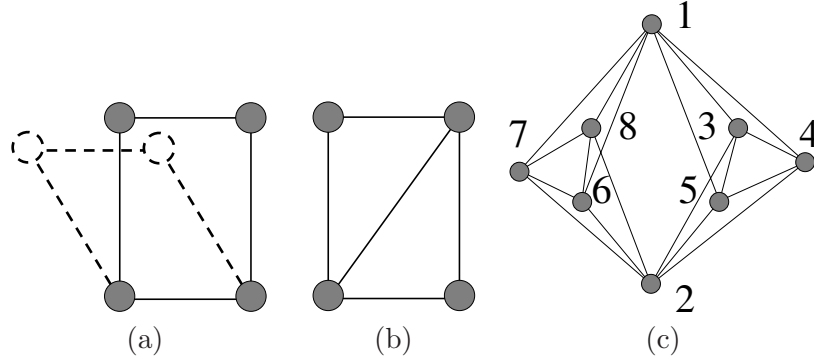


FIGURE 1. In $\mathbb{R}^2$, the graph represented in (a) is not rigid because it can be deformed (dashed line), while the one in (b) is rigid. The graph (c) satisfies the first two conditions of Theorem 2 but not the third one, and is therefore not rigid in $\mathbb{R}^3$: the two parts of the graph can rotate around the axis defined by 1 and 2.

In this paper, we treat the problem of determining whether a given meta-formation obtained by merging several persistent formations is persistent. For this purpose, we first consider the above mentioned problem of determining whether a meta-formation obtained by merging rigid formations is rigid. We also analyze the conditions under which a collection of persistent formations can be merged into a persistent meta-formation. Conditions are then given on the minimal number of additional links that are needed to achieve such a merging. Note that throughout all the paper, we always assume that the internal structure of the formations cannot be modified. Moreover, we use a convenient graph theoretical formalism, abstracting agents by vertices and (unilateral) distance constraints by (directed) edges.

After reviewing some properties of rigidity and persistence of graphs in Section 2, we examine in Section 3 the issues mentioned above for agents evolving in a two-dimensional space. We show in Section 4 how our results can be generalized in a three-dimensional space, and explain why this generalization can only partially be achieved. Note that some proofs are omitted for three-dimensional space when they are direct generalization of results on two-dimensional space. The paper ends with the concluding remarks in Section 5.

This paper is an extended version of [12] in which some proofs are omitted for space reasons. Some preliminary results have also been published in [11] without proofs, and are included here at a greater level of details. Moreover, Propositions 7 and 8 correct the unproven Proposition 5 in [11], which did not take the case described in Proposition 7 into account.

2. REVIEW OF RIGIDITY AND PERSISTENCE

2.1. Rigidity. As explained in Section 1, the rigidity of a graph has the following intuitive meaning: Suppose that each vertex represents an agent in a formation, and each edge represents an inter-agent distance constraint enforced by an external observer. The graph is rigid if for almost every such structure, the only possible continuous moves are those which preserve every inter-agent distance, as shown in Fig. 1(a) and (b). For a more formal definition, the reader is referred to [9, 22]. In $\mathbb{R}^2$, that is, if the agents represented by the vertices of the graph evolve in two dimensions, there exists a combinatorial criterion to check if a given graph is rigid:

Theorem 1 (Laman [15, 24]). *A graph $G = (V, E)$, with $|V| > 1$, is rigid in $\mathbb{R}^2$ if and only if there is a sub-set $E' \subseteq E$ such that*
(i) $|E'| = 2|V| - 3$.

(ii) For all non-empty $E'' \subseteq E'$ there holds
 $|E''| \leq 2|V(E'')| - 3$,
 where $V(E'')$ is the set of vertices incident to edges of E'' .

Unfortunately, the analogous criterion in $\mathbb{R}^3$ is only necessary.

Theorem 2. *If a graph $G = (V, E)$, with $|V| > 2$, is rigid in $\mathbb{R}^3$, there exists $E' \subseteq E$ such that*
 (i) $|E'| = 3|V| - 6$.
 (ii) *For all non-empty $E'' \subseteq E'$, there holds*
 $|E''| \leq 3|V(E'')| - 6$, *where $V(E'')$ is the set of vertices incident to edges of E'' .*
 (iii) *The graph $G'(V, E')$ is 3-connected (i.e. remains connected after removal of any pair of vertices).*

Condition (iii), which also implies the 3-connectivity of G , is not usually stated but is independently necessary even if the two first conditions are satisfied. Fig. 1(c) shows for example a non-rigid graph for which (i) and (ii) are satisfied, but not (iii). Intuitively, the graph G' in the theorem needs to be sufficient to ensure “alone” the rigidity of G . 3-connectivity is then needed as otherwise two or more parts of the graph could rotate around the axis defined by any pair of vertices whose removal would disconnect the graph. Note that such connectivity condition is not necessary in 2-dimensional spaces, as the counting conditions (i) and (ii) of Theorem 1 imply the 2-connectivity. For more information on necessary conditions for rigidity in three-dimensional spaces, we refer the reader to [16].

We say that a graph is *minimally rigid* if it is rigid and if no single edge can be removed without losing rigidity. It follows from the results above that a graph is minimally rigid in $\mathbb{R}^2$ (resp. in $\mathbb{R}^3$) if and only if it is rigid and contains $2|V| - 3$ (resp. $3|V| - 6$) edges [22]. Therefore we have the following characterization of minimal rigidity in $\mathbb{R}^2$.

Theorem 3 (Laman [15, 24]). *A graph $G = (V, E)$, with $|V| > 1$, is minimally rigid in $\mathbb{R}^2$ if and only if it is rigid and contains $2|V| - 3$ edges, or equivalently if and only if*
 (i) $|E| = 2|V| - 3$.
 (ii) *For all non-empty $E'' \subseteq E$ there holds*
 $|E''| \leq 2|V(E'')| - 3$, *where $V(E'')$ is the set of vertices incident to edges of E'' .*

The notion of rigidity can also be described from a linear algebraic point of view, using the so-called rigidity matrix. Suppose that a position $p_i \in \mathbb{R}^d$ (with $d = 2, 3$) is given to each vertex i of a graph $G = (V, E)$, and let $p \in \mathbb{R}^{d|V|}$ be the juxtapositions of all positions. For each vertex, consider now an infinitesimal displacement δp_i , and let δp be a vector obtained by juxtaposing these displacements. Since with infinitesimal displacements one can neglect higher order terms, the distance between the positions of two vertices i and j is preserved by the set of infinitesimal displacements if

$$(1) \quad (p_i - p_j)^T (\delta p_i - \delta p_j) = 0.$$

Hence, if each edge represents a distance constraint, a set of infinitesimal displacements is allowed if and only if (1) is satisfied for any edge $(i, j) \in E$. This set of linear constraints can be conveniently re-expressed in a condensed form as $R_G \delta p = 0$ where $R_G \in \mathbb{R}^{|E| \times d|V|}$ is the rigidity matrix, which contains one row for each edge and d columns for each vertex. In the row corresponding to the edge (i, j) , the $d(i-1) + 1^{st}$ to di^{th} columns are $(p_i - p_j)^T$, the $d(j-1) + 1^{st}$ to dj^{th} columns are $(p_j - p_i)^T$, and all other columns are 0. A graph G is rigid if for almost all position assignment its rigidity matrix has a rank $d|V| - f(d, |V|)$, where $f(d, |V|)$ is the number

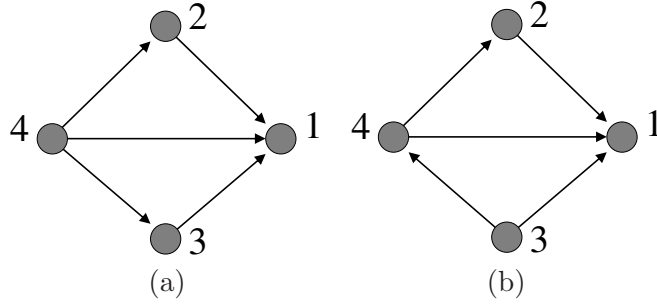


FIGURE 2. In $\mathbb{R}^2$, the graph represented in (a) is rigid but not persistent. For almost all uncoordinated displacements of 2, 3 and 4 (even if they satisfy their constraints), 4 is indeed unable to satisfy its three constraints. This problem cannot happen for the graph represented in (b), which is persistent.

of degrees of freedom in a d -dimensional space of a $\min(|V| - 1, d)$ -dimensional rigid body (Observe that $\min(|V| - 1, d)$ is the largest possible dimension of a graph on $|V|$ vertices embedded in a d -dimension space). In a 2-dimensional space, a single point has two DOFs $f(2, 1) = 2$, and any one or two-dimensional body has three DOFs. In a three-dimensional space, a single point has three DOFs, a one-dimensional object has five DOFs, and any other object has six DOFs. A subgraph $G'(V', E') \subseteq G(V, E)$ is rigid if the restriction $R_{G'}$ of R_G to the rows and columns corresponding to E' and V' has a rank $d|V'| - f(d, |V'|)$. Note that the rank $d|V'| - f(d, |V'|)$ is the maximal that can be attained by a rigidity (sub-)matrix. In a minimally rigid (sub-)graph, this rank is attained with a minimal number of edges and all rows of the rigidity matrix are thus linearly independent. For more information on the rigidity matrix, we refer the reader to [22].

2.2. Persistence. Consider now that the constraints are not enforced by an external entity, but that each constraint is the responsibility of one agent to enforce. To each agent, one assigns a (possibly empty) set of unilateral distance constraints represented by directed edges: the notation $\overrightarrow{(i, j)}$ for a directed edge connotes that the agent i has to maintain its distance to j constant during any continuous move. As explained in the Introduction, the *persistence* of the directed graph means that provided that each agent is trying to satisfy its constraints, the distance between any pair of connected or non-connected agents is maintained constant during any continuous move, and as a consequence the shape of the formation is preserved. Note though that the assignments given to an agent may be impossible to fulfill, in which case persistence is not achieved. An example of a persistent and a non-persistent graph having the same underlying undirected graph is shown in Fig. 2. For a more formal definition of persistence, the reader is referred to [9, 28], where are also proved the rigidity of all persistent graphs and the following criterion to check persistence:

Theorem 4. *A graph G is persistent in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) if and only if every subgraph obtained from G by removing edges leaving vertices whose out-degree is greater than 2 (resp. 3) until no such vertex is present anymore in the graph is rigid.*

A key result in the proof of Theorem 4 [9, 28] is the following:

Proposition 1. *A persistent graph $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) remains persistent after removal of an edge leaving a vertex whose out-degree is larger than 2 (resp. 3).*

We use the term *number of degrees of freedom* of a vertex i to denote the (generic) dimension of the set in which the corresponding agent can choose its position (all the other agents being fixed). Thus it represents in some sense the decision power of this agent. In a three-dimensional space, an agent being responsible for one distance constraint can for example freely move on the surface of a sphere centered on the agent from which the distance needs to be maintained, and has thus two degrees of freedom. The number of degrees of freedom of a vertex i in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is given by $\max(0, 2 - d^+(i))$ (resp. $\max(0, 3 - d^+(i))$), where $d^+(i)$ represent the out-degree of the vertex i . A vertex having a maximal number of degrees of freedom (i.e. an out-degree 0) is called a *leader* since the corresponding agent does not have any distance constraint to satisfy. We call *the number of degrees of freedom of a graph* the sum of the numbers of degrees of freedom over all its vertices. It is proved in [9, 28] that this quantity cannot exceed 3 in $\mathbb{R}^2$ and 6 in $\mathbb{R}^3$. Note that those numbers correspond to the number of independent translations and rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$. In the sequel we abbreviate degree of freedom by DOF.

As explained in [28], although the concept of persistence is applicable in three and larger dimensions, it is not sufficient to imply the desired stability of the formation shape. For the shape stability, the graph corresponding to a three-dimensional formation needs in addition to be *structurally persistent*. In $\mathbb{R}^3$, a graph is structurally persistent if and only if it is persistent and contains at most one leader, i.e. at most one vertex with no outgoing edge. In $\mathbb{R}^2$, persistence and structural persistence are equivalent.

Similarly to minimal rigidity, we say that a graph is *minimally persistent* if it is persistent and if no single edge can be removed without losing persistence. It is proved in [9, 28] that a graph is minimally persistent if and only if it is persistent and minimally rigid. The number of edges of such a graph is thus uniquely determined by the number of its vertices as it is the case for minimally rigid graphs.

3. RIGIDITY AND PERSISTENCE OF 2D META-FORMATIONS

3.1. Rigidity. Consider a set N of disjoint rigid (in $\mathbb{R}^2$) graphs $G_1, \dots, G_{|N|}$ having at least two vertices each, and a set S of single-vertex graphs $G_{|N|+1}, \dots, G_{|N|+|S|}$. In the sequel, those graphs are called *meta-vertices*, and it is assumed that no modification can be made on their internal structure: no internal edge or vertex can be added to or removed from a meta-vertex. We define the merged graph G by taking the union of all the meta-vertices, and of some additional edges E_M each of which has end-points belonging to different meta-vertices.

The conditions under which the merging of two meta-vertices leads to a rigid graph are detailed in [26]: If both meta-vertices contain more than one vertex, the merged graph is rigid if and only if E_M contains at least three edges, the aggregate of which are incident to at least two vertices of each meta-vertex. This is actually a particular case of the following result for an arbitrary number of graphs (analogous to a result in [18] which is obtained under the assumption that no vertex of any meta-vertex is incident to more than one edge of E_M):

Theorem 5. *If it contains at least two vertices, $G = \left(\bigcup_{N,S} G_i\right) \cup E_M$ (with N and S as defined at the beginning of this section) is rigid if and only if there exists $E'_M \subseteq E_M$ such that*

- (i) $|E'_M| = 3|N| + 2|S| - 3$.
- (ii) *For all non-empty $E''_M \subseteq E'_M$, there holds*

$$|E''_M| \leq 3|I(E''_M)| + 2|J(E''_M)| - 3,$$

where $I(E''_M)$ is the set of meta-vertices such that there are at least two vertices within the meta-vertex all incident to edges of E''_M , and $J(E''_M)$ is the set of meta-vertices such that there is precisely one vertex within the meta-vertex that is incident to one or several edges of E''_M . Note

that in each case, there can be an arbitrary number of vertices in the meta-vertex which are not incident on any edge of E''_M .

To prove this theorem, we first need the following lemma, which we shall prove the both for $\mathbb{R}^2$ and $\mathbb{R}^3$, intending to use the $\mathbb{R}^3$ result in the next section.

Lemma 1. *Let $G(V, E)$ be a rigid graph (in $\mathbb{R}^2$ or $\mathbb{R}^3$), and $G'_1, \dots, G'_N$ be minimally rigid subgraphs of G having distinct vertices. Then there exists a minimally rigid subgraph $G'(V, E')$ of G containing all vertices of G and all subgraphs G_i .*

Proof. For simplicity, let us first consider the 2-dimensional case. Consider the rigidity matrix R_G of G . Since G is rigid, it has (for almost all positions) a rank $2|V| - 3$. Since each G'_i is minimally rigid, the restriction $R_{G'_i}$ of R_G to the rows and columns corresponding to the edges and vertices of G'_i has $2|V_i| - 3$ linearly independent rows (or is an empty matrix if $|V_i| = 1$). Also, since the vertices of the different G'_i are distinct, there can be no dependence between rows corresponding to edges of different subgraphs G'_i . Therefore, all rows of $R_{\bigcup G'_i}$, corresponding to all edges of $\bigcup G'_i$, are linearly independent. Since the rank of R_G is $2|V| - 3$, it is a standard result in linear algebra that $R_{\bigcup G'_i}$ can be completed by the addition of further rows of R_G to obtain a subset of $2|V| - 3$ linearly independent rows of R_G . Letting E' be the set of edges corresponding to this set of rows, the graph $G'(V, E')$ is a minimally rigid subgraph of G containing all G'_i . This completes the proof for the 2-dimensional case. The proof for the 3-dimensional case is established following the same steps above, but replacing $2|V| - 3$ by $3|V| - 6$ and adding a special case for $|V_i| = 2$ in addition to the case where $|V_i| = 1$. $\square$

We can now prove Theorem 5.

Proof. For every G_i , let G'_i be a minimally rigid subgraph of G_i on the same vertices (The existence of such subgraphs follows directly from the definition of minimal rigidity, and they can be obtained by successively removing edges from the initial graph). Since they are minimally rigid, they contain $2|V_i| - 3$ edges if $G_i \subseteq N$ and no edge if $G_i \subseteq S$.

We first suppose that there exists a set E'_M as described in the theorem and prove the rigidity of G , by proving the minimal rigidity of one of its subgraph viz., $G' = (V, E') = \left(\bigcup_{N,S} G'_i\right) \cup E'_M$ which contains all its vertices. The number of edges in G' is

$$\begin{aligned} |E'| &= |E'_M| + \sum_{G_i \in N} |E'_i| \\ &= 3|N| + 2|S| - 3 + \sum_{G_i \in N} (2|V_i| - 3) \\ &= 2|V| - 3, \end{aligned}$$

since $|V| = |S| + \sum_{G_i \in N} |V_i|$. To show that G' satisfies the second condition of Theorem 1, suppose that there exists a subset of edges $E'' \subset E'$ such that $|E''| > 2|V(E'')| - 3$, let I be the set of meta-vertices containing at least two vertices of $V(E'')$ and J the set of meta-vertices containing only one vertex of $V(E'')$. Let now $E''_M = E_M \cap E''$ and for each i , $V''_i = V(E'') \cap V_i$ and $E''_i = E'' \cap E'_i$. There holds $V(E'') = \sum_{G_i \in I} |V''_i| + |J|$, and $E'' = E''_M + \sum_{G_i \in I} |E''_i|$. Moreover, since each G'_i is minimally rigid, it follows from Theorem 3 that $|E''_i| \leq 2|V''_i| - 3$. We have then

$$\begin{aligned} |E''_M| &= |E''| - \sum_{G_i \in I} |E''_i| \\ &> 2|V''| - 3 - \sum_{G_i \in I} (2|V''_i| - 3) \\ &= 3|I| + 2|J| - 3, \end{aligned}$$

so that this $E''_M \subseteq E'_M$ does not satisfy condition (ii) in the theorem.

We now suppose that G is rigid. It follows from Lemma 1 that there is a minimally rigid subgraph $G'(V, E') \subseteq G$ containing all G'_i . Let $E'_M = E' \cap E_M$; we prove that E'_M satisfies the condition of this theorem. Since G' is minimally rigid, there holds $|E'| = 2|V| - 3$. Moreover, we have $|E'| = |E'_M| + \sum_{i \in N} |E'_i|$, and $|V| = \sum_{G_i \in N} |V'_i| + |S|$, so that

$$|E'_M| = 2|V| - 3 - \sum_{G_i \in N} (2|V_i| - 3) = 3|N| + 2|S| - 3.$$

E'_M contains thus the predicted number of edges. We suppose now that there is a set E''_M such that $|E''_M| > 3|I(E''_M)| + 2|J(E''_M)| - 3$ and show that this contradicts the minimal rigidity of G' . Let us build E'' by taking the union of E''_M and all E'_i for which $G_i \in I(E''_M)$. There holds $|V(E'')| = |J(E''_M)| + \sum_{G_i \in I(E''_M)} |V_i|$. Therefore, we have

$$\begin{aligned} |E''| &= |E''_M| + \sum_{i \in I(E''_M)} |E'_i| \\ &> 3|I(E''_M)| + 2|J(E''_M)| - 3 \\ &\quad + \sum_{G_i \in I(E''_M)} (2|V_i| - 3) \\ &= 2V(E'') - 3. \end{aligned}$$

By Theorem 3, this contradicts the minimal rigidity of $G'(V, E')$ as $E'' \subseteq E'$. $\square$

This criterion can be checked by a quadratic time algorithm (with respect to the number of meta-vertices) which would be a simple adaptation of the pebble game algorithm that is used for rigid graphs (see [14]), or even faster [18].

For a given collection of meta-vertices, we say that G is an *edge-optimal rigid merging* if no single edge of E_M can be removed without losing rigidity. Notice that a single graph can be an edge-optimal rigid merging with respect to a certain collection of meta-vertices, and not with respect to another one, as shown in Fig. 3. If all meta-vertices are minimally rigid, then an edge-optimal rigid merging is also a minimally rigid graph. From Theorem 5, one can deduce the following characterization of edge-optimal rigid merging.

Theorem 6. $G = \left(\bigcup_{N,S} G_i\right) \cup E_M$ (with N and S as defined at the beginning of this section) containing at least two vertices is an edge-optimal rigid merging if and only if it is rigid and satisfies $|E_M| = 3|N| + 2|S| - 3$. Moreover, each rigid merging contains an edge-optimal rigid merging on the same set of meta-vertices.

Proof. Observe first that Theorem 5 requires a rigid merged graph G to satisfy $E_M \geq 3|N| + 2|S| - 3$. Therefore a rigid merged graph for which $E_M = 3|N| + 2|S| - 3$ is an edge-optimal merging. Let now G be a rigid merged graph. By Theorem 5 there exists $E'_M \subset E_M$ with $|E'_M| = 3|N| + 2|S| - 3$ satisfying condition (ii) of this same theorem. One can see, again using Theorem 5, that $G' = \left(\bigcup_{N,S} G_i\right) \cup E'_M$ is rigid, as the set E'_M trivially contains itself and satisfies both conditions (i) and (ii). It follows then from the size of E'_M and from the discussion above that G' is an edge-optimal rigid merging. We have thus proved that any rigid merged graph G contains an edge-optimal rigid merged graph G' on the same meta-vertices satisfying $|E'_M| = 3|N| + 2|S| - 3$. Therefore it cannot contain less than $3|N| + 2|S| - 3$ edges, and if it contains more of them, it is not edge-optimal. It is thus edge-optimal if and only if $E_M = 3|N| + 2|S| - 3$. $\square$

3.2. Persistence. Next we analyze the case where the meta-vertices G_i are directed persistent graphs, and adapt the definitions of N and S in consequence. If it is possible to merge them into a persistent graph, then it is possible to do so in such a way that all the edges of E_M leave vertices which have an out-degree not greater than 2 in G : a set of edges E_M that would make

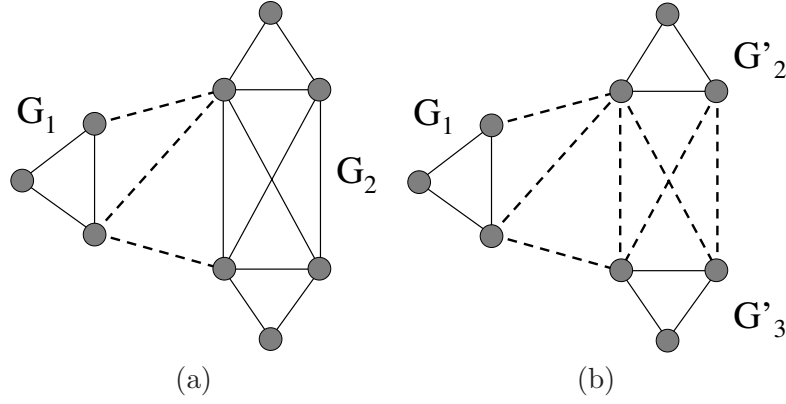


FIGURE 3. The graph represented in (a) and (b) is an edge-optimal rigid merge if it is obtained by merging G_1 and G_2 (a) but not if it is obtained by merging G_1 , G'_2 and G'_3 (b). The dashed edges represent the edges of E_M

G persistent but that would not satisfy this property could indeed be reduced by Proposition 1 until it satisfies it. Moreover, we have the following proposition.

Proposition 2. *Let $G = \left(\bigcup_{N,S} G_i\right) \cup E_M$ with N and S as defined at the beginning of this section, and with all G_i persistent. If no vertex left by an edge of E_M has an out-degree larger than 2, then G is persistent if and only if it is rigid.*

Proof. Rigidity is a necessary condition for persistence, so we just have to prove that it is here sufficient. Let G' be a (directed) graph obtained from G by removing edges leaving vertices with out-degree larger than 2 until no such vertex exists in the graph. It follows from Theorem 4 that we just need to prove the rigidity of any such G' . For every i , let G'_i be the restriction of G' to the meta-vertex G_i . Since in G , every edge of E_M leaves a vertex with an out-degree at most 2, there holds $G' = \left(\bigcup_{N,S} G'_i\right) \cup E_M$ as no edge of E_M is removed when building G' . Moreover, for every i , G'_i can be obtained from G_i by removing edges leaving vertices with an out-degree larger than 2 until no such vertex exists in the graph anymore. The only vertices that are not left by exactly the same edges in G as in G_i are indeed those left by edges of E_M , which by hypothesis have an out-degree at most 2 and are therefore unaffected by the edge-removal procedure. It follows then from the persistence of all G_i and from Theorem 4 that all G'_i are rigid. And since G is rigid, E_M satisfies the necessary and sufficient conditions of Theorem 5. Therefore, the graph $G' = \left(\bigcup_{N,S} G'_i\right) \cup E_M$ is also rigid, as the conditions of Theorem 5 do not depend on the edges inside the different meta-vertices. As explained above, this implies the persistence of G' . $\square$

The condition on the out-degrees of the vertices with an outgoing edge of E_M can be conveniently re-expressed in terms of degrees of freedom: To each DOF (within a single meta-vertex) of any vertex there corresponds at most one outgoing edge of E_M . By an abuse of language, we say that such edges leave a vertex with one or more *local* DOFs, i.e. a vertex which inside its meta-vertex has one or more DOFs and *which is then left by no more edges of E_M than the number of DOFs it has*. This allows reformulating Proposition 2, the proof of which can directly be extended to any dimension, in a dimension-free way:

Theorem 7. *A collection of persistent meta-vertices can be merged into a persistent graph if and only if it can be merged into a persistent graph by adding edges leaving vertices with one or more local DOFs, the number of added edges not exceeding the number of local DOFs. In that*

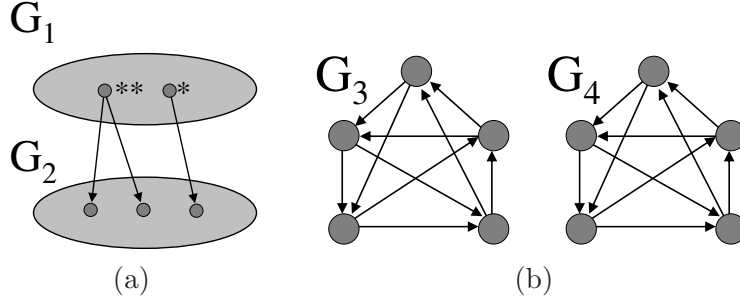


FIGURE 4. Merging of the persistent meta-vertices G_1 and G_2 into a persistent graph in $\mathbb{R}^2$ (a). The symbol “*” represents one DOF (with respect to the meta-vertex). (b) represents two persistent meta-vertex that cannot be merged into a persistent graph in $\mathbb{R}^2$ by addition of interconnecting edges because none of their vertices has a DOF.

case, the merged graph is persistent if and only if it is rigid.

If one or more edges of E_M do leave a vertex with an out-degree larger than 2, no criterion has been found yet to determine whether the merged graph is persistent or not, while also taking advantage of the fact that the graph is obtained by merging several persistent meta-vertices.

Tying Theorem 7 together with what is known and reviewed above regarding the merging of two rigid meta-vertices, we conclude: two persistent meta-vertices G_a and G_b each having two or more vertices can be merged into a persistent graph if and only if three edges leaving vertices with local DOFs can be added in such a way that they are incident to at least two vertices in each meta-vertex. There must thus be at least three local DOFs available among the vertices in G_a and G_b . Conversely, if there are available three local DOFs among the vertices of G_a and G_b , since no vertex can have more than two DOFs, it is possible to add a total of at least three edges leaving at least two vertices of $G_a \cup G_b$. The vertices to which those edges arrive can then be chosen in such a way that at least two vertices of both G_a and G_b are incident to edges of E_M , as in the example shown in Fig. 4. It follows then from Theorem 5 that this graph is rigid, which by Theorem 7 implies that the merged graph is persistent:

Proposition 3. *Two persistent meta-vertices each having two or more vertices can be merged into a persistent graph if and only if the sum of their DOF numbers is at least 3. At least three edges are needed to perform this merging, and merging can always be done with exactly three edges.*

If one or two of the meta-vertices are single vertex graphs, the result still holds, but the minimal number of added edges (and therefore the number of needed DOFs) are then respectively 2 and 1. We define the *number of missing DOFs* (m_{DOF}) to be the maximal number of DOFs that any graph with the same number of vertices can have, less the number of DOFs the graph actually has. In $\mathbb{R}^2$, this maximal number is 2 for the single vertex graphs, and 3 for other persistent graphs. There is an interesting consequence: when the minimal number of edges is used to merge two meta-vertices G_a and G_b , the number of missing DOFs is preserved through the process, i.e. $m_{DOF}(G_a \cup G_b \cup E_M) = m_{DOF}(G_a) + m_{DOF}(G_b)$.

Consider now an arbitrary number of persistent meta-vertices, possibly containing single-vertex graphs, but such that the total number of vertices is at least 2. If the sum of their number of missing DOFs is no greater than 3, it follows from Proposition 3 that any two of

them can be merged in such a way that the obtained graph is persistent and that the total number of missing DOFs remains unchanged. Any pair of those meta-vertices would indeed contain at least the required number of DOFs. Doing this recursively, it is possible to merge all these meta-vertices into a single persistent graph. In case there are more than 3 missing DOFs, the total DOF number is by definition smaller than $3|N| + 2|S| - 3$, which is the minimal number of edges required to make the merged graph rigid. It follows then from Theorem 7 that such meta-vertices cannot be merged in a persistent graph by addition of interconnecting edges. We have thus proved the following result:

Proposition 4. *A collection of persistent meta-vertices $N \cup S$ (with N and S as defined in the beginning of this section) can be merged into a persistent graph if and only if the total number of missing DOFs is no greater than 3, or equivalently if the total number of local DOF in $N \cup S$ is at least $3|N| + 2|S| - 3$. At least $3|N| + 2|S| - 3$ edges are needed to perform this merging, and merging can always be done with exactly this number of edges.*

As when merging rigid meta-vertices, we say that G is an *edge-optimal persistent merging* if no single edge of E_M can be removed without losing persistence. Again, if all meta-vertices are minimally persistent, then G is an edge-optimal persistent merging if and only if it is minimally persistent.

Theorem 8. *$G = \left(\bigcup_{N,S} G_i\right) \cup E_M$ (with N and S as defined at the beginning of this section and with all G_i persistent) is an edge-optimal persistent merging if and only if it is an edge-optimal rigid merging and all edges of E_M leave vertices with local DOFs.*

Proof. Let G be a persistent merging. If there is an edge that lies in E_M leaving a vertex with no local DOF, then it follows from Proposition 1 that the graph obtained by removing this edge would also be persistent, and thus that G is not an edge-optimal persistent merging.

Now if G is a persistent merging for which all edges of E_M leave local DOFs but which is not an edge-optimal rigid merging, then by removing one edge of E_M it is possible to obtain a rigid graph which by Proposition 2 is also persistent, so that G is not an edge-optimal persistent merging.

There remains to prove that an edge-optimal rigid merging G where all edges of E_M leave local DOFs is an edge-optimal persistent merging. Since such G is rigid, it follows from Proposition 2 that it is also persistent. Moreover, since it is an edge-optimal rigid merging, removing any edge of E_M destroys rigidity and therefore persistence. □

Tying Theorem 8 with Theorem 6 leads to the following more explicit characterization of edge-optimal persistent merging.

Theorem 9. *$G = \left(\bigcup_{N,S} G_i\right) \cup E_M$ (with N and S as defined at the beginning of this section and with all G_i persistent) containing at least two vertices is an edge-optimal persistent merging in $\mathbb{R}^2$ if and only if the following conditions all hold:*

- (i) $|E_M| = 3|N| + 2|S| - 3$.
- (ii) For all non-empty $E''_M \subseteq E'_M$, there holds
$$|E''_M| \leq 3|I(E''_M)| + 2|J(E'_M)| - 3$$
with $I(E''_M)$ and $J(E'_M)$ as defined in Theorem 5
- (iii) All edges of E_M leave vertices with local DOFs.

Notice that an efficient way to obtain such a merging is provided in the discussion immediately preceding Proposition 4.

$ V_a $	1	1	1	2	2	≥ 3
$ V_b $	1	2	≥ 3	2	≥ 3	≥ 3
$\min E_M $	1	2	3	4	5	6

TABLE 1. Minimal number of edges required to merge two rigid graphs G_a and G_b into a single rigid graph in $\mathbb{R}^3$.

4. RIGIDITY AND PERSISTENCE OF 3D META-FORMATIONS

4.1. Rigidity. We now consider a set N of disjoint rigid (in $\mathbb{R}^3$) graphs $G_1, \dots, G_{|N|}$ having at least three vertices each, a set D of graphs containing two (connected) vertices $G_{|N|+1}, \dots, G_{|N|+|D|}$, and a set S of single-vertex graphs $G_{|N|+|D|+1}, \dots, G_{|N|+|D|+|S|}$. As in Section 3, these graphs are called meta-vertices, and we define the merged graph G by taking the union of all the meta-vertices, and of some additional edges E_M each of which has end-points belonging to different meta-vertices.

The merging of two rigid meta-vertices, each containing more than two vertices, is treated in [26]: At least six edges are needed, and they must be incident to at least three vertices of each meta-vertex (which is necessary for 3-connectivity). But these conditions are only necessary, as they do not imply 3-connectivity. For example, the so-called “double-banana” graph in Fig. 1(c) can be obtained by merging two distinct rigid tetrahedral meta-vertices (1,3,4,5) and (2,5,7,8) using a total of six edges incident to four vertices of each meta-vertex. However, it is always possible to achieve a rigid merging using exactly six edges incident to exactly three vertices of each meta-vertex, with no single vertex having more than three incident edges out of the six. With a minor modification, the merging result above holds in the cases where at least one meta-vertex has less than 3 vertices: The required number of edges is different, as summarized in Table 1 where $\min |E_M|$ represents the minimal number of edges required to merge the meta-vertices $G_a(V_a, E_a)$ and $G_b(V_b, E_b)$ into a rigid graph. Also, if a meta-vertex has less than 3 vertices, all of them should be incident to edges of E_M , otherwise at least 3 of them should be. When merging several meta-vertices, there is no available necessary and sufficient condition for the rigidity of G . Determining whether a merged graph is rigid in $\mathbb{R}^3$ is indeed a more general problem than determining whether a given graph is rigid (for which it suffices to take $N = D = \emptyset$) and there is no known set of combinatorial necessary and sufficient conditions for this. We can however prove that the rigidity of the merged graph G only depends on E_M , on the vertices to which nodes of E_M are incident and on the belonging of the G_i to N , D or S .

Proposition 5. *Let $G = \left(\bigcup_{N,D,S} G_i\right) \cup E_M$ with N, D, S as defined at the beginning of this section. Suppose that a meta-vertex G_i is replaced by a meta vertex G'_i with the same set of vertices incident to E_M , with the same set membership, N , S or D , as G_i , but otherwise with different internal structure. Let G' be the graph so obtained. Then G' is rigid if and only if G is rigid.*

Proof. This could be proved using algebraic arguments based on the rigidity matrix, but we prefer the following more intuitive argument.

The result is trivial for meta-vertices of D and S as they are entirely determined by their belonging to these classes; we assume therefore that $G_i \in N$. We also assume that the set $V_i(E_M)$ of vertices of G_i (and G'_i) that are incident on edges of E_M contains at least three vertices. In case this assumption is not verified, both G and G' fail to be 3-connected and therefore rigid (by Theorem 2), so that the result is also trivial. We then prove that the non-rigidity of G implies the non-rigidity of G' . Since the roles of G and G' can be exchanged, this is sufficient to prove

the theorem.

Suppose that G is not rigid, and give positions in $\mathbb{R}^3$ to its vertices. Then there is a smooth motion M (satisfying the distance constraints corresponding to edges in G) of the vertices of G apart from pure translation or rotation. Because G_i is rigid, the restriction of M to the vertices of G_i is a rigid motion, that is a translation and/or rotation, which we call T . Therefore, the restriction of M to $(G \setminus G_i) \cup V_i(E_M)$ is not a rigid motion. Otherwise all distances would be preserved by M apart from some distances between vertices of $G_i \setminus V_i(E_M)$ and vertices of $G \setminus G_i$. We would then have two vertices whose relative distance is not preserved while their relative distance with respect to all the three or more vertices of $V_i(E_M)$ are preserved, which is impossible. We call M^* this restriction to $(G \setminus G_i) \cup V_i(E_M)$. Let now M' be a smooth motion of the vertices of G' , which for the vertices of G'_i is the translation and/or rotation T , and for the vertices of $(G' \setminus G'_i) \cup V_i(E_M)$ is the motion M^* (observe that that the two motions are identical on $V(E_M)$ which is the intersection of the two sets on which M' is defined). Since M^* is a non-rigid motion (not preserving all distances), so is M' . Therefore, we just need to prove that M' satisfies all distance constraints on vertices connected by edges in G' to prove the non-rigidity of G' . Consider a pair of vertices. If they both belong to $(G' \setminus G'_i) \cup V_i(E_M)$, their constraint in G' is the same as in G , and their motion is defined by M^* which satisfies all distance constraints. If they do not both belong to $(G' \setminus G'_i) \cup V_i(E_M)$, then due to the structure of the graph they necessarily both belong to G'_i , and their motion is the rotation and/or translation which by essence preserve all distances. $\square$

Moreover, we have the following necessary condition:

Theorem 10. *Let G_i for $i = 1, 2, \dots, |N| + |D| + |S|$ be rigid meta-vertices, and suppose $G = \left(\bigcup_{N,D,S} G_i\right) \cup E_M$ (with N, D, S as defined at the beginning of this section) is rigid in $\mathbb{R}^3$ and contains at least three vertices. Then there exists $E'_M \subseteq E_M$ such that*

$$(i) |E'_M| = 6|N| + 5|D| + 3|S| - 6$$

(ii) *For all non-empty $E''_M \subseteq E'_M$, there holds*

$$|E''_M| \leq 6|I(E''_M)| + 5|J(E''_M)| + 3|K(E''_M)| - 6,$$

where $I(E''_M)$ is the set of meta-vertices such that either there are at least three vertices within the meta-vertex all incident to edges of E''_M , or precisely two vertices within the meta-vertex which are unconnected and both incident to edges of E''_M . $J(E''_M)$ is the set of meta-vertices such that there are precisely two vertices within the meta-vertex which are connected and both incident to edges of E''_M ; $K(E''_M)$ is the set of meta vertices such that there is precisely one vertex within the meta-vertex that is incident to one or several edges of E''_M . Note that in each case, there can be an arbitrary number of vertices in the meta-vertex which are not incident on any edge of E''_M .

Moreover, the graph $\left(\bigcup_{N,D,S} G_i\right) \cup E'_M$ is rigid.

Proof. The proof is similar to the one of Theorem 5 (necessary part). For every G_i , let G'_i be a minimally rigid subgraph of G_i on the same vertices, which therefore contains $3|V_i| - 6$ edges if $G_i \in N$, one edge if $G_i \in D$ and no edge if $G_i \subseteq S$. As mentioned in its proof, Lemma 1 can also be applied in a three-dimensional space. So if G is rigid, there is a minimally rigid subgraph $G'(V, E') \subseteq G$ containing all G'_i . Let $E'_M = E' \cap E_M$; we shall prove that E'_M satisfies the condition of this theorem. Since G' is minimally rigid, there holds $|E'| = 3|V| - 6$. Moreover, we have $|E'| = |E'_M| + \sum_{G_i \in N} |E'_i| + |D|$, and $|V| = \sum_{G_i \in N} |V_i| + 2|D| + |S|$, so that

$$\begin{aligned} |E'_M| &= 3|V| - 6 - \sum_{G_i \in N} (3|V_i| - 6) - |D| \\ &= 6|N| + 5|D| + 3|S| - 6. \end{aligned}$$

E'_M contains thus the predicted number of edges. We suppose now that there is a set E''_M such that $|E''_M| > 6|I(E''_M)| + 5|J(E''_M)| + 3|K(E''_M)| - 6$ and show that this contradicts the minimal rigidity of G' . Let us then build E'' by taking the union of E''_M and all E'_i for which $i \in I(E''_M)$, and the edge connecting the two vertices incident to E''_M in all meta-vertices in $J(E''_M)$. There holds $V(E'') = |K(E''_M)| + 2|J(E''_M)| + \sum_{G_i \in I(E''_M)} |V_i|$. Therefore, we have

$$\begin{aligned} |E''| &= |E''_M| + \sum_{G_i \in I(E''_M)} |E'_i| + |J(E''_M)| \\ &> 6|I(E''_M)| + 5|J(E''_M)| + 3|K(E''_M)| - 6 \\ &\quad + \sum_{G_i \in I(E''_M)} (3|V_i| - 6) + |J(E''_M)| \\ &= 3|V(E'')| - 6. \end{aligned}$$

This however contradicts the minimal rigidity of G' as $E'' \subseteq E'$. Finally, since $G' = \left(\bigcup_{N,D,S} G_i\right) \cup E'_M$ is rigid, it follows from several applications of Proposition 5 that $\left(\bigcup_{N,D,S} G_i\right) \cup E'_M$ is also rigid. $\square$

Note that the rigidity of $\left(\bigcup G_i\right) \cup E'_M$ is explicitly mentioned here and not in Theorem 5, because in a two-dimensional space it follows directly from sufficiency of the counting conditions. But, the counting conditions of Theorem 10 are not sufficient for rigidity, as the non-rigid graph of Fig. 1(c) which can be obtained by merging two rigid tetrahedral meta-vertices (1,3,4,5) and (2,6,7,8) would indeed satisfy them. Nevertheless, one can deduce from Theorem 10 that G is an edge-optimal rigid merging in $\mathbb{R}^3$ if and only if it is rigid and $|E_M| = 6|N| + 5|D| + 3|S| - 6$, using E'_M exactly in the same way as in Theorem 6.

4.2. Persistence. We consider now that all meta-vertices G_i are persistent graphs, and adapt the definitions of N , D and S in consequence. Theorem 7 can be generalized to three dimensions, as it follows from Proposition 2, the proof of which can be immediately extended to three dimensions.

Theorem 11. *A collection of (structurally) persistent meta-vertices can be merged into a (structurally) persistent graph if and only if it can be merged into a (structurally) persistent graph by adding edges leaving vertices with one or more local DOFs. In that case, the merged graph is persistent if and only if it is rigid.*

Proof. Suppose first that a collection of persistent meta-vertices can be merged into a persistent graph in such a way that some edges do not leave local DOFs. Then, it follows from Proposition 1 that these edges can be removed without destroying the persistence of the merged graph, so that the same collections of meta-vertices can be merged without having connecting edges that do not leave local DOFs. In case the meta-vertices are structurally persistent and are merged into a structurally persistent graph, the result still holds as removing edges that do not leave local DOFs never destroys structural persistence. The reverse implication is trivial.

The proof of the rest of the result is done exactly as in Theorem 2, using Proposition 5 instead of Theorem 5. $\square$

Merging two meta-vertices into a persistent graph is however a more complicated problem in $\mathbb{R}^3$ than in $\mathbb{R}^2$. Consider indeed a meta-vertex G_a without any DOF, and a meta-vertex G_b which is not structurally persistent, i.e. which is persistent and contains two vertices (leaders) having three DOFs. The number of available DOFs is equal to the minimal number of edges that should be added to obtain a rigid merged graph. However, the only way to add six edges leaving local DOFs is to add three edges leaving each leader of G_b and arriving in G_a , as represented by the example in Fig. 5(a). Only two vertices of G_b would thus be incident to the added

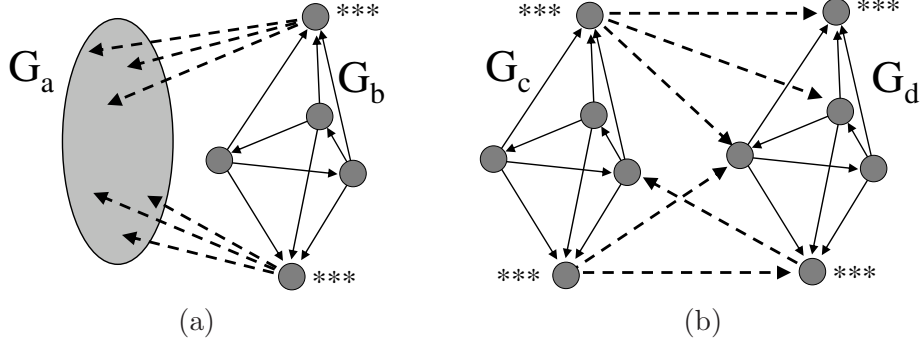


FIGURE 5. Example of a persistent but not structurally persistent meta-vertex G_b which cannot be merged into a persistent or rigid graph with the meta-vertex G_a , the latter being persistent but having no DOF. (b) shows how two non-structurally persistent meta-vertices can be merged into a structurally persistent graph. The symbol “*” represents one DOF, and the dashed edges are the edges of E_M .

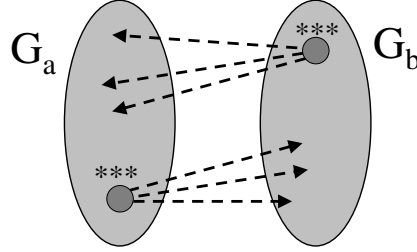


FIGURE 6. G_a and G_b both have all their DOFs concentrated on one leader. As a result they cannot be merged into a persistent graph. The only way to add 6 edges leaving local DOFs is depicted and does not lead to a rigid graph, because the overall graph is not 3-connected. The symbol “*” represents one DOF, and the dashed edges are the edges of E_M .

edges, which prevents the merged graph from being rigid and therefore persistent as it is thus not 3-connected. We have thus proved the following condition:

Proposition 6. *If two persistent meta-vertices are such that one is not structurally persistent and the other does not have any DOF, they cannot be merged into a persistent graph by addition of interconnecting edges.*

Another problem appears when G_a and G_b each have one leader (having three DOFs) and no other vertex has DOFs. Again, the number of available DOFs is equal to the minimal number of edges that should be added to obtain a rigid merged graph, but the only way to add six edges leaving local DOFs does not lead to a rigid graph. One can indeed only add three edges leaving each leader as shown in Fig. 6. This results in a graph that is not 3-connected and therefore not rigid by Theorem 2, as the removal of the two ex-leaders would render the graph unconnected. We have thus proved the following condition:

Proposition 7. *If two persistent meta-vertices have each one leader (with 3 DOFs) and no other DOF, they cannot be merged into a persistent graph by addition of interconnecting edges.*

However, these are the only cases for which the argument used in establishing Proposition 3 cannot be generalized to establish an analogous property in $\mathbb{R}^3$:

Proposition 8. *Two persistent meta-vertices (each with three or more vertices) can be merged into a persistent graph by addition of directed connecting edges if and only if the sum of their DOFs is at least 6 and the DOFs are located on more than two vertices. At least six edges are needed to perform this merging, and merging can always be done with exactly six edges and in such a way that the graph obtained is structurally persistent and does not have all its DOFs located on leaders.*

Proof. Consider two meta-vertices each having more than 2 vertices. It follows from Theorem 11 that they can be merged into a persistent graph if and only if it is possible to add directed edges leaving local DOFs in such a way that the obtained graph is rigid.

Suppose first that the total number of available DOFs is 6. If all these DOFs are located on two leaders, the two graphs satisfy the conditions of either Proposition 6 or Proposition 7, so that they cannot be merged into a persistent graph. If the 6 DOFs are located on more than 2 vertices, an exhaustive verification (see Appendix) show that the two graphs can always be merged into a rigid graph by adding 6 edges, each leaving a vertex with a local DOF, with at least one DOF for each edge. Note that this exhaustive verification is needed as no sufficient condition for rigidity of a graph obtained by connecting two rigid graphs is known which is sufficiently weak to be helpful for this proof.

If the total number of DOFs is larger than 6, they are located on at least 3 vertices, as a vertex has at most 3 DOFs. It is therefore possible to select a subset of 6 DOFs located on at least 3 vertices, and to apply the result obtained above for 6 DOFs.

There remains to prove that the merging can always be done in such a way that the obtained graph does not have all its DOFs located on leaders, or in other words the obtained graph has only vertices with 0 or 3 DOFs (This also implies that the graph obtained is structurally persistent, as the only persistent graphs that are not structurally persistent are those with two leaders and therefore no other DOF). Such a situation, i.e. the obtained graph has only vertices with 0 or 3 DOFs, could only happen if this graph has exactly 3 or 6 DOFs, and thus if 9 or 12 DOFs are initially available, as the merge is done by addition of 6 edges. A simple way of avoiding having all remaining DOFs on leaders is then to select the 6 DOFs that are going to be removed in the merging process in such a way that a number of DOFs different from 3 and 6 is left in each of the initial graphs. At least one vertex has then indeed one or two DOFs. $\square$

In case at least one of the two meta-vertices has less than 3 vertices, an exhaustive consideration of all possible cases (see Appendix) shows that the result still holds, but with a different required number of edges in E_M and therefore of available DOFs: these minimal numbers are both equal to $\min |E_M|$ in Table 1 (for the merging of a graph $G_a(V_a, E_a)$ with a graph $G_b(V_b, E_b)$). Observe that as in the 2-dimensional case, the merge can be done in such a way that the number of missing DOFs is preserved, the number of missing DOFs being defined in the same way as in Section 3.2, with maximal number of DOFs being 6, 5 and 3 for meta-vertices of respectively N , D and S . It is worth noting that even if one or both of the meta-vertices are not structurally persistent, it is possible to obtain a structurally persistent merged graph, as represented in Fig. 5(b). This has already been observed in [28] for the case where one meta-vertex is a single vertex graph.

Consider now a collection of meta-vertices such that the total number of vertices is at least 3. Unless the collection consists in two meta-vertices satisfying the hypotheses of Proposition 6 or 7, all the graphs that compose it can be merged into one large persistent graph by addition of edges.

Proposition 9. *A collection of persistent meta-vertices $N \cup D \cup S$ (with N, D, S as defined in the beginning of this section) containing in total at least three vertices and that does not consist of only two meta-vertices satisfying the condition of Proposition 6 or 7 can be merged into a persistent graph if and only if the total number of missing DOFs is no greater than 6, or equivalently if the total number of local DOFs in $N \cup D \cup S$ is at least $6|N| + 5|D| + 3|S| - 6$. At least $6|N| + 5|D| + 3|S| - 6$ edges are needed to perform this merging. Merging can always be done with exactly this number of edges, and in such a way that the merged graph is structurally persistent.*

Proof. The proof is similar to the one of Proposition 4. If a pair of meta-vertices can be merged into a persistent graph, this merging can be done in such a way that the number of missing DOFs is preserved, and by adding only edges leaving vertices with local DOFs (with at most one edge for each DOF). Doing this recursively, we eventually obtain a single persistent graph that has the same number of missing DOFs as the initial collection of graphs. The number of added edges is then equal to the number of DOFs that have disappeared during the merging process, that is $6|N| + 5|D| + 3|S| - 6$.

There remains to prove that these mergings can actually be done, and that the obtained graph is structurally persistent. By Proposition 8 (and its extension to graphs with 1 or 2 vertices), when their number of missing DOFs is smaller than 6, two persistent graphs can always be merged into a structurally persistent graph, unless either one of them is not structurally persistent while the other has no DOF (case of Proposition 6), or both of them have one leader and no other DOF (case of Proposition 7). In these two cases, the two “problematic” meta-vertices have at least three vertices each.

Suppose first that one meta-vertex has no DOF (and that the rest of the meta-vertices collection does not consist in one single non structurally persistent meta-vertex). Then since the total number of missing DOF is 6, no other meta-vertex has a missing DOF, and by hypothesis there are at least two other meta-vertices (or possibly exactly one structurally persistent meta-vertex). It follows then from successive applications of Proposition 8 that they all can be merged into a structurally persistent graph that still does not have any missing DOF. This latter graph can then be merged with the graph that has no DOF, and the graph obtained is also structurally persistent.

Suppose now that two meta-vertices have exactly one leader and no other DOF. It follows then from the hypotheses that there is at least one other meta-vertex in the collection. And again, no other meta-vertex has any missing DOF. Temporarily isolating one of the meta-vertices with one leader and no other DOF, it follows again from successive applications of Proposition 8 that all other graphs can be merged into a persistent graph that does not have all its DOFs located on one single leader, and this graph can then be merged with the temporarily isolated graph into a structurally persistent graph.

□

As in the two-dimensional case, a merged graph is an edge-optimal persistent merging if and only if it is an edge-optimal rigid merging and all edges in E_M (such as defined in the beginning of this subsection) leave local DOFs. The proof of this is an immediate generalization of Theorem

8. However, due to the absence of necessary and sufficient conditions allowing a combinatorial checking of the rigidity of a graph or of a merged graph in $\mathbb{R}^3$, the result cannot be expressed in a purely combinatorial way. Since the number of edges in E_M in an edge-optimal rigid merging is fixed, the above criterion can be re-expressed as

Theorem 12. $G = \left(\bigcup_{N,D,S} G_i\right) \cup E_M$ (with N, D, S as defined at the beginning of this section and with all G_i persistent) containing in total at least three vertices is an edge-optimal persistent merging in $\mathbb{R}^3$ if and only if the following conditions all hold:

- (i) G is rigid.
- (ii) All edges of E_M leave local DOFs.
- (iii) $|E_M| = 6|N| + 5|D| + 3|S| - 6$.

Again, an efficient way to obtain an edge-optimal persistent merging from a collection of meta-vertices satisfying the hypotheses of Proposition 9 is to first merge two of them and then to iterate, as in the discussion of Propositions 4 and 9.

5. CONCLUSIONS

We have analyzed the conditions under which a formation resulting from the merging of several persistent formations is itself persistent. Necessary and sufficient conditions were found to determine which collections of persistent formations could be merged into a larger persistent formation. We first treated these issues in $\mathbb{R}^2$. Our analysis was then generalized to $\mathbb{R}^3$ and to structural persistence, leading to somewhat less powerful results. This is especially the case for those which rely on the sufficient character of Laman's conditions for rigidity in $\mathbb{R}^2$ (Theorem 1), no equivalent condition being known in $\mathbb{R}^3$. Following this work, we plan to develop systematic ways to build all possible optimally merged persistent formations, similarly to what has been done for minimally persistent formations [10] and for minimally rigid merged formations [27]. These references canvas generalizations of the Henneberg sequence concept [13, 22] for building all minimally rigid graphs in two dimensions.

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In this appendix, we complete the proof of Proposition 8 on the merging of persistent meta-vertices in a 3-dimensional space, and extend this proposition to cases where one of the meta-vertices has less than 3 vertices. We have to prove that two persistent graphs (in a three-dimensional space) G_a and G_b having in total 6 DOFs located on at least three vertices can always be merged into a rigid graph by addition of six edges leaving vertices with local DOFs, with at least one local DOF for each added edge.

For this purpose, we use the following lemma, which summarizes results obtained in [26].

Lemma 2. *Let G_a and G_b be two (initially distinct) rigid graphs each with three or more vertices. Performing a sequence of three or more operations selected among the two following types of operations results in merging G_a and G_b into a rigid graph by addition of 6 edges.*

Operation (v): *Taking a vertex i of G_a not connected yet to any vertex of G_b , and connecting it to $3 - t$ vertices of G_b , where t is the number of operations already performed.*

Operation (e): *Taking a vertex i of G_a not connected yet to any vertex of G_b , and an edge (k, j) with $k \in V_A$ and $j \in V_B$. Replacing the edge (k, j) by (i, j) and connecting i to $2 - t$ other*

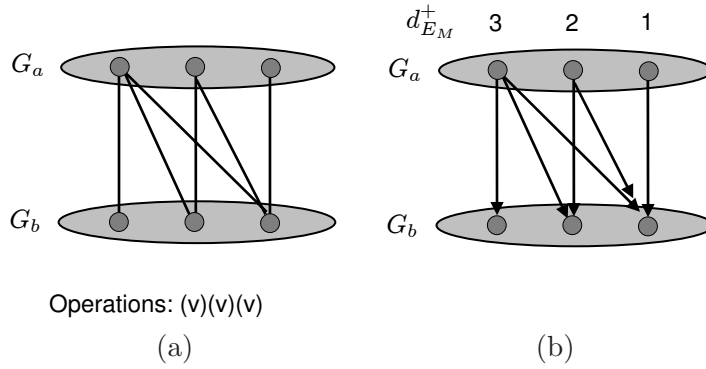


FIGURE 7. (a) represents a rigid merged graph obtained by performing three operations (v). It is shown in (b) how directions can be given to the edges in such a way that three vertices of G_a are left by respectively 3, 2 and 1 edges.

vertices in G_b , where t is the number of operations already performed.

Without loss of generality, we suppose that G_a has at least as many DOFs as G_b . The partition of DOFs can thus be 6-0, 5-1, 4-2 or 3-3. In the sequel, we prove the result for each of these particular cases, starting with G_a having 6 DOFs.

It follows from Lemma 2 that the merged graph represented in Fig. 7(a) is rigid. It can indeed be obtained by three applications of the operation (v). Moreover, one can see in Fig. 7(b) that directions can be given to the connecting edges in such a way that the out-degree distribution (with respect to the connecting edges) is (3, 2, 1), that is one vertex of G_a is left by three connecting edges, one by two, and one by one. Suppose now that G_a is a persistent graph with 6 DOFs with a DOF allocation (3, 2, 1), that is a persistent graph having one vertex having 3 DOFs, one 2 DOFs, and one 1 DOF. Then it can be merged with G_b into a rigid graph by adding 6 edges leaving vertices with local DOF (with one DOF for each edge). It suffices indeed to take the edges represented in Fig. 7(b), identifying each vertex with δ DOFs with a vertex left by δ connecting edges.

We now treat a DOF allocation (2, 2, 2). It follows again from Lemma 2 that the merged graph represented in Fig. 8(a) is rigid, as it can be obtained by two applications of the operation (v) followed by one application of operation (e). Moreover, Fig. 8(b) shows that directions can be assigned to the edges in such a way that the out-degree distribution (with respect to the connecting edges) is (2, 2, 2). For the same reason as above, G_a can thus be merged with G_b into a rigid graph by adding 6 edges leaving vertices with local DOF (with one DOF for each edge) if its DOF distribution is (2, 2, 2).

Next we show that such construction can be obtained in all other cases, except those where the 6 DOFs are all located on two vertices. When G_a has 6 DOFs, the remaining possible DOF distributions are (3, 1, 1, 1), (2, 2, 1, 1), (2, 1, 1, 1, 1) and (1, 1, 1, 1, 1, 1), the case (3, 3) does not satisfy the hypotheses. The construction in these four cases are obtained by performing the operation (e) of Lemma 2 (up to three times) on the constructions detailed above for (3, 2, 1) and (2, 2, 2). They are represented in Fig. 9

If G_a has 5 DOFs and G_b one DOF, the required construction can always be obtained from one of the construction for the case where G_a has 6 DOFs. It suffices indeed to use one of the

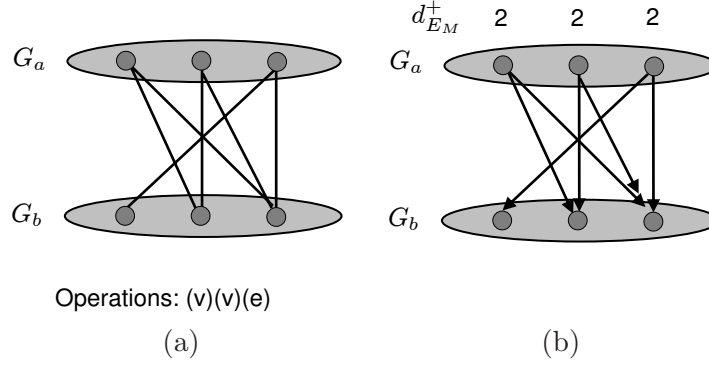


FIGURE 8. (a) represents a rigid merged graph obtained by performing three operations (v). It is shown in (b) how directions can be given to the edges in such a way that the three vertices of G_a are each left by 2 edges.

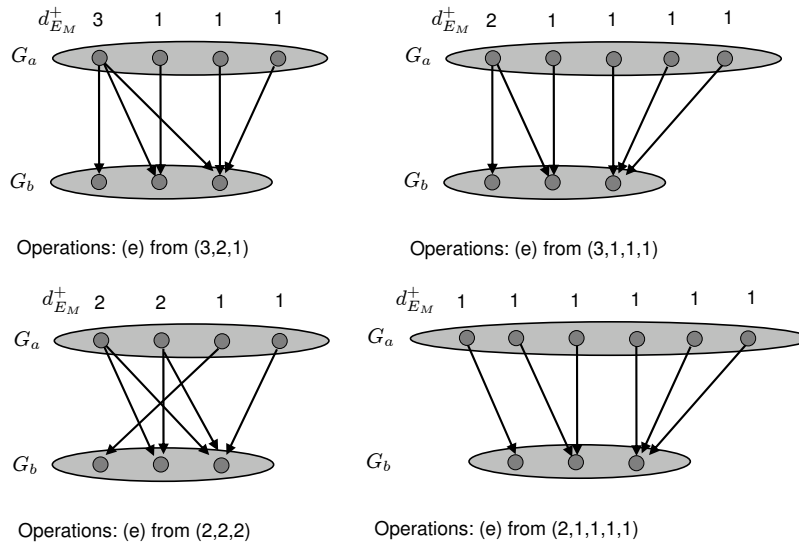


FIGURE 9. Representations of how a rigid graph can be obtained by merging two persistent graphs G_a and G_b where G_b has no DOF and where the DOF allocation of G_a is $(3, 1, 1, 1)$, $(2, 2, 1, 1)$, $(2, 1, 1, 1, 1)$ or $(1, 1, 1, 1, 1, 1)$. The operations (v) and (e) used to obtain the structure are also mentioned.

constructions already provided by temporarily adding one vertex with one DOF to the distribution of 5 DOFs in G_a , executing the appropriate construction from the group above, and then reversing the direction of the edge leaving a vertex with one DOF, as shown in Fig. 10 for a DOF distribution $(3, 2)$.

Suppose now that G_a has 4 DOFs, and G_b 2 DOFs. Then the possible DOF distribution for G_a are $(3, 1)$, $(2, 2)$, $(2, 1, 1)$ and $(1, 1, 1, 1)$. For G_b , they are (2) and $(1, 1)$. The construction proving the result for these eight cases are shown in Fig. 11.

Finally, if both graphs have 3 DOFs, the possible distribution for each are (3) , $(2, 1)$ and $(1, 1, 1)$. The case where they both have a distribution (3) does not satisfy the hypotheses of this Proposition, and three other cases do not need to be treated for symmetry reasons. The

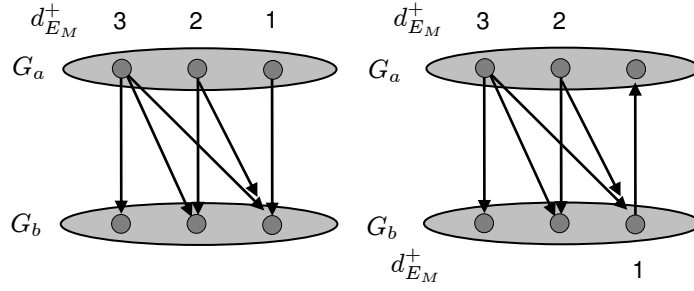


FIGURE 10. Representation of how the construction for a DOF partition 5-1 between G_a and G_b can be obtained from a construction for a partition 6-0.

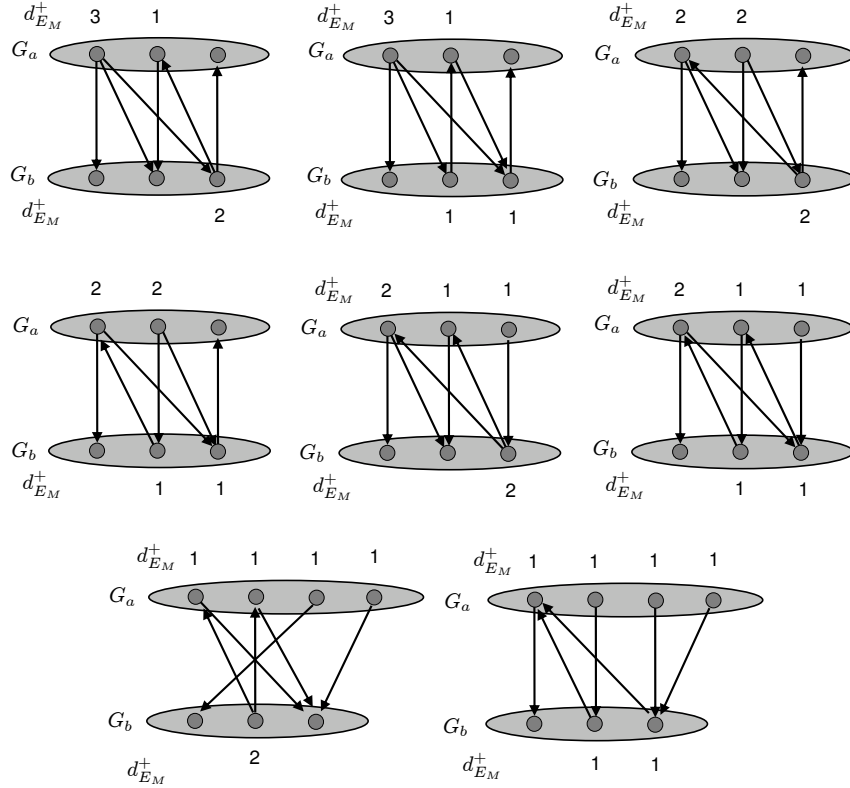


FIGURE 11. Constructions for the eight possible DOF allocations when G_a has 4 DOFs and G_b 2 DOFs. The graphs are all rigid as they have the same undirected underlying graphs as construction in Fig. 7, 8 or 9.

construction for the remaining 5 cases is shown in Fig. 12.

We now suppose that at least one of the graphs has less than 3 vertices, and show that a rigid graph can be obtained by adding directed edges leaving vertices with local DOFs, the number of these edges being provided in Table 1. Observe that a graph consisting of one single vertex always has 3 DOFs, and thus that it is never needed to use any DOF of the other graph. Similarly, each vertex of a graph containing two vertices has at least 2 DOFs, so that at most one DOF of the other graph needs to be used, and only when the other graph has three or more vertices. Fig. 13 shows how these mergings can be performed. Note that the rigidity of

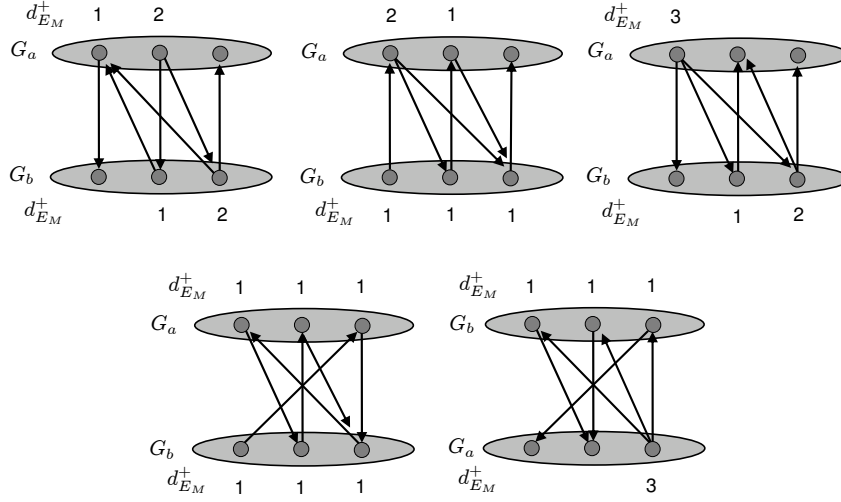


FIGURE 12. Constructions for the five different DOF allocations satisfying the hypothesis of Proposition 8 when each of G_a and G_b has 3 DOFs. The graphs are all rigid as they have the same undirected underlying graphs as construction in Fig. 7, 8 or 9 or rotated versions of them.

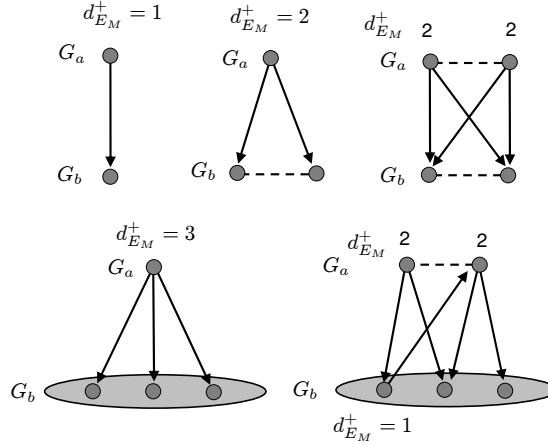


FIGURE 13. Illustration of the merging between two graphs, one of which at least has less than 3 vertices. The dashed line represent the internal edge(s) of graphs with two vertices, the orientation of which is not relevant for our purpose. The vertex count in G_b is precisely 1, 2 and 2 for the first three and a minimum of 3 for the last two.

the three first graphs is immediate as they are complete graphs. The rigidity of the other two follows from the fact that they can be obtained from G_b by performing one of two operations (v), which guarantees the rigidity of the graph obtained [22].