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John John Ketelbuters, Donatien Hainaut

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A Recursive Method for Fractional Hawkes Intensities and the Potential Approach of Credit Risk

John-John Ketelbuters^{a,b,*}, Donatien Hainaut^{a,c}

^aUCLouvain, LIDAM, Voie du Roman Pays 20, 1348 Louvain-la-Neuve (Belgium)

^bjohn-john.ketelbuters@uclouvain.be

^cdonatien.hainaut@uclouvain.be

*corresponding author

Abstract

This article explores the potential approach for credit risk, which is an alternative to structural and reduced models. In the context of credit risk, it consists in assuming that the survival probability of a company is equal to the ratio of the expected value of a supermartingale divided by its initial value. This approach, that was previously used for modeling the term structure of interest rates, is extended by the use of a fractional self-exciting process, or fractional Hawkes process. This is a self-exciting process that is time-changed by the inverse of an alpha-stable subordinator. A self-exciting process allows to take into account the clustering of economic events such as defaults, whereas the time-change allows to properly fit highly convex survival probability curves. We derive a new recursive method that allows to compute all the moments of a self-exciting process intensity, as well as its time-changed counterpart. We show that this method can be used to approximate the survival probabilities in the potential approach. More specifically, we prove that the approximation converges and we provide a bound on the numerical error. Finally, we calibrate the model and show that it allows to properly fit survival probability curves that are highly convex.

Keywords: Credit risk, potential approach, fractional Hawkes processes.

1 Introduction

The most common paradigms in credit risk management are structural and reduced models. In the first approach, the balance sheet of a firm is explicitly modeled and the default occurs when the assets fall below the liabilities, as e.g. in [1], [2], [3] or [4]. In the reduced approach, the default is caused by the first jump of a point process, see e.g. [5], [6] or [7]. [8] unify discrete structural and reduced models whereas [9] propose an hybrid approach. In this paper, we investigate a third alternative called the potential approach. This approach dates back to [10] and has been frequently used in the literature for interest rate modeling.

In interest rate modeling, the most frequent approaches are respectively to model the spot rate, which is done in e.g. [11] and [12], or to model the forward rate. Well known articles that use the forward rate approach are [13] and [14]. The spot rate approach consists in specifying a

stochastic process $(r_t)_{t \geq 0}$, called the spot rate or instantaneous rate, so that for a contingent claim Y paid at time $T \geq 0$, the price at time $t \in [0, T]$ is given by

$$\mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ - \int_t^T r_s ds \right\} Y \middle| \mathcal{F}_t \right],$$

where $\mathbb{Q}$ is a risk-neutral measure and $(\mathcal{F}_t)_{t \geq 0}$ is the relevant filtration. In case the contingent claim Y equals one with probability one, we obtain the important particular case of a zero-coupon bond, whose price is usually denoted by $P(t, T)$,

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ - \int_t^T r_s ds \right\} \middle| \mathcal{F}_t \right]. \quad (1)$$

[10] proposed to directly model the state price process, also called the pricing kernel. This is a stochastic process $(\xi_t)_{t \geq 0}$ such that for a contingent claim Y , the price is given by

$$\frac{\mathbb{E}_{\mathbb{Q}} [\xi_T Y | \mathcal{F}_t]}{\xi_t}.$$

This was called the potential approach by [15], since $(\xi_t)_{t \geq 0}$ should be a potential¹ to guarantee satisfying properties² for the model. With the potential approach, [16] derived valuation formulas for interest rate caps and swaptions. Since then, the potential approach for interest rates has been used in numerous studies, including [17], [18], [19] and [20].

In the reduced approach to credit risk, it appears that the credit model has the same structure as a spot rate model for interest rates. More precisely, the reduced approach to credit risk uses a positive process $(\lambda_t)_{t \geq 0}$, called the default intensity, such that

$$\mathbb{Q}(\tau > T | \tau > t) = \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ - \int_t^T \lambda_s ds \right\} \middle| \mathcal{F}_t \right], \quad (2)$$

where τ is the time of default. Based on the similarity between Equations (1) and (2), we propose to use the potential approach for credit risk by computing the survival probabilities from a potential $(\xi_t)_{t \geq 0}$. Equation (2) would then become

$$\mathbb{Q}(\tau > T | \tau > t) = \frac{\mathbb{E}_{\mathbb{Q}} [\xi_T | \mathcal{F}_t]}{\xi_t}.$$

This article extends the potential approach to credit risk in two directions. Firstly, we work with a self-exciting process that allows sudden and repeated jumps of default probabilities, as those caused by contagion between defaults of several firms. In this framework, the intensity of default arrival depends on recent past shocks. This approach finds its origin in the articles of [21], [22] and [23]. More recently, [24] and [25] use a self-exciting approach for modeling defaults in a credit portfolio. [26] use a similar model to measure contagion between international markets. [27] and [28] propose a model explaining the dynamics of interest rates by self-exciting jump-diffusions.

The second way in which we extend the potential approach is the use of the technique of time-change. As shown in [29], this technique allows to obtain a better fit for survival probability

¹A positive supermartingale whose expectation converges to zero as the time goes to infinity.

²Such as the positivity of interest rates.

curves that exhibit a high degree of convexity. Such curves are observed for companies that face financial difficulties in the short-term, but has a good long-term viability provided that they overcome their difficulties. Airline companies during the Covid-19 crisis were given as an example above. Obtaining such convex shapes involves a non-Markov time-change built with the inverse of an α -stable subordinator. This approach is applied to diffusions in physics for describing the movements of heavy particles that can get immobilized (see e.g. [30] or [31]). This type of time-changed Brownian motions, called subdiffusions, are applied to financial derivatives by [32] and [33] for illiquid markets. As shown e.g. in [34], the probability density function of sub-diffusions is solution of a Fractional Fokker-Planck equation. This equation is proposed in [35] and [36]. Articles of [37] and [38] go a step further and explore fractional Pearson diffusions and their correlation.

Computing default probabilities in such a setting requires to be able to approximate the moment generating function of (fractional) Hawkes process intensities. To do this, the most common approach is to derive a (fractional partial) differential equation and to solve it numerically, as in e.g. [29]. Since this approach is rather complicated both mathematically and numerically, we propose instead to use the moments of the Hawkes process intensity to approximate the moment generating function via the Maclaurin series of the exponential function. In a recent article, [39] describe a method to derive the moments of Hawkes processes and their intensities. In this paper, we go a step further and prove that the moments of the Hawkes process intensity always share the same form and can be computed recursively. This recursive procedure is new to the best of our knowledge and allows to efficiently compute default probabilities in the model we propose.

The structure of this paper is as follows. Sections 2 and 3 review the potential approach and the properties of a Hawkes jump-diffusion. In Section 4, we extend the Hawkes jump-diffusion to the subdiffusive setting by using the inverse of an α -stable subordinator as time-change. Section 5 introduces our new numerical method to compute all the moments of a Hawkes process intensity. We show that this method also works in the subdiffusive time-changed setting. We end Section 5 by showing how this result can be used to approximate default probabilities and provide upper bounds on the numerical error. Finally, Section 6 presents a numerical analysis and shows in particular that our model performs as well as in [29], while being much simpler to deal with numerically.

2 The Potential Approach for Credit Risk Modeling

All stochastic processes considered in this paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where $\mathbb{Q}$ is a risk-neutral measure. The default time of a firm is a random time denoted by τ . As mentioned in the introduction, the two most common approaches in credit risk management are the structural and reduced models. In the first one, the default is triggered when the market value of assets falls below debts. In the second approach, the default is caused by the first jump of a point process. In this paper, we investigate an alternative way, called the potential approach. Instead of modeling an event triggering the default, we propose a dynamics for the probability that the company survives until a certain maturity. This probability is of course in the interval $[0, 1]$ and is a decreasing function of the time horizon. The potential approach is based upon an $\mathbb{F}^\xi = (\mathcal{F}_t^\xi)_{t \geq 0}$ -adapted stochastic process $(\xi_t)_{t \geq 0}$. The filtration $\mathbb{F}^\xi$ is defined precisely in the next section. If the company is still in activity at time s , the survival probability until time $t \geq s$

is denoted by $\mathbb{Q}(\tau > t | \tau > s)$ and defined as

$$\mathbb{Q}(\tau > t | \tau > s) = \frac{1}{\xi_s} \mathbb{E}_{\mathbb{Q}}[\xi_t | \mathcal{F}_s^{\xi}].$$

To ensure that $\frac{1}{\xi_s} \mathbb{E}_{\mathbb{Q}}[\xi_t | \mathcal{F}_s^{\xi}]$ takes its values in $[0, 1]$ and decreases towards 0 as t increases, $(\xi_t)_{t \geq 0}$ should be a potential. This is a stochastic process defined as follows:

Definition 2.1. Let $(\xi_t)_{t \geq 0}$ be a stochastic process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. Assume that $(\xi_t)_{t \geq 0}$ is adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. We say that $(\xi_t)_{t \geq 0}$ is a *potential* if it satisfies the following conditions:

- (i) $\xi_t > 0$ for all t , $\mathbb{Q}$ -a.s.;
- (ii) $(\xi_t)_{t \geq 0}$ is a supermartingale, i.e. for every $0 \leq s \leq t$ it holds that $\mathbb{E}_{\mathbb{Q}}[\xi_t | \mathcal{F}_s^{\xi}] \leq \xi_s$, $\mathbb{Q}$ -a.s. ;
- (iii) the expectation satisfies $\lim_{t \rightarrow +\infty} \mathbb{E}_{\mathbb{Q}}[\xi_t] = 0$.

There exist various ways to build such a process in a Brownian setting. We refer to e.g. [15] for examples. In this paper, we consider a process $(\xi_t)_{t \geq 0}$ that is a Hawkes jump-diffusion supermartingale. The construction and properties of this process are developed in the next sections.

We emphasize that the model is directly specified under the risk-neutral measure $\mathbb{Q}$. It is the case because our model will be calibrated on default probabilities inferred from CDS quotes on the market. As a consequence, the resulting default probabilities do not correspond to the physical measure $\mathbb{P}$ but to the risk-neutral measure $\mathbb{Q}$. In particular, the numerical values for these probabilities can't be interpreted as being the probabilities of default in the real world.

3 Hawkes Jump-diffusion Supermartingales

This section is devoted to the construction of Hawkes jump-diffusion supermartingales. We consider a self-exciting counting process $(N_t)_{t \geq 0}$ whose intensity is given by

$$d\lambda_t = \kappa(\theta - \lambda_{t-})dt + \sigma\sqrt{\lambda_{t-}}dW_t + \eta dP_t$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion and

$$P_t = \sum_{k=1}^{N_t} J_k.$$

The jump sizes $J_1, J_2, \dots =^{\text{dist}} J$ are i.i.d. random variables that are assumed to be distributed according to a measure ν defined as

$$\nu(B) := \int_{B \cap \mathbb{R}^+} \rho e^{-\rho x} dx, \quad B \in \mathcal{B}_{\mathbb{R}}$$

with $\rho > 0$. As a consequence, $J_1, J_2, \dots$ have an exponential distribution with parameter ρ (and thus expected value $1/\rho$). The process $(N_t)_{t \geq 0}$ is called a *marked Hawkes process* (see Sections 2 and 3 in [40] or Sections 5 and 7 in [41]). The stochastic process $(\lambda_t)_{t \geq 0}$ has a mean reverting behaviour, as it tends to be attracted towards the parameter $\theta > 0$. The parameter $\kappa > 0$ corresponds to the speed of the mean reversion towards θ . The intensity $(\lambda_t)_{t \geq 0}$ is also

affected by two types of events: very frequent events of small magnitude and rare events of large magnitude. The first type of events is represented by the term $\sigma\sqrt{\lambda_{t-}}dW_t$, where $\sigma \geq 0$ represents the sensitivity to such events. The second type of events is represented by the term ηdP_t , where $\eta \geq 0$ is again a sensitivity parameter. The sizes of the larger events are described by an exponential distribution of parameter $\rho > 0$. When a jump occurs in the stochastic process $(P_t)_{t \geq 0}$, its jump intensity $(\lambda_t)_{t \geq 0}$ increases, so that further jumps become more likely in $(P_t)_{t \geq 0}$. This phenomenon is known as a self-exciting behaviour. The condition $\kappa > \eta/\rho$ guarantees that $\lim_{t \rightarrow +\infty} \mathbb{E}_{\mathbb{Q}}[\lambda_t] < +\infty$. It is assumed to hold and is referred to as the non-explosion condition.

The intensity $(\lambda_t)_{t \geq 0}$ is known to be a CBI process (continuous state branching processes with immigration), for which we refer again the reader to Sections 2 and 3 in [40] or Sections 5 and 7 in [41]. In particular, it guarantees that the intensity is a positive process. It is even a strictly positive process if and only if $2\kappa\theta \geq \sigma^2$, according to Proposition 3.4 in [40].

We denote by $\mathbb{F}^\xi = (\mathcal{F}_t^\xi)_{t \geq 0}$ the filtration generated by the process $(\lambda_t, P_t, N_t, W_t)_{t \geq 0}$. We assume that the random variables $J_1, J_2, \dots$ are in some sense independent of the process $(\lambda_t, P_t, N_t, W_t)_{t \geq 0}$. More precisely, if we denote by $T_1, T_2, \dots$ the stopping times that represent the jump times of $(\lambda_t, N_t, P_t)_{t \geq 0}$, that is

$$T_k = \inf\{\tau > 0 : N_\tau = k\}$$

then we assume that J_k is independent of $\mathcal{F}_{T_k-}^\xi$ for any $k = 1, 2, \dots$. In other words, nothing can be said about the jump size J_k until just before the associated jump time T_k . Of course, the jump sizes $J_1, J_2, \dots$ are not independent from the process $(N_t)_{t \geq 0}$, as the size of a jump influences the occurrence of subsequent jumps of $(N_t)_{t \geq 0}$. However, past jumps do not have any impact on subsequent jump sizes. Using our assumptions, we can obtain an expression for the expected value of the exponential of $(\lambda_t)_{t \geq 0}$.

Define the random measure Ξ on $(\mathbb{R}^+ \times \mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+ \times \mathbb{R}^+})$ as

$$\Xi(A, B)(\omega) = \sum_{k \geq 1} \delta_{(T_k(\omega), J_k(\omega))}(A, B),$$

where $\delta_{(a,b)}$ is the Dirac measure located at $(a, b) \in \mathbb{R}^2$. This is the random measure associated with the marked Hawkes process $(N_t)_{t \geq 0}$. The compensated measure $\tilde{\Xi}$ is

$$\tilde{\Xi}(A, B) = \Xi(A, B) - \int_{s \in A} \int_{y \in B} \lambda_{s-} \nu(dy) ds.$$

We start with two technical lemmas. They will be used to prove that the random variables $(e^{z\lambda_t})_{t \geq 0}$ have finite expectation for some $z > 0$. The condition $z < \rho/\eta$ in Lemma 3.1 is essential for the existence of the Laplace transform of the jump distribution ν . Note that as a consequence, it is assumed to hold throughout this paper.

Lemma 3.1. *Let $a : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuously differentiable function and fix $z < \rho/\eta$. It holds that*

$$e^{a(t)+z\lambda_t} = e^{a(s)+z\lambda_s} + (A_t - A_s) + (M_t - M_s)$$

with

$$A_t = \int_0^t (a'(s) + z\kappa\theta + q(z)\lambda_{s-}) e^{a(s)+z\lambda_{s-}} ds$$

and

$$M_t = \int_0^t z e^{a(s)+z\lambda_s-} \sigma \sqrt{\lambda_{s-}} dW_s + \int_0^t e^{a(s)+z\lambda_{s-}} \int_{y \in \mathbb{R}^+} (e^{z\eta y} - 1) \tilde{\Xi}(dy, ds).$$

The function $q : (-\infty, \rho/\eta) \rightarrow \mathbb{R}$ that appears in the definition of $(A_t)_{t \geq 0}$ is given by

$$q(z) = -z\kappa + \frac{\sigma^2}{2} z^2 + \frac{\rho}{\rho - \eta z} - 1. \quad (3)$$

Moreover, the stochastic process $(M_t)_{t \geq 0}$ is a local martingale.

Proof. Fix some $z < \rho/\eta$. To shorten the notations, let us write

$$Y_t = e^{a(t)+z\lambda_t}.$$

Ito's lemma applied on the process $(Y_t)_{t \geq 0}$ yields

$$\begin{aligned} Y_t &= Y_s + \int_s^t a'(u) Y_{u-} du + \int_s^t z\kappa(\theta - \lambda_{u-}) Y_{u-} du \\ &\quad + \int_s^t z\sigma \sqrt{\lambda_{u-}} Y_{u-} dW_u + \frac{1}{2} \int_s^t z^2 \sigma^2 \lambda_{u-} Y_{u-} du \\ &\quad + \int_s^t \int_{y \in \mathbb{R}^+} \left(e^{a(u)+z(\lambda_{u-}+\eta y)} - Y_{u-} \right) \Xi(dy, du) \\ &= Y_s + \int_s^t (a'(u) + z\kappa\theta) Y_{u-} du + \int_s^t z\sigma \sqrt{\lambda_{u-}} Y_{u-} dW_u \\ &\quad + \int_s^t \left(-z\kappa + \frac{\sigma^2}{2} z^2 + \int_{y \in \mathbb{R}^+} (e^{z\eta y} - 1) \nu(dy) \right) \lambda_{u-} Y_{u-} du \\ &\quad + \int_s^t Y_{u-} \int_{y \in \mathbb{R}^+} (e^{z\eta y} - 1) \tilde{\Xi}(dy, du) \\ &= Y_s + \int_s^t (a'(u) + z\kappa\theta + q(z)\lambda_{u-}) Y_{u-} du \\ &\quad + \int_s^t z\sigma \sqrt{\lambda_{u-}} Y_{u-} dW_u + \int_s^t Y_{u-} \int_{y \in \mathbb{R}^+} (e^{z\eta y} - 1) \tilde{\Xi}(dy, du) \\ &= Y_s + (A_t - A_s) + (M_t - M_s), \end{aligned}$$

which is the formula announced in the statement. Since stochastic integrals with respect to local martingales are local martingales and sums of local martingales are also local martingales, $(M_t)_{t \geq 0}$ is a local martingale. $\square$

As mentioned before Lemma 3.1, the goal is to prove that $(e^{z\lambda_t})_{t \geq 0}$ has finite expectation for some $z > 0$. To this end, our strategy is to study the function q that appears in Lemma 3.1. More precisely, determining the values z such that $q(z) \leq 0$ will allow us to obtain an upper bound for the expected value of $(e^{z\lambda_t})_{t \geq 0}$. This is done in Lemma 3.2. Obviously we can restrict our attention to $z \in (0, \rho/\eta)$, as $e^{z\lambda_t} \leq 1$ implies that the expectation is finite when $z \leq 0$.

Lemma 3.2. *If $\sigma > 0$ and if we define the constants*

$$\zeta = \left(\frac{\sigma^2}{2} \rho + \eta \kappa \right)^2 + 2\eta\sigma^2(\eta - \kappa\rho) \quad (4)$$

and

$$\gamma_{\pm} = \frac{\left(\frac{\sigma^2}{2} \rho + \eta \kappa \right) \pm \sqrt{\zeta}}{\eta\sigma^2}, \quad (5)$$

then $0 < \gamma_- < \rho/\eta < \gamma_+$. Moreover, for $z \in (0, \rho/\eta)$,

$$q(z) < 0 \iff z \in (0, \gamma_-),$$

$$q(z) = 0 \iff z = \gamma_-,$$

and

$$q(z) > 0 \iff z \in (\gamma_-, \rho/\eta).$$

If $\sigma = 0$ then $\frac{\kappa\rho - \eta}{\eta\kappa} > 0$ and for $z \in (0, \rho/\eta)$,

$$q(z) < 0 \iff z \in (0, \frac{\kappa\rho - \eta}{\eta\kappa}),$$

$$q(z) = 0 \iff z = \frac{\kappa\rho - \eta}{\eta\kappa},$$

and

$$q(z) > 0 \iff z \in (\frac{\kappa\rho - \eta}{\eta\kappa}, \rho/\eta). \quad (6)$$

Proof. Since $\rho - \eta z > 0$ and $z > 0$, multiplying the inequality $q(z) \leq 0$ by $(\rho - \eta z)$ and dividing it by z yields

$$q(z) \leq 0 \iff -\frac{\sigma^2}{2}\eta z^2 + \left(\frac{\sigma^2}{2}\rho + \eta\kappa \right) z + \eta - \kappa\rho \leq 0. \quad (7)$$

Assume that $\sigma > 0$. Then the determinant of the second order equation in (7) is given by ζ in Equation (4). Some straightforward computations lead to

$$\zeta = \left(\frac{\sigma^2}{2}\rho + \eta\kappa \right)^2 + 2\eta^2\sigma^2, \quad (8)$$

so that ζ is strictly positive. As a consequence, $q(\gamma_{\pm}) = 0$, where $\gamma_{\pm}$ is given in Equation (5). Recall that the non-explosion condition for the Hawkes process intensity is the inequality $\eta - \kappa\rho < 0$. Therefore,

$$\zeta = \left(\frac{\sigma^2}{2}\rho + \eta\kappa \right)^2 + 2\eta\sigma^2(\eta - \kappa\rho) < \left(\frac{\sigma^2}{2}\rho + \eta\kappa \right)^2,$$

which implies that both quantities of Equation (5) are strictly positive. Moreover, Equation (8) implies that

$$\sqrt{\zeta} > \left| \frac{\sigma^2}{2}\rho + \eta\kappa \right| \geq \eta\kappa - \frac{\sigma^2}{2}\rho$$

and thus

$$\gamma_- = \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) - \sqrt{\zeta}}{\eta\sigma^2} < \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) - \left(\eta\kappa - \frac{\sigma^2}{2}\rho\right)}{\eta\sigma^2} = \frac{\rho}{\eta}. \quad (9)$$

In addition,

$$\begin{aligned} \gamma_+ &= \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) + \sqrt{\left(\frac{\sigma^2}{2}\rho - \eta\kappa\right)^2 + 2\eta^2\sigma^2}}{\eta\sigma^2} \\ &> \frac{\left(\frac{\sigma^2}{2}\rho + \eta\kappa\right) + \left|\frac{\sigma^2}{2}\rho - \eta\kappa\right|}{\eta\sigma^2} \\ &\geq \frac{\sigma^2\rho}{\eta\sigma^2} = \frac{\rho}{\eta}. \end{aligned}$$

All the claims of the $\sigma > 0$ case are proved. In the case $\sigma = 0$, the claims easily follow from Equation (7) and the non-explosion condition $\eta - \kappa\rho < 0$. $\square$

These lemmas allow to prove the integrability of $e^{z\lambda_t}$, as well as an upper bound for its expected value.

Corollary 3.1. *Define*

$$\gamma = \begin{cases} \gamma_- & \text{if } \sigma > 0, \\ \frac{\kappa\rho - \eta}{\eta\kappa} & \text{if } \sigma = 0. \end{cases}$$

Then for any $z \leq \gamma$, the stochastic process $(e^{-z\kappa\theta t + z\lambda_t})_{t \geq 0}$ is a supermartingale. In particular, for any $t \geq 0$, the random variable $e^{z\lambda_t}$ is $\mathbb{Q}$ -integrable and we have the upper bound

$$\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t} | \mathcal{F}_s^{\xi}] \leq 1 \vee e^{z\lambda_s + z\kappa\theta(t-s)}$$

whenever $t \geq s \geq 0$.

Proof. We note first that if $z \leq 0$, then the inequality $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t} | \mathcal{F}_s^{\xi}] \leq 1$ is clear. Thus let $0 < z \leq \gamma$. Define the positive process $(Y_t)_{t \geq 0}$ as $Y_t = e^{-z\kappa\theta t + z\lambda_t}$. The proof consists in showing that $(Y_t)_{t \geq 0}$ is a supermartingale. By taking the function $a(t) = -z\kappa\theta t$, Lemma 3.1 gives us $Y_t = Y_s + (A_t - A_s) + (M_t - M_s)$, with

$$A_t - A_s = \int_s^t q(z)\lambda_{s-} Y_{s-} ds$$

and $(M_t)_{t \geq 0}$ a local martingale. From Lemma 3.2, we know that $q(z) \leq 0$. Since both the processes $(\lambda_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are positive, we clearly have $(A_t - A_s) \leq 0$ for every $t \geq s \geq 0$ $\mathbb{Q}$ -a.s., and thus $Y_t \leq Y_s + (M_t - M_s)$. The process $(M_t)_{t \geq 0}$ being a local martingale, there exists a sequence of stopping-times $(\tau_n)_{n \geq 1}$ with $\tau_n \uparrow +\infty$ $\mathbb{Q}$ -a.s. as $n \uparrow +\infty$ such that, for every $n \geq 1$, the stopped process $(M_{t \wedge \tau_n})_{t \geq 0}$ is a martingale. Since the process $(Y_t)_{t \geq 0}$ is positive, Fatou's lemma yields

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[Y_t | \mathcal{F}_s^{\xi}] &\leq \mathbb{E}_{\mathbb{Q}}[Y_s + (M_t - M_s) | \mathcal{F}_s^{\xi}] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\liminf_{n \rightarrow +\infty} (Y_s + (M_{t \wedge \tau_n} - M_{s \wedge \tau_n})) | \mathcal{F}_s^{\xi} \right] \\ &\leq \liminf_{n \rightarrow +\infty} (Y_s + \mathbb{E}_{\mathbb{Q}} [M_{t \wedge \tau_n} - M_{s \wedge \tau_n} | \mathcal{F}_s^{\xi}]) \\ &= Y_s. \end{aligned} \quad (10)$$

From the definition of $(Y_t)_{t \geq 0}$, we obtain that $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t} | \mathcal{F}_s^\xi] \leq e^{z\kappa\theta(t-s) + z\lambda_s}$. $\square$

With the help of our previous results, we build a *Hawkes jump-diffusion supermartingale*. This process is introduced in the next corollary.

Corollary 3.2. *For any $z \in (0, \gamma]$ and $w > \beta = \gamma\kappa\theta$, the process $(\xi_t)_{t \geq 0}$ defined by $e^{-wt+z\lambda_t}$ is a potential.*

Proof. We check the 3 conditions of Definition 2.1. The positivity of $(\xi_t)_{t \geq 0}$ is straightforward. We show now that it is a supermartingale. Corollary 3.1 implies that when $t \geq s \geq 0$, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\xi_t | \mathcal{F}_s^\xi] &= \mathbb{E}_{\mathbb{Q}}[e^{-wt+z\lambda_t} | \mathcal{F}_s^\xi] \\ &\leq e^{-wt+z\lambda_s+\beta(t-s)} \\ &= \xi_s e^{-(w-\beta)(t-s)} \\ &< \xi_s \end{aligned} \tag{11}$$

so that $(\xi_t)_{t \geq 0}$ is a supermartingale. The convergence of its expectation to 0 follows from Equation (11) and $e^{-(w-\beta)(t-s)} \rightarrow 0$ when $t \rightarrow +\infty$, since $w - \beta > 0$. $\square$

If we consider the supermartingale $(\xi_t)_{t \geq 0}$ and write τ for the time of default, the potential approach consists in assuming that the term structure of default probabilities is determined by

$$\mathbb{Q}(\tau > t | \tau > s) = \frac{\mathbb{E}[\xi_t | \mathcal{F}_s^\xi]}{\xi_s} = \frac{e^{-wt} \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t} | \mathcal{F}_s^\xi]}{e^{-ws+z\lambda_s}}. \tag{12}$$

The next section introduces the fractional version of the model.

4 Fractional Hawkes Jump-diffusion Supermartingales

In this section, we introduce a fractional Hawkes jump-diffusion supermartingale. This process is obtained by time-changing the supermartingale $(\xi_t)_{t \geq 0}$ by the inverse of an α -stable subordinator. As we have seen in the introduction, such a time-change allows to obtain a better fit for highly convex survival probability curves. We start by introducing the time-change process and the considered filtrations. Afterwards, we prove that the supermartingale $(\xi_t)_{t \geq 0}$ remains a supermartingale under the time-change and that it keeps all its desired properties.

An α -stable subordinator and its inverse are defined as follows.

Definition 4.1. Let $\alpha \in (0, 1)$. A stochastic process $(U_t)_{t \geq 0}$ is an α -stable subordinator if it is an increasing Lévy process that satisfies

$$\mathbb{E}[e^{-uU_t}] = e^{-tu^\alpha}$$

for all $t \geq 0$ and $u \in \mathbb{R}$. The *inverse of the α -stable subordinator* $(U_t)_{t \geq 0}$ is the stochastic process $(S_t)_{t \geq 0}$ defined as

$$S_t := \inf\{\tau \geq 0 : U_\tau \geq t\}.$$

For more material about α -stable subordinators and processes, we refer the reader to [42]. Let $(U_t)_{t \geq 0}$ be an α -stable subordinator on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, with $\alpha \in (0, 1)$. The parameter α will be referred to as the *fractional order* in the sequel. We denote its inverse by $(S_t)_{t \geq 0}$ and its natural filtration by $\mathbb{F}^U = (\mathcal{F}_t^U)_{t \geq 0}$. The natural filtration $\mathbb{F}^U$ of $(U_t)_{t \geq 0}$ is assumed to be independent from $\mathbb{F}^\xi$. The supermartingale $(\xi_t)_{t \geq 0}$ will be time-changed by the inverse process $(S_t)_{t \geq 0}$. The inverse process $(S_t)_{t \geq 0}$ possesses the property of "randomly freezing time". This property³ can be observed on Figure 1. This figure⁴ displays corresponding sample paths of the α -stable process $(U_t)_{t \geq 0}$ and its inverse $(S_t)_{t \geq 0}$ for some levels of the fractional order α . One can see that the frozen time periods of $(S_t)_{t \geq 0}$ (i.e. the periods of time for which the process remains constant) correspond to the jumps of the original process $(U_t)_{t \geq 0}$. Also note that by definition of $(S_t)_{t \geq 0}$, representing the path $[0, 10] \ni t \mapsto S_t(\omega)$ only requires the values of $t \mapsto U_t(\omega)$ until it reaches the value 10. This is why the graphs on the right-side are truncated at $y = 10$. Finally, note that the values of the fractional order close to 0 give the wildest behaviour for $(U_t)_{t \geq 0}$, i.e. paths with very large jumps. This translates into a lot of long motionless periods for the inverse $(S_t)_{t \geq t}$ when α is close to 0.

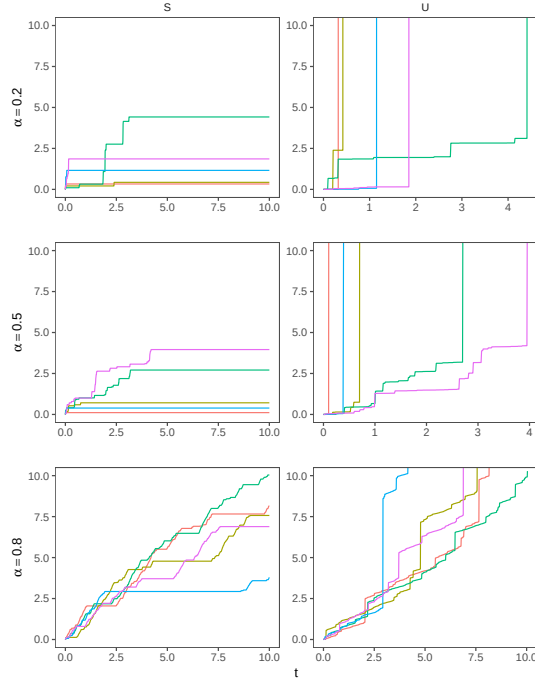


Figure 1: Five simulated paths of $(U_t)_{t \geq 0}$ and corresponding paths of $(S_t)_{t \geq 0}$ for fractional orders $\alpha \in \{0.2, 0.5, 0.8\}$.

Let us now introduce precisely the filtration we consider for the time-changed supermartingale $(\xi_{S_t})_{t \geq 0}$. Define the filtration $\mathbb{F}^{\xi, U} = (\mathcal{F}_t^{\xi, U})_{t \geq 0}$ as $\mathcal{F}_t^{\xi, U} = \mathcal{F}_t^\xi \vee \mathcal{F}_t^U$. By definition of $(S_t)_{t \geq 0}$, for any $u, s \geq 0$ it holds that $\{S_u \leq s\} = \{U_s \geq u\}$, which clearly belongs to $\mathcal{F}_s^U$. As a consequence,

³By "randomly freezing time", we mean that the paths of this stochastic process exhibit motionless periods.

⁴The algorithm used to obtain the simulated sample paths is from [34]. It is also described in details in [29] (Appendix B).

for any fixed $u \geq 0$, S_u is a $\mathbb{F}^U$ -stopping-time, and thus also a $\mathbb{F}^{\xi, U}$ -stopping-time. We can therefore define the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ as the collection of stopped σ -algebras

$$\mathcal{G}_t := \mathcal{F}_{S_t}^{\xi, U} = \{A \in \mathcal{F}_{\infty}^{\xi, U} : A \cap \{S_t \leq s\} \in \mathcal{F}_s^{\xi, U} \text{ for all } s \geq 0\},$$

with $\mathcal{F}_{\infty}^{\xi, U} = \bigvee_{t \geq 0} \mathcal{F}_t^{\xi, U}$. We show now that the obtained process $(\xi_{S_t})_{t \geq 0}$ remains a supermartingale and can therefore be used as in the previous section to define a term structure of default probabilities.

Proposition 4.1. *The process $(\xi_{S_t})_{t \geq 0}$ is a positive $(\mathbb{Q}, \mathbb{G})$ -supermartingale that satisfies $\lim_{t \rightarrow +\infty} \mathbb{E}_{\mathbb{Q}}[\xi_{S_t}] = 0$.*

Proof. By Corollary 3.2, $(\xi_t)_{t \geq 0}$ is a $(\mathbb{Q}, \mathbb{F}^{\xi})$ -supermartingale. Since the filtrations $\mathbb{F}^{\xi}$ and $\mathbb{F}^U$ are independent, it holds that

$$\mathbb{E}_{\mathbb{Q}}[\xi_t | \mathcal{F}_s^{\xi} \vee \mathcal{F}_s^U] = \mathbb{E}_{\mathbb{Q}}[\xi_t | \mathcal{F}_s^{\xi}]$$

for any $s, t \geq 0$. The property we use in the previous equation can be found in the chapter 9.7 of [43], at property (k). As a consequence, $(\xi_t)_{t \geq 0}$ is also a $(\mathbb{Q}, \mathbb{F}^{\xi, U})$ -supermartingale. Then, since for any $s \leq t$, we have $S_s \leq S_t$ $\mathbb{Q}$ -a.s., it follows from the Optional Stopping Theorem for supermartingales (see e.g. Theorem 3.3 in the chapter 2 of [44]) that

$$\xi_{S_s} \geq \mathbb{E}_{\mathbb{Q}}[\xi_{S_t} | \mathcal{F}_{S_s}^{\xi, U}] = \mathbb{E}_{\mathbb{Q}}[\xi_{S_t} | \mathcal{G}_s].$$

This proves that $(\xi_{S_t})_{t \geq 0}$ is a $(\mathbb{Q}, \mathbb{G})$ -supermartingale. Recall that from [45],

$$\mathbb{E}_{\mathbb{Q}}[e^{-\omega S_t}] = E_{\alpha}(-\omega t^{\alpha}),$$

where the function E_{α} is the Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + 1)}.$$

Let us recall that $\xi_t = e^{-wt + z\lambda_t}$. Let $v \in (\beta, w)$ and let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined as $g(t) = e^{-(v-\beta)t + z\lambda_0}$. From Equation (11), we obtain

$$\mathbb{E}_{\mathbb{Q}}[\xi_t] \leq e^{-(w-\beta)t + z\lambda_0} \leq g(t).$$

Fix $t \geq s \geq 0$. Using again Equation (11), we find

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\xi_t - g(t) | \mathcal{F}_s^{\xi, U}] &= \mathbb{E}_{\mathbb{Q}}[\xi_t - g(t) | \mathcal{F}_s^{\xi}] \\ &\leq \xi_s e^{-(w-\beta)(t-s)} - e^{-(w-\beta)t + z\lambda_0} \\ &= e^{-(w-\beta)(t-s)} [\xi_s] \end{aligned}$$

As a consequence, from Corollary 3.1 and Lemma A.1 (see the appendix)

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\xi_{S_t}] &\leq \mathbb{E}_{\mathbb{Q}}[g(S_t)] \\ &= e^{z\lambda_0} E_{\alpha}((\beta - w)t^{\alpha}) \end{aligned}$$

where the equality follows from a result from [45], which states that

$$\mathbb{E}_{\mathbb{Q}}[e^{-\omega S_t}] = E_{\alpha}(-\omega t^{\alpha}).$$

This proves the convergence of the expectation to 0 and ends the proof. $\square$

The supermartingale $(\xi_{S_t})_{t \geq 0}$ inherits from $(S_t)_{t \geq 0}$ the property of having motionless periods, during which the time appears to be frozen. The term structure of default probabilities is defined as follows: let $\tau^{(\alpha)}$ be the time of default in the fractional model. Then we assume that the term structure of default probabilities satisfies

$$\mathbb{Q}(\tau^{(\alpha)} > t | \tau^{(\alpha)} > s) = \frac{\mathbb{E}_{\mathbb{Q}}[e^{-wS_t + z\lambda_{S_t}} | \mathcal{G}_s]}{e^{-wS_s + z\lambda_{S_s}}}, \quad (13)$$

for $t \geq s \geq 0$.

The next section addresses the problem of computing the default probabilities, in both the non-fractional and fractional settings.

5 Computation of the Default Probabilities

The main challenge in computing the default probabilities of Equations (12) and (13) is the computation of $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t} | \mathcal{F}_s^{\xi}]$ and $\mathbb{E}_{\mathbb{Q}}[e^{-wS_t + z\lambda_{S_t}} | \mathcal{G}_s]$, $t \geq s \geq 0$. To this end, several approaches can be thought of. A first possible approach is the one of [46]. It consists in showing that the expected value $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t} | \mathcal{F}_s]$ can be written as $\exp\{a(t-s) + \lambda_s b(t-s)\}$ for some functions a, b and that these functions satisfy a system of differential equations. The expected value can then be found by solving this system numerically. In our context, the main drawback of this approach is that it does not transpose very well to the fractional model. As a matter of fact, obtaining the default probabilities in the fractional setting would require to compute numerically the integral

$$\int_0^{+\infty} \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_{\tau}}] g(t, \tau) d\tau = \int_0^{+\infty} \exp\{a(\tau, z) + \lambda_0 b(\tau, z)\} g(t, \tau) d\tau,$$

where $\tau \mapsto g(t, \tau)$ is the probability density function of S_t . Then one could use Proposition 3.2 in [47] that says that

$$g(t, \tau) = \frac{1}{\pi} \sum_{k=0}^{+\infty} \frac{(-\tau)^k}{k!} \Gamma((1+k)\alpha) t^{-(k+1)\alpha} \sin(\pi\alpha(k+1)), \quad \tau > 0$$

to evaluate the integral. This first approach seems to be computationally heavy, as it requires several layers of numerical approximations. A second approach is to derive a partial differential equation for the transform $(t, z) \mapsto \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}]$. This approach is used in e.g. [29] and has the benefit of having a natural extension to the fractional case. As a matter of fact, the transform $(t, z) \mapsto \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_{S_t}}]$ satisfies a fractional partial differential equation. Compared to the non-fractional case, the only difference is that the partial derivative with respect to time is a Caputo fractional derivative of order α . In both cases, the (fractional) partial differential equation can be solved numerically with a finite difference method. However, this second approach suffers from two drawbacks. The first drawback is that we have to use fractional calculus, which is rather complicated from a mathematical standpoint. The second drawback is that fractional partial differential equations are far from being the easiest equations to solve numerically. Indeed, performing finite difference approximations for fractional derivatives requires, at each iteration, to use the computed values for all the previous iterations. This substantially increases the computation time. This is why we propose here a third approach that rely on a recursive method to compute all the moments of λ_t . To the best of our knowledge, this recursive method is new. Once the moments $\mathbb{E}_{\mathbb{Q}}[\lambda_t^k]$ are computed for all nonnegative integers $k \leq n$, we can approximate

the expectation $\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}]$ with the help of the MacLaurin series expansion of the exponential function, that is

$$\mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}] \approx \sum_{k=0}^n \frac{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^k]}{k!}.$$

This third approach avoids the shortcomings of the two other ones. Firstly, it is much faster computationally. Secondly, it has an easy extension to the fractional case, without relying on the complicated machinery of fractional calculus. That is, once the moments of λ_t are obtained, the moments of the time-changed process λ_{S_t} can be found very easily. Thirdly, the approximation can be shown to be convergent and a bound on the numerical error can be provided in both the non-fractional and fractional cases.

This section is organized as follows: we start by introducing and proving the recursive method that allows to compute the moments of λ_t . Then, as a corollary to this method, we show that the moments of the time-changed process λ_{S_t} can be obtained in a similar manner. Finally, after introducing the approximations for the survival probabilities of Equations (12) and (13), we show that these approximations are convergent and give an upper bound on the associated numerical error.

After a technical lemma, the recursive method for the moments of λ_t will be given in the next proposition.

Lemma 5.1. *Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a twice continuously differentiable function that satisfies*

$$\mathbb{E}_{\mathbb{Q}}[|f(\lambda_t)|] \vee \mathbb{E}_{\mathbb{Q}}\left[\int_0^t (f'(\lambda_{s-}))^2 \lambda_{s-} ds\right] \vee \mathbb{E}_{\mathbb{Q}}\left[\int_0^t |f''(\lambda_{s-})| \lambda_{s-} ds\right] < +\infty$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^t \int_{y \in \mathbb{R}^+} |f(\lambda_{s-} + \eta y) - f(\lambda_{s-})| \lambda_{s-} \nu(dy) ds\right] < +\infty$$

for every $t \geq 0$. Then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[f(\lambda_t)] - f(\lambda_0) &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^t f'(\lambda_{s-}) \kappa(\theta - \lambda_{s-}) ds\right] + \frac{\sigma^2}{2} \mathbb{E}_{\mathbb{Q}}\left[\int_0^t f''(\lambda_{s-}) \lambda_{s-} ds\right] \\ &\quad + \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \int_{y \in \mathbb{R}^+} (f(\lambda_{s-} + \eta y) - f(\lambda_{s-})) \lambda_{s-} \nu(dy) ds\right]. \end{aligned}$$

Proof. Applying Ito's lemma, we obtain

$$\begin{aligned} f(\lambda_t) - f(\lambda_0) &= \int_0^t f'(\lambda_{s-}) \kappa(\theta - \lambda_{s-}) ds + \sigma \int_0^t f'(\lambda_{s-}) \sqrt{\lambda_{s-}} dW_s \\ &\quad + \frac{\sigma^2}{2} \int_0^t f''(\lambda_{s-}) \lambda_{s-} ds + \int_0^t \int_{y \in \mathbb{R}^+} (f(\lambda_{s-} + \eta y) - f(\lambda_{s-})) \Xi(dy, ds). \end{aligned} \tag{14}$$

From our assumptions, the quadratic variation of the brownian integral has finite expectation, which implies that it is a martingale. Its expected value is therefore 0. Note that

$$(M_t)_{t \geq 0} = \left(\int_0^t \int_{y \in \mathbb{R}^+} |f(\lambda_{s-} + \eta y) - f(\lambda_{s-})| \tilde{\Xi}(dy, ds) \right)_{t \geq 0}$$

is a local martingale, so there exists a sequence of stopping times $(\tau_n)_{n \geq 1}$ with $\tau_n \uparrow +\infty$ $\mathbb{Q}$ -a.s. such that the stopped process $(M_{t \wedge \tau_n})_{t \geq 0}$ is a martingale for every n . From the definition of the compensated measure $\tilde{\Xi}$, we get for each n

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[\int_0^{t \wedge \tau_n} \int_{y \in \mathbb{R}^+} |f(\lambda_{s-} + \eta y) - f(\lambda_{s-})| \tilde{\Xi}(dy, ds) \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^{t \wedge \tau_n} \int_{y \in \mathbb{R}^+} |f(\lambda_{s-} + \eta y) - f(\lambda_{s-})| \lambda_{s-} \nu(dy) ds \right] \end{aligned}$$

and since the two processes involved are non-negative and increase with n , the monotone convergence theorem yields

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[\int_0^t \int_{y \in \mathbb{R}^+} |f(\lambda_{s-} + \eta y) - f(\lambda_{s-})| \tilde{\Xi}(dy, ds) \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^t \int_{y \in \mathbb{R}^+} |f(\lambda_{s-} + \eta y) - f(\lambda_{s-})| \lambda_{s-} \nu(dy) ds \right] \end{aligned}$$

which is finite, by assumption. Now, using a similar approach and the dominated convergence theorem yields

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[\int_0^t \int_{y \in \mathbb{R}^+} (f(\lambda_{s-} + \eta y) - f(\lambda_{s-})) \tilde{\Xi}(dy, ds) \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^t \int_{y \in \mathbb{R}^+} (f(\lambda_{s-} + \eta y) - f(\lambda_{s-})) \lambda_{s-} \nu(dy) ds \right]. \end{aligned}$$

Using our results and taking expectations on Equation (14), we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} [f(\lambda_t)] - f(\lambda_0) &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^t f'(\lambda_{s-}) \kappa(\theta - \lambda_{s-}) ds \right] + \frac{\sigma^2}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^t f''(\lambda_{s-}) \lambda_{s-} ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^t \int_{y \in \mathbb{R}^+} (f(\lambda_{s-} + \eta y) - f(\lambda_{s-})) \lambda_{s-} \nu(dy) ds \right], \end{aligned}$$

as announced. $\square$

Proposition 5.1. *Let $z > 0$. For each $n \geq 1$, the moment $\mathbb{R}^+ \ni t \mapsto \mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^n]$ satisfies*

$$\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^n] = \left(\sum_{j=1}^n d_{j,n} e^{-jat} \right) + c_n$$

with $a = \kappa - (\eta/\rho)$, for all $n \in \mathbb{N}$. Moreover, the constants $d_{k,n}, c_n$ can be computed recursively with the relations

$$\begin{aligned} d_{1,1} &= z\lambda_0 - \frac{\kappa z \theta}{a}, \quad c_1 = \frac{\kappa z \theta}{a}, \\ d_{j,n} &= \begin{cases} - \left(\frac{b_n d_{j,n-1}}{(j-n)a} + \sum_{p=2}^{n-j+1} z^{-1} m_p \binom{n}{p} \frac{d_{j,n-p+1}}{(j-n)a} \right) & j < n, \\ (z\lambda_0)^n - \left(\sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na} \right) - \frac{b_n c_{n-1}}{na} \\ \quad + \sum_{k=1}^{n-1} \left(\frac{b_n d_{k,n-1}}{(k-n)a} + \sum_{p=2}^{n-k+1} z^{-1} m_p \binom{n}{p} \frac{d_{k,n-p+1}}{(k-n)a} \right) & j = n, \end{cases} \end{aligned}$$

$$c_n = \frac{b_n c_{n-1}}{na} + \sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na}.$$

The symbol b_n is used as a shorthand for

$$b_n = n\kappa z\theta + \frac{n(n-1)z\sigma^2}{2}$$

and $m_p = \left(\frac{z\eta}{\rho}\right)^p p!$.

Proof. Define the process $(\tilde{\lambda}_t)_{t \geq 0}$ as $\tilde{\lambda}_t := z\lambda_t$. This process satisfies the SDE

$$d\tilde{\lambda}_t = \kappa(\tilde{\theta} - \tilde{\lambda}_{t-})dt + \tilde{\sigma}\sqrt{\tilde{\lambda}_{t-}}dW_t + \tilde{\eta}dP_t$$

where $\tilde{\theta} = z\theta$, $\tilde{\sigma} = \sqrt{z}\sigma$ and $\tilde{\eta} = z\eta$. Note that the jump intensity of $(P_t)_{t \geq 0}$ remains $(\lambda_t)_{t \geq 0} = (z^{-1}\tilde{\lambda}_t)_{t \geq 0}$. For each n , define the function $g_n : \mathbb{R}^+ \rightarrow \mathbb{R}$ as $g_n(x) = x^n$. We now check that each g_n satisfy the conditions of Lemma 5.1. Firstly, Corollary 3.1 implies that all the moments of λ_t exist, and thus $\mathbb{E}_{\mathbb{Q}}[|g_n(\lambda_t)|] < +\infty$ for every t and n . Moreover,

$$\mathbb{E}_{\mathbb{Q}} \left[\int_0^t (g'_n(\lambda_{s-}))^2 \lambda_{s-} ds \right] = n^2 \int_0^t \mathbb{E}_{\mathbb{Q}} [\lambda_s^{2n-1}] ds < +\infty$$

and

$$\mathbb{E}_{\mathbb{Q}} \left[\int_0^t |g''_n(\lambda_{s-}) \lambda_{s-}| ds \right] = n(n-1) \int_0^t \mathbb{E}_{\mathbb{Q}} [\lambda_s^{n-1}] ds < +\infty,$$

where the integrals have been swapped by applying Tonelli's theorem, as the integrands are positive. Finally, using again Tonelli's theorem,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[\int_0^t \int_{y \in \mathbb{R}^+} |g_n(\lambda_{s-} + \eta y) - g_n(\lambda_{s-})| \lambda_{s-} \nu(dy) ds \right] \\ &= \sum_{p=1}^n \binom{n}{p} \mathbb{E}_{\mathbb{Q}} \left[\int_0^t \lambda_{s-}^{n-p+1} \int_{y \in \mathbb{R}^+} (\eta y)^p \nu(dy) ds \right] \\ &= \sum_{p=1}^n \binom{n}{p} \left(\frac{\eta}{\rho}\right)^p \int_0^t \mathbb{E}_{\mathbb{Q}} [\lambda_{s-}^{n-p+1}] ds < +\infty. \end{aligned}$$

As a consequence, by applying Lemma 5.1 on $(\tilde{\lambda}_t)_{t \geq 0}$, rearranging the terms and taking the derivative with respect to time, we get

$$\frac{d}{dt} \mathbb{E}_{\mathbb{Q}} [\tilde{\lambda}_t^n] = \mathbb{E}_{\mathbb{Q}} [\tilde{\lambda}_t^{n-1}] \left(n\kappa\tilde{\theta} + \frac{n(n-1)\tilde{\sigma}^2}{2} \right) - n\kappa \mathbb{E}_{\mathbb{Q}} [\tilde{\lambda}_t^n] + \sum_{p=1}^n \binom{n}{p} z^{-1} m_p \mathbb{E}_{\mathbb{Q}} [\tilde{\lambda}_t^{n-p+1}],$$

where m_p is defined in the statement of this proposition. Therefore, if we define the functions $(f_n)_{n \in \mathbb{N}}$, $f_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as $f_n(t) = \mathbb{E}_{\mathbb{Q}} [\tilde{\lambda}_t^n]$, then each of these functions satisfies the first order linear differential equation

$$f'_n(t) = -na f_n(t) + h_{n-1}(t), \quad (15)$$

where

$$h_{n-1}(t) = \begin{cases} b_n f_{n-1}(t) & \text{if } n = 1, \\ b_n f_{n-1}(t) + \sum_{p=2}^n \binom{n}{p} z^{-1} m_p f_{n-p+1}(t) & \text{if } n \geq 2. \end{cases}$$

Equation (15) is easy to solve. Multiplying both sides by e^{nat} leads to

$$e^{nat} f'_n(t) + nae^{nat} f_n(t) = e^{nat} h_{n-1}(t)$$

and since the left-hand side is $(e^{nat} f_n(t))'$, an integration yields

$$f_n(t) = e^{-nat} \tilde{\lambda}_0^n + e^{-nat} \int_0^t e^{nas} h_{n-1}(s) ds. \quad (16)$$

We prove now by induction that the functions $(f_n)_{n \in \mathbb{N}}$ are of the form

$$f_n(t) = \left(\sum_{j=1}^n d_{j,n} e^{-jat} \right) + c_n \quad (17)$$

and that the numbers $d_{j,n}, c_n$ can be computed recursively, as in the statement. From Equation (16), we find that

$$f_1(t) = \left(\tilde{\lambda}_0 - \frac{\kappa \tilde{\theta}}{a} \right) e^{-at} + \frac{\kappa \tilde{\theta}}{a}$$

so that Equation (17) holds for $n = 1$ with $d_{1,1} = \tilde{\lambda}_0 - \kappa \tilde{\theta}/a$ and $c_1 = \kappa \tilde{\theta}/a$. Assume that Equation (17) holds for $1, \dots, n-1$ and that $n \geq 2$. Using Equations (16) and (17), we can compute

$$\begin{aligned} \int_0^t e^{nas} h_{n-1}(s) ds &= b_n \int_0^t e^{nas} \left[\left(\sum_{j=1}^{n-1} d_{j,n-1} e^{-jas} \right) + c_{n-1} \right] ds \\ &\quad + \sum_{p=2}^n z^{-1} m_p \binom{n}{p} \int_0^t e^{nas} \left[\left(\sum_{j=1}^{n-p+1} d_{j,n-p+1} e^{-jas} \right) + c_{n-p+1} \right] ds \\ &= - \sum_{j=1}^{n-1} e^{-(j-n)at} \left(\frac{b_n d_{j,n-1}}{(j-n)a} + \sum_{p=2}^{n-j+1} z^{-1} m_p \binom{n}{p} \frac{d_{j,n-p+1}}{(j-n)a} \right) \\ &\quad + e^{nat} \left[\frac{b_n c_{n-1}}{na} + \sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na} \right] - \sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na} \\ &\quad + \sum_{j=1}^{n-1} \left[\frac{b_n d_{j,n-1}}{(j-n)a} + \sum_{p=2}^{n-j+1} z^{-1} m_p \binom{n}{p} \frac{d_{j,n-p+1}}{(j-n)a} \right] - \frac{b_n c_{n-1}}{na}. \end{aligned}$$

As a consequence, we find

$$\begin{aligned}
f_n(t) &= - \sum_{j=1}^{n-1} e^{-jat} \left(\frac{b_n d_{j,n-1}}{(j-n)a} + \sum_{p=2}^{n-j+1} z^{-1} m_p \binom{n}{p} \frac{d_{j,n-p+1}}{(j-n)a} \right) \\
&\quad + e^{-nat} \left[\tilde{\lambda}_0^n - \frac{b_n c_{n-1}}{na} + \sum_{j=1}^{n-1} \left(\frac{b_n d_{j,n-1}}{(j-n)a} + \sum_{p=2}^{n-j+1} z^{-1} m_p \binom{n}{p} \frac{d_{j,n-p+1}}{(j-n)a} \right) \right. \\
&\quad \left. - \left(\sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na} \right) \right] + \left[\frac{b_n c_{n-1}}{na} + \sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na} \right] \\
&= \left(\sum_{j=1}^n d_{j,n} e^{-jat} \right) + c_n,
\end{aligned}$$

where

$$d_{j,n} = \begin{cases} - \left(\frac{b_n d_{j,n-1}}{(j-n)a} + \sum_{p=2}^{n-j+1} z^{-1} m_p \binom{n}{p} \frac{d_{j,n-p+1}}{(j-n)a} \right) & \text{if } j < n \\ \tilde{\lambda}_0^n - \left(\sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na} \right) - \frac{b_n c_{n-1}}{na} & \text{if } j = n \\ + \sum_{k=1}^{n-1} \left(\frac{b_n d_{k,n-1}}{(k-n)a} + \sum_{p=2}^{n-k+1} z^{-1} m_p \binom{n}{p} \frac{d_{k,n-p+1}}{(k-n)a} \right) & \text{if } j = n \end{cases}$$

and

$$c_n = \frac{b_n c_{n-1}}{na} + \sum_{p=2}^n z^{-1} m_p \binom{n}{p} \frac{c_{n-p+1}}{na}.$$

This concludes the proof. $\square$

Note that in the proof of the previous result, the method of using differential equations to find the moments of a Hawkes process intensity is not new, as this approach is used in e.g. [39]. The originality of the result lies in the observation that all the moments share the same form, i.e. a linear combination of exponential functions plus a constant, and that the associated coefficients can be computed recursively. According to the next corollary, it turns out that this result extends particularly well to the fractional case.

Corollary 5.1. *The moment $\mathbb{R}^+ \ni t \mapsto \mathbb{E}_{\mathbb{Q}}[(z\lambda_{S_t})^n]$ satisfy*

$$\mathbb{E}_{\mathbb{Q}}[(z\lambda_{S_t})^n] = \left(\sum_{j=1}^n d_{j,n} \mathbb{E}_{\alpha}(-jat^{\alpha}) \right) + c_n,$$

for all $n \in \mathbb{N}$. The function $\mathbb{E}_{\alpha}$ is the Mittag-Leffler function

$$\mathbb{E}_{\alpha}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + 1)}.$$

The constants $a, c_n, d_{j,n}$ are given in Proposition 5.1.

Proof. The main tool of this proof is a result from [45], which states that

$$\mathbb{E}_{\mathbb{Q}}[e^{-\omega S_t}] = \mathbb{E}_{\alpha}(-\omega t^{\alpha}).$$

As a consequence, by Proposition 5.1 and the law of iterated expectations, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[(z\lambda_{S_t})^n] &= \left(\sum_{j=1}^n d_{j,n} \mathbb{E}_{\mathbb{Q}}[e^{-j\alpha S_t}] \right) + c_n \\ &= \left(\sum_{j=1}^n d_{j,n} \mathbb{E}_{\alpha}(-j\alpha t^{\alpha}) \right) + c_n, \end{aligned}$$

as announced. $\square$

We now have the tools to approximate the survival probabilities. In the non-fractional case, we approximate the survival probabilities $t \mapsto q(t) := \mathbb{Q}(\tau > t)$ by the sequence of functions $(q_n)_{n \in \mathbb{N}}$ defined as

$$q_n(t) := \sum_{k=0}^n \frac{e^{-\omega t} \mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^k]}{e^{z\lambda_0} k!}. \quad (18)$$

We prove now the convergence of this approximation and give a bound for the numerical error.

Proposition 5.2. *Let $z \in [0, \gamma]$. For each $t \geq 0$, the approximation $q_n(t)$ converges towards $q(t)$ as $n \rightarrow +\infty$. Moreover, if $z \leq \gamma/2$, the error $q(t) - q_n(t)$ satisfies the following bounds*

$$\begin{aligned} 0 &\leq q(t) - q_n(t) \\ &\leq \frac{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}] e^{\gamma(\lambda_0 + \kappa\theta t)} + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2} \sqrt{e^{\gamma(\lambda_0 + \kappa\theta t)}}}{e^{\omega t + z\lambda_0} (n+1)!}. \end{aligned}$$

Proof. Note that by positivity of $(\tilde{\lambda}_t)_{t \geq 0}$, the sequence of random variables $(\sum_{j=0}^n \tilde{\lambda}_t^j)_{n \in \mathbb{N}}$ is increasing. As a consequence, the monotone convergence theorem states that

$$\mathbb{E}_{\mathbb{Q}}[e^{\tilde{\lambda}_t}] = \mathbb{E}_{\mathbb{Q}} \left[\lim_{n \rightarrow +\infty} \sum_{j=0}^n \frac{\tilde{\lambda}_t^j}{j!} \right] = \lim_{n \rightarrow +\infty} \sum_{j=0}^n \frac{\mathbb{E}_{\mathbb{Q}}[\tilde{\lambda}_t^j]}{j!},$$

which already proves the convergence of the approximation. We now derive a bound on the error we commit by keeping the $n+1$ first terms in the series. This error is the expected value of $\sum_{k \geq n+1} ((z\lambda_t)^k / k!)$, which satisfies

$$\begin{aligned} \sum_{k=n+1}^{+\infty} \frac{(z\lambda_t)^k}{k!} &= \sum_{k=0}^{+\infty} \frac{(z\lambda_t)^{k+n+1}}{(k+n+1)!} \\ &\leq \frac{(z\lambda_t)^{n+1}}{(n+1)!} \sum_{k=0}^{+\infty} \frac{(z\lambda_t)^k}{k!} \\ &= \frac{(z\lambda_t)^{n+1} e^{z\lambda_t}}{(n+1)!} \end{aligned}$$

where the inequality is a consequence of $(k+n+1)! \geq k!(n+1)!$. The proof is completed by observing that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}e^{z\lambda_t}] &= \mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}] \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}] + \text{Cov}_{\mathbb{Q}}((z\lambda_t)^{n+1}, e^{z\lambda_t}) \\
&\leq \mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}] \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}] + \sqrt{\text{Var}_{\mathbb{Q}}((z\lambda_t)^{n+1})\text{Var}_{\mathbb{Q}}(e^{z\lambda_t})} \\
&\leq \mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}] \mathbb{E}_{\mathbb{Q}}[e^{z\lambda_t}] + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2} \sqrt{\mathbb{E}_{\mathbb{Q}}[e^{2z\lambda_t}]} \\
&\leq \mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}] \mathbb{E}_{\mathbb{Q}}[e^{\gamma\lambda_t}] + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2} \sqrt{\mathbb{E}_{\mathbb{Q}}[e^{\gamma\lambda_t}]} \\
&\leq \mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}] e^{\gamma(\lambda_0 + \kappa\theta t)} + \sqrt{\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}] - (\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}])^2} \sqrt{e^{\gamma(\lambda_0 + \kappa\theta t)}},
\end{aligned} \tag{19}$$

where the last equality follows from Corollary 3.1. $\square$

Note that the moments $\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{2n+2}]$ and $\mathbb{E}_{\mathbb{Q}}[(z\lambda_t)^{n+1}]$ that appear in the upper bound of Proposition 5.2 can be computed with the help of the recursive method of Proposition 5.1.

In the fractional case, we approximate the survival probabilities $t \mapsto \mathbb{Q}(\tau^{(\alpha)} > t)$ by the functions $(q_n^{(\alpha)})_{n \in \mathbb{N}}$ as

$$\begin{aligned}
q_n^{(\alpha)}(t) &:= \sum_{k=0}^n \frac{\mathbb{E}_{\mathbb{Q}}[e^{-wS_t}(z\lambda_{S_t})^k]}{e^{z\lambda_0}k!} \\
&= \sum_{k=0}^n \frac{1}{e^{z\lambda_0}k!} \mathbb{E}_{\mathbb{Q}}[e^{-wS_t} \mathbb{E}_{\mathbb{Q}}[(z\lambda_{S_t})^k | \mathcal{F}_t^S]] \\
&= \sum_{k=0}^n \frac{1}{e^{z\lambda_0}k!} \mathbb{E}_{\mathbb{Q}} \left[e^{-wS_t} \left(c_k + \sum_{j=1}^k d_{j,k} e^{-jaS_t} \right) \right] \\
&= \sum_{k=0}^n \frac{1}{e^{z\lambda_0}k!} \mathbb{E}_{\mathbb{Q}} \left[c_k e^{-wS_t} + \sum_{j=1}^k d_{j,k} e^{-(w+ja)S_t} \right] \\
&= \sum_{k=0}^n \frac{1}{e^{z\lambda_0}k!} \left(c_k \mathbb{E}_{\alpha}(-wt^{\alpha}) + \sum_{j=1}^k d_{j,k} \mathbb{E}_{\alpha}(-(w+ja)t^{\alpha}) \right).
\end{aligned} \tag{20}$$

In a similar way, we can also prove the convergence of the approximation in the fractional case, and give a bound on the error.

Proposition 5.3. *Let $z \in [0, \gamma]$. For each $t \geq 0$, the approximation $q_n^{(\alpha)}(t)$ converges towards $q^{(\alpha)}(t)$ as $n \rightarrow +\infty$. Moreover, if $z \leq \gamma/2$, the error $q^{(\alpha)}(t) - q_n^{(\alpha)}(t)$ satisfies the following bounds*

$$\begin{aligned}
0 &\leq q^{(\alpha)}(t) - q_n^{(\alpha)}(t) \\
&\leq \frac{e^{-z\lambda_0}}{(n+1)!} \left(\mathbb{E}_{\mathbb{Q}}[(z\lambda_{S_t})^{n+1}] e^{\gamma\lambda_0} \mathbb{E}_{\alpha}((\gamma\kappa\theta - w)t^{\alpha}) \right. \\
&\quad \left. + \sqrt{\text{Var}_{\mathbb{Q}}((z\lambda_{S_t})^{n+1}) e^{\gamma\lambda_0} \mathbb{E}_{\alpha}((\gamma\kappa\theta - 2w)t^{\alpha})} \right).
\end{aligned}$$

Proof. The convergence of $q_n^{(\alpha)}(t)$ towards $q^{(\alpha)}(t)$ follows from the monotone convergence theorem.

As in the proof of Proposition 5.2,

$$\sum_{k=n+1}^{+\infty} \frac{(z\lambda_{S_t})^k}{k!} \leq \frac{(z\lambda_{S_t})^{n+1} e^{z\lambda_{S_t}}}{(n+1)!}.$$

It follows that

$$\begin{aligned} q^{(\alpha)}(t) - q_n^{(\alpha)}(t) &\leq \frac{e^{-z\lambda_0}}{(n+1)!} \mathbb{E}_{\mathbb{Q}} [(z\lambda_{S_t})^{n+1} e^{-wS_t + z\lambda_{S_t}}] \\ &\leq \frac{e^{-z\lambda_0}}{(n+1)!} \left(\frac{\mathbb{E}_{\mathbb{Q}} [(z\lambda_{S_t})^{n+1}] \mathbb{E}_{\mathbb{Q}} [e^{-wS_t + z\lambda_{S_t}}]}{+ \sqrt{\text{Var}_{\mathbb{Q}}((z\lambda_{S_t})^{n+1}) \text{Var}_{\mathbb{Q}}(e^{-wS_t + z\lambda_{S_t}})}} \right) \\ &\leq \frac{e^{-z\lambda_0}}{(n+1)!} \left(\frac{\mathbb{E}_{\mathbb{Q}} [(z\lambda_{S_t})^{n+1}] \mathbb{E}_{\mathbb{Q}} [e^{-wS_t + z\lambda_{S_t}}]}{+ \sqrt{\text{Var}_{\mathbb{Q}}((z\lambda_{S_t})^{n+1}) \mathbb{E}_{\mathbb{Q}}(e^{-2wS_t + 2z\lambda_{S_t}})}} \right) \\ &\leq \frac{e^{-z\lambda_0}}{(n+1)!} \left(\frac{\mathbb{E}_{\mathbb{Q}} [(z\lambda_{S_t})^{n+1}] \mathbb{E}_{\mathbb{Q}} [e^{-wS_t + \gamma\lambda_{S_t}}]}{+ \sqrt{\text{Var}_{\mathbb{Q}}((z\lambda_{S_t})^{n+1}) \mathbb{E}_{\mathbb{Q}}[e^{-2wS_t + \gamma\lambda_{S_t}}]}} \right). \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} [e^{-wS_t + \gamma\lambda_{S_t}}] &= \int_0^{+\infty} e^{-w\tau} \mathbb{E}_{\mathbb{Q}} [e^{\gamma\lambda_{\tau}}] g(t, \tau) d\tau \\ &\leq e^{\gamma\lambda_0} \int_0^{+\infty} e^{(\gamma\kappa\theta - w)\tau} g(t, \tau) d\tau \\ &= e^{\gamma\lambda_0} \mathbb{E}_{\alpha}((\gamma\kappa\theta - w)t^{\alpha}) \end{aligned}$$

and similarly

$$\mathbb{E}_{\mathbb{Q}} [e^{-2wS_t + \gamma\lambda_{S_t}}] \leq e^{\gamma\lambda_0} \mathbb{E}_{\alpha}((\gamma\kappa\theta - 2w)t^{\alpha}).$$

The result follows. $\square$

Again, the moments that appear in the upper bound of Proposition 5.3 can be computed with the recursive method of Corollary 5.1.

6 Numerical Experiments

This section presents the numerical results we obtained with the methods presented above. The model is calibrated on real data taken from Bloomberg on 25 January 2021. These data consist of survival probabilities for American Airlines. Note that these data are the same as in [29], so that we can compare the fit for both models. The model is calibrated in both settings, i.e. non-fractional and fractional. The parameters are found by minimizing the sum of squared errors, i.e.

$$\sum_{T=1}^{10} (Prob_T^{\text{Model}} - Prob_T^{\text{Market}})^2$$

where $Prob_T^{\text{Market}}$ denotes the market survival probability up to time T taken from Bloomberg and $Prob_T^{\text{Model}}$ denotes the corresponding survival probability implied by our model. The optimal

Table 1: Parameters used for the computations. NF stands for the non-fractional model whereas F stands for the fractional model.

	θ	σ	η	ρ	κ	λ_0	γ	z	w	α
NF	0.0551	0.1051	9.5599	3.1870	7.7522	0.0551	0.2044	0.1022	0.2434	\
F	0.6751	0.0999	9.9308	2.3752	6.1281	0.6751	0.0760	0.0380	0.3144	0.9083

parameters obtained for both models are given in Table 1 whereas Figure 2 shows the graphical result of the calibration. In particular, Figure 2 shows that the fit is as good as in [29]. It is actually slightly better: when computing the sum of absolute errors

$$\sum_{T=1}^{10} |Prob_T^{\text{Model}} - Prob_T^{\text{Market}}|,$$

we obtain a value⁵ of 0.0989, whereas this value is 0.1055 in [29].

As explained above, this is achieved with a model that is easier to deal with both mathematically and numerically. Table 2 shows the numerical convergence of our method and Table 3 shows the values of the upper bounds that were given at Propositions 5.2 and 5.3. Finally, Figure 3 shows the different shapes we can achieve through the choice of the fractional order α , when all the other parameters are kept unchanged and equal to their values in Table 1.

Regarding the calibration, we proceeded as follows. The parameters $\theta, \eta, \rho, \kappa$ and σ were calibrated with the only constraint of being strictly positive. During the computations, the initial value λ_0 was always kept equal to θ . The parameter α is present in the fractional case only and was calibrated with the constraint of being between 0 and 1. The parameter z was calibrated under the constraint $z \leq \gamma/2$. This constraint was chosen because it allows to obtain upper bounds on the numerical error. We can observe that this constraint is saturated here. Finally, the parameter w was calibrated under the constraint that $w > \gamma\kappa\theta$.

To conclude this section about numerical results, we propose an additional example of calibration to real data. We show that the fractional model is also particularly suited for CDS on countries, and especially countries during war times. This second example is a calibration on default probabilities for the Russian Federation, taken from Bloomberg on the 20th September 2022. The results are displayed at Tables 4 and 5, as well as on Figure 4. We can observe here that the fractional model performs much better than the non-fractional model, the latter being completely unable to reproduce the survival probability shape.

⁵This can be computed from Table 2, with the market data and the line $n = 20$ in the fractional case.

Table 2: Convergence of the computed probabilities and market survival probabilities.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$	$t = 9$	$t = 10$
Market data	0.795400	0.567300	0.457600	0.363300	0.301100	0.249200	0.203200	0.166100	0.133900	0.106100
Non-fractional										
$n = 2$	0.787450	0.617377	0.484019	0.379467	0.297500	0.233237	0.182856	0.143358	0.112392	0.088114
$n = 5$	0.787809	0.617661	0.484242	0.379642	0.297636	0.233345	0.182941	0.143424	0.112443	0.088155
$n = 10$	0.787835	0.617681	0.484258	0.379654	0.297646	0.233352	0.182947	0.143429	0.112447	0.088158
$n = 15$	0.787835	0.617682	0.484258	0.379655	0.297646	0.233353	0.182947	0.143429	0.112447	0.088158
$n = 20$	0.787835	0.617682	0.484258	0.379655	0.297646	0.233353	0.182947	0.143429	0.112447	0.088158
Fractional										
$n = 2$	0.766549	0.588248	0.458572	0.362939	0.291225	0.236678	0.194678	0.161986	0.136285	0.115891
$n = 5$	0.771007	0.592472	0.462013	0.365698	0.293448	0.238485	0.196162	0.163218	0.137318	0.116765
$n = 10$	0.771297	0.592780	0.462269	0.365904	0.293615	0.238621	0.196273	0.163310	0.137395	0.116830
$n = 15$	0.771301	0.592785	0.462274	0.365908	0.293618	0.238623	0.196276	0.163312	0.137396	0.116832
$n = 20$	0.771301	0.592785	0.462274	0.365908	0.293618	0.238623	0.196276	0.163312	0.137396	0.116832

Table 3: Bounds on the numerical error.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$	$t = 9$	$t = 10$
Non-fractional										
$n = 5$	0.0096	0.0079	0.0065	0.0053	0.0043	0.0036	0.0029	0.0024	0.0020	0.0016
$n = 10$	0.0061	0.0050	0.0041	0.0034	0.0028	0.0023	0.0019	0.0015	0.0012	0.0010
$n = 20$	0.0038	0.0031	0.0026	0.0021	0.0017	0.0014	0.0011	0.0009	0.0008	0.0006
$n = 30$	0.0028	0.0023	0.0019	0.0016	0.0013	0.0011	0.0009	0.0007	0.0006	0.0005
$n = 40$	0.0023	0.0019	0.0016	0.0013	0.0010	0.0009	0.0007	0.0006	0.0005	0.0004
Fractional										
$n = 5$	0.0338	0.0377	0.0350	0.0317	0.0286	0.0259	0.0236	0.0216	0.0199	0.0184
$n = 10$	0.0192	0.0239	0.0226	0.0205	0.0185	0.0168	0.0153	0.0140	0.0128	0.0119
$n = 20$	0.0098	0.0146	0.0142	0.0130	0.0118	0.0107	0.0097	0.0089	0.0082	0.0076
$n = 30$	0.0062	0.0108	0.0107	0.0099	0.0090	0.0081	0.0074	0.0068	0.0062	0.0058
$n = 40$	0.0064	0.0087	0.0078	0.0080	0.0073	0.0072	0.0060	0.0055	0.0049	0.0045

Table 4: Russia: market survival probabilities and results of the calibration. M, F and NF respectively stand for the market data, the fractional model and the non-fractional model.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$	$t = 9$	$t = 10$
M	0.3919	0.3735	0.3563	0.3401	0.3247	0.3099	0.2956	0.2818	0.2685	0.2555
F	0.4115	0.3675	0.3427	0.3257	0.3129	0.3026	0.2940	0.2868	0.2804	0.2749
NF	0.7785	0.6049	0.4700	0.3652	0.2837	0.2204	0.1713	0.1331	0.1034	0.0803

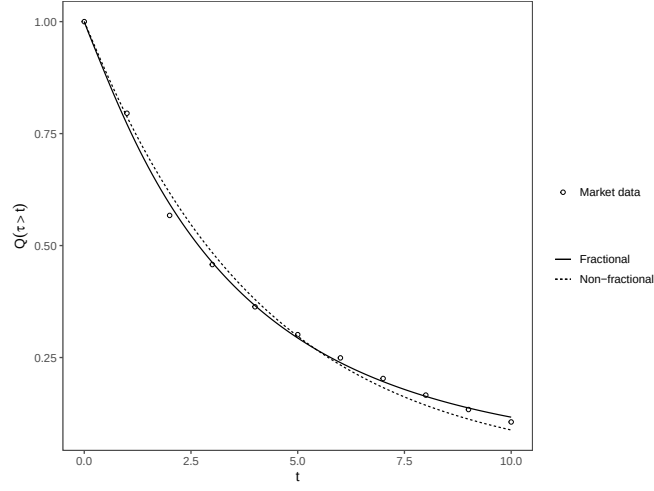


Figure 2: Results of the calibration.

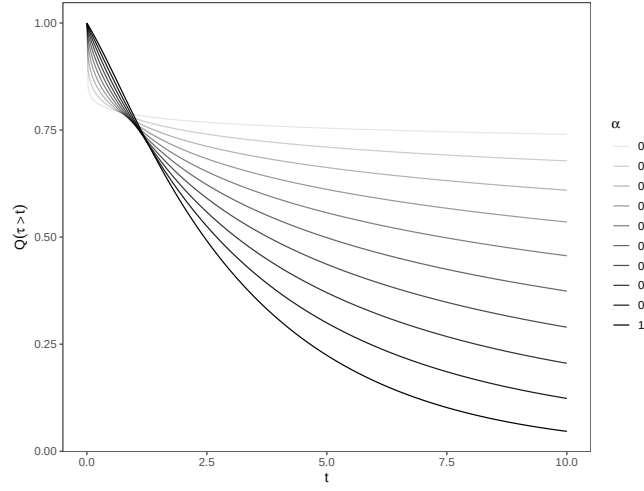


Figure 3: Survival probabilities for various fractional orders α .

Table 5: Calibration on survival probabilities for Russia. NF stands for the non-fractional model and F for the fractional model.

	θ	σ	η	ρ	κ	λ_0	γ	z	w	α
NF	0.0223	0.0999	9.3351	5.2387	7.6154	0.0223	0.4298	0.2149	0.2524	1.000
F	0.9574	0.1014	9.7745	3.6976	6.4232	0.9574	0.2226	0.1113	1.3687	0.253

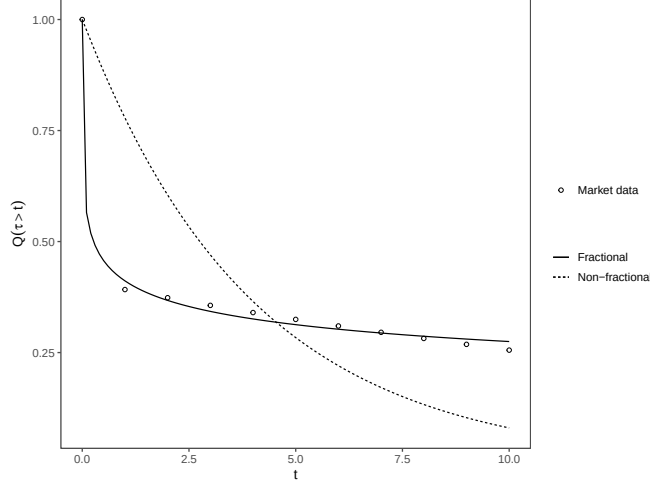


Figure 4: Survival probabilities for Russia and calibrated models.

Conclusions

In this paper, we proposed to adapt the potential approach of interest rates to credit risk. This has been done with a fractional Hawkes jump-diffusion supermartingale. In order to compute the survival probabilities implied by the model, we proposed a new recursive method to compute all the moments of a (fractional) Hawkes process intensity. We provided a proof that our numerical method converges and gave an upper bound on the numerical error. Finally, our numerical experiments showed that our model gives a good fit compared to similar models, but with a better numerical tractability.

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A Appendix

Lemma A.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let T be a positive random variable and $(X_t)_{t \geq 0}$ a positive càdlàg stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a decreasing and continuous function. Denote by $(\mathcal{F}_t^X)_{t \geq 0}$ the natural filtration of $(X_t)_{t \geq 0}$ and by $\sigma(T)$ the σ -algebra generated by T . If the σ -algebras $\bigvee_{t \geq 0} \mathcal{F}_t^X$ and $\sigma(T)$ are independent, and if $\mathbb{E}[X_t] \leq g(t)$ for every $t \geq 0$, then $\mathbb{E}[X_T] \leq \mathbb{E}[g(T)]$.*

Proof. First note that $g(X_T)$ is a well-defined random variable, since it can be written as a composition of measurable maps. As a first step, we show that the result holds if T is a simple random variable. Let $t_1, \dots, t_n$ be its distinct values and $A_i = T^{-1}(\{t_i\})$ for each i . Then we have

$$\begin{aligned}
\mathbb{E}[X_T] &= \sum_{i=1}^n \mathbb{E}[\mathbb{I}_{A_i} X_{t_i}] \\
&= \sum_{i=1}^n \mathbb{P}(A_i) \mathbb{E}[X_{t_i}] \\
&\leq \sum_{i=1}^n \mathbb{P}(A_i) g(t_i) = \mathbb{E}[g(T)],
\end{aligned}$$

where the second equality follows from the independence of $\bigvee_{t \geq 0} \mathcal{F}_t^X$ and $\sigma(T)$. This shows that the statement holds when T is a simple random variable. The second step is to show that the statement holds when T is bounded. Let $(T_n)_{n \in \mathbb{N}}$ be a decreasing sequence of simple $\sigma(T)$ -measurable random variables such that $\lim_{n \rightarrow +\infty} T_n = T$ $\mathbb{P}$ -a.s.. Since $(X_t)_{t \geq 0}$ is right-continuous, we have that $\lim_{n \rightarrow +\infty} X_{T_n} = X_T$ $\mathbb{P}$ -a.s.. As a consequence, using Fatou's lemma, we find

$$\mathbb{E}[X_T] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}[X_{T_n}] \leq \lim_{n \rightarrow +\infty} \mathbb{E}[g(T_n)] = \mathbb{E}[g(T)],$$

where we used the fact that $(g(T_n))_{n \in \mathbb{N}}$ is an increasing sequence of random variables to obtain the existence of $\lim_{n \rightarrow +\infty} \mathbb{E}[g(T_n)]$. The equality was obtained by applying the monotone convergence theorem and using the continuity of g . This shows that the statement holds for bounded random variables and concludes the second step of the proof. The third step is to extend the statement to a random variable T that is not necessarily bounded. For such a T , we have $\lim_{n \rightarrow +\infty} X_{T \wedge n} = X_T$ $\mathbb{P}$ -a.s. so that Fatou's lemma yields

$$\mathbb{E}[X_T] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}[X_{T \wedge n}] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}[g(T \wedge n)],$$

where the second inequality follows from the second step of this proof. The sequence $(\mathbb{E}[g(T \wedge n)])_{n \in \mathbb{N}}$ is clearly decreasing, so that we can write $\mathbb{E}[X_T] \leq \lim_{n \rightarrow +\infty} \mathbb{E}[g(T \wedge n)]$. Finally, since $g(T \wedge n) \leq g(0)$ $\mathbb{P}$ -a.s. for every n , the dominated convergence theorem gives $\mathbb{E}[X_T] \leq \mathbb{E}[g(T)]$, concluding the proof. $\square$

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