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An atypical flat arch in Prato Cathedral: some remarks on the non-standard plastic behaviour of stone structures

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Abstract. *In this contribution the equilibrium of an intriguing case study is studied, which was inspired by the flat arch that crowns the transept portal in the Prato Cathedral. The 'wrong' shape of some voussoirs invites to reflect on the difficulties for the application of limit analysis techniques to determine the collapse conditions.*

Investigations on the minimum friction coefficient compatible with an equilibrium condition are conducted by means of different methods.

At first, a semi-analytical approach combines static and kinematic analyses exploiting both a modern version of the 'stability area method' and the virtual work theorem. Then, the results thus obtained are compared with those derived from a FEA, which aims at simulating the construction process of the flat arch, and from DEM, which capture the collapse mechanisms; results are in a very good agreement. Finally, inconsistencies in adopting a limit state analysis based only on the lower limit theorem are highlighted.

This research stresses the need for a critical viewpoint in the examination of the collapse behavior of masonry structures.

1. Introduction

This study is part of an on-going research project into historical masonry constructions made of interlocking blocks [1]. This project investigates from a mechanical point of view non-conventional structural masonry elements (see for example Fig. 1) in order to understand to which extent different methods are able to provide an accurate description of their structural behaviour.

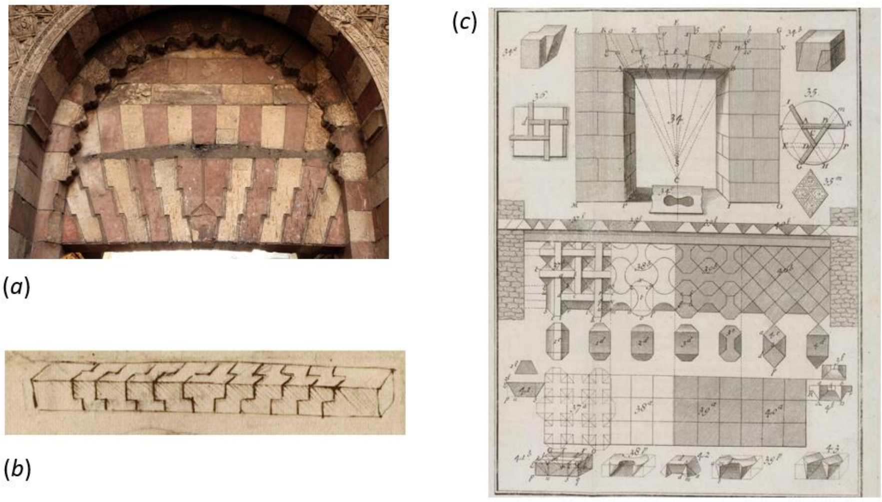


Figure 1: Construction elements with interlocking blocks:
Bāb al-Futuh, Cairo (*a*); Leonardo da Vinci
(Codex Atlanticus) (*b*); A. Frézier [2] (*c*).

Among the examples examined, an intriguing flat arch of architectural interest draws attention due to its seemingly static absurdity. It belongs to the Prato Cathedral's transept portal and is characterized by the peculiar shape of its voussoirs, which consist of two trapezoidal prisms equal and mirrored with respect to each other on the horizontal plane, and joined to form a single block. Ideally, the front and rear trapezoidal prisms can be regarded as two layers that make up the unconventional flat arch (Fig. 2).

Figure 2: The flat arch crowning the transept portal in the
Prato Cathedral (Photo: courtesy of *Piacenti S.p.a.*).

Interlocking of the voussoirs ensures the equilibrium of the actual flat arch (Fig. 2); however, what might happen if the flat arch were instead composed of blocks cut according to the stereotomy of the frontal layer? Such a 'wrong' design invites to reflect on the concept of 'voussoir' and the use of limit analysis approaches to evaluate the collapse conditions when a finite value of the friction coefficient is taken into account.
This contribution focuses on the collapse behaviour of a theoretical flat arch corresponding to the front layer and adopts different approaches to investigate the alleged static illogicality resulting from the atypical voussoirs.

2. Statement of the problem

The flat arch is assumed to be subjected to its self-weight and the weight of a wall above, of height H = 1 m. The constant out-of-plane thickness, the depth of the flat arch, and the middle span are $t = 0.135$ m, $h = 0.505$ m, and $l = 1.075$ m, respectively, see Fig. 3 (left). As regards the mechanical parameters, a compressive strength $\sigma_c = 45$ MPa, and a weight density $\gamma = 26\ ^{\mathrm{kN}}/_{\mathrm{m}^3}$ are assumed. Moreover, investigations are performed by considering a finite value of Coulomb's friction; the friction coefficient is denoted by $\mu = \tan\psi$, where ψ is the friction angle.

The case study asks for insights on the non-standard plastic behaviour of stone structures. Indeed, when sliding occurs according to a non-associative friction model, i.e., in the direction parallel to the internal shear force, the normality rule does not hold, and there can be a gap between the results obtained by means of the upper and lower bound theorems of Limit Analysis, as Drucker [3] first observed.

Given the above, this contribution aims at finding the minimum friction coefficient, μ, able to prevent the arising of collapse mechanisms, by developing non-standard analyses.

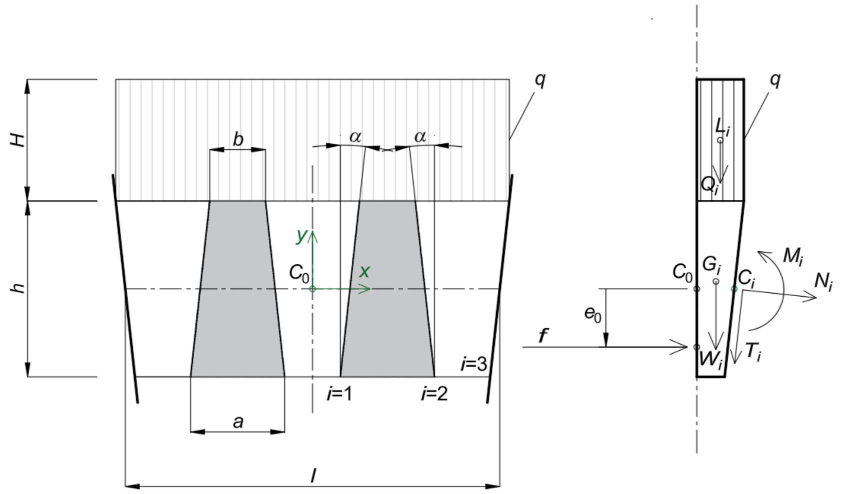


Figure 3: Front layer of the flat arch of the Prato Cathedral: geometric parameters (left); equilibrium of a portion of the flat arch comprised between the ideal midspan section and generic joint i (right).

The unusual shape of the voussoirs is another challenge in the equilibrium analysis of such a structure, since mutual detachment between the blocks could arise.

In order to better visualize the problem, let assume that blocks (*b*) are fixed (Fig. 4, left). It is apparent that block (*a*), delimited by the 'wrong' joints identified by i = 2, could move downward without frictional dissipation at the interfaces, unless a precompression action is applied to the structure.

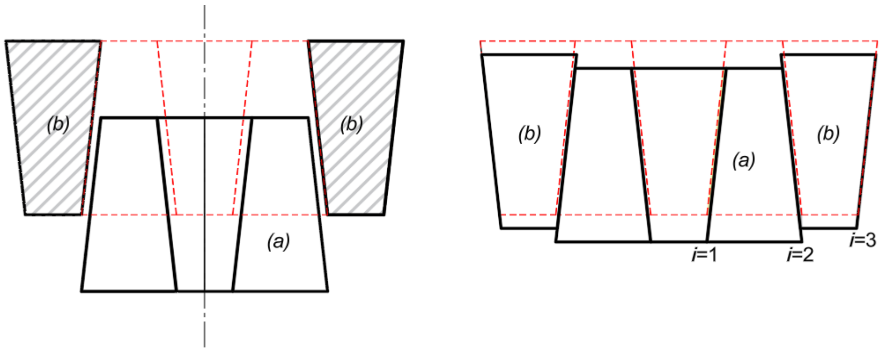


Figure 4: Possible collapse mode when blocks (*b*) are fixed (left); sliding collapse mode (right).

When blocks (b) are not fixed, a mechanism of the type of that plotted in Fig. 4 (right) may arise. In this case, the sliding movement of the outer blocks could have a stabilizing effect. Thus, the minimum friction coefficient that is consistent with the equilibrium of the flat arch is searched: the collapse condition is defined as the one separating equilibrium from starting motion.

3. Analytical and numerical analyses

A first analysis is conducted by means of a semi-analytical integrated approach combining both static and kinematic methods.

The stability area static method, first introduced by Alfred Durand-Claye [4] is revisited to retrieve the statically admissible solutions. Given a finite value for the friction coefficient and for the compressive strength of the material, it is assumed that the collapse results from the reaching of a limit condition at some of the block interfaces, such that a mechanism is identified.

By adapting the procedure and the notation proposed in previous works [5], the rotational domain, A_i^{rot}, and the sliding domain, A_i^{sl}, related to generic i-th joint, are obtained in the (f, e_0) plane, see Fig. 5 (left), where f is the thrust acting at the ideal midspan vertical section, while e_0 is its eccentricity with respect to the center of gravity, C_0 , of such section, see Fig. 3 (right). The rotational domain and the sliding domain associated with the entire flat arch, represented by A^{rot} (see the dashed area), A^{sl} (see the light yellow area), respectively, can be obtained by imposing the equilibrium on the portion of the flat arch comprised between the ideal midspan section and generic joint i (Fig. 3, right), so that the A_i^{rot}, A_i^{sl} domains are obtained for each joint, and, then, accounting for the intersections between all A_i^{rot}, A_i^{sl} domains. In Fig. 5 (left) the statically admissible solutions correspond to the points (f, e_0) included in the stability area A, which is the dashed region in the light yellow one and is delimited by the segment ab, belonging to the boundary of A^{sl}, i.e., to the straight line r identified by the maximum value of the minimum thrust.

For $\mu \geq 1.7772$ the rotational domain A^{rot} is entirely included in the sliding domain A^{sl}; in this case, the straight line delimiting A^{sl} passes through point c (see the dashed vertical line r' in Fig. 5, left), and corresponds to the attainment of a limit condition at joint i = 3; by decreasing the friction coefficient, so that $0.4389 < \mu < 1.7772$, the straight line r moves toward right and cuts the rotational domain, see points a and b; it identifies, again, a local limit condition at joint i = 3. When $\mu = 0.4389$, a limit condition arises: the points included in the segment ab correspond to sliding both at joints i = 2 and i = 3, i.e., to the purely sliding collapse mechanism plotted in Fig. 4 (right). Note that in Fig. 5 (right) the funicular polygons corresponding to points a and b of the stability area are tangent to the friction cone at joints i = 2 and i = 3. When $\mu < 0.4389$, a collapse condition arises. The results are verified by means of kinematic analyses, here omitted for the sake of brevity, and exploiting the virtual work theorem.

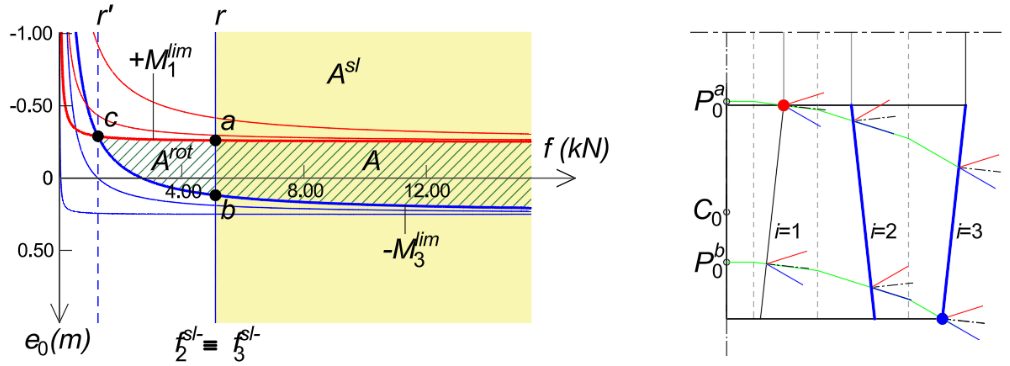


Figure 5: Stability area (left) and thrust lines (right) corresponding to points a and b for H = 1 m, $\mu = 0.4389$. The friction coefficient corresponds to a purely sliding collapse mechanism.

To investigate whether the above conditions could be reached in construction, Finite Element Analyses (FEA) are implemented using the software Abaqus.

The FEA procedure attempts to simulate the flat arch's construction and loading; it takes into account the potential application of a stabilizing pre-compression force by introducing suitable restriction devices; stability is then evaluated by determining whether equilibrium can be reached even after eliminating the pre-compression force. The procedure shows that for $\mu \geq 0.4389$ all the precompression action can be removed. This value of the friction coefficient is in excellent agreement with the one identified by the analytical method. Fig. 6 shows the deformed configuration corresponding to an equilibrium condition after all precompression action has been removed.

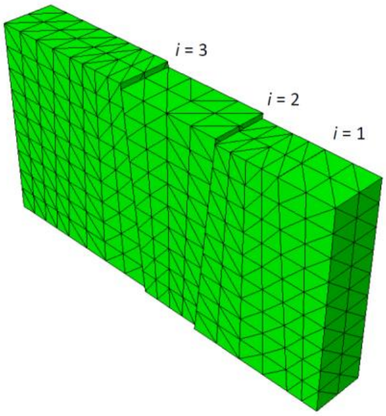
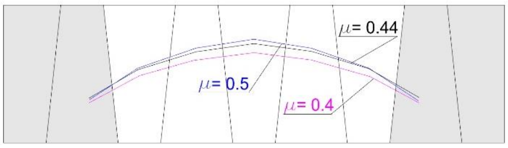


Figure 6: Deformed configuration corresponding to the equilibrium condition after all precompression action has been removed.

Eventually, Discrete Element Modelling (DEM) in a custom masonry formulation integrated with the Non-Smooth Contact Dynamics (NSCD) ProjectChrono simulation engine is used to visualize the equilibrium problem and its consequences on the failure [6]. The governing parameters, timestep Δt = 0.001 s and compliance C = 5e-9 m^2/N, were calibrated based on the case of Figure 4 (left). Analyses are conducted for noteworthy friction values. Fig. 7 (left) shows a comparison of the thrust lines derived from the simulation at the same simulation time T = 0.5 s, for three different values of the friction coefficient, i.e., μ = 0.5, μ = 0.44, μ = 0.4. All friction coefficients resulted in some differential displacement between the central and the end free blocks, as exemplified in Figure 7 (right). The scenario with μ = 0.4 leads to failure, while when $\mu \geq 0.44$, no further displacement is seen. Also, this result is in good agreement with the previous analyses.



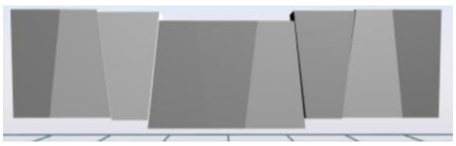


Figure 7. Comparison of the thrust lines corresponding to μ = 0.5, μ = 0.44, μ = 0.4 at T = 0.5 s (left). Snapshot for μ = 0.4 at T = 3 s (right).

Finally, a limit state analysis based on the lower bound theorem was conducted. A multi-objective optimization is set which minimizes the friction coefficient μ and the sum of internal normal forces, under the same external loads. The formulation, disregarding the effects of displacement to the static equilibrium, documents the possibility of equilibrium for the limit case where the friction angle and the tapering angle $\alpha = 6.216°$ of the blocks tend to converge, returning however extremely large precompression forces, Figure 8.

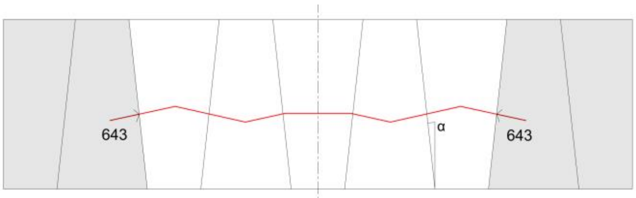


Figure 8. The thrust line resulting from the mathematical programming, friction minimization problem, imposing equilibrium, and resulting forces at supports, expressed in kN.

4. Concluding remarks

The results offer food for thought regarding the non-standard plastic behaviour in the presence of Coulomb friction between voussoirs. Challenges related to the use of a lower bound approach emerge, underlining the need to combine static analyses with kinematic investigations. To find the minimum friction coefficient that guarantees stability, both the domain of statically admissible solutions and the potential onset of collapse mechanisms are examined. The results provided by FEA and DEM help better understand the collapse behaviour of the unconventional flat arch, by resulting in excellent agreement with the outcomes of the analytical method.

Finally, it is noted that the unusual stereotomy of the voussoirs, which may be prone to mutual detachment, is also responsible for the anomalous mechanical behaviour, in addition to the assumption of a finite Coulomb friction. These aspects and their relevance for analysis as well as the mentioned contribution of kinematics will be presented in a forthcoming paper.

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